RIGOROUS RESULTS FOR THE HOPFIELD MODEL WITH MANY PATTERNS

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Abstract. We perform a thorough investigation of the main aspects of the Hopfield model with many patterns. Advances are made toward the validity of the "replica symmetric" solution. Strong evidence of the validity of this solution is given over the entire domain where this validity is conjectured; Complete proof is given in a subregion that contains strictly the ergodic region.

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1. Introduction.

The Hopfield model centers on a certain random function defined on the space $\Sigma_N = \{-1, 1\}^N$. An element $\boldsymbol{\epsilon}$ of Σ_N will be called a configuration (because physically it describes a configuration of N spins). The randomness is brought by an independent sequence $(\eta_{i,k})_{i \le N, k \le M}$ of Bernoulli random variables ($P(\eta_{i,k} = 1) = P(\eta_{i,k} = -1) = 1/2$). For $k \le M, \boldsymbol{\eta}_k = (\eta_{i,k})_{i \le N}$ represents a certain configuration. These M random configurations play a special role and are called the *prototypes*. We will consider the quantities (called the *overlaps*).

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$$m_k(\boldsymbol{\epsilon}) = (\frac{1}{N}\sum_{i\le N}\eta_{i,k}\epsilon_i)^2$$

that measure how close $\boldsymbol{\epsilon}$ is from $\boldsymbol{\eta}_k$. The random function of interest (called the Hamiltonian) is

$$H(\boldsymbol{\epsilon}) = -\frac{N}{2}\sum_{k\le M} m_k(\boldsymbol{\epsilon})^2. \tag{1.1}$$

Of course, $H(\boldsymbol{\epsilon})$ depends upon N, M, and the variables $(\eta_{i,k})$. The variables $(\eta_{i,k})$ are thought to be fixed at the beginning of any study of H, and are called the *quenched* variables. All the quantities we will write depend upon the quenched variables; but the dependence almost always remains implicit. On the other hand, when necessary we will indicate the dependence of H in N and M.

The factor $\frac{N}{2}$ is a convenient normalization. The function $H(\boldsymbol{\epsilon})$ physically represents the energy of the configuration $\boldsymbol{\epsilon}$. When $\boldsymbol{\epsilon} = \boldsymbol{\eta}_k$, we have $m_k(\boldsymbol{\epsilon})^2 = 1$, and the corresponding term gives a large contribution. Then (within normalization) $H(\boldsymbol{\epsilon})$ is a particularly simple choice of a function that tries to be small at each prototype. It should be clear then that somehow a system governed by (1.1) "remembers" the prototypes, and, while the present model was apparently introduced by Pastur and Figotin, it is its rediscovery and interpretation by Hopfield as a model for memory that made it popular. (While a discussion of the actual relevance of this model to the inner workings of our brains is better left, say, to [T-D-C], it should be obvious that anything as simple as (1.1) can at best be an extreme simplification).

It will turn out to be necessary to have one of the prototypes play a special role, and for this reason, given $h > 0$, we will generalize (1.1) into

$$H(\boldsymbol{\epsilon}) = -\frac{N}{2}\sum_{k\le M} m_k(\boldsymbol{\epsilon})^2 - hNm_1(\boldsymbol{\epsilon}). \tag{1.2}$$

One could of course distinguish p prototypes, by replacing the last term by $N\sum_{k\le p} h_k m_k(\boldsymbol{\epsilon})$; one could also introduce a term $-h\sum_{i\le N}\epsilon_i$ to represent an "external field". These variations, however, require no new idea, so we feel more appropriate to stick to the simplest case (1.2).

The system governed by (1.2) will be subject to "thermal noise", that is, its properties will be described by the Gibbs measure

$$G(\boldsymbol{\epsilon}) = \frac{2^{-N}}{Z}\exp(-\beta H(\boldsymbol{\epsilon})) \tag{1.3}$$

where $Z = 2^{-N}\sum_{\boldsymbol{\epsilon}}\exp(-\beta H(\boldsymbol{\epsilon}))$ and where the summation is over $\boldsymbol{\epsilon} \in \Sigma_N$. The parameter β in (1.3) physically represents the inverse of the temperature. The lower the temperature, the larger is β, and the more the specific properties of H influence G. This Gibbs measure is the main object of the study of this paper.

The Hopfield model is somewhat connected to a famous model for spin glasses, the Sherrington-Kirkpatrick (SK) model. Both models exhibit, at low temperature, a mysterious "spin glass" phase. An important difference however is that the extra parameter M makes the high temperature phase of the Hopfield model richer, and hence more worthy of study. While writing the present paper, the author realized that some of his methods were already of interest when applied to the technically simpler SK model. This prompted the writing of [T4]. The present paper is almost self-contained; however the key ideas underlying several section are already present in a simplified and more accessible form in [T4].

With the exception of Section 9, all the results of the present paper concern the "physically trivial" range of the parameters of the model (i.e., outside the spin glass phase). For these values, the physicists have been able to discover beautiful formulas [A-G-S] that agree with numerical simulations, and are believed to be correct. The derivation of these formulas relied upon the replica method, that is remarkably far from being mathematically rigorous. Providing rigorous proofs for these results is a challenge, some of which is met in the present paper.

A number of properties of the Hopfield model are better studied as $N \to \infty$. The most interesting case (and the only one that will be studied in this paper) is when $M = M(N)$ grows with N by staying "proportional" to N, the so called case of many patterns. We will follow the tradition to consider the ratio $\alpha = M/N$ as a parameter of the system, even though this notation creates an irresistible urge to treat α as a continuous parameter (and at times to write formally incorrect statements).

We now turn to a detailed description of our main results. Beside the Gibbs measure, another object of prime importance is the *free energy* $F = \log Z$ (although a physicist might use instead $\beta^{-1} \log 2^N Z$). This is a random function, of course, and when need arises to clear ambiguity, we may write $F_N(\alpha, \beta, h)$ rather than F (here, as always, $\alpha = M/N$), or we may specify only some of the parameters. The importance of F stems from the fact that taking derivatives makes Z appear as a denominator. Thus quantities actually physically measurable appear as partial derivatives of F, e.g. $\frac{\partial F}{\partial \beta} = \langle -H(\boldsymbol{\epsilon}) \rangle$ is the average energy of a configuration. Average here means for the Gibbs measure, and, for a function $A : \Sigma_N \to \mathbb{R}$, $\langle A(\boldsymbol{\epsilon}) \rangle$ denotes its integral for the Gibbs measure, i.e.

$$\langle A(\boldsymbol{\epsilon}) \rangle = \frac{1}{2^N Z} \sum_{\epsilon} A(\boldsymbol{\epsilon}) \exp(-\beta H(\boldsymbol{\epsilon})).$$

For simplicity, a quantity such as $\langle A(\boldsymbol{\epsilon}) \rangle$ will be called a *bracket*.

In Section 2, we consider the case $h = 0, \beta(1 + \sqrt{\alpha}) < 1$. Arguments that are specific to this case allow a detailed study.

The free energy, and most of the quantities we study depend upon the quenched variables. To study this dependence we denote by E and P expectation and probability relative to these.

Our first result bears on the fluctuations of F_N.

Theorem 1.1. *Consider α_0, β_0 with $\beta_0(1+\sqrt{\alpha_0}) < 1$, and assume $h = 0$. Then there is a constant K, depending only upon α_0, β_0, with the following property. If $\alpha \le \alpha_0, \beta \le \beta_0$, then we have, for $u > 0$:*

$$P(F_N(\beta) > \frac{M}{2}\log(\frac{1}{1-\beta}) + u) \le e^{-u} \tag{1.4}$$

$$P(F_N(\beta) < \frac{M}{2}\log(\frac{1}{1-\beta}) - u) \le K \exp -\frac{u^2}{K}. \tag{1.5}$$

The proof of this result parallels the proof of [T4], Theorem 2.1. The main ingredients are a second moment calculation (after truncation) and concentration of measure arguments.

It is claimed in [Sca-T] that $F_N(\beta) + \frac{M}{2}\log(1-\beta)$ converges in distribution as $N \to \infty$ to normal (non standard) r.v. This result goes in a somewhat different direction than Theorem 1.1, which presents inequalities true for all N (a formulation better adapted to the potential physical content of the theorem). It was also pointed out to me by two colleagues that the complicated estimates of [Sca-T] are not easy to validate.

Ever present in our topic is the idea of replicas. A *p-replica* is simply product space (Σ_N^p, G^p) (for the same realization of the quenched variables). A prime use of replicas is the possibility to write a product of two brackets as a single bracket by the formula

$$\langle A(\boldsymbol{\epsilon})\rangle\langle B(\boldsymbol{\epsilon})\rangle = \langle A(\boldsymbol{\epsilon})B(\boldsymbol{\epsilon}')\rangle. \tag{1.6}$$

There the bracket on the right represents an integral on (Σ_N^2, G^2), and the generic point of Σ_N^2 is $(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}')$. Formula (1.6) will be called the *replica trick.* It is nothing else than the formula $EXY = EXEY$ valid for independent r.v. The notation (1.6) does not attempt to distinguish whether the bracket represents an integral on Σ_N or Σ_N^p; this should be clear from the context.

Another use of replicas is to define important parameters of the system, such as

$$\tau_N = N^{-2}\langle(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\rangle \tag{1.7}$$

where of course $\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}' = \sum_{i\le N} \epsilon_i \cdot \epsilon_i'$.

There the bracket again means $\int(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2 dG(\boldsymbol{\epsilon})dG(\boldsymbol{\epsilon}')$. For simplicity, we will say that $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}'$ are thermally independent. The idea under (1.7) is (as all great ideas) basically simple. Suppose that it happens that $\boldsymbol{\epsilon}$ points mostly in one single direction (when distributed for G). Then an independent copy $\boldsymbol{\epsilon}'$ will point in the same direction, so $N^{-2}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2$ will often be of order one. On the other hand, τ_N being small means lack of polarization. Quite naturally, this is the case at high temperature, as the following result shows.

Theorem 1.2. *For $\beta_0(1+\sqrt{\alpha_0})<1$, $h=0$, there exists K depending upon α_0,β_0 only such that if $\alpha\le\alpha_0,\beta\le\beta_0$ we have*

$$E\langle\exp\frac{(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2}{KN}\rangle\le K \tag{1.8}$$

and in particular

$$E\exp\frac{N}{K}\tau_N\le K. \tag{1.9}$$

Since the Hamiltonian (1.1) is defined in terms of the overlaps, it is natural to consider the overlap vector $\boldsymbol{m}(\boldsymbol{\epsilon})=(m_k(\boldsymbol{\epsilon}))_{k\le M}$, and the parameter

$$\langle(\boldsymbol{m}(\boldsymbol{\epsilon})\cdot\boldsymbol{m}(\boldsymbol{\epsilon}'))^2\rangle.$$

Theorem 1.3. *For $\beta_0(1+\sqrt{\alpha_0})<1$, $h=0$, there is K depending only upon α_0,β_0 and an event Ω_0 of probability $\ge 1-2^{-N}$ such that*

$$E1_{\Omega_0}<\exp\frac{(\boldsymbol{m}(\boldsymbol{\epsilon})\cdot\boldsymbol{m}(\boldsymbol{\epsilon}'))^2}{KN}>\le K.$$

As the temperature decreases, so does the thermal noise, and at some point (for α small) the influence of the prototype appears. To state our result, for $h\ge 0$, $\beta>0$, we consider the largest root $m^*=m^*(\beta,h)$ of the equation

$$m^*=\mathrm{th}\beta(m^*+h) \tag{1.10}$$

where th denotes the hyperbolic tangent. Thus $m^*=0$ only if $h=0,\beta\le 1$. We denote the canonical basis of $\mathbb{R}^M$ by $(\boldsymbol{e}_k)_{k\le 1}$.

Theorem 1.4. *There exist two numbers L_1,L_2 with the following property. Consider $\beta>0$ and $\alpha\le m^{*4}/L_1$.*

a) If $h=0$, $\beta>1$, consider the set C of configurations $\boldsymbol{\epsilon}$ such that $\boldsymbol{m}(\boldsymbol{\epsilon})$ is NOT within distance $L_2(\alpha/m^{*2})^{1/2}$ of a point $\pm m^*\boldsymbol{e}_k(k\le M)$

b) If $h>0$, consider the set C of configurations $\boldsymbol{\epsilon}$ such that $\boldsymbol{m}(\boldsymbol{\epsilon})$ is NOT within distance $L_2(\alpha/m^{*2})^{1/2}$ of $m^*\boldsymbol{e}_1$.

Then, for some constant K independent of N,

$$E(G(C)) \le K \exp(-N/K).$$

In some sense Theorem 1.4 describes a memory effect since the Gibbs measure is then supported by the union of $2M$ small balls (or even one small ball for $h > 0$).

A result of the same nature (but with worse estimates) was first proved in [B-G-P]. The correct estimates, in the case $h = 0$, were independently announced in [T3] in the case $\beta > 1, \beta - 1$ small and proved for all $\beta > 1$ in [B-G 2]. (This requires an additional simple argument compared to the case $\beta - 1$ small). Actually, Bovier and Gayrard prove Theorem 1.4 by deducing it from deeper and more precise facts. The approach we will use (which is essentially the approach of our first proof) succeeds in avoiding a number of the obstacles that Bovier and Gayrard have to conquer.

Theorem 1.4 will be proved in Section 3. In Section 4, we start to discuss the main topic of the paper, the so called replica-symmetric (RS) solution of the Hopfield model. This "solution" is a set of equation between the main parameters of the model, relations that will be described below. These relations were discovered in [A-G-S] using the replica method. It is a priori not clear what really lies behind these remarkable formulas, and the first purpose of Section 4 is to draw the overall picture, as we see it. The second purpose of Section 4 is to explain what are the underlying ideas of the technical work ahead, and in some sense this section consists in a considerable amplification of the part of the present introduction up to (1.18)

Before proceeding any further, let us write the basic equations of the RS solution.

Consider a standard normal r.v. g, and the system of equations

$$\mu = E\mathrm{th}\beta(g\sqrt{r} + \mu + h) \tag{1.11}$$

$$q = E\mathrm{th}^2\beta(g\sqrt{r} + \mu + h) \tag{1.12}$$

where we have set $r = \alpha q(1 - \beta(1 - q))^{-2}$. Then (hopefully) these equations define two functions μ, q of α, β, h (a fact that is not so obvious and for which we know no reference). The RS solution predicts that

$$\lim_{N\to\infty} N^{-1} EF_N(\alpha, \beta, h) = RS(\alpha, \beta, h) \tag{1.13}$$

where

$$\begin{aligned} RS(\alpha, \beta, h) = -\frac{\mu^2\beta}{2} + \frac{\alpha}{2}\Big(\frac{\beta q}{1 - \beta + \beta q} - \log(1 - \beta + \beta q)\Big) \\ - \beta^2 \frac{r}{2}(1 - q) + E \log \mathrm{ch}\ \beta(g\sqrt{r} + \mu + h) \end{aligned} \tag{1.14}$$

Given a domain D of $\mathbb{R}^3$, we will say that "the RS solution holds in the limit in D" if (1.13) holds for $(\alpha, \beta, h) \in D$.

The main results of the paper rely upon induction over the number N of spins, and iterative use of certain estimates. In Section 5, we learn how to relate a system with $(N+1)$ spins to a system with N spins. We then make a first use of iteration to obtain the following.

Theorem 1.5. *There is $\beta_0 > 0$ such that (for each h) the RS solution holds in the limit for $\beta < \beta_0$.*

The method of proof of Theorem 1.5 is based on iterative estimates of the quantity

$$E_N = E(N^{-2} \sum_{1 \leq i,j \leq N} (\langle \epsilon_i \epsilon_j \rangle - \langle \epsilon_i \rangle \langle \epsilon_j \rangle)^2). \tag{1.15}$$

The basic idea is simply to prove that

$$E_{N+1} \leq C E_N + \text{small term}$$

for $C < 1$; Iteration then yield that E_N is small, a key step in establishing the validity of the RS solution. The method unfortunately produces an irretrievable loss of information that forces restrictive conditions on β. To go beyond Theorem 1.5, one needs rather to estimate iteratively a quantity such as

$$E \sum_{1 \leq k,\ell \leq M} (\langle m_k m_\ell \rangle - \langle m_k \rangle \langle m_\ell \rangle)^2. \tag{1.16}$$

This turns out to be a task of an entirely different magnitude. The main effort of the author went into developing techniques to do this; these techniques are presented in Section 6. These estimates identify leading terms and smaller order (error) terms. The problem then is to control the error terms. Quite interestingly, Theorem 1.4 is of a great help in this direction. The culmination of these efforts will result in the following:

Theorem 1.6. *There exists a number L with the following property. If $h > 0$, and either*

$$\beta \leq 2,\ \alpha \leq \frac{1}{L}(m^{*4} + (1-\beta)^2) \ \textit{or}\ \beta \geq 2,\ \alpha \leq \frac{1}{L \log \beta} \tag{1.17}$$

then the RS solution holds in the limit.

While we do not know how to prove the validity of the RS solution outside the domain of Theorem 1.6, we have succeeded in proving that the Almeida-Thouless conditions

$$\alpha \beta^2 E \mathrm{ch}^{-4} \beta (g\sqrt{r} + \mu + h) < (1 - \beta(1-q))^2 \tag{1.18}$$

occurs in a very natural way. Unfortunatly, it does not seem possible at this stage to give even an informal version of the result that would be intelligible (such an informal version is given in Theorem 4.2), and we urge the reader who has reached this point, but is not interested in proofs to at least glance at Section 4. In fact, we consider the exact identification of condition (1.18) by mathematical methods (rather than by analysis of the eigenvalues of matrices of dimension $\to 0 \cdots$) as the greatest success of the approach that we develop. Most remarkable is the fact that (1.18) occurs as the result of a long computation where over a dozen of terms rather miraculously combine into (1.18).

In Section 9, we investigate the zero temperature case. We give short proofs of (improved versions of) several results of [Lou], concerning the existence of energy barriers, for small α and the collapse of these as $\alpha \to \infty$. While these results apparently are today's state of the art, they rely on somewhat ad-hoc methods and are rather unsatisfactory. (Thus, while it could happen that some methods presented elsewhere in the paper will be of long-lasting use, this is less likely for the results of this section, and the proofs therefore are less detailed). The most frustrating questions concern the evaluation of the minimum of $H(\boldsymbol{\epsilon})$, for which only extremely crude results are known. For example, there is overwhelming numerical evidence that for, some values of α (say $\alpha = \cdot 1$) there are local minima near the prototypes, while the global minimum is not near any prototype, a fact we could not prove.

Estimates for the norm of certain random matrices play an important technical role in the Hopfield model. There is a well established and deep theory of these [S]. On the other hand, one could get confused by the fact that some papers on the Hopfield model have made use of complicated results that are not quite as good as those of [S]. For clarification, we give in an appendix a short self-contained proof of all what we need in this direction.

