Resultado

$\underset{\sim}{X}_1, \dots, \underset{\sim}{X}_n$ a.a. de uma distribuição com $E(\underset{\sim}{X}_i) = \underset{\sim}{\mu}$ e $Var(\underset{\sim}{X}_i) = \Sigma$ para $i = 1, \dots, n$

Então

1. $\overline{\underset{\sim}{X}}$ é ENV de $\underset{\sim}{\mu}$
2. $Var(\overline{\underset{\sim}{X}}) = \dfrac{\Sigma}{n}$
3. $\left(\dfrac{n}{n-1}\right) S$ é ENV de Σ.

1. $$E(\underset{\sim}{\bar{X}}) = E\left(\sum_{j=1}^{n} \frac{\underset{\sim}{X}_j}{n}\right) = \frac{1}{n}\sum_{j=1}^{n} E(\underset{\sim}{X}_j) = \frac{n}{n}\underset{\sim}{\mu} = \underset{\sim}{\mu}$$

2. $$\begin{aligned}
Var(\underset{\sim}{\bar{X}}) &= E\left((\underset{\sim}{\bar{X}}-\underset{\sim}{\mu})(\underset{\sim}{\bar{X}}-\underset{\sim}{\mu})^T\right) \\
&= E\left[\left(\sum_{j=1}^{n}\frac{\underset{\sim}{X}_j}{n} - \frac{n}{n}\underset{\sim}{\mu}\right)\left(\sum_{i=1}^{n}\frac{\underset{\sim}{X}_i}{n} - \frac{n}{n}\underset{\sim}{\mu}\right)^T\right] \\
&= \frac{1}{n^2} E\left[\sum_{j=1}^{n}(\underset{\sim}{X}_j-\underset{\sim}{\mu}) \sum_{i=1}^{n}(\underset{\sim}{X}_i-\underset{\sim}{\mu})^T\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \underbrace{E\left[(\underset{\sim}{X}_i-\underset{\sim}{\mu})(\underset{\sim}{X}_j-\underset{\sim}{\mu})^T\right]}_{Cov(\underset{\sim}{X}_i,\,\underset{\sim}{X}_j) \begin{cases} Var(\underset{\sim}{X}_i) & \text{se } i=j \\ 0 & \text{se } i\neq j \end{cases}}
\end{aligned}$$

2. $\operatorname{Var}(\bar{\underset{\sim}{X}}) = \frac{1}{n^2} \sum_{j=1}^{n} \operatorname{Var}(\underset{\sim}{X}_j) = \frac{n\Sigma}{n^2} = \frac{\Sigma}{n}.$

3. $E\left(\frac{n}{n-1} S\right)$

$$S = \frac{1}{n} \sum_{j=1}^{n} (\underset{\sim}{X}_j - \bar{\underset{\sim}{X}})(\underset{\sim}{X}_j - \bar{\underset{\sim}{X}})^T$$

$$= \frac{1}{n}\left[\sum_{j=1}^{n} (\underset{\sim}{X}_j - \bar{\underset{\sim}{X}})\underset{\sim}{X}_j^T - \underbrace{\sum_{j=1}^{n} (\underset{\sim}{X}_j - \bar{\underset{\sim}{X}})}_{=0}\bar{\underset{\sim}{X}}^T\right]$$

$$= \frac{1}{n}\left[\sum_{j=1}^{n} \underset{\sim}{X}_j \underset{\sim}{X}_j^T - n\bar{\underset{\sim}{X}}\bar{\underset{\sim}{X}}^T\right]$$

$\sum_{j=1}^{n} \underset{\sim}{X}_j - n\bar{\underset{\sim}{X}}$ mas $\bar{\underset{\sim}{X}} = \frac{\sum_{j=1}^{n} \underset{\sim}{X}_j}{n}$

$$E(S) = \frac{1}{n} E\left(\sum_{j=1}^{n} \underset{\sim}{x}_j \underset{\sim}{x}_j^T - n \bar{\underset{\sim}{X}} \bar{\underset{\sim}{X}}^T \right) = \frac{1}{n} \left[\sum_{j=1}^{n} \underbrace{E(\underset{\sim}{x}_j \underset{\sim}{x}_j^T)} - n \underbrace{E(\bar{\underset{\sim}{X}} \bar{\underset{\sim}{X}}^T)} \right] = *$$