Now, a few words concerning the style. This paper attempts to be a fully rigorous mathematical paper. There is, however a basic difficulty in the topic: a number of secondary obstacles occur a great many times. These are easy to pass, but the sheer accumulation of routine work needed to handle them in complete detail every time would make the paper impossible to read (and to write). The strategy has been to address in complete detail every such obstacle at its first occurrence. After some point, when it is felt that the reader should be convinced that handling the obstacle is now routine, the obstacle is ignored altogether.

Throughout the paper, we will say that an event occurs with overwhelming probability if the probability that it does not occur is bounded by $\exp(-N/C)$, where C does not depend upon N. We denote by L a universal constant, that may change at each occurrence. When it helps to distinguish these constants they are labeled $L_0, L_1, \cdots$; this labeling remains valid for a few lines only (thus the several constants L_1 occurring at various places are not the same.) In contrast, constants that do not depend upon N, but might depend upon $\alpha, \beta, h, \cdots$ are denoted by K.

Acknowledgments. I am indebted to Pierre Picco for sending the paper [B-G-P] to me, a paper that started my interest in the Hopfield Model, and to D. Loukianova for communicating her thesis, that inspired much of Section 9. And, above all, it must be said that this paper would not have been written without the encouragement of Erwin Bolthausen. (The reader will observe that, as what should have been a three months project ended up only after over a year of very intense struggle, the word "grateful" was omitted from the Acknowledgment).

2. High temperature, no external field.

In this section, we will prove Theorems 1.1 to 1.3. We assume $h = 0$ unless specified otherwise. We fix α_0, β_0 with $\beta_0(1+\sqrt{\alpha_0}) < 1$, and we assume $\alpha \leq \alpha_0, \beta \leq \beta_0$. For simplicity we do not attempt to track the dependence of our bounds upon α_0, β_0, so we denote by K a constant depending only upon α_0, β_0, that may vary at each occurrence.

A large part of the proof of Theorem 1.1 is devoted to elementary moment estimates, that are presented in a series of lemmas that ends with Corollary 2.5.

Lemma 2.1. *For each $\boldsymbol{\epsilon} \in \Sigma_N$ we have*

$$\frac{1}{K}(1-\beta)^{-M/2} \leq E\exp -\beta H(\boldsymbol{\epsilon}) \leq (1-\beta)^{-M/2}. \tag{2.1}$$

Proof. First, we observe that by independence we have

$$E\exp(-\beta H(\boldsymbol{\epsilon})) = \left(E\exp\frac{\beta}{2N}\Big(\sum_{i\leq N}\eta_{i,1}\epsilon_i\Big)^2\right)^M.$$

We now use the fact that if g is $N(0,1)$, for $a \in \mathbb{R}$ we have

$$\exp\frac{a^2}{2} = E\exp ag \tag{2.2}$$

so that

$$\begin{aligned} E\exp\frac{\beta}{2N}\Big(\sum_{i\leq N}\eta_{i,1}\epsilon_i\Big)^2 &= E\exp\sqrt{\frac{\beta}{N}}\sum_{i\leq N}\eta_{i,1}\epsilon_i g \\ &= E\exp N\log \mathrm{ch} g\sqrt{\frac{\beta}{N}} \end{aligned} \tag{2.3}$$

assuming, as we may, that g is independent of the $\eta_{i,1}$, and averaging over these first. Here, of course $\mathrm{ch}x = (e^x + e^{-x})/2$. The elementary inequality $\mathrm{ch}x \leq \exp x^2/2$ yields a bound $E\exp\beta^2 g^2/2 = (1-\beta)^{-1/2}$.

To prove the lower bound in (2.1), we first observe the following elementary result, that we state for further reference.

Lemma 2.2. *The derivatives of the function* $\log chx$ *satisfy*

$$\begin{aligned} &(\log chx)' = thx, (\log chx)'' = (chx)^{-2}, (\log chx)^{(3)} = -\frac{2thx}{ch^2x}, \\ &(\log chx)^{(4)} = 4(chx)^{-2} - 6(chx)^{-4} \leq 4. \end{aligned}$$

In particular, Taylor's formula show that

$$\log \mathrm{ch} x \geq \frac{x^2}{2} - \frac{x^4}{6} \tag{2.4}$$

so that (2.3) yields

$$\begin{aligned}
E \exp \frac{\beta}{2N} \Big(\sum_{i \leq N} \eta_{i,1} \epsilon_i \Big)^2 &\geq E \exp \Big(\frac{\beta g^2}{2} - \frac{\beta^2 g^4}{6N} \Big) \\
&= \frac{1}{\sqrt{2\pi}} \int \exp \Big(- \frac{1-\beta}{2} t^2 - \frac{\beta^2}{6N} t^4 \Big) dt \\
&= \frac{1}{\sqrt{1-\beta}} \int \frac{1}{\sqrt{2\pi}} \exp \Big(- \frac{t^2}{2} - \frac{\beta^2 t^4}{6N(1-\beta)^2} \Big) dt \\
&\geq \frac{1}{\sqrt{1-\beta}} \int \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2}) \Big(1 - \frac{\beta^2 t^4}{6N(1-\beta)^2} \Big) dt \\
&\geq \frac{1}{\sqrt{1-\beta}} \Big(1 - \frac{K}{N} \Big) \geq \frac{1}{\sqrt{1-\beta}} \exp - \frac{K}{N}
\end{aligned}$$

where we have used in the forth line the inequality $e^{-x} \geq 1 - x$. □

Proof of Theorem 1.1. (Upper bound)

The upper bound of (2.1) implies $EZ \leq (1-\beta)^{-M/2}$, from which (1.4) follows by Markov inequality, since $F_N = \log Z$. □

The following lemmas prepare the proof of (1.5) that is much harder.

Since $\beta_0(1+\sqrt{\alpha_0}) < 1$, we have $\alpha_0(\frac{1}{1-\beta_0} - 1)^2 < 1$. Thus we can consider ρ such that $\alpha_0(\rho-1)^2 < 1$ and $\rho > 1/(1-\beta)$. We set $t_0 = M\rho/2$, our truncation level

Lemma 2.3. *We have, for each $\boldsymbol{\epsilon} \in \Sigma_N$*

$$E \exp(-\beta H(\boldsymbol{\epsilon})) 1_{\{-H(\boldsymbol{\epsilon}) \geq t_0\}} \leq (1-\beta)^{-M/2} \exp(-\frac{N}{K}). \tag{2.5}$$

Proof. Using Markov inequality and Lemma 2.1 we get, for $\beta' > \beta$ that the left hand side is bounded by

$$(1-\beta)^{-M/2} \exp - \frac{M}{2} \Big[- \log(1-\beta) + \log(1-\beta') + (\beta' - \beta)\rho \Big].$$

The exponent is

$$\frac{M}{2} \Big(\log \Big(1 - \frac{\beta - \beta'}{1 - \beta'} \Big) + \rho(\beta - \beta') \Big).$$

Taking β' with $\rho = 1/(1-\beta')$ finish the proof, since $\log(1-x) + x < 0$ for $0 < x < 1$.

□

Lemma 2.4. *Consider $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}'$ in Σ_N, and $u = N^{-1}\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}'$.*

Then

$$E\exp(-\beta(H(\boldsymbol{\epsilon}) + H(\boldsymbol{\epsilon}')))1_{\{-H(\boldsymbol{\epsilon})-H(\boldsymbol{\epsilon}')\leq 2t_0\}} \leq (1-\beta)^{-M}\exp\frac{M}{2}(\rho-1)^2u^2. \tag{2.6}$$

Proof. We set $I = \{i \leq N; \epsilon_i = \epsilon'_i\}$; $J = \{i \leq N; \epsilon_i = -\epsilon'_i\}$. Thus

$$\mathrm{card} I = N(1+u)/2, \mathrm{card} J = N(1-u)/2.$$

Use of the formula

$$(x+y)^2 + (x-y)^2 = 2x^2 + 2y^2$$

yields

$$H(\boldsymbol{\epsilon}) + H(\boldsymbol{\epsilon}') = -\frac{1}{N}\left[\sum_{k\leq M}\Big(\sum_{i\in I}\epsilon_i\eta_{i,k}\Big)^2 + \sum_{k\leq M}\Big(\sum_{i\in J}\epsilon_i\eta_{i,k}\Big)^2\right].$$

We write

$$\frac{1}{N} = \frac{1+u}{2\mathrm{card} I} = \frac{1-u}{2\mathrm{card} J}$$

and we use Lemma 2.1 with $\lambda(1+u)$ rather than β and $\mathrm{card} I$ rather than N to get, by independence

$$E\exp -\lambda(H(\boldsymbol{\epsilon}) + H(\boldsymbol{\epsilon}')) \leq \exp M\varphi(\lambda, u) \tag{2.7}$$

where

$$\varphi(\lambda, u) = \frac{1}{2}\log\frac{1}{1-\lambda(1+u)} + \frac{1}{2}\log\frac{1}{1-\lambda(1-u)}.$$

Now, using (2.7) for $\lambda \leq \beta$, and recalling that $2t_0 = \rho M$, we get

$$E\exp -\beta(H(\boldsymbol{\epsilon}) + H(\boldsymbol{\epsilon}'))1_{\{-H(\boldsymbol{\epsilon})-H(\boldsymbol{\epsilon}')\leq 2t_0\}} \leq \exp M((\beta-\lambda)\rho + \varphi(\lambda, u)).$$

Thereby, to finish the proof it suffices to show that

$$\inf_{\lambda\leq\beta}((\beta-\lambda)\rho + \varphi(\lambda, u)) \leq \log\frac{1}{1-\beta} + \frac{1}{2}(\rho-1)^2u^2. \tag{2.8}$$

Let us denote by $h(u)$ the left hand side of (2.8).

Thus

$$h(u) = \rho(\beta - \lambda(u)) + \varphi(\lambda(u), u)$$

where $\lambda(u)$ is given by

$$\rho = \frac{\partial \varphi}{\partial \lambda}(\lambda(u), u). \tag{2.9}$$

Hence

$$h'(u) = \frac{\partial \varphi}{\partial u}(\lambda(u), u).$$

To prove (2.8), it suffices to show that $h'(u) \leq u(\rho - 1)^2$. Recalling (2.9), it suffices to show that

$$\frac{\partial \varphi}{\partial u} \leq u(\frac{\partial \varphi}{\partial \lambda} - 1)^2. \tag{2.10}$$

Setting $D = 1 - 2\lambda + \lambda^2(1 - u^2)$, algebra shows that

$$\frac{\partial \varphi}{\partial u} = \frac{\lambda^2 u}{D}; \quad \frac{\partial \varphi}{\partial \lambda} = \frac{1 - \lambda(1 - u^2)}{D}$$

so that (2.10) becomes

$$\frac{\lambda^2 u}{D} \leq \frac{u\lambda^2}{D^2}(1 + u^2 - \lambda(1 - u^2))^2.$$

i.e.

$$D \leq (1 + u^2 - \lambda(1 - u^2))^2.$$

But this is true because

$$D = (1 - \lambda)^2 - u^2\lambda^2 \leq (1 - \lambda)^2 \leq (1 - \lambda + u^2(1 + \lambda))^2. \qquad \square$$

For a function A on Σ_N, we write

$$E_\epsilon A = 2^{-N} \sum_{\boldsymbol{\epsilon}} A(\boldsymbol{\epsilon})$$

so that in particular $Z = E_\epsilon \exp(-\beta H(\boldsymbol{\epsilon}))$.

Corollary 2.5. *For some $\delta = \delta(\alpha_0, \beta_0)$, we have*

$$E E_\epsilon E_{\epsilon'} \exp(-\beta H(\boldsymbol{\epsilon})) 1_{\{-H(\boldsymbol{\epsilon}) \leq t_0\}} \exp(-\beta H(\boldsymbol{\epsilon}')) 1_{\{-H(\boldsymbol{\epsilon}') \leq t_0\}} \exp \frac{\delta}{N} (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2 \\ \leq K \left(\frac{1}{1-\beta} \right)^M .$$

Proof. We fix $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ in Σ_N; we then see from Lemma 2.4 that

$$E \exp(-\beta H(\boldsymbol{\epsilon})) 1_{\{-H(\boldsymbol{\epsilon}) \leq t_0\}} \exp(-\beta H(\boldsymbol{\epsilon}')) 1_{\{-H(\boldsymbol{\epsilon}') \leq t_0\}} \\ \leq \left(\frac{1}{1-\beta} \right)^M \exp \frac{\alpha}{2} (\rho - 1)^2 \frac{(\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2}{N}$$

because

$$1_{\{-H(\boldsymbol{\epsilon}) \leq t_0\}} 1_{\{-H(\boldsymbol{\epsilon}') \leq t_0\}} \leq 1_{\{-H(\boldsymbol{\epsilon}) - H(\boldsymbol{\epsilon}') \leq 2t_0\}}.$$

We see now that we can choose δ such that $\delta + \alpha(\rho - 1)^2/2 < 1/2$. Then the proof of Lemma 2.1 and the fact that $\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}'$ is distributed like $\sum_{i \leq N} \eta_i$ show that

$$E_\epsilon E_{\epsilon'} \exp \frac{\gamma (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2}{2N} \leq \frac{1}{\sqrt{1-\gamma}}. \qquad \square$$

The next two lemmas prepare to the use of concentration of measure arguments. As we are dealing with Bernoulli r.v., these arguments require a convexification procedure; that is, we extend the definition of H as follows. For $\boldsymbol{y} = (y_{i,k}) \in [-1,1]^{N \times M}$, we define

$$H(\boldsymbol{y}, \boldsymbol{\epsilon}) = -\frac{1}{2N} \sum_{k \leq M} \left(\sum_{i \leq N} y_{i,k} \epsilon_i \right)^2 .$$

We denote by $\|\boldsymbol{y}\|_2$ the euclidean norm of $\boldsymbol{y}$, i.e. $\|\boldsymbol{y}\|_2^2 = \sum_{k,i} y_{i,k}^2$ and by $\|\boldsymbol{y}\|$ the norm of $\boldsymbol{y}$ seen as an operator from $\mathbb{R}^N$ to $\mathbb{R}^M$, i.e.

$$\|\boldsymbol{y}\| = \sup \left\{ \sum_{k,i} u_k v_i y_{i,k}; \sum_{k \leq M} u_k^2 = 1, \sum_{i \leq N} v_i^2 \leq 1 \right\}.$$

Lemma 2.6. *For $\boldsymbol{x}, \boldsymbol{y} \in [-1,1]^{N \times M}$ we have*

$$H(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{\epsilon}) - H(\boldsymbol{y}, \boldsymbol{\epsilon}) = \sum_{i,j \leq N} \epsilon_i \epsilon_j w_{i,j}$$

where $w_{i,j} = w_{j,i}$ *and where*

$$\sum_{i,j \le N} w_{i,j}^2 \le \frac{K}{N^2} \|\boldsymbol{x}\|_2^2 (\|\boldsymbol{y}\|^2 + \|\boldsymbol{x}\|_2^2).$$

Proof. We write

$$H(\boldsymbol{y}, \boldsymbol{\epsilon}) = -\frac{1}{2N} \sum_{i,j \le N} \epsilon_i \epsilon_j \Big(\sum_{k \le M} y_{i,k} y_{j,k} \Big)$$

so that

$$w_{i,j} = -\frac{1}{2N} \sum_{k \le M} (y_{i,k} x_{j,k} + x_{i,k} y_{j,k} + x_{i,k} x_{j,k}).$$

Now

$$\sum_{i,j \le N} \Big(\sum_{k \le M} x_{i,k} y_{j,k} \Big)^2 \le \|\boldsymbol{y}\|^2 \|\boldsymbol{x}\|_2^2$$

and, using Cauchy-Schwarz

$$\begin{aligned} \sum_{i,j \le N} \Big(\sum_{k \le M} x_{j,k} x_{i,k} \Big)^2 &\le \sum_{i,j \le N} \Big(\sum_{k \le M} x_{i,k}^2 \Big) \Big(\sum_{k \le M} x_{j,k}^2 \Big) \\ &= \|\boldsymbol{x}\|_2^2. \end{aligned}$$

□

Given $\boldsymbol{y}$ in $[-1,1]^{N \times M}$, we can consider the corresponding Gibbs measure $G_{\boldsymbol{y}}$ on Σ_N given by

$$G_{\boldsymbol{y}}(\{\boldsymbol{\epsilon}\}) = 2^{-N} Z^{-1} \exp(-\beta H(\boldsymbol{y}, \boldsymbol{\epsilon}))$$

where $Z = Z(\boldsymbol{y}) = E_\epsilon \exp(-\beta H(\boldsymbol{y}, \boldsymbol{\epsilon}))$. Integrals with respect to $G_{\boldsymbol{y}}$ are denoted by $\langle \cdot \rangle_{\boldsymbol{y}}$.

Lemma 2.7. *We have*

$$\langle \exp \beta \sum_{i,j \le N} \epsilon_i \epsilon_j w_{i,j} \rangle_{\boldsymbol{y}} \ge \exp\Big(-\beta \Big(\sum_{i,j \le N} w_{i,j}^2 \Big)^{1/2} \langle (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2 \rangle_{\boldsymbol{y}}^{1/2} \Big).$$

Proof. We use Jensen's inequality in the space $(\Sigma_N, G_{\boldsymbol{y}})$ to get

$$\langle \exp \beta \sum_{i,j \le N} \epsilon_i \epsilon_j w_{i,j} \rangle_{\boldsymbol{y}} \ge \exp \beta \langle \sum_{i,j \le N} \epsilon_i \epsilon_j w_{i,j} \rangle_{\boldsymbol{y}}.$$

and we use Cauchy-Schwarz, since $\sum_{i,j \le N} \langle \epsilon_i \epsilon_j \rangle_{\boldsymbol{y}}^2 = \langle (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')^2 \rangle_{\boldsymbol{y}}$. □

Proof of Theorem 1.1, **Lower bound**

Step 1. We decompose Z as $Z_1 + Y$ where

$$Y = E_\epsilon \exp(-\beta H(\boldsymbol{\epsilon}))1_{\{-H(\boldsymbol{\epsilon})\le t_0\}}$$

is the "main part" of Z and

$$Z_1 = E_\epsilon \exp(-\beta H(\boldsymbol{\epsilon}))1_{\{-H(\boldsymbol{\epsilon})> t_0\}}$$

is small (but badly behaved). To see that Z_1 is small we use that from Lemma 2.5

$$EZ_1 \le (1-\beta)^{-M/2} \exp\Big(-\frac{N}{K}\Big). \tag{2.11}$$

It follows from Lemma 2.1 that

$$EY = EZ - EZ_1 \ge (1-\beta)^{-M/2}\Big(\frac{1}{K} - \exp(-\frac{N}{K})\Big)$$

Since $M/N \le \alpha_0$, considering separately the case where N is small yield

$$EY \ge \frac{1}{K}(1-\beta)^{-M/2}. \tag{2.12}$$

On the other hand, Corollary 2.5 implies that

$$EY^2 \le K(1-\beta)^{-M}. \tag{2.13}$$

We then appeal to the following elementary fact ("Paley-Sygmund inequality"): for any r.v. $Y \ge 0$, we have

$$P(Y \ge \frac{EY}{2}) \ge \frac{1}{4}\frac{(EY)^2}{EY^2}. \tag{2.14}$$

With (2.12), (2.13) this yields

$$P(Y \ge \frac{1}{K}(1-\beta)^{-M/2}) \ge \frac{1}{K}. \tag{2.15}$$

Step 2. In order to use concentration of measure arguments, it is necessary to think of the quenched variables $\boldsymbol{\eta} = (\eta_{i,k})$ as a point of the space $\{-1,1\}^{N\times M}$ provided with the uniform probability. The aim of this step is to show that we can choose K large enough that $P(A) \ge 1/K$, where

$$A = \{\boldsymbol{\eta} \in \{-1,1\}^{N\times M}; Z \ge \frac{1}{K}(1-\beta)^{-M/2}, \langle(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\rangle \le KN;$$
$$\|\boldsymbol{\eta}\| \le K\sqrt{N}\}$$

Here, as before, $\|\boldsymbol{\eta}\|$ is the operator norm.

We consider $\delta > 0$, and we write

$$\begin{aligned}\langle \exp \frac{\delta(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2}{N}\rangle &= Z^{-2}E_\epsilon E_{\epsilon'} \exp(-\beta H(\boldsymbol{\epsilon}) - \beta H(\boldsymbol{\epsilon}') + \frac{\delta}{N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2) \\ &\le Z^{-2}(U_1+U_2)\end{aligned} \tag{2.18}$$

where

$$U_1 = E_\epsilon E_{\epsilon'} 1_{\{-H(\boldsymbol{\epsilon}')\le t_0\}} 1_{\{-H(\boldsymbol{\epsilon})\le t_0\}} \exp(-\beta H(\boldsymbol{\epsilon}) - \beta H(\boldsymbol{\epsilon}') + \frac{\delta}{N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2)$$
$$U_2 = E_\epsilon E_{\epsilon'} (1_{\{-H(\boldsymbol{\epsilon})\ge t_0\}} + 1_{\{-H(\boldsymbol{\epsilon}')\ge t_0\}}) \exp(-\beta H(\boldsymbol{\epsilon}) - \beta H(\boldsymbol{\epsilon}') + \frac{\delta}{N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2).$$

Thus

$$U_2 \le 2Ze^{\delta N} E_\epsilon 1_{\{-H(\boldsymbol{\epsilon})\ge t_0\}} \exp(-\beta H(\boldsymbol{\epsilon})).$$

It then follows from Lemma 2.3 and Corollary 2.5 that we can choose $\delta > 0$ depending only upon α_0, β_0 such that (2.18) implies

$$\langle \exp \frac{\delta}{N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\rangle \le Z^{-2}U_1 + Z^{-1}V_1 \tag{2.19}$$

where $EU_1 \le K(1-\beta)^{-M}, EV_1 \le K(1-\beta)^{-M/2}$.

Now, we know from (2.15) that we can find K_0 such that $P(B) \ge 1/K_0$, where

$$B = \{Z \ge (1-\beta)^{-M/2}/K_0\}$$

and (2.19) implies

$$E(1_B\langle \exp \frac{\delta}{N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\rangle) \le K.$$

Since

$$\exp \frac{\delta}{N}\langle(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\rangle \le \langle \exp \frac{\delta(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2}{N}\rangle$$

and since $\|\boldsymbol{\eta}\| \le K\sqrt{N}$ with probability $\ge 1 - e^{-N/K}$, (see Lemma 10.3) we have shown that $P(A) \ge 1/K$. □

Step 3. Consider $u > 0$ and the set

$$C = \{\boldsymbol{z} \in [-1,1]^{N\times M}; Z(\boldsymbol{z}) \le e^{-u}(1-\beta)^{-M/2}\}.$$

The definition of $Z(\boldsymbol{z})$ shows the all important fact that C is convex. It follows from general principles [T2] that we can find $\boldsymbol{y}$ in $A, \boldsymbol{z}$ in C such that $\boldsymbol{x} = \boldsymbol{z} - \boldsymbol{y}$ satisfies

$$\|\boldsymbol{x}\|_2 \le K\sqrt{\log\frac{1}{P}} \tag{2.20}$$

where

$$P = P(B \cap \{-1,1\}^N) = P(F \leq \frac{M}{2}\log(1-\beta) - u).$$

With the notation of Lemma 2.6 we have

$$\sum_{i,j\leq N} w_{i,j}^2 \leq K\left(\frac{\|\boldsymbol{x}\|_2^2}{N} + \frac{\|\boldsymbol{x}\|_2^4}{N^2}\right).$$

The key observation is that

$$\begin{aligned} Z(\boldsymbol{z}) &= E_\epsilon \exp -\beta H(\boldsymbol{y},\boldsymbol{\epsilon}) \exp \beta(H(\boldsymbol{y},\boldsymbol{\epsilon}) - H(\boldsymbol{z},\boldsymbol{\epsilon})) \\ &= Z(\boldsymbol{y})\langle \exp \beta(H(\boldsymbol{y},\boldsymbol{\epsilon}) - H(\boldsymbol{z},\boldsymbol{\epsilon}))\rangle_{\boldsymbol{y}} \end{aligned}$$

Combining Lemmas 2.6 and 2.7, and using the properties of A, we have

$$Z(\boldsymbol{z}) \geq K^{-1}(1-\beta)^{-M/2}\exp\left(-K\left(\|\boldsymbol{x}\|_2 + \frac{\|\boldsymbol{x}\|_2^2}{\sqrt{N}}\right)^2\right).$$

Since $\boldsymbol{z} \in C$, this implies

$$u - K \leq K\left(\|\boldsymbol{x}\|_2 + \frac{\|\boldsymbol{x}\|_2^2}{\sqrt{N}}\right).$$

Combining with (2.20),

$$u \leq K + K\sqrt{\log\frac{1}{P}} + \frac{K}{N}\log\frac{1}{P}$$

so that $P \leq \max\left(\exp -\left(\frac{u-K}{K}\right)^2, \exp -\frac{N(u-K)}{K}\right)$.

But, since obviously $Z \geq 1$, only the values of $u \leq KN$ matter. Theorem 1.1. follows easily. □

Proof of Theorem 1.2.

We write

$$\begin{aligned} \langle \exp\frac{\delta}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\rangle &\leq \langle \exp\frac{\delta}{N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2\rangle^{1/2} \\ &\leq Z^{-1}U_1^{1/2} + Z^{-1/2}V_1^{1/2} \end{aligned}$$

where U_1, V_1 are as in (2.19). Now (1.5) implies that $EZ^{-2} \leq K(1-\beta)^M$, so use of Cauchy-Schwarz finish the proof. □

Proof of Theorem 1.3.

For further purposes, we will prove a bit more than what is needed for the proof of Theorem 1.3, by allowing the case $h > 0$. Throughout the paper, for a function $A : \Sigma_N \to \mathbb{R}^d (d \leq 1, N, \cdots)$, we write $\dot{A} = A - \langle A \rangle$. Thus $\dot{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon} - \langle \boldsymbol{\epsilon} \rangle$. Note that, by symmetry, $\dot{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}$ when $h = 0$.

Consider the symmetric bilinear form on $(\mathbb{R}^N)^2$ given by $\boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{m}(\boldsymbol{y})$ where

$$\boldsymbol{m}(\boldsymbol{x}) = \left(\frac{1}{N} \sum_{i \leq N} \eta_{i,k} x_i \right)_{k \leq M}.$$

We find an orthonormal basis $(\boldsymbol{v}_p)_{p \leq N}$ of $\mathbb{R}^N$ and numbers $(\lambda_p)_{p \leq N}$ such that

$$\boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{m}(\boldsymbol{y}) = \sum_{p \leq N} \lambda_p (\boldsymbol{v}_p \cdot \boldsymbol{x})(\boldsymbol{v}_p \cdot \boldsymbol{y}). \tag{2.21}$$

Thus for any natural number ℓ, we have

$$\begin{aligned} (\boldsymbol{m}(\dot{\boldsymbol{\epsilon}}) \cdot \boldsymbol{m}(\dot{\boldsymbol{\epsilon}}'))^\ell &= \left(\sum_{p \leq N} \lambda_p (\boldsymbol{v}_p \cdot \dot{\boldsymbol{\epsilon}})(\boldsymbol{v}_p \cdot \dot{\boldsymbol{\epsilon}}') \right)^\ell \\ &= \sum \lambda_{p_1} \cdots \lambda_{p_\ell} \prod_{r \leq \ell} (\boldsymbol{v}_{p_r} \cdot \dot{\boldsymbol{\epsilon}})(\boldsymbol{v}_{p_r} \cdot \dot{\boldsymbol{\epsilon}}'). \end{aligned}$$

Here and below, the summation is over all choices of indexes $p_1, \cdots, p_\ell \leq N$. Using the replica trick, we get

$$\begin{aligned} \langle (\boldsymbol{m}(\dot{\boldsymbol{\epsilon}}) \cdot \boldsymbol{m}(\dot{\boldsymbol{\epsilon}}'))^\ell \rangle &= \sum \lambda_{p_1} \cdots \lambda_{p\ell} \langle (\boldsymbol{v}_{p_1} \cdot \dot{\boldsymbol{\epsilon}}) \cdots (\boldsymbol{v}_{p_\ell} \cdot \dot{\boldsymbol{\epsilon}}) \rangle^2 \\ &\leq \Lambda^\ell \sum \langle (\boldsymbol{v}_{p_1} \cdot \dot{\boldsymbol{\epsilon}}) \cdots (\boldsymbol{v}_{p_\ell} \cdot \dot{\boldsymbol{\epsilon}}) \rangle^2 \end{aligned}$$

where $\Lambda = \max\limits_{p \leq M} |\lambda_p|$. To handle the summation in the last term, we perform the same computation of before, taking now $\lambda_p = 1$ for all p, to see that this sum is

$$\langle (\sum_k (\boldsymbol{v}_k \cdot \dot{\boldsymbol{\epsilon}})(\boldsymbol{v}_k \cdot \dot{\boldsymbol{\epsilon}}))^\ell \rangle = \langle (\dot{\boldsymbol{\epsilon}} \cdot \dot{\boldsymbol{\epsilon}}')^\ell \rangle$$

Thus, for all ℓ,

$$\langle (\boldsymbol{m}(\dot{\boldsymbol{\epsilon}}) \cdot \boldsymbol{m}(\dot{\boldsymbol{\epsilon}}'))^\ell \rangle \leq \Lambda^\ell \langle (\dot{\boldsymbol{\epsilon}} \cdot \dot{\boldsymbol{\epsilon}}')^\ell \rangle \tag{2.22}$$

and power series expansion show that

$$\langle \exp \frac{\delta}{N\Lambda^2} (\boldsymbol{m}(\dot{\boldsymbol{\epsilon}}) \cdot \boldsymbol{m}(\dot{\boldsymbol{\epsilon}}'))^2 \rangle \leq \langle \exp \frac{\delta}{N} (\dot{\boldsymbol{\epsilon}} \cdot \dot{\boldsymbol{\epsilon}}')^2 \rangle.$$

Now, using (2.21) for $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{v}_p$ show that

$$\Lambda \leq \sup\{ \|\boldsymbol{m}(\boldsymbol{x})\|^2; \|\boldsymbol{x}\|_2 \leq 1 \}$$

and, with probability $\geq 1 - 2^{-N}$, this is at most $N^{-1}(1 + K\sqrt{\alpha})^2$. Theorem 1.3 follows. □

We will use the following corollary of the proof.

Corollary 2.8. *There exists a number K such that for all α, β, we have*

$$E\langle(\boldsymbol{m}(\dot{\boldsymbol{\epsilon}}) \cdot \boldsymbol{m}(\dot{\boldsymbol{\epsilon}}'))^2\rangle \leq (1+K\sqrt{\alpha})^4 EN^{-2}\langle(\dot{\boldsymbol{\epsilon}} \cdot \dot{\boldsymbol{\epsilon}}')^2\rangle + \alpha^2 2^{-N} N^2 \tag{2.23}$$

Proof. We take expectation in (2.22) for $\ell = 2$; on the exceptional event $\Lambda \geq N^{-1}(1+K\sqrt{\alpha})^2$ we use the trivial bounds $\Lambda \leq \alpha, \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' \leq N$. □

More results can be proved. For example, Theorem 1.7 of [T4] extends immediately to the present setting. An interesting question is whether Theorem 1.8 of [T4] can be adapted too.

To conclude this section we show that when $\beta < 1$ (whatever the value of α), the overlaps $m_k(\boldsymbol{\epsilon})(k \geq 2)$ are small. In this result, we again allow the case $h \neq 0$.

Lemma 2.9. *If $\beta < 1, h \geq 0$, for $2 \leq k \leq M$ and $u > 0$ we have*

$$E(G(\{m_k(\cdot) \geq u\})) \leq \frac{1}{\sqrt{1-\beta}} \exp(-\frac{N}{4}(1-\beta)u^2). \tag{2.24}$$

Proof. We consider

$$Z_k = E_\epsilon \exp\left(\frac{\beta N}{2} \sum_{\ell \neq k} m_\ell^2(\boldsymbol{\epsilon}) + \beta h N m_1(\boldsymbol{\epsilon})\right)$$

so that $Z \geq Z_k$. Denoting by E_k exception at the variables $\eta_{i,\ell}$ fixed for $\ell \neq k$, it suffices to prove that

$$E_k\left(E_\epsilon 1_{\{m_k(\boldsymbol{\epsilon}) \geq u\}} \exp(-\beta H(\boldsymbol{\epsilon}))\right) \leq \frac{2Z_k}{\sqrt{1-\beta}} \exp(-\frac{N}{4}(1-\beta)u^2). \tag{2.25}$$

Indeed, after dividing by Z_k, since Z_k does not depend upon $(\eta_{k,i})_{i \leq N}$, we see that the left-hand side dominates $E_k(G(\{m_k(\cdot) \geq u\})$. To prove (2.25), for $\beta' > \beta$, we write, using Lemma 2.1

$$\begin{aligned} &E_k 1_{\{m_k(\boldsymbol{\epsilon}) \geq u\}} \exp \frac{\beta N}{2} m_k(\boldsymbol{\epsilon})^2 \\ &\quad \leq \exp(\beta - \beta')\frac{N}{2} E_k \exp \frac{\beta' N}{2} m_k(\boldsymbol{\epsilon})^2 \leq (1-\beta')^{-1/2} \exp(-\frac{N}{2}(\beta' - \beta)) \end{aligned}$$

and we take $\beta' = (1+\beta)/2$. □

3. Emergence of Memory.

In this section, we prove Theorem 1.4.. In order to avoid a number of trivial but confusing difficulties, we assume $\beta \geq 1/2$, and we leave the case $\beta \leq 1/2$ to the reader. One should observe that the case $\beta \leq 1/2, \alpha \leq 1$ is in fact (in principle) completely understood because we will calculate the limit of the free energy in Section 5.

On $\mathbb{R}^M$, we consider the Gaussian probability γ of density $W \exp(-\beta N \|\boldsymbol{z}\|^2/2)$, where W is the normalizing factor $W = (N\beta/2\pi)^{M/2}$. We will use the Hubbard-Stratonovich transformation, that is, we will consider the measure $\overline{G} = G' * \gamma$, where G' is the image of G under the map $\boldsymbol{\epsilon} \to \boldsymbol{m}(\boldsymbol{\epsilon})$. Since γ is sharply concentrated on a ball of radius $\sqrt{\alpha/\beta}$, it suffices to prove that $E\overline{G}(C) \leq K \exp(-N/K)$ for the sets C of Theorem 1.4.

Considering the vectors $\boldsymbol{\eta}_i = (\eta_{i,k})_{k \leq M}$ of $\mathbb{R}^M$, we define the function (depending upon the quenched variables)

$$\psi(\boldsymbol{z}) = -\frac{\beta N}{2}\|\boldsymbol{z}\|^2 + \sum_{i \leq N} \log \operatorname{ch} \beta(\boldsymbol{\eta}_i \cdot \boldsymbol{z} + h\eta_{i,1}) \tag{3.1}$$

where $\|\boldsymbol{z}\|^2 = \sum_{k \leq M} z_k^2$. The following lemma occurs already in [P-F1], and the simple proof is reproduced for the convenience of the reader.

Lemma 3.1. *The density of $\overline{G}$ at $\boldsymbol{z}$ with respect to Lebesgue measure is $WZ^{-1} \exp \psi(\boldsymbol{z})$.*

Proof. This density is

$$\begin{aligned} &WZ^{-1} E_\epsilon \exp\left(\frac{\beta N}{2}\|\boldsymbol{m}(\boldsymbol{\epsilon})\|^2 - \frac{\beta N}{2}\|\boldsymbol{z} - \boldsymbol{m}(\boldsymbol{\epsilon})\|^2 + \beta h N m_1(\boldsymbol{\epsilon})\right) \\ &= WZ^{-1} \exp -\frac{\beta N}{2}\|\boldsymbol{z}\|^2 E_\epsilon \exp(\beta N \boldsymbol{z} \cdot \boldsymbol{m}(\boldsymbol{\epsilon}) + \beta h N m_1(\boldsymbol{\epsilon})) \\ &= WZ^{-1} \exp \psi(\boldsymbol{z}) \end{aligned}$$

where we used the fact that

$$N(\boldsymbol{m}(\boldsymbol{\epsilon}) \cdot \boldsymbol{z} + hNm_1(\boldsymbol{\epsilon})) = \sum_{i \leq N} \epsilon_i(\boldsymbol{\eta}_i \cdot \boldsymbol{z} + h\eta_{i,1}) \qquad \square$$

Our first task is to find lower bounds for Z. We observe that, from Lemma 3.1

$$Z = W \int_{\mathbb{R}^N} \exp \psi(\boldsymbol{z}) d\boldsymbol{z}. \tag{3.2}$$

Consider a number a to be determined later, and $b = \beta(a+h)$. We make the change of variables $\boldsymbol{z} = a\boldsymbol{e}_1 + \boldsymbol{v}$. We make an expansion of $\log \operatorname{ch}(b + \beta t)$ at order 4 using

Lemma 2.2, to get

$$\begin{aligned}(3.3)\qquad \psi^{\sim}(\boldsymbol{v}) =: \psi(\boldsymbol{z}) = & -\frac{\beta N}{2}a^2 + N\log \mathrm{ch} b - \frac{\beta N}{2}\|\boldsymbol{v}\|^2 - \beta N a\boldsymbol{v}\cdot\boldsymbol{\epsilon}_1 \\ & + \mathrm{th} b \sum_{i\le N} \beta \eta_{i,1}(\boldsymbol{\eta}_i\cdot\boldsymbol{v}) + \frac{\beta^2}{2\mathrm{ch}^2 b}\sum_{i\le N}(\boldsymbol{\eta}_i\cdot\boldsymbol{v})^2 \\ & - \frac{\beta^3}{3}\frac{\mathrm{th} b}{\mathrm{ch}^2 b}\sum_{i\le N}\eta_{i,1}(\boldsymbol{\eta}_i\cdot\boldsymbol{v})^3 + \frac{\beta^4}{6}\sum_{i\le N}R_i(\boldsymbol{v})(\boldsymbol{\eta}_i\cdot\boldsymbol{v})^4\end{aligned}$$

where $|R_i(\boldsymbol{v})| \le 1$.

We observe that, for any rotation U of $\mathbb{R}^M$, we have

$$Z = W\int \exp\psi^{\sim}(\boldsymbol{v})d\boldsymbol{v} = W\int \exp\psi^{\sim}(U(\boldsymbol{v}))d\boldsymbol{v}.$$

If we denote by dU the Haar measure on the group of rotations, by Jensen's inequality we see that

$$(3.4)\qquad Z \ge W\int \exp\Big(\int \psi^{\sim}(U(\boldsymbol{v}))dU\Big)d\boldsymbol{v}.$$

The idea there is that the inner integral greatly simplifies the expression (3.4). Indeed, for any vector $\boldsymbol{x}$ of $\mathbb{R}^M$, and $p\in\mathbb{N}$

$$(3.5)\qquad \int (\boldsymbol{x}\cdot U(\boldsymbol{v}))^p dU = c_p\|\boldsymbol{x}\|^p\|\boldsymbol{v}\|^p$$

where the number c_p does not depend on $\boldsymbol{x}$ and $\boldsymbol{v}$. To estimate c_p we apply (3.5) to a Gaussian vector $\boldsymbol{X}$ of covariance matrix the identity and we take expectation, to get

$$c_p E\|\boldsymbol{X}\|^p = Eg^p$$

where g is $N(0,1)$. Thus, $c_1 = c_3 = 0, c_2 = M^{-1}$, and, since $E\|\boldsymbol{X}\|^4 \ge (E\|\boldsymbol{X}\|^2)^2$, we have $c_4 \le 3M^{-2}$.

Thereby since $\|\boldsymbol{\eta}_i\| = M^{1/2}$ for each i we have proved that

$$Z \ge W\exp N\Big(-\frac{a^2\beta}{2} + \log \mathrm{ch} b\Big)\int \exp\Big(-\frac{\beta N}{2}\Big(1-\frac{\beta}{\mathrm{ch}^2 b}\Big)\|\boldsymbol{v}\|^2 - \beta^4\frac{N}{2}\|\boldsymbol{v}\|^4\Big)d\boldsymbol{v}.$$

By change of variable, we see that if $\boldsymbol{X}$ is as above

$$Z \ge \Big(2\pi\Big(1-\frac{\beta}{\mathrm{ch}^2 b}\Big)\Big)^{-M/2}\exp N\Big(-\frac{a^2\beta}{2} + \log \mathrm{ch} b\Big)E\exp\Big(-\frac{\beta^2\|\boldsymbol{X}\|^4}{2N(1-\beta/\mathrm{ch}^2 b)^2}\Big).$$

We apply again Jensen's inequality to the last term, and we use that $E\|\boldsymbol{X}\|^4 \leq LM^2$. As for choosing a, it is appropriate to choose it to maximize the main term

$$-\frac{a^2\beta}{2} + \log \mathrm{ch}\beta(a+h)$$

i.e. $a = \mathrm{th}\beta(a+h)$ that is $a = m^* = m^*(\beta, h)$.

We observe that

$$\frac{1}{\mathrm{ch}^2 b} = 1 - \mathrm{th}^2 b = 1 - \mathrm{th}^2\beta(m^* + h) = 1 - m^{*2}.$$

Also, the derivative of the function $a \to \mathrm{th}\beta(a+h)$ decreases, so that if $m^* > 0$ at $a = m^*$ this derivative must be < 1, so that $\beta(1 - m^{*2}) < 1$. To simplify notation, we set

$$a^* = 1 - \beta(1 - m^{*2}); \quad b^* = -\frac{\beta m^{*2}}{2} + \log \mathrm{ch}\beta(m^* + h) \tag{3.6}$$

(it might provide some insight to observe that b^* is the free energy per site for the Curie-Weiss model).

Throughout the section we consider only the case $m^* > 0$ (i.e. either $\beta > 1$ or $h \neq 0$) so that $a^* > 0$. We have shown the following.

Proposition 3.2. *We have*

$$Z \geq \left(\frac{1}{La^*}\right)^{M/2} \exp\left(Nb^* - \frac{L\beta^2\alpha M}{a^{*2}}\right). \tag{3.7}$$

It should be observed that this bound holds for all values of α, β, and of the quenched variables.

Corollary 3.3. *If*

$$\alpha \leq \frac{a^{*2}}{\beta^2} \tag{3.8}$$

we have

$$Z \geq \left(\frac{1}{La^*}\right)^{M/2} \exp Nb^*.$$

To understand a^* better we note the following.

Lemma 3.4. *We have*

$$\frac{m^{*2}}{L} \le a^*. \tag{3.9}$$

If $\beta > 1$, we have

$$a^* \le 2m^{*2}; \min(1, \beta - 1) \le Lm^{*2}. \tag{3.10}$$

Proof. If $\beta \le 1$, we have $m^{*2} \le 2a^*$. Indeed this means

$$m^{*2} \le 2(1-\beta) + 2\beta m^{*2}.$$

Since $m^{*2} \le 1$, it suffices to distinguish the cases $\beta \le 1/2, \beta \ge 1/2$. We now assume $\beta \ge 1$. First, it is obvious that $m^*(\beta, h)$ increases with h. Next, as $\beta \to 1, m^{*2}(\beta, 0) \sim 3(\beta - 1)$ so that

$$1 - \beta(1 - m^*(\beta, 0)^2) \sim 2(\beta - 1).$$

Also, $1 - \beta(1 - m^*(\beta, 0)^2)$ stays away from zero as β stays away from one. To prove the second part of (3.10), one can assume $h = 0$, and the result is obvious. To prove (3.9), i.e.

$$\frac{m^{*2}}{L} \le 1 - \beta + \beta m^{*2} = a^*$$

it suffices again (if $L \ge 1$) to consider the case $h = 0$, where this is obvious. Now, to prove the first part of (3.10),

$$a^* = 1 - \beta(1 - m^{*2}) \le 2m^{*2}$$

is true if $2m^{*2} \ge 1$. If $m^{*2} \le 1/2$, then

$$a^* = 1 - \beta(1 - m^{*2}) \le 1 - \beta/2$$

so it suffices to consider the case $\beta \le 2$. But then

$$a^* = 1 - \beta(1 - m^{*2}) = 1 - \beta + \beta m^{*2} \le 2m^{*2}. \qquad \square$$

In conclusion, for $\beta \ge 1, a^*$ is of order m^{*2}. (On the other hand, for $\frac{1}{2} < \beta < 1, a^* = 1 - \beta + \beta m^{*2}$ is of order $1 - \beta + m^{*2}$, and, possibly $m^{*2} << 1 - \beta$).

In trying to find upper bounds for ψ, we write, with some loss of information

$$\begin{aligned} \log \mathrm{ch}\beta(\boldsymbol{\eta}_i \cdot \boldsymbol{z} + h\eta_{i,1}) &\le \log \mathrm{ch}\beta(|\boldsymbol{\eta}_i \cdot \boldsymbol{z}| + h) \\ &= \varphi((\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2) \end{aligned} \tag{3.11}$$

where $\varphi(x) = \log \mathrm{ch}\beta(\sqrt{x} + h)$.

Lemma 3.5. *We have*

$$\varphi''(x) \leq -\frac{\beta}{L}\min(1, x^{-3/2}).$$

Proof. If $\varphi(x) = f(\sqrt{x})$, it is straightforward that

$$\varphi''(x) = \frac{1}{4x^{3/2}}(\sqrt{x}f''(\sqrt{x}) - f'(\sqrt{x})).$$

Here $f'(y) = \beta \mathrm{th}\beta(y+h)$, $f''(y) = \beta^2/\mathrm{ch}^2\beta(y+h)$, so that

$$\begin{aligned}\varphi''(x) &= \frac{\beta}{4x^{3/2}}\left(\frac{\beta\sqrt{x}}{\mathrm{ch}^2\beta(\sqrt{x}+h)} - \mathrm{th}\beta(\sqrt{x}+h)\right)\\ &= \frac{\beta}{8x^{3/2}}\left(\frac{2\beta\sqrt{x} - \mathrm{sh}2\beta(\sqrt{x}+h))}{\mathrm{ch}^2\beta(\sqrt{x}+h)}\right)\\ &\leq \frac{\beta}{8x^{3/2}}\left(\frac{2t - \mathrm{sh}2t}{\mathrm{ch}^2 t}\right)\end{aligned}$$

for $t = \beta(\sqrt{x}+h)$. Now, distinguishing the cases $t \leq 1$ and $t \geq 1$, we see that

$$\frac{2t - \mathrm{sh}2t}{\mathrm{ch}^2 t} \leq -\frac{1}{L}\min(t^3, 1)$$

so that

$$\varphi''(x) \leq -\frac{\beta}{L}\min(1, x^{-3/2}) \qquad \square$$

Since $m^* \leq 1$, it follows from Lemma 3.5 and Taylor's formula that

$$\begin{aligned}\varphi((\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2) \leq \varphi(m^{*2}) &+ \varphi'(m^{*2})((\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2 - m^{*2})\\ &- \frac{\beta}{L}\min(1, ((\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2 - m^{*2})^2)\end{aligned} \tag{3.12}$$

Quite conveniently, we have

$$\varphi'(m^{*2}) = \frac{\beta}{2m^*}\mathrm{th}\beta(m^*+h) = \frac{\beta}{2}$$

so that, by summation of the inequalities (3.12), and after regrouping the terms we get

$$\begin{aligned}\psi(\boldsymbol{z}) \leq Nb^* &+ \frac{\beta}{2}\left(\sum_{i\leq N}(\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2 - N\|\boldsymbol{z}\|^2\right)\\ &- \frac{\beta}{L}\sum_{i\leq N}\min(1, ((\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2 - m^{*2})^2).\end{aligned} \tag{3.13}$$

The last term there is crucial. In order to study its influence, we fix $\boldsymbol{z}$ and we write

$$(\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2 = \|\boldsymbol{z}\|^2 + X_i$$

where

$$X_i = \sum_{k \neq \ell} \eta_{i,k}\eta_{i,\ell}z_k z_\ell. \tag{3.14}$$

Thus, if $Y_i = Y_i(\boldsymbol{z}) = ((\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2 - m^{*2})^2$, we have

$$Y_i = (X_i + b)^2$$

for $b = \|\boldsymbol{z}\|^2 - m^{*2}$.

We observe that $EX_i = 0$, so that $EY_i = b^2 + EX_i^2$. We also observe that $EX_i^2 = \sum_{k \neq \ell} z_k^2 z_\ell^2$.

It is general fact that for a r.v. of the type (3.14) we have $EX_i^4 \leq L(EX_i^2)^2$ [Bo]. Thus

$$\begin{aligned} EY_i^2 &= EX_i^4 + 4EX_i^3 b + 6EX_i^2 b^2 + 4EX_i b + b^4 \\ &\leq L[(EX_i^2)^2 + (EX_i^2)^{3/2} b + 6EX_i^2 b^2 + b^4] \\ &\leq L(EX_i^2 + b^2)^2 = L(EY_i)^2. \end{aligned}$$

Use of the Paley-Zygmund inequality (2.14) yields

$$P(Y_i \geq EY_i/2) \geq L^{-1} \tag{3.15}$$

To simplify notation, we set

$$R(\boldsymbol{z})(= EY_i) = (\|\boldsymbol{z}\|^2 - m^{*2})^2 + \sum_{k \neq \ell} z_k^2 z_\ell^2.$$

Lemma 3.6. *Assume $\alpha \leq a^{*2}/\beta, \beta \geq 1/2$. Consider the event Ω_0 of Appendix 2 (Lemma 11-3), and L_0 such that on Ω_0 we have $\|\boldsymbol{m}(\boldsymbol{\epsilon})\| \leq L_0$ for all $\boldsymbol{\epsilon} \in \Sigma_N$ (as provided by (11.4)). Consider a subset A of $\mathbb{R}^M$. In order to prove that*

$$E(\overline{G}(A)) \leq K \exp(-N/K) \tag{3.16}$$

(where K does not depend upon N) we can assume that A is within distance L_0 of the origin, and it suffices to prove that $I(A) \leq K \exp(-N/K)$, where

$$I(A) = (L\beta N a^*)^{M/2} \int_A \exp\left[\frac{\beta}{2}(\sup_{\Omega_0}(\sum_{i \leq N}(\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2 - N\|\boldsymbol{z}\|^2)) - \frac{NR(\boldsymbol{z})}{L}\right] d\boldsymbol{z}. \tag{3.17}$$

Proof. To prove (3.16) it suffices to prove that

$$E(1_{\Omega_0}\overline{G}(A)) \le K\exp(-N/K). \tag{3.18}$$

The ball B of $\mathbb{R}^{\mathbb{N}}$ consisting of points within distance L_0 of the origin satisfies $\overline{G}(B) \ge 1-\exp(-N/K)$, so that we can assume $A \subset B$. Using Lemma 3.1

$$\overline{G}(A) = WZ^{-1}\int_A \exp\psi(\boldsymbol{z})d\boldsymbol{z}.$$

Using Corollary 3.3 and (3.13),

$$1_{\Omega_0}\overline{G}(A) \le (L\beta Na^*)^{M/2}\int_A \exp U(\boldsymbol{z})d\boldsymbol{z}$$

where

$$U(\boldsymbol{z}) = \frac{\beta}{2}\sup_{\Omega_0}\left(\sum_{i\le N}(\boldsymbol{\eta}_i\cdot\boldsymbol{z})^2 - N\|\boldsymbol{z}\|^2\right) - \frac{\beta}{L}\sum_{i\le N}\min(1,Y_i(\boldsymbol{z})).$$

Now, using (3.15)

$$E\exp(-\frac{\beta}{L}\min(1,Y_i(\boldsymbol{z})) \le (1-\frac{1}{L}) + \frac{1}{L}\exp(-\frac{\beta}{L}\min(1,EY_i(\boldsymbol{z})/2)).$$

Also,

$$EY_i(\boldsymbol{z}) = R(\boldsymbol{z}) \le 3\|\boldsymbol{z}\|^4 + 2m^{*4} \le L_1 \tag{3.19}$$

because $A \subset B$. Thus, as $\beta \ge 1/2$

$$E\exp(-\frac{\beta}{L}\min(1,Y_i(\boldsymbol{z}))) \le 1-\frac{1}{L}R(\boldsymbol{z}) \le \exp-\frac{R(\boldsymbol{z})}{L_2}.$$

Thus

$$E(1_{\Omega_0}\overline{G}(A)) \le (L\beta Na^*)^{M/2}\int_A \exp\left[\frac{\beta}{2}\left(\sup_{\Omega_0}(\sum_{i\le N}(\boldsymbol{\eta}_i\cdot\boldsymbol{z})^2 - N\|\boldsymbol{z}\|^2)\right) - \frac{NR(\boldsymbol{z})}{L}\right]d\boldsymbol{z}.$$

□

We will say that a set A that satisfies (3.16) is *negligible*. For a moment we will use the estimate

$$\sup_{\Omega_0}\sum_{i\le N}((\boldsymbol{\eta}_i\cdot\boldsymbol{z})^2 - N\|\boldsymbol{z}\|^2) \le NL\sqrt{\alpha}\|\boldsymbol{z}\|^2 \tag{3.20}$$

of Lemma 10.3, so that

$$I(A) \le (L\beta Na^*)^{M/2}\int_A \exp N\left(\beta L\sqrt{\alpha}\|\boldsymbol{z}\|^2 - \frac{R(\boldsymbol{z})}{L}\right)d\boldsymbol{z}. \tag{3.21}$$

Lemma 3.7. *If $\alpha < a^{*2}/\beta$, the set*

$$A = \{\boldsymbol{z};\ \|\boldsymbol{z}\| \geq c\}$$

is negligible, where

$$c = \max\left(2m^*, L\beta^{1/2}\left(\alpha \log\left(\frac{L\beta a^*}{\sqrt{\alpha}}\right)\right)^{1/4}\right). \tag{3.22}$$

Comment. Here and below, the dependence in β are not important; only crude and simple bounds are used for this.

Proof. For $\|\boldsymbol{z}\| \geq c$, we have $\|\boldsymbol{z}\|^2 - m^{*2} \geq c^2 - m^{*2} \geq 3c^2/4$, so that $R(\boldsymbol{z}) \geq c^2\|\boldsymbol{z}\|^2/2$. Thus, if $c^2 \geq L\sqrt{\alpha}\beta$, we have

$$\begin{aligned} I(A) &\leq (L\beta N a^*)^{M/2} \int_{\|\boldsymbol{z}\| \geq c} \exp -\frac{N}{L} c^2 \|\boldsymbol{z}\|^2 d\boldsymbol{z} \\ &\leq (L\beta N a^*)^{M/2} \exp(-\frac{Nc^4}{2L}) \int \exp -\frac{N\|\boldsymbol{z}\|^2 c^2}{2L} d\boldsymbol{z} \\ &\leq \left(\frac{L\beta a^*}{c^2}\right)^{M/2} \exp\left(-\frac{Nc^4}{2L}\right). \end{aligned}$$

The result follows easily. □

Here we see the importance of the critical number

$$\alpha^* = \frac{m^{*4}}{L_1 \beta^2 \log L_1\left(\frac{a^*\beta^2}{m^{*2}}\right)} \tag{3.23}$$

which is the value of α below which $c = 2m^*$.

To understand α^* better, we note that by (3.10) we could also define

$$\alpha^* = \frac{m^{*4}}{L_1\beta^2 \log L_1\beta^2} \text{ for } \beta > 1$$

$$\alpha^* = \frac{m^{*4}}{L_1} \text{ for } \beta < 1,\ m^{*2} \geq 1 - \beta$$

$$\alpha^* = \frac{m^{*4}}{L_1 \log(L_1(1-\beta)/m^{*2})} \text{ for } \beta < 1,\ m^{*2} \leq 1 - \beta$$

The last two claims result from the fact that, for $\frac{1}{2} < \beta < 1$, $a^* = 1 - \beta + \beta m^{*2}$ is of order $\max(1-\beta, m^{*2})$.

Corollary 3.8. *If $\alpha \le \alpha^*$, the set $\{\boldsymbol{z}; \|\boldsymbol{z}\| \ge 2m^*\}$ is negligible.*

¿From now on, we assume $\alpha \le \alpha^*$. To decide if a set A is negligible, we can assume $\|\boldsymbol{z}\| \le 2m^*$ on A.

Lemma 3.9. *Consider $0 < \xi < 1/2$. Then, if $\|\boldsymbol{z}\| \le 2m^*$,*

$$\forall k \le M,\ \|\boldsymbol{z} \pm m^* \boldsymbol{e}_k\| \ge \xi m^* \Rightarrow R(\boldsymbol{z}) \ge \frac{\xi^2}{256} m^{*4}$$

Proof. If $|\|\boldsymbol{z}\| - m^*| \ge \xi m^*/16$, then

$$\begin{aligned} R(\boldsymbol{z}) \ge (\|\boldsymbol{z}\|^2 - m^{*2})^2 &= (\|\boldsymbol{z}\| - m^*)^2 (\|\boldsymbol{z}\| + m^*)^2 \\ &\ge \xi^2 m^{*4}/256. \end{aligned}$$

Thus we can assume $|\|\boldsymbol{z}\| - m^*| \le \xi m^*/16$. Now

$$\sum_{k \ne \ell} z_k^2 z_\ell^2 = \|\boldsymbol{z}\|^4 - \sum_{k \le M} z_k^4.$$

Assume that for each k we have $z_k^2 \le (1-\delta)\|\boldsymbol{z}\|^2$, where $\delta = \xi^2/8$. Then

$$\sum_{k \le M} z_k^4 \le (1-\delta)\|\boldsymbol{z}\|^2 \sum_{k \le M} z_k^2 = (1-\delta)\|\boldsymbol{z}\|^4$$

so that $\sum_{k \ne \ell} z_k^2 z_\ell^2 \ge \delta \|\boldsymbol{z}\|^4 \ge \delta m^{*4}/4$ and the proof is finished. Now, if $z_k^2 \ge (1 - \xi^2/8)\|\boldsymbol{z}\|^2$, we have $\sum_{\ell \ne k} z_\ell^2 \le \xi^2 \|\boldsymbol{z}\|^2/8$, so that $\|\boldsymbol{z} - z_k \boldsymbol{e}_k\| \le \xi \|\boldsymbol{z}\|/2\sqrt{2} \le \xi m^*/\sqrt{2}$.

Also, since $\|\boldsymbol{z}\| \ge |z_k| \ge (1-\xi^2/8)^{1/2}\|\boldsymbol{z}\|$, we have $\||z_k| - m^*| \le \xi m^*/4$, so that $\|\boldsymbol{z} - m^* \mathrm{sign} z_k \boldsymbol{e}_k\| < \xi m^*$, a contradiction that finishes the proof. □

Lemma 3.10. *We have*

$$\int_{\|\boldsymbol{y}\| \le \theta} d\boldsymbol{y} \le \left(\frac{L\theta^2}{M} \right)^{M/2}$$

Proof. Of course we could use the formula for the volume of a ball. It is however easier to write for $\lambda > 0$

$$\begin{aligned} \int_{\|\boldsymbol{y}\| \le \theta} d\boldsymbol{y} &\le \exp \lambda\theta^2 \int \exp -\lambda \|\boldsymbol{y}\|^2 d\boldsymbol{y} \\ &= \left(\frac{\pi}{\lambda}\right)^{M/2} \exp \lambda\theta^2 \end{aligned}$$

and to take $\lambda = M/2\theta^2$. $\square$

We consider now the set

$$A = \{\boldsymbol{z} \in \mathbb{R}^M ; \|\boldsymbol{z}\| \le 2m^*, \forall k \le M, \|\boldsymbol{z} \pm m^* \boldsymbol{e}_k\| \ge \frac{1}{2} m^*\}. \tag{3.24}$$

Using Lemma 3.9 we now see that when $\alpha \le \alpha^*$ (and the constant L_1 of (3.23) is large enough), (3.21) yield

$$I(A) \le (L\beta N a^*)^{M/2} \exp\Big(-\frac{Nm^{*4}}{L}\Big)\int_A d\boldsymbol{z}.$$

Using Lemma 3.10 for $\theta = 2m^*$, we then get

$$\begin{aligned} I(A) &\le \Big(\frac{L\beta m^{*2} a^*}{\alpha}\Big)^{M/2} \exp\Big(-\frac{Nm^{*4}}{L}\Big) \\ &\le K \exp(-KN) \end{aligned}$$

if the constant of (3.23) is large enough.

The set A of (3.24) is the complement of a union of balls. To be able to reduce the radius of these, we need to improve upon (3.20).

Lemma 3.11. *On the event Ω_0 of Appendix 2, for $\boldsymbol{v}, \boldsymbol{z}$ in $\mathbb{R}^M$ we have*

$$\begin{aligned} \sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2 - N\|\boldsymbol{z}\|^2 &\le L\sqrt{\alpha} N \|\boldsymbol{z} - \boldsymbol{v}\| \|\boldsymbol{v}\| + L\sqrt{\alpha} N \|\boldsymbol{z} - \boldsymbol{v}\|^2 \\ &\quad + \sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{v})^2 - N\|\boldsymbol{v}\|^2. \end{aligned}$$

Proof. Setting $\boldsymbol{y} = \boldsymbol{z} - \boldsymbol{v}$, we have

$$\begin{aligned} \sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2 - N\|\boldsymbol{z}\|^2 &= \sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{y})^2 - N\|\boldsymbol{y}\|^2 + \sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{v})^2 - N\|\boldsymbol{v}\|^2 \\ &\quad + 2\Big(\sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{y})(\boldsymbol{\eta}_i \cdot \boldsymbol{v}) - N\boldsymbol{y} \cdot \boldsymbol{v}\Big). \end{aligned}$$ $\square$

For $p \ge 1$, we consider

$$A_{k,p} = \{\boldsymbol{z} \in \mathbb{R}^M, 2^{-p-1} m^* \le \|\boldsymbol{z} - m^* \boldsymbol{e}_k\| \le 2^{-p} m^*\}.$$

Using Lemma 3.11 for $\boldsymbol{v} = m^* \boldsymbol{e}_k$, and observing that $(\boldsymbol{\eta}_i \cdot \boldsymbol{v})^2 = \|\boldsymbol{v}\|^2$ for all i, we see that

$$\sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{z})^2 - N\|\boldsymbol{z}\|^2 \le L\sqrt{\alpha} N 2^{-p} m^{*2}$$

so that, using (3.17) and Lemma 3.9,

$$I(A_{k,p}) \le (L\beta N a^*)^{M/2} \exp N\Big(L\beta\sqrt{\alpha}2^{-p}m^{*2} - \frac{2^{-2p}m^{*4}}{L}\Big)\int_{\|\boldsymbol{v}\|\le\theta} d\boldsymbol{v}$$

for $\theta = 2^{-p}m^*$. Use of Lemma 3.10 yield

$$I(A_{k,p}) \le \Big(\frac{L\beta a^* 2^{-2p}m^{*4}}{\alpha}\Big)^{M/2} \exp\Big(-\frac{2^{-2p}m^{*4}}{2L}\Big)$$

when $2^{-p} \ge L\beta(\alpha/m^{*2})^{1/2}$. We then see that $I(A_{k,p}) \le K\exp(-N/K)$ whenever

$$2^{-2p} \ge L\beta^2\frac{\alpha}{m^{*2}}\log\frac{L\beta a^*}{m^{*2}}.$$

Thus we have proved the following.

Proposition 3.12. *If the constant L_1 of (3.23) is large enough, and if $\alpha \le \alpha^*$, then the set*

$$C = \{\boldsymbol{z}; \forall k \le M, \|\boldsymbol{z} \pm m^*\boldsymbol{e}_k\| \ge \Big(L\frac{\beta^2\alpha}{m^{*2}}\log\frac{L\beta a^*}{m^{*2}}\Big)^{1/2}\}$$

is negligible.

Since $a^*/m^{*2} \le 2$, this proves Theorem 4.1 when $h = 0, \beta$ stays bounded. The weakest point of this result is the poor dependence in β; but this is unimportant, since we will use a different argument for β large.

In the case $h > 0$, we will prove that the set C can be replaced by

$$C_0 = \{\|\boldsymbol{z}\|; \|\boldsymbol{z} - m^*\boldsymbol{e}_1\| \ge \Big(L\frac{\beta^2\alpha}{m^{*2}}\log\frac{L\beta a^*}{m^{*2}}\Big)^{1/2}\}. \tag{3.25}$$

The difficulty there is that when $\beta \ge 1$, h can be arbitrarily small compared to m^*. As discovered in [B-G-P], concentration of measure provides the answer.

Lemma 3.13. *Assume $h = 0$. Then for any set $A \subset \mathbb{R}^M$, there exists a number μ such that*

$$0 \le t \le N \Rightarrow P(|\log\overline{G}(A) - \mu| \ge t) \le 2\exp\Big(-\frac{Nt^2}{K}\Big).$$

To provide motivation, we first prove the following.

Proposition 3.14. *If $0 < h < m^*/L$, we can replace C by C_0 in Proposition 3.12.*

Proof. Consider, for $k \le M, \eta \in \{-1, 1\}$, the ball $B_{k,\eta}$ centered at $\eta m^* \boldsymbol{e}_k$ of radius δ. Let us denote by $\overline{G}_0$ the measure corresponding to the case $h = 0$ (for the same value of the quenched variables). Then, by symmetry, the distribution of $\overline{G}_0(B_{k,\eta})$ does not depend upon k, η.

Thus, there is μ such that

$$P(\forall k, \eta, |\log \overline{G}_0(B_{k,\eta}) - \mu| \ge t) \le 2N \exp\left(-\frac{Nt^2}{K}\right). \tag{3.26}$$

Let us now try to compare $\overline{G}_0(B_{k,\eta})$ with $\overline{G}(B_{k,\eta})$. Given $\boldsymbol{y}$ with $\|\boldsymbol{y}\| \le m^*/6$, we have, assuming the radius of $B_{k,\eta}$ to be at most $m^*/6$ that

$$\begin{aligned} \boldsymbol{z} \in B_{1,1} + \boldsymbol{y} &\Rightarrow z_1 \ge 2m^*/3 \\ \boldsymbol{z} \in B_{1,-1} + \boldsymbol{y} &\Rightarrow z_1 \le -2m^*/3 \\ k \ne 1, \boldsymbol{z} \in B_{k,\eta} + \boldsymbol{y} &\Rightarrow z_1 \le m^*/3. \end{aligned}$$

The influence of the term $Nhm_1(\boldsymbol{\epsilon})$ on the Hamiltonian implies that

$$(k,\eta) \ne (1,1) \Rightarrow G'(B_{k,\eta} + \boldsymbol{y}) \le \exp(Nhm^*/3) G'_0(B_{k,\eta} + \boldsymbol{y}) \tag{3.27}$$

$$G'(B_{1,1} + \boldsymbol{y}) \ge \exp(2Nhm^*/3) G'_0(B_{1,1} + \boldsymbol{y}). \tag{3.28}$$

where G' (resp. G'_0) is the image of G (resp. G_0) under the map $\boldsymbol{\epsilon} \to \boldsymbol{m}(\boldsymbol{\epsilon})$.

Now, we recall the Gaussian measure γ defined at the beginning of this section. We have

$$\gamma(\|\boldsymbol{y}\| \ge m^*/6) \le \exp(-\frac{Mm^{*2}}{L\alpha}) = \exp(-\frac{Nm^{*2}}{L})$$

and thus, by integration of (3.27), (3.28) with respect to γ we have

$$(k,\eta) \ne (1,1) \Rightarrow \overline{G}(B_{k,\eta}) \le \exp(Nhm^*/3) \overline{G}_0(B_{k,\eta}) + \exp\left(-\frac{Nm^{*2}}{L}\right). \tag{3.29}$$

$$\overline{G}(B_{1,1}) \ge \exp(2Nhm^*/3)\left(\overline{G}_0(B_{1,1}) - \exp\left(-\frac{Nm^{*2}}{L}\right)\right). \tag{3.30}$$

Now, by (3.26), the event

$$\forall k, \eta, |\log \overline{G}_0(B_{k,\eta}) - \mu| \le Nhm^*/12 \tag{3.31}$$

has a probability at least $1 - K\exp(-N/K)$. Under (3.31), we have from (3.29), (3.30)

$$(k,\eta) \ne (0,1) \Rightarrow \overline{G}(B_{k,\eta}) \le \exp\left(\frac{5Nhm^*}{12}\right) e^{\mu} + \exp\left(-\frac{Nm^{*2}}{L}\right) \tag{3.32}$$

$$\overline{G}(B_{1,1}) \ge \exp\left(\frac{7Nhm^*}{12}\right)\left(e^{\mu} - \exp\left(-\frac{Nm^{*2}}{L}\right)\right). \tag{3.33}$$

Thus, from (3.33), since $\overline{G}$ is a probability,

$$e^{\mu} \leq \exp\left(\frac{-7Nhm^*}{12}\right) + \exp\left(-\frac{Nm^{*2}}{L}\right)$$

and, using $h \leq m^*/L$, this is at most $2\exp(-Nhm^*/2)$.

Plugging in (3.32) we get

$$\bigcup_{(k,\eta)\neq(1,1)} \overline{G}(B_{k,\eta}) \leq 4N\exp\left(-\frac{Nhm^*}{L}\right)$$

and this finishes the proof. □

Proof of Lemma 3.13. We first observe that it has been shown in [T1], Theorem 6.8 that if $t \leq N$, we have (when $\alpha \leq 1$)

$$P(|\beta^{-1}\log Z - \mu_0| \geq t) \leq 12\exp\left(-\frac{t^2}{LN}\right)$$

for a certain μ_0. Thus, it suffices to show that for any subset A of $\mathbb{R}^N$, such that $\|\boldsymbol{z}\| \leq L$ for $\boldsymbol{z} \in A$, we have, for a certain μ_A,

$$P(|\beta^{-1}\log\int_A \exp\psi(\boldsymbol{z})d\boldsymbol{z} - \mu_A| \geq t) \leq \exp\left(-\frac{t^2}{NL}\right).$$

For $\boldsymbol{w} \in [-1,1]^{N\times M}$, we define

$$\psi(\boldsymbol{w},\boldsymbol{z}) = -\frac{\beta N}{2}\|\boldsymbol{z}\|^2 + \sum_{i\leq N}\log\operatorname{ch}\beta\Big(\sum_{k\leq M} w_{i,k}z_k + hw_{i,1}\Big).$$

This is a convex function of $\boldsymbol{w}$, and so is

$$\boldsymbol{w} \to \int_A \exp\psi(\boldsymbol{w},\boldsymbol{z})d\boldsymbol{z}$$

so that

$$f(\boldsymbol{w}) = \beta^{-1}\log\int_A \exp\psi(\boldsymbol{w},\boldsymbol{z})d\boldsymbol{z}$$

is such that the sets $\{f(\cdot) \leq u\}$ are convex. Moreover, since $\log\operatorname{ch}$ has a derivative ≤ 1, we have

$$|\psi(\boldsymbol{w},\boldsymbol{z}) - \psi(\boldsymbol{w}',\boldsymbol{z})| \leq (\|\boldsymbol{z}\| + h)\sqrt{N}\|\boldsymbol{w} - \boldsymbol{w}'\|_2$$

and thus

$$f(\boldsymbol{w}) - f(\boldsymbol{w}') \leq (L+h)\sqrt{N}\|\boldsymbol{w} - \boldsymbol{w}'\|_2.$$

The conclusion then follows from [T2], Theorem 6.6. □

Unfortunately, Proposition 3.14 requires $h < m^*/L$. However, the difficulty was the case h small. The case where h is comparatively large can be handled by separate arguments, that we start now. We make the change of variable $\boldsymbol{z} = m^*\boldsymbol{e}_1 + \boldsymbol{v}$, and from (3.1) we get

$$\begin{aligned}\psi(\boldsymbol{z}) = &-\frac{\beta N}{2} m^{*2} - \frac{\beta N}{2}\|\boldsymbol{v}\|^2 - \beta N m^* \boldsymbol{v}\cdot\boldsymbol{e}_1 \\ &+ \sum_{i\le N} \log \mathrm{ch}\beta(m^* + h + \eta_{i,1}\boldsymbol{\eta}_i\cdot\boldsymbol{v})\end{aligned} \tag{3.34}$$

To take care of the last summation, we observe the elementary inequality, true for $b > 0$ and all x

$$\log \mathrm{ch} x \le x^2 \frac{\mathrm{th} b}{2b} + \log \mathrm{ch} b - \frac{b}{2}\mathrm{th} b$$

that we rewrite as

$$\log \mathrm{ch}(b + \beta x) \le \log \mathrm{ch} b + x\beta \mathrm{th} b + \frac{x^2\beta^2}{2b}\mathrm{th} b. \tag{3.35}$$

We use this for $b = \beta(m^* + h)$ so that $\mathrm{th} b = m^*$, and we get

$$\begin{aligned}\psi(\boldsymbol{z}) \le N b^* &- \frac{\beta N}{2}\|\boldsymbol{v}\|^2 + \beta m^* \Big(\sum_{i\le N} \eta_{i,1}\boldsymbol{\eta}_i\cdot\boldsymbol{v} - N\boldsymbol{e}_1\cdot\boldsymbol{v}\Big) \\ &+ \frac{\beta m^*}{2(m^* + h)}\sum_{i\le N}(\boldsymbol{\eta}_i\cdot\boldsymbol{v})^2,\end{aligned}$$

where b^* is defined in (3.6), so that on the event Ω_0 of Appendix 2,

$$\begin{aligned}\psi(\boldsymbol{z}) &\le -N b^* + L\beta m^* \sqrt{\alpha}\|\boldsymbol{v}\| \\ &\quad - \frac{\beta N}{2}\Big[L\sqrt{\alpha} - \frac{h}{m^* + h}\Big]\|\boldsymbol{v}\|^2.\end{aligned}$$

Consider, for $R > 0$, the set

$$C = \{\boldsymbol{z}; R \le \|\boldsymbol{v}\| = \|\boldsymbol{z} - m^*\boldsymbol{e}_p\| \le 2R\}.$$

Proceeding as usual, and since we assume $h \ge m^*/L$, we see that

$$E(1_{\Omega_0}\overline{G}(C)) \le (\frac{La^*\beta R^2}{\alpha})^{M/2} \exp N\Big(L\beta m^*\sqrt{\alpha}R - \frac{\beta R^2}{2L}\Big)$$

so that we can take R as small as $Lm^*\sqrt{\alpha}$, and have C negligible.

It turns out that

$$m^*\sqrt{\alpha} \le \Big(\frac{\alpha}{m^{*2}} \log \frac{La^*}{m^{*2}}\Big)^{1/2} \tag{3.36}$$

since $m^* \le 1$. Thus we have proved the following.

Proposition 3.15. *If $h > 0, \alpha \le \alpha^*$ the set*

$$C = \{\boldsymbol{z}, \|\boldsymbol{z} - m^* \boldsymbol{e}_1\| \le L \left(\frac{\alpha \beta^2}{m^{*2}} \log \frac{L\beta a^*}{m^{*2}} \right)^{1/2} \}$$

is negligible.

Proof of Theorem 1.4. Proposition 9.2 (to be proved in Section 9) shows that given a constant L_0, that there is β_0, α_0, such that if $\beta > \beta_0, \alpha < \alpha_0$, then $E(\overline{G}(C)) \le K \exp -N/K$, where

$$C = \{\boldsymbol{z}; \forall k \le M, \|\boldsymbol{z} \pm m^* \boldsymbol{e}_k\| \ge \frac{1}{L_0}\}. \tag{3.37}$$

Moreover, for $\beta \ge \beta_0$, we can assume $m^* \ge 1/2$.

We observe that if $\beta \le \beta_0$, Proposition 3.12 (when $h = 0$) and 3.13 (when $h \ne 0$) prove Theorem 1.4. We consider the case $\beta \ge \beta_0$, and $h = 0$. It suffices to show that in (3.37) the radius L_0^{-1} can be replaced by $L\sqrt{\alpha}$. We have to find a sustitute for (3.3) (where $a = m^*, b = \beta(m^* + h)$). For a lower bound, we simply use the convexity of $\log \mathrm{ch} x$. For an upper bound, we observe that

$$\begin{aligned} \log \mathrm{ch}(b + \beta t) \le \log \mathrm{ch} b + \beta t \mathrm{th} b + \frac{\beta^2}{2\mathrm{ch}^2(\beta m^*/2)} t^2 \\ + \frac{\beta}{2} t^2 1_{\{t| \ge m^*/2\}}. \end{aligned} \tag{3.38}$$

The case $|t| \ge m^*/2$ follows from (3.35), while the case $|t| \le m^*/2$ follows from Taylor's formula, since, if $|t| \le m^*/2$, we have $(\log \mathrm{ch}(b + \beta t))'' \le \beta^2 \mathrm{ch}^{-2}(\beta m^*/2)$. We use this for $t = \eta_{i,1}(\boldsymbol{\eta}_i \cdot \boldsymbol{v})$ to get by summation from (3.34) that

$$\begin{aligned} \psi(m^* \boldsymbol{e}_1 + \boldsymbol{v}) \le N b^* - \frac{\beta N}{2} \|\boldsymbol{v}\|^2 + N \beta m^* \Big(\sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{e}_1)(\boldsymbol{\eta}_i \cdot \boldsymbol{v}) - \boldsymbol{e}_1 \cdot \boldsymbol{v} \Big) \\ + \frac{\beta^2}{2\mathrm{ch}^2(\beta m^*/2)} \sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{v})^2 + \frac{\beta}{2} \sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{v})^2 1_{\{(|\boldsymbol{\eta}_i \cdot \boldsymbol{v}| \ge m^*/2\}}. \end{aligned}$$

Since $\beta \ge \beta_0, m^* \ge 1/2$, we can assume $\beta/\mathrm{ch}^2(\beta m^*/2) \le 1/4$. On the event Ω_0 of Appendix 1, we have

$$\sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{v})^2 \le 2N \|\boldsymbol{v}\|^2 \le 2N/L_0^2$$

where L_0 is the constant of (3.36). Thus if

$$J = \{i \le N; (\boldsymbol{\eta}_i \cdot \boldsymbol{v})^2 \ge \frac{m^{*2}}{4} \ge \frac{1}{16}\},$$

we have card$J \le 32N/L_0^2$. If $32/L_0^2$ is smaller than the constant δ_0 of Lemma 11.4, on the event Ω_1 of this lemma we then have

$$\sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{v})^2 1_{\{|\boldsymbol{\eta}_i \cdot \boldsymbol{v}| \ge m^*/2\}} \le N\|\boldsymbol{v}\|^2/2.$$

Combining these gives

$$\psi(m^*\boldsymbol{e}_1 + \boldsymbol{v}) \le Nb^* - \frac{\beta N}{8}\|\boldsymbol{v}\|^2 + L\sqrt{\alpha}\|\boldsymbol{v}\|.$$

By a routine computation already done many times, it follows that we can replace L_0 by $L\sqrt{\alpha}$ in (3.37).

It now remains only to consider the case $h \ne 0, \beta > \beta_0$. If $h \le m^*/L$, we deduce it from the case $h = 0$, using the proof of Proposition 3.14, and (3.36) if $h \ge m^*/L$.

Theorem 1.4 is proved. □

4. The issues of the Replica-Symmetric solution.

The first attempt to justify the RS solution by rigorous means is to be found in [P-S-T]. The authors assume that

$$\mathrm{Var} N^{-1} \sum_{i \le N} \langle \epsilon_i \rangle^2 \to 0 \tag{4.1}$$

and try to derive from this condition the equations (1.11), (1.12), where μ, q, r are natural parameters of the system. The rather subjective matter of as to which extend the arguments given there are correct, complete and rigorous is better left to the reader's own appreciation, but, in order to avoid confusion, we are at least obliged to say that in our opinion this paper misses a number of points. Even some of the lesser problems are not so easily addressed, and taking care of these will require a rather significant part of our future effort.

Why consider condition (4.1)? It seems to us that the main reason is historical; the main motivation is the physicists' prediction that (4.1) **fails** at low temperature (in the "spin glass" region). Thus it is natural from this point of view to assume that (4.1) hold as a condition to ensure that we are outside the spin glass region and then prove that the RS solution holds. (The much more delicate question of deciding when (4.1) actually hold is then left by [P-S-T] for future research.)

Let us now consider the condition

$$E \frac{1}{N^2} \langle (\dot{\boldsymbol{\epsilon}} \cdot \dot{\boldsymbol{\epsilon}}')^2 \rangle \to 0. \tag{4.2}$$

We recall that here, as well as as in the rest of the paper we use the notation $\dot{A}$ to mean $A - \langle A \rangle$, where A is a map from Σ_N to $\mathbb{R}^d_N$.

Even though this is certainly not apparent at this stage, there is a very close link between (4.1) and (4.2). It does not seem to be known how to show the equivalence of (4.1) and (4.2) unless one uses the (somewhat mysterious) technique of "perturbated Hamiltonians". It is simply for this reason that the authors of [P-S-T] consider the Hamiltonian

$$H(\boldsymbol{\epsilon}) = H_0(\boldsymbol{\epsilon}) - \gamma_1 \sqrt{N} \sum_{2 \le k \le M} g_k m_k(\boldsymbol{\epsilon}) - \gamma_2 \sum_{i \le N} g'_i \epsilon_i \tag{4.3}$$

where $H_0(\boldsymbol{\epsilon})$ is given by (1.1), and where $(g_k)_{k \le M}, (g'_i)_{i \le N}$ are independent Gaussian sequences. The last two terms are intended to be small perturbation terms with limited influence on the limit of the free energy per site. The last perturbation term of (4.3) allows to prove the equivalence of (4.1) and (4.2), following a technique that will be used on several occasions in Appendix 1.

Consider now the condition

$$E \langle (\dot{\boldsymbol{m}} \cdot \dot{\boldsymbol{m}}')^2 \rangle \to 0. \tag{4.4}$$

Here, as well as in the rest of the paper, we simplify notation by thinking of $\boldsymbol{m}(\boldsymbol{\epsilon})$ as a function $\boldsymbol{m}$ on Σ_N, of which $\boldsymbol{m}'$ is a (thermally) independent copy. It is proved in [P-S-T] (and in Corollary 2.8) that (4.2) implies (4.4), and a simple argument will be given in Section 5 to show the converse. The reason for the first perturbation term in (4.3) is that it allows to prove the equivalence of (4.4) and of the technically useful fact that

$$\operatorname{Var}(\sum_{k\leq M} \dot{m}_k^2) \to 0, \tag{4.5}$$

(with the same method as for the equivalence of (4.1) and (4.2)). However, we must insist that:

(4.5) Condition (4.4) (or, equivalently, (4.2)) **is the central feature of the RS solution**.

Let us now explain this statement. The basis of our approach is to try to calculate all quantities of interest by induction upon N. The first step is that program, Proposition 5.1 below, is simple algebra. This proposition brings to light the importance of the quantities $\langle \exp t\boldsymbol{\eta}\cdot\boldsymbol{m}\rangle$, where $\eta = \langle \eta_k\rangle_{k\leq M}$ is independent of all the other random sequences. It is obviously very helpful to know how to approximate such a quantity, an idea already central to [P-S-T] (lemma 2.2 there)

Lemma 4.1. *For $t \leq 1$, we have*

$$\langle \exp t\boldsymbol{\eta}\cdot\boldsymbol{m}\rangle = \exp(t\boldsymbol{\eta}\cdot\langle\boldsymbol{m}\rangle + \frac{t^2}{2}\langle\|\dot{\boldsymbol{m}}\|^2\rangle + R_N) \tag{4.6}$$

where

$$\text{(4.7)}\quad E|R_N| \leq K((E\langle(\dot{\boldsymbol{m}}\cdot\dot{\boldsymbol{m}}')^2\rangle)^{1/2} + E\langle\sum_{k\leq M}\dot{m}_k^4\rangle + E\langle(\|\dot{\boldsymbol{m}}\|^2 - \langle\|\dot{\boldsymbol{m}}\|^2\rangle)^2\rangle + K\exp(-N/K)$$

With a little more effort one could replace the term $(E\langle(\dot{\boldsymbol{m}}\cdot\dot{\boldsymbol{m}}')^2\rangle)^{1/2}$ by its square, but this makes little difference under (4.4). It turns out from general principles (also used in [P-S-T]) that the term $E\langle(\|\dot{\boldsymbol{m}}\|^2 - \langle\|\dot{\boldsymbol{m}}\|^2\rangle)^2\rangle$ has a vanishing contribution so that, under (4.4), $E|R_N| \to 0$ provided

$$E\langle\sum_{k\leq M}\dot{m}_k^4\rangle \to 0. \tag{4.8}$$

It turns out that $E\dot{m}_1^2 \to 0$ is easy to get, and we will consider the condition

$$E\langle\sum_{2\leq k\leq M} m_k^4\rangle \to 0 \tag{4.9}$$

that is thus essentially stronger than (4.8).

Not only (4.9) allows to show that

$$\langle \exp t\boldsymbol{\eta}\cdot\boldsymbol{m}\rangle \simeq \exp(t\boldsymbol{\eta}\cdot\langle\boldsymbol{m}\rangle + \frac{t^2}{2}\langle\|\dot{\boldsymbol{m}}\|^2\rangle)$$

it implies that conditionally upon all the other r.v. variables other than $\boldsymbol{\eta}$, the quantity $\sum_{k\geq 2}\eta_k\langle m_k\rangle$ is essentially gaussian, and (4.7) gives us a really complete description of the r.v. $\langle \exp t\boldsymbol{\eta}\cdot\boldsymbol{m}\rangle$.

One can then use Proposition 4.1 to obtain a relation of the type

$$(\mu_{N+1}, U_{N+1}, R_{N+1}) = \Psi_{\alpha,\beta,h}(\mu_N, U_N, R_N) + \text{ small error} \tag{4.10}$$

where μ_N, R_N, U_N are some important parameters of the system (e.g. $\mu_N = E\langle m_1\rangle$), where N denotes the number of sites, and where $\Psi_{\alpha,\beta,h}$ is some explicit function.

Suppose now that we know that there is a point (μ, u, r) (depending upon α, β, h) such that the following occurs

$$(\mu, u, r) = \Psi_{\alpha,\beta,h}(\mu, u, r). \tag{4.11}$$

(4.12) There is a neighborhood V of (μ, u, r), such that if $(\mu_1, u_1, r_1) \in V$ and $(\mu_{n+1}, u_{n+1}, r_{n+1}) = \Psi_{\alpha,\beta,h}(\mu_n, u_n, r_n)$, the sequence (μ_n, u_n, r_n) converges to (μ, u, r).

Assume moreover that for some N we can prove $(\mu_N, U_N, R_N) \in V$. Then we are in very good position to iterate (4.10), and to be able to *prove* the convergence of (μ_N, U_N, R_N) towards (μ, u, r). Equations (1.11) and (1.12) are then a transcription of (4.11). There is then a simple heuristic argument (given after Proposition 7.10) to understand the full result (1.14).

While the function $\Psi_{\alpha,\beta,h}$ is rather explicit, it is not simple, and the range of values of (μ, u, r) where (4.11) and (4.12) hold is by no means obvious. In the range of Theorem 1.5, it is easily checked that $\Psi_{\alpha,\beta,h}$ is a contraction. In the range of Theorem 1.6, we will provide an ad-hoc (infinitely tedious) argument. Condition (4.12) is of particular interest. It amounts to say that the eigenvalues of the differential of Ψ at the fixed point (4.11) are of absolute value < 1. While deciding for which values of the parameters this holds is in principle elementary mathematics, I could not muster the energy to do it.

Before we discuss the critical condition (4.8) and (4.9), we prove Lemma 4.1. We will not explicitly use this lemma, because the relation (4.10) will follow almost immediately from the machinery developed to prove (4.4). Still the proof is very instructive, and contains a simple occurrence of many of the basic ideas of this paper. It relies upon a simple second moment computation that allows considerable simplification of previous arguments, such as those used in [P-S-T]. (The reader might rather find, at first glance, that what we do is more complicated than the argument of [P-S-T], but must keep in mind that we aim at a different level of rigor).

Proof of Lemma 4.1. We consider an independent copy $\boldsymbol{\eta}'$ of $\boldsymbol{\eta}$, and we set

$$X = \langle \exp t\boldsymbol{\eta} \cdot \dot{\boldsymbol{m}} \rangle, X' = \langle \exp t\boldsymbol{\eta}' \cdot \dot{\boldsymbol{m}} \rangle, \tag{4.13}$$

so that

$$E_{\eta,\eta'}(X - X')^2 = 2E_\eta(X - E_\eta X)^2. \tag{4.14}$$

We consider a (thermally) independent copy $\boldsymbol{m}'$ of $\boldsymbol{m}$ and we use the replica trick to write

$$\begin{aligned}(X - X')^2 &= \langle \exp t\boldsymbol{\eta} \cdot (\dot{\boldsymbol{m}} + \dot{\boldsymbol{m}}') \rangle + \langle \exp t\boldsymbol{\eta}' \cdot (\dot{\boldsymbol{m}} + \dot{\boldsymbol{m}}') \rangle \\ &\quad - 2\langle \exp t(\boldsymbol{\eta} \cdot \dot{\boldsymbol{m}} + \boldsymbol{\eta}' \cdot \dot{\boldsymbol{m}}') \rangle\end{aligned}$$

so that

$$E_{\eta,\eta'}(X - X')^2 = 2\langle \exp \frac{t^2}{2}(\|\dot{\boldsymbol{m}}\|^2 + \|\dot{\boldsymbol{m}}'\|^2)(\exp(t^2\dot{\boldsymbol{m}} \cdot \dot{\boldsymbol{m}}' + t^4 B_1) - \exp t^4 B_2)) \rangle$$

where, for $j = 1, 2$, $|B_j| \le L\Big(\sum\limits_{k \le M} \dot{\boldsymbol{m}}_k^4 + \dot{\boldsymbol{m}}_k'^4\Big)$.

Now, there is an event Ω_0 (described in Lemma 11.3) in the quenched variables such that $\|\dot{\boldsymbol{m}}\|^2, \|\dot{\boldsymbol{m}}\|^2 \le K$ when this event occurs, while $P(\Omega_0) \ge 1 - Ke^{-N/K}$. Using the bound $|e^x - 1| \le |x|e^{|x|}$, we then see that on Ω_0

$$E_{\eta,\eta'}(X - X')^2 \le K\Big(\langle |\dot{\boldsymbol{m}} \cdot \dot{\boldsymbol{m}}'| \rangle + \langle \sum_{k \le M} \dot{\boldsymbol{m}}_k^4 \rangle\Big). \tag{4.15}$$

Now,

$$E_\eta X = \langle \exp \frac{t^2}{2}\|\dot{\boldsymbol{m}}\|^2 + t^4 B \rangle$$

where $|B| \le L\Big(\sum\limits_{k \le M} \dot{m}_k^4\Big)$, and thus, on Ω_0,

$$|E_\eta X - \exp \frac{t^2}{2}\langle \|\dot{\boldsymbol{m}}\|^2 \rangle| \le K\Big[\langle \sum_{k \le M} \dot{m}_k^4 \rangle + \langle |\|\dot{\boldsymbol{m}}\|^2 - \langle \|\dot{\boldsymbol{m}}\|^2 \rangle| \rangle\Big].$$

Combining with (4.14), (4.15) we see that on Ω_0, using Cauchy-Schwarz, and the fact that $\langle \sum_{k \le M} \dot{m}_k^4 \rangle \le K$,

$$E_\eta (X - \exp \frac{t^2}{2} \langle \|\dot{\boldsymbol{m}}\|^2 \rangle)^2 \le K \left[\langle \sum_{k \le M} \dot{m}_k^4 \rangle + \langle |\dot{\boldsymbol{m}} \cdot \dot{\boldsymbol{m}}'| \rangle + \langle (\|\dot{\boldsymbol{m}}\|^2 - \langle \|\dot{\boldsymbol{m}}\|^2 \rangle)^2 \rangle \right].$$

Writing $x = y \exp R$ for $R = \log \frac{x}{y}$, we see that $|R| \le K|x - y|$ when $y \ge 1$ and x stays bounded. Using crude estimates (such as $|\boldsymbol{\eta} \cdot \dot{\boldsymbol{m}}| \le M$) when Ω_0 does not occur, the result follows taking expectations. $\square$

Remark. This proof is the first occurrence of a general fact. In our estimates, the influence of the fact that $\sup \|\boldsymbol{m}(\boldsymbol{\epsilon})\| \le K$ holds only outside an event of probability $\exp(-N/K)$ rather that holding always results only in exponentially small permutation terms (while the main terms are of order at least N^{-1}). In order to make the proofs easier to read we will from this point on ignore these small effects, and behave, in all further estimates as if it were true that $\|\boldsymbol{m}(\boldsymbol{\epsilon})\| \le K$ for all choices of quenched variables.

Let us now turn to the study of condition (4.2). Again, this is the crucial point, and the relations (4.10) represent the end rather than the beginning of the "real" proof, but of course the reader that is mainly motivated by the fanciful formula (1.14) should jump directly to (7.14).

To prove that a quantity E_N depending of N (and possibly of the parameters $\alpha, \beta, h, \cdots$) is small, we will simply try to prove that

$$E_{N+1} < \theta E_N + \text{ small term} \tag{4.17}$$

where $\theta < 1$, where the value of the parameter on the right is not too much different from the values on the left. It then suffices to iterate this relation a few times to prove that E_N is small.

We first put this idea to use in Section 5. We prove a bound of the type (4.17) for $D_N = E\langle (\dot{\boldsymbol{\epsilon}} \cdot \dot{\boldsymbol{\epsilon}}')^2 \rangle$, and where N is of course the number of spins. This is rather easy to do, and establishes (4.2) in the range of Theorem 1.5. As in this range of parameters (4.9) is automatic, we have then already passed the main obstacles towards the proof of this theorem.

The problem with this first attempt is that there seems to be no way to make a precise estimation of E_{N+1} as a function of E_N. On the other hand, it will be possible to estimate the quantity (4.4) by induction over N. As the term m_1 is better handled separately, throughout the paper we will use the vector $\boldsymbol{u} = (m_k)_{2 \le k \le N}$ of $\mathbb{R}^{M-1}$; we will always denote by $\boldsymbol{v}$ a thermally independent copy of $\boldsymbol{u}$. Thus, we wish to study $E\langle (\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{v}})^2 \rangle$ rather than $E\langle (\dot{\boldsymbol{m}} \cdot \dot{\boldsymbol{m}}')^2 \rangle$. The centering implicitly contained in the notation $\dot{\boldsymbol{u}}$ is not amenable to easy computations so we replace it by symmetrization $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}'$, where $\boldsymbol{u}'$ is a new thermally independent copy of $\boldsymbol{u}$. At an early stage of the writing of this paper, we observed that computation of

$E\langle(\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{v}})^2\rangle$ by induction involve the quantity $E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle$. As it was not obvious to relate these two quantities except by the trivial inequality

$$E\langle(\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{v}})^2\rangle \leq 4E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle$$

it appeared a better bet to study the larger one namely $C_N = E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle$. We know now how to relate these two quantities by applying a beautiful idea of F. Guerra (Proposition 10.9 below) but we see no reason not to keep studying C_N.

The first step in the study of C_{N+1} is algebraic: one makes an expansion and separate the occurrences of ϵ_{N+1} (or its copies) from the other terms. This is done in Proposition 6.1 below, that yields a representation of C_{N+1} as a sum of eight terms. The most dangerous of these is a sum $E\sum_{2\leq k,\ell\leq M}\eta_k H_{k,\ell}$ where η_k is an independent Bernoulli sequence. Each term $H_{k,\ell}$ is potentially of the same order as C_N. Fortunately $H_{k,\ell}$ depends only rather little of η_k, so there is huge cancellation. If the variables η_k were gaussian rather than Bernoulli, use of integration by parts as in $E(gf(g)) = Ef'(g)$ would take care of the situation. In Proposition 6.2, we develop a substitute to integration by part, a substitute that expresses C_{N+1} as a combination of reasonable quantities such as $\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})\|\boldsymbol{v}\|^2\rangle$, etc. and that produces a reasonable looking error term.

Even if at that stage we could use Lemma 4.1 to approximate quantities of the type $\langle\exp t\boldsymbol{\eta}\cdot\boldsymbol{m}\rangle$ by $C(\exp t\boldsymbol{\eta}\cdot\langle\boldsymbol{m}\rangle)$, where C does not depend upon $\boldsymbol{\eta}$, we would still face the fact that these quantities, when they occur in a denominator (as in Proposition 5.1) are not easy handled. The only way we could imagine was a conditioning argument upon the variables $\boldsymbol{\eta}\cdot\langle\boldsymbol{m}\rangle$. Such an argument was successfully used in a similar but technically much simpler situation, in the last section of [T4]. Unfortunately we do not know how to make conditioning with respect to a sum $\sum\eta_k a_k$ where the η_k are Bernoulli. To go around the problem we show that the variables $(\eta_k)_{k\geq 2}$ can be replaced by standard normal variables with not essential worsening of the previous error terms. This is the purpose of Proposition 6.3. Of course an essential point is that this can be done *before* we have succeeded in getting any real information about the system, and is very different from saying (as becomes obvious much later in the proof) that the variables $\boldsymbol{\eta}\cdot\langle\boldsymbol{m}\rangle$ are nearly Gaussian. This essential technical step opens the way to conditioning arguments, and to appropriate expressions. This is the part that requires real care, because for the purpose of identifying the line (1.18) one must carefully account for each first-order contribution. This is the purpose of Proposition 6.4, and the reader who finds this complicated should try to imagine the kind of energy it took to realize that (6.19) is a successful attack.

At the end of Section 6, we will have the tools to express C_{N+1} (and in fact a lot of other parameters of a $N+1$ spin system) as a sum of terms depending of a N spin systems. Some of these look like main terms; the others look like error terms, of a lower order. Of course it is natural to try to handle as many of the error terms through general principles. For this reason, rather than the Hamiltonian (4.3), we will use the Hamiltonian

$$H(\boldsymbol{\epsilon}) = H_0(\boldsymbol{\epsilon}) - \gamma\varphi(N) \sum_{2\leq k\leq M} g_k m_k(\boldsymbol{\epsilon}) \tag{4.16}$$

where $\gamma \geq 0, (g_k)_{k\leq M}$ are as in (4.3) and where $\varphi(N)$ is a certain function of N (chosen for example as $N^{1/3}$). To distinguish this Hamiltonian from (1.1) it will be called the perturbed Hamiltonian, while (1.1) is the original Hamiltonian. The miracle of this perturbation term is that, when $\varphi(N)^2/N \to 0$ as $N \to \infty$, it has a vanishing influence on the value of the free energy per site. Yet the existence of this term allows to prove strong regularity conditions, as will be shown in Appendix 1. There are certainly reasons to feel uneasy about what can appear as unnatural "tricks". Possibly this uneasiness will disappear when our understanding deepens. More importantly is must be said that this perturbating term in the Hamiltonian should simply be seen as a labour saving device rather than as an essential tool; and we feel that, with extra work, (involving no new ideas or techniques) one should be able to dispense from using it. We have however felt that in the present stage, it was better to present the shortest possible proofs. The reader observes that the perturbation term in (4.16) does not include the case $k = 1$. There is no compelling reason for including or not this term. It makes no difference.

We have felt that it would be confusing to present now the special techniques that take advantage of the perturbation term (4.16), so these are relegated to an Appendix, to which the reader will be referred when the need arises. It is the use of these techniques that requires a technical device (that was already used in [T4]), namely smoothing by integration of the parameters over a small domain.

General principles however do not allow to control all the error terms. In particular to control some of the error terms in the computation of C_N, one must (not surprisingly) control

$$A_N = E\langle \sum_{2\leq k\leq N} m_k^4 \rangle$$

and show first that this is small. The study of A_N is then undertaken also by induction over N, fortunately requiring no new techniques, and turning out to be only a side story . This story has an amusing twist, as computation of A_N by inductions forces to consider an auxiliary quantity B_N. As the computation of B_N by induction uses A_N, one is led to consider combinations $\rho A_N + B_N$. These behave very well, because for *all* values of the parameters, one (almost) have a relation

$$\rho A_{N+1} + B_{N+1} \leq \theta(\rho A_N + B_N) + \text{ error terms,} \tag{4.17}$$

where $\theta < 1$. This does not say that the error terms are easy to control; but one is certainly led to conjecture that $\lim A_N = 0$ whatever the value of the parameters.

Despite all this work, in order to make the iteration succeed one seem to require some extra information. In Section 7, we exploit the information obtained in Section 3 for this purpose and we complete the proof of Theorem 1.6.

The purpose of Section 8 is to identify the Almeida-Thouless line, and it seems worthwhile to explain in detail what we do there. Assuming that the N spin system is close to what the RS solution predicts, we compute C_{N+1} as a function of parameters of the N spin system. The results reads

$$C_{N+1} = \theta C_N + \text{ Error terms } + \text{ terms involving higher moments.} \tag{4.18}$$

The most striking feature is that $\theta < 1$ if and only if (1.18) holds. The error terms of (4.18) are small from general principles. The higher moment terms are just that; terms such as $E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^4\rangle$, etc. We observe now that for a r.v. $0 \leq X \leq 1$, with EX small, the only way that EX^2 is not much smaller than EX is if a significant part of EX comes form values of X close to 1. The only way it could fail that $E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^4\rangle = o(E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle)$ is if a large part of $E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle$ would come from values where $|\tilde{\boldsymbol{u}}\cdot\boldsymbol{v}|$ is of order one. Rather it is to be expected that the tails of $\tilde{\boldsymbol{u}}\cdot\boldsymbol{v}$ look like Gaussian, and that the sets where $\tilde{\boldsymbol{u}}\cdot\boldsymbol{v}$ is of order 1 have exponentially small contribution. We formalize in Definition 8.1 the fact that higher moments should be of small order and Conjecture 8.2 asserts that this is the case for the function $(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2$ and $\sum\limits_{2\leq k\leq M} m_k^2$. It must be emphasized in the strongest possible way that this conjecture is *extremely weak*. Should it fail, the corresponding pathology would be considerably more surprising to me than the wildest predictions of the Parisi solution. This of course does not mean that we see how this conjecture could be proved. Certainly the related conjectures of [T4], in a technically much simpler situation, should be studied first. Going back to (4.18), the meaning is as follows. If for N spins, the system is close to the RS solution, then (unless extreme pathology occurs) the condition (1.18) means that the (fundamental) constraint "C_N small" is stable against addition of one extra spin. It is unfortunately necessary to give slightly more complicated statement, because, while the condition "C_N small" is certainly the crucial one, the condition "A_N small " is also important.

Theorem 4.2. (Informal version) *Under (1.18), if the N spin system is close to the RS solution, the conditions "C_N and A_N small" are stable against addition of one spin (unless extraordinary pathology occurs).*

Even though we have not written it in complete detail, the reader should be convinced after reading Section 8 that, under the same conditions, condition (1.18) is necessary for the stability of the condition "C_N small" against addition of one spin.

In order to find the exact domain of parameters where the RS solution is stable against addition of one extra spin, the "only" work to do is the analysis of (4.12), a task with little relationship with the area of interest of the author, and that is thus better left to others.

As an excuse for studying the stability of the RS solution against addition of one spin, it must be pointed out that apparently the physisists do not have arguments that the RS solution is the true solution: they only show a type of "stability". The relationship between their notion of stability and stability against addition of one spin is unclear to me.

Upon reading our proof of Theorem 1.7, A. Bovier and V. Gayrard [B-G3] discovered a very beautiful different proof of the fact that, in most of the range of that theorem, C_N and A_N are small, (a fact that, as already mentioned, is the cornerstone of (1.14)). This proof is quite simpler than ours, at least if one assumes a certain rather delicate convexity property of the function Ψ of (3.1) they had proved earlier. This raises the question of whether this convexity property is a central feature or a lucky coincidence. It should be at least said that our simple minded method (they simply consists of computing everything by induction on N!) worked quite well in the case of the SK model, where no convexity is apparent, and that convexity properties do not yet appear able to reach either Theorem 4.2 or even the full range of Theorem 1.7. We also hope to demonstrate in further work the wide range of uses of the iteration method.

5. A first look at iteration.

A first purpose of this section is to learn how to relate a situation for $N+1$ (or $N+2$) sites with a situation for N sites. As a first application of the iteration method, we then show that (4.2) hold in the range of Theorem 1.5. The function φ of this section is that of (4.16). In the sequel, given $\boldsymbol{\epsilon} \in \Sigma_N, \epsilon_{N+1} \in \{-1,1\}$, we identify $(\boldsymbol{\epsilon}, \epsilon_{N+1})$ with an element of Σ_{N+1}.

Proposition 5.1. *Consider a fresh sequence $\boldsymbol{\eta} = (\eta_k)$ of Bernoulli r.v., and set*

$$\begin{aligned}&\beta' = \beta N/(N+1), h' = h(N+1)/N, \gamma' = \gamma\varphi(N+1)/\varphi(N)\\ &\overline{\gamma} = \gamma\varphi(N+1)/N, \Gamma = h'\eta_1 + \overline{\gamma} \sum_{2\le k\le M} g_k \eta_k.\end{aligned}$$

Then, for a function A on $\Sigma_{N+1}, \langle A(\boldsymbol{\epsilon}, \epsilon_{N+1})\rangle$ has the same distribution as

$$Z^{-1} \sum_{\epsilon_{N+1}=\pm 1} \langle A(\boldsymbol{\epsilon}, \epsilon_{N+1}) \exp \epsilon_{N+1}\beta'(\boldsymbol{\eta}\cdot\boldsymbol{m} + \Gamma)\rangle_0 \tag{5.1}$$

where

$$Z = \sum_{\epsilon_{N+1}=\pm 1} \langle \exp \epsilon_{N+1}\beta'(\boldsymbol{\eta}\cdot\boldsymbol{m} + \Gamma)\rangle_0.$$

Here $\langle\cdot\rangle$ denotes thermal average with respect to the Hamiltonian

$$\begin{aligned}H_{N+1}(\boldsymbol{\epsilon}, \epsilon_{N+1}) = &- \frac{1}{2(N+1)} \sum_{k\le M} \Big(\sum_{i\le N+1} \eta_{i,k}\epsilon_i\Big)^2 - h \sum_{i\le N+1} \eta_{i,1}\epsilon_i\\ &- \gamma \frac{\varphi(N+1)}{N+1} \sum_{2\le k\le M} \Big(\sum_{i\le N+1} \eta_{i,k}\epsilon_i\Big) g_k\end{aligned} \tag{5.2}$$

at inverse temperature β, while $\langle\cdot\rangle_0$ denote thermal average for the Hamiltonian

$$\begin{aligned}H_N(\boldsymbol{\epsilon}) = &- \frac{1}{2N} \sum_{k\le M} \Big(\sum_{i\le N} \eta_{i,k}\epsilon_i\Big)^2 - h' \sum_{i\le N} \eta_{i,1}\epsilon_i\\ &- \frac{\gamma'\varphi(N)}{N} \sum_{2\le k\le M} \Big(\sum_{i\le N} \eta_{i,k}\epsilon_i\Big) g_k\end{aligned}$$

at inverse temperature β'.

Comment. What this means is that we can reduce the computation of the distribution of thermal averages for a system with $N+1$ spins to the computation of thermal averages for a system with N spins. In doing so, the inverse temperature changes from β to β', and the parameters α, h, γ change slightly. It will become apparent later that these shifts in parameters play no role whatsoever.

Proof. We have

$$\langle A(\boldsymbol{\epsilon}, \epsilon_{N+1})\rangle = \frac{\sum\limits_{\epsilon_{N+1}\pm 1} E_\epsilon A(\boldsymbol{\epsilon}, \epsilon_{N+1}) \exp(-\beta H_{N+1}(\boldsymbol{\epsilon}, \epsilon_{N+1}))}{\sum\limits_{\epsilon_{N+1}=\pm 1} E_\epsilon \exp(-\beta H_{N+1}(\boldsymbol{\epsilon}, \epsilon_{N+1}))}.$$

Now, straightforward algebra shows that, setting $\eta_k = \eta_{N+1,k}$

$$\begin{aligned} -\beta H_{N+1}(\boldsymbol{\epsilon}, \epsilon_{N+1}) &= -\beta' H_N(\boldsymbol{\epsilon}) + \beta M/2(N+1) \\ &\quad + \epsilon_{N+1}\beta' \left[\sum_{k\le M} \eta_k \frac{1}{N} \sum_{i \le N} \eta_{i,k}\epsilon_i + \Gamma \right]. \end{aligned}$$

The result follows. (The reader will note that the value of $\boldsymbol{m}$ in (5.1) is indeed the value corresponding to an N-spin system.) □

In the sequel we will use formulas corresponding to Proposition 5.1 in the case where the bracket is not an average over Σ_{N+1} but over Σ_{N+1}^p. Generalization is immediate.

Our next task is to relate the quantities in (4.2) and (4.4). Setting $\rho_N = N^{-2}\langle(\dot{\boldsymbol{\epsilon}} \cdot \dot{\boldsymbol{\epsilon}}')^2\rangle$, and expending the dot product, we have

$$\rho_N = N^{-2} \sum_{i,j \le N} \langle \dot\epsilon_i \dot\epsilon_j \rangle^2$$

so that, by symmetry

$$E\rho_N \le 4N^{-1} + E\langle \dot\epsilon_1 \dot\epsilon_2\rangle^2$$

and, consequently

$$E\rho_{N+2} \le 4N^{-1} + E\langle \dot\epsilon_{N+1}\dot\epsilon_{N+2}\rangle^2.$$

In order to evaluate the last term, we need a version of Proposition 5.1 to relate a system of $N+2$ spins with a system of N spins. With obvious notation, the reader will check that $\langle A(\boldsymbol{\epsilon}, \epsilon_{N+1}, \epsilon_{N+2})\rangle$, at inverse temperature β, and for parameters h, γ, has the same distribution as

$$Z^{-1} \sum_{\epsilon_{N+1},\epsilon_{N+2}=\pm 1} \langle A(\boldsymbol{\epsilon}, \epsilon_{N+1}, \epsilon_{N+2}) V(\epsilon_{N+1}, \epsilon_{N+2})\rangle_0 \tag{5.3}$$

where

$$Z = \sum_{\epsilon_{N+1},\epsilon_{N+2}=\pm 1} \langle V(\epsilon_{N+1}, \epsilon_{N+2})\rangle_0.$$

Here, $\langle\cdot\rangle_0$ denotes thermal average of a system with N spins, inverse temperature $\beta' = \beta N/(N+2)$, parameters $\alpha' = \alpha N/(N+2), h' = h(N+2)/N, \gamma' = \gamma\varphi(N+2)/\varphi(N)$, and

$$\begin{aligned} V(\epsilon_{N+1}, \epsilon_{N+2}) = \exp(&\epsilon_{N+1}\beta'(\boldsymbol{\eta}\cdot\boldsymbol{m} + \Gamma) + \epsilon_{N+2}\beta'(\boldsymbol{\eta}'\cdot\boldsymbol{m} + \Gamma') \\ &\frac{\beta}{N+2}\epsilon_{N+1}\epsilon_{N+2}\boldsymbol{\eta}\cdot\boldsymbol{\eta}') \end{aligned} \tag{5.4}$$

where $\boldsymbol{\eta}, \boldsymbol{\eta}'$ are independent fresh Bernoulli sequences, where $\beta' = \beta N/(N+2)$,

$$\Gamma = h'\eta_1 + \gamma\frac{\varphi(N+2)}{N}\sum_{2\le k\le M} g_k\eta_k$$
$$\Gamma' = h'\eta_1' + \gamma\frac{\varphi(N+2)}{N}\sum_{2\le k\le M} g_k\eta_k'.$$

We now leave the reader perform the simple algebra needed to obtain from (5.3) that

$$E\langle\dot\epsilon_{N+1}\dot\epsilon_{N+2}\rangle^2 = E(\frac{16}{Z^2}(X-Y)^2),$$

where

$$X = \langle V(1,1)\rangle_0\langle V(-1,-1)\rangle_0;\ Y = \langle V(1,-1)\rangle_0\langle V(-1,1)\rangle_0.$$

Using the inequality $e^x \ge 1+x$ to find a lower bound for Z, we then get

$$E\langle\dot\epsilon_{N+1}\dot\epsilon_{N+2}\rangle^2 \le \frac{1}{16}E(X-Y)^2.$$

To lighten the presentation, we will leave to the reader to show (arguing as in Proposition 3.3 of [T4]) that the last term in (5.4) has a smaller order influence, and we will pretend that it is not there. Using 2-replicas we write

$$X = \langle\exp\beta'(\boldsymbol{\eta}\cdot(\boldsymbol{m}-\boldsymbol{m}') + \boldsymbol{\eta}'\cdot(\boldsymbol{m}-\boldsymbol{m}'))\rangle_0$$
$$Y = \langle\exp\beta'(\boldsymbol{\eta}\cdot(\boldsymbol{m}-\boldsymbol{m}') - \boldsymbol{\eta}'\cdot(\boldsymbol{m}-\boldsymbol{m}'))\rangle_0.$$

We set $\boldsymbol{n} = \boldsymbol{m}-\boldsymbol{m}'$, and we consider a (thermally) independent copy $\boldsymbol{n}^*$ of $\boldsymbol{n}$. Thus, using 4-replicas

$$X^2 = \langle\exp\beta'(\boldsymbol{\eta}\cdot(\boldsymbol{n}+\boldsymbol{n}^*) + \boldsymbol{\eta}'\cdot(\boldsymbol{n}+\boldsymbol{n}^*))\rangle_0$$
$$XY = \langle\exp\beta'(\boldsymbol{\eta}\cdot(\boldsymbol{n}+\boldsymbol{n}^*) + \boldsymbol{\eta}'\cdot(\boldsymbol{n}-\boldsymbol{n}^*))\rangle_0.$$

We now denote by $E_{\eta\eta'}$ expectation as only $\boldsymbol{\eta}, \boldsymbol{\eta}'$ vary; we assume $\beta \le 1/2$, which is not a restriction to prove Theorem 1.5. Then we have

$$\begin{aligned} E_{\eta\eta'}(X-Y)^2 &= 2E_{\eta\eta'}(X^2 - XY) \\ &= \langle\exp\frac{\beta'^2}{2}(2\|\boldsymbol{n}+\boldsymbol{n}^*\|^2 + B)\rangle_0 \\ &\quad - \langle\exp\frac{\beta'^2}{2}(\|\boldsymbol{n}+\boldsymbol{n}^*\|^2 + \|\boldsymbol{n}-\boldsymbol{n}^*\|^2 + B')\rangle_0 \end{aligned}$$

where $|B|, |B'| \le L\left[\sum_{k\le M} n_k^4 + n_k^{*4}\right]$. Thus

$$E_{\eta\eta'}(X-Y)^2 = \langle\exp\beta'^2(\|\boldsymbol{n}\|^2 + \|\boldsymbol{n}^*\|^2)(\exp(2\beta'^2\boldsymbol{n}\cdot\boldsymbol{n}^* + B) - \exp(B'))\rangle_0.$$

We use the inequality $|e^x - x - 1| \le L_1 x^2$ for $x \le L$ (Since $\alpha \le 1$ we can pretend by Lemma 11.3 that $\|\boldsymbol{m}\| \le L$, and hence $\|\boldsymbol{n}\|, \|\boldsymbol{n}^*\| \le L$). We observe that $B^2 \le L|B|$, and that

$$\langle \boldsymbol{n} \cdot \boldsymbol{n}^* \exp \beta'^2 (\|\boldsymbol{n}\|^2 + \|\boldsymbol{n}^*\|^2) \rangle_0 = 0$$

by symmetry (say, exchange of the first two replicas). Thus

$$E_{\eta\eta'}(X-Y)^2 \le L\beta'^2 (\langle (\boldsymbol{n} \cdot \boldsymbol{n}^*)^2 \rangle_0 + \langle \sum_k n_k^4 \rangle_0).$$

We now observe that it is an easy consequence of Lemma 2.9 that $E\langle m_k^4 \rangle_0 \le L/N^2$. Thus, since $\langle n_k^4 \rangle_0 \le L \langle m_k^4 \rangle_0$ and since $\langle (\boldsymbol{n} \cdot \boldsymbol{n}^*)^2 \rangle_0 = 4 \langle (\dot{\boldsymbol{m}} \cdot \dot{\boldsymbol{m}}')^2 \rangle_0$ we have shown the following, where $\tau_N = \langle (\dot{\boldsymbol{m}} \cdot \dot{\boldsymbol{m}}')^2 \rangle_0$.

Proposition 5.2. *If $\alpha \le 1, \beta \le 1/2$, we have*

$$E\rho_{N+2}(\alpha, \beta, \gamma, h) \le \frac{L}{N} + L\beta^2 E\tau_N(\alpha', \beta', \gamma', h')$$

where $\alpha' = \alpha N/(N+2)$, and β', γ', h' are as before.

Combining with Corollary 2.8, we then get that

$$E\rho_{N+2}(\alpha, \beta, \gamma, h) \le \frac{L}{N} + \beta^2 L E\rho_N(\alpha', \beta', \gamma', h').$$

Thus if we take β_0 such that $\beta_0^2 L \le 1/2$, use of iteration conclude that $E\rho_N \le L/N$ (for $\alpha \le 1$). This information is the main step in the proof of the validity of the RS solution. The other arguments are by no means trivial. To avoid repetition they are presented in complete detail only in the more difficult situation of Theorem 1.6. The reader who does not wish to use the tools of Section 6 to establish the recursion relation 7.22 can do so using Lemma 4.1 (and some tools from Section 10). Analysis of this recursion relation is made simpler than in the case of Lemma 7.9 because the matrix V there can be shown to be a contraction for β small.

6. The basic techniques of iteration.

The inefficient part of the approach of Section 5 apparently lies in the estimate

$$E\langle(\dot{\boldsymbol{m}}\cdot\dot{\boldsymbol{m}}')^2\rangle \le (1+K\sqrt{\alpha})^2 E\langle(\dot{\boldsymbol{\epsilon}}\cdot\dot{\boldsymbol{\epsilon}}')^2\rangle. \tag{6.1}$$

If we inspect the proof of Corollary 2.8, we observe that (with the notation of the proof of Theorem 1.3), by rank consideration, at most M of the numbers λ_p are different from zero, so that $N^{-1}\sum\limits_{p\le N}\lambda_p \le \alpha\Lambda$. One could then hope in (2.22) the terms $\lambda_{p_1},\cdots,\lambda_{p_\ell}$ would contribute like their average rather than like their maximal value. If this were the case we could improve (6.1) by a factor $L\alpha^2$ on the right, and we could hope to extend the argument of Section 5 to the region $\alpha\beta$ small rather than β small. The above heuristic argument appears however to be plain nonsense, and the situation to be considerably more subtle.

Rather than trying to improve upon (6.1), we will directly study by induction the quantity

$$\begin{aligned} C_N = C_N(\alpha,\beta,h,\gamma) &= E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle \\ &= E\sum_{2\le k,\ell\le N}\langle\tilde{m}_k\tilde{m}_\ell\rangle\langle m_k m_\ell\rangle. \end{aligned} \tag{6.2}$$

Here the last equality follows as usual by expending the dot product. The notation $\tilde{\boldsymbol{u}}$ holds for "symmetrization"; that is, $\tilde{\boldsymbol{u}} = \boldsymbol{u}-\boldsymbol{u}'$, where $\boldsymbol{u},\boldsymbol{u}'$ are thermally independent. So the bracket in $\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle$ in an average in a 3-replica. Similarly, $\tilde{m}_k = m_k - m'_k$.

We observe that we can write

$$\tilde{m}_k = \frac{1}{N+1}\sum_{i\le N+1}\eta_{i,k}\tilde{\epsilon}_i$$

where $\tilde{\epsilon}_i$ is the difference of two independent copies of ϵ_i, or, more precisely, the i^{th} component of $\tilde{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}-\boldsymbol{\epsilon}'$, where $\boldsymbol{\epsilon},\boldsymbol{\epsilon}'$ are thermally independent. We substitute in (6.2) to get, writing η_k rather than $\eta_{N+1,k}$,

$$\begin{aligned} C_{N+1} &= \frac{1}{N+1}E\sum_{i\le N+1}\sum_{2\le k,\ell\le M}\eta_{i,k}\langle\tilde{\epsilon}_i\tilde{m}_\ell\rangle\langle m_k m_\ell\rangle \\ &= E\sum_{2\le k,\ell\le M}\eta_k\langle\tilde{\epsilon}_{N+1}\tilde{m}_\ell\rangle\langle m_k m_\ell\rangle \end{aligned} \tag{6.3}$$

using the symmetry between the sites. We now want to reduce to a system of N spins, so we must remove the dependence of $\tilde{m}_\ell, m_k, m_\ell$ upon the last spin. Also, the normalization factor $N+1$ in m_k must be changed into N. So we write

$$m_\ell = \frac{1}{N+1}\epsilon_{N+1}\eta_\ell + \frac{N}{N+1}u_\ell \tag{6.4}$$

where $u_\ell = \frac{1}{N}\sum\limits_{i\le N}\eta_{i,\ell}\epsilon_\ell$, and similarly from $m_k, \tilde{m}_\ell$. We find eight terms.

Proposition 6.1. *We have* $(\frac{N+1}{N})^3 C_{N+1} = \sum_{p=1}^{8} C_{N+1}^{(p)}$ *where*

$$C_{N+1}^{(1)} = E \sum_{2\le k,\ell\le M} \eta_k \langle \tilde{\epsilon}_{N+1}\tilde{u}_\ell\rangle\langle u_k u_\ell\rangle$$

$$C_{N+1}^{(2)} = E \sum_{2\le k,\ell\le M} \frac{1}{N}\langle \tilde{\epsilon}_{N+1}\tilde{u}_\ell\rangle\langle \epsilon_{N+1} u_\ell\rangle$$

$$C_{N+1}^{(3)} = E \sum_{2\le k,\ell\le M} \frac{1}{N}\eta_k\eta_\ell\langle \tilde{\epsilon}_{N+1}\tilde{u}_\ell\rangle\langle u_k \epsilon_{N+1}\rangle$$

$$C_{N+1}^{(4)} = E \sum_{2\le k,\ell\le M} \frac{1}{N^2}\eta_\ell\langle (\tilde{\epsilon}_{N+1})^2\rangle\langle \epsilon_{N+1} u_\ell\rangle$$

$$C_{N+1}^{(5)} = E \sum_{2\le k,\ell\le M} \frac{1}{N^2}\eta_k\langle (\tilde{\epsilon}_{N+1})^2\rangle\langle \epsilon_{N+1} u_\ell\rangle$$

$$C_{N+1}^{(6)} = E \sum_{2\le k,\ell\le M} \frac{1}{N^2}\eta_k\langle (\tilde{\epsilon}_{N+1})^2\rangle\langle u_k\epsilon_{N+1}\rangle$$

$$C_{N+1}^{(7)} = E \sum_{2\le k,\ell\le M} \frac{1}{N^2}\eta_\ell\langle \tilde{\epsilon}_{N+1}\tilde{u}_\ell\rangle$$

$$C_{N+1}^{(8)} = E \sum_{2\le k,\ell\le M} \frac{1}{N^3}\langle (\tilde{\epsilon}_{N+1})^2\rangle$$

We would be in trouble if these terms were equally important. Fortunately this is not the case. Roughly speaking, what happens is that the correlation between the terms η_k, η_ℓ and the brackets following them is weak enough that, (as far as order of magnitudes are concerned) these terms play the role of a factor $1/N$. For this reason, the leading terms of the previous decomposition are for $p = 1, 2$, and all the others are of lower order. Certainly this is not obvious now. What we will do is to study in great detail the most dangerous term, that is $C_{N+1}^{(1)}$, through general estimates. We will then sketch how to deal with $C_{N+1}^{(2)}$; by that time the reader will most likely agree that the other terms are much easier to handle.

We want to transform the product of brackets in $C_{N+1}^{(1)}$ into a single bracket. For this, we consider (on Σ_{N+1}) the vector $(u_\ell)_{1\le\ell\le M}$ and two thermally independent copies $(u_\ell)_{1\le\ell\le M}, (v_\ell)_{1\le\ell\le M}$. We write $\boldsymbol{u} = (u_\ell)_{2\le\ell\le M}$ (omitting the first coordinate), and similarly we define $\boldsymbol{u}', \tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}'$, and $\boldsymbol{v}$.

To simplify the vocabulary, we describe the above procedure by saying that $\boldsymbol{u}', \boldsymbol{v}$ are "thermally independent copies of $\boldsymbol{u}$", being understood that it is in fact the pairs $(u_1', \boldsymbol{u}'), (v_1, \boldsymbol{v})$ that are independent copies of $(u_1, \boldsymbol{u})$.

Thus we have

$$C_{N+1}^{(1)} = E \sum_{2\le k,\ell\le M} \eta_k\langle \tilde{\epsilon}_{N+1}\tilde{u}_\ell v_k v_\ell\rangle = E \sum_{2\le k\le M} \eta_k\langle \tilde{\epsilon}_{N+1} v_k(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})\rangle.$$

The quantity $v_k(\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})$ does not depend upon the last spin, so that we can appeal to the version of Proposition 5.1 for a 3-replica.

First, we observe that our sequence (η_k) is indeed the same as occurs in that Proposition.

Next, we observe that in (5.1) the product denoted there by $\boldsymbol{\eta} \cdot \boldsymbol{m}$ includes the term $\eta_1 m_1$ (which is now denoted $\eta_1 u_1$). We will now abuse notation and still write $\boldsymbol{\eta}$ for the vector $(\eta_k)_{2 \leq k \leq M}$; The product $\boldsymbol{\eta} \cdot \boldsymbol{u}$ does not include the term $\eta_1 u_1$, and this term has to be included separately. We set

$$\Gamma = h' \eta_1 + \overline{\gamma} \sum_{2 \leq k \leq M} g_k \eta_k$$

where $\overline{\gamma} = \gamma \varphi (N+1)/N$. Then we have

$$(6.5) \quad C_{N+1}^{(1)} = E \sum_{2 \leq k \leq M} \frac{\eta_k}{8Z^3} \langle \sum_{\epsilon_{N+1}, \epsilon'_{N+1}, \epsilon''_{N+1} = \pm 1} (\epsilon_{N+1} - \epsilon'_{N+1}) v_k(\tilde{\boldsymbol{u}} \cdot \boldsymbol{v}) V(\epsilon_{N+1}, \epsilon'_{N+1}, \epsilon''_{N+1}) \rangle_0$$

where

$$(6.6) \quad \begin{aligned} V(\epsilon_{N+1}, \epsilon'_{N+1}, \epsilon''_{N+1}) = \exp \beta' (&\epsilon_{N+1} \boldsymbol{\eta} \cdot \boldsymbol{u} + \epsilon'_{N+1} \boldsymbol{\eta} \cdot \boldsymbol{u}' + \epsilon''_{N+1} \boldsymbol{\eta} \cdot \boldsymbol{v} \\ &+ \eta_1 (\epsilon_{N+1} u_1 + \epsilon'_{N+1} u'_1 + \epsilon''_{N+1} v_1) + (\epsilon_{N+1} + \epsilon'_{N+1} + \epsilon''_{N+1}) \Gamma) \end{aligned}$$

and where

$$(6.7) \qquad Z = \frac{1}{8} \langle \sum_{\epsilon_{N+1} = \pm 1} \exp \beta' \epsilon_{N+1} (\boldsymbol{\eta} \cdot \boldsymbol{u} + \eta_1 u_1 + \Gamma) \rangle_0$$

The meaning of $\langle \cdot \rangle_0$ is as in Proposition 5.1.

We note that $\epsilon_{N+1} - \epsilon'_{N+1} = 0$ unless $\epsilon'_{N+1} = -\epsilon_{N+1}$. Thus, (6.5) reduces to

$$(6.8) \qquad C_{N+1}^{(1)} = E \sum_{2 \leq k \leq M} \frac{\eta_k}{4Z^3} \langle \sum_{\xi, \theta = \pm 1} \xi v_k (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v}) \mathcal{E} \rangle_0$$

where

$$(6.9) \qquad \mathcal{E} = \mathcal{E}(\xi, \theta) = \exp \beta' (\xi \boldsymbol{\eta} \cdot \tilde{\boldsymbol{u}} + \theta \boldsymbol{\eta} \cdot \boldsymbol{v} + \xi \eta_1 \tilde{u}_1 + \theta \eta_1 v_1 + \theta \Gamma).$$

Now, we have to address the problem of understanding the correlation of η_k with the other terms. If the r.v. η_k were to be Gaussian, we would use integration by parts. Let us denote by $f_k(t)$ the function obtained by replacing η_k by t wherever η_k occurs in the expression

$$\frac{1}{4} Z^{-3} \langle \sum_{\xi, \theta = \pm 1} \xi v_k (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v}) \mathcal{E} \rangle_0.$$

Thus

$$C_{N+1}^{(1)} = E \sum_{2 \le k \le M} \eta_k f_k(\eta_k). \tag{6.10}$$

If we could integrate by parts, we would find $C_{N+1}^{(1)} = E \sum_{2 \le k \le M} f_k'(\eta_k)$. We will show that this is true modulo a small error.

For any smooth function f on $\mathbb{R}$, we have (integration by parts)

$$f(1) - f(-1) = f'(1) + f'(-1) + \int_{-1}^{1} \frac{1}{2}(t^2 - 1) f^{(3)}(t)dt$$

and thus, using (6.10)

$$|C_{N+1}^{(1)} - E \sum_{2 \le k \le M} f_k'(\eta_k)| \le \sum_{2 \le k \le M} \int_{-1}^{1} E|f_k^{(3)}(t)|dt. \tag{6.11}$$

We turn to the evaluation of the last term. Writing $f_k(t) = YZ^{-3}$, $f_k^{(p)}(t)$ is sum of terms of the type

$$\text{Const. } Y^{(\ell_0)} Z^{(\ell_1)} \cdots Z^{(\ell_p)} Z^{-p-3}$$

where $\ell_0 + \ell_1 + \cdots + \ell_p = p$. (Here we need only $p = 3$; the case $p = 4$ will be needed later.)

Using the replica trick, each of these terms can be written

$$(2Z)^{-p-3} \beta'^p \langle \sum_{\xi, \theta_0, \cdots, \theta_p = \pm 1} \xi v_k (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v}^0) x_1 \cdots x_p \mathcal{E} \rangle_0. \tag{6.12}$$

Here

$$\mathcal{E} = \exp \beta' (\boldsymbol{\eta} \cdot \tilde{\boldsymbol{u}} + \boldsymbol{\eta} \cdot (\sum_{0 \le \ell \le p} \theta_\ell \boldsymbol{v}^\ell) + \eta_1 (\sum_{0 \le \ell \le p} \