$$\text{Var}(\underset{\sim}{X}) = E(\underset{\sim}{X} \underset{\sim}{X}^T) - \underset{\sim}{\mu} \underset{\sim}{\mu}^T \Longrightarrow \boxed{E(\underset{\sim}{X} \underset{\sim}{X}^T) = \text{Var}(\underset{\sim}{X}) + \underset{\sim}{\mu} \underset{\sim}{\mu}^T}$$

$$E(\underset{\sim}{x}_j \underset{\sim}{x}_j^T) = \Sigma + \underset{\sim}{\mu} \underset{\sim}{\mu}^T$$

$$E(\bar{\underset{\sim}{X}} \bar{\underset{\sim}{X}}^T) = \frac{\Sigma}{n} + \underset{\sim}{\mu} \underset{\sim}{\mu}^T .$$

$$* = \frac{1}{n} \left[n \Sigma + \cancel{n \underset{\sim}{\mu} \underset{\sim}{\mu}^T} - \cancel{n} \frac{\Sigma}{\cancel{n}} - \cancel{n \underset{\sim}{\mu} \underset{\sim}{\mu}^T} \right] = \frac{1}{n} (n-1) \Sigma .$$

3. $E(S) = \frac{n-1}{n}\Sigma$. S é viesado para Σ.

Logo

$$E\left(\frac{n}{n-1}S\right) = \frac{n}{n-1}E(S) = \frac{\cancel{n}}{\cancel{n-1}}\cdot\frac{\cancel{n-1}}{\cancel{n}}\Sigma = \Sigma.$$

$$E\left(\frac{n}{n-1}S\right) = \Sigma$$

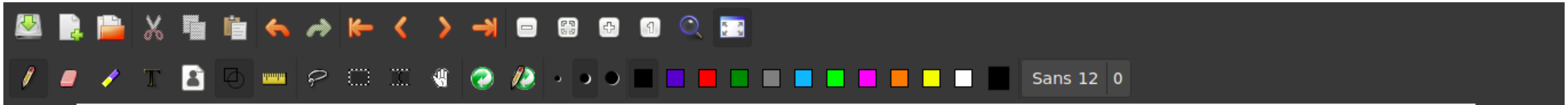
$\therefore \left(\frac{n}{n-1}S\right)$ é ENV para Σ.

Resultado

$\underset{\sim}{X}_1, \dots, \underset{\sim}{X}_n$ a.a. de uma distribuição com $E(\underset{\sim}{X}_i) = \mu$ e $Var(\underset{\sim}{X}_i) = \Sigma$ para $i = 1, \dots, n$

Então

1. $\overline{\underset{\sim}{X}}$ é ENV de μ ✓
2. $Var(\overline{\underset{\sim}{X}}) = \dfrac{\Sigma}{n}$ ✓
3. $\left(\dfrac{n}{n-1}\right) S$ é ENV de Σ. ✓



$\frac{n}{n-1} S$?

$$S = \frac{1}{n}\sum_{j=1}^{n} (\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})^T$$

$$\frac{n}{n-1} S = \frac{\cancel{n}}{n-1} \cdot \frac{1}{\cancel{n}} \sum_{j=1}^{n} (\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})^T$$

De agora em diante,

$$S \doteq \frac{1}{n-1} \sum_{j=1}^{n} (\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})^T \text{ é ENV para } \Sigma .$$

Resultado (revisitado)

$\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ a.a. com $E(\underset{\sim}{X}_j) = \underline{\mu}$ e $Var(\underset{\sim}{X}_j) = \Sigma$, $j = 1, \ldots, n$

1. $\bar{\underset{\sim}{X}} = \frac{1}{n} \sum_{j=1}^{n} \underset{\sim}{X}_j$ é ENV para $\underline{\mu}$

2. $Var(\bar{\underset{\sim}{X}}) = \frac{\Sigma}{n}$

3. $S = \frac{1}{n-1} \sum_{j=1}^{n} (\underset{\sim}{X}_j - \bar{\underset{\sim}{X}})(\underset{\sim}{X}_j - \bar{\underset{\sim}{X}})^T$ é ENV para Σ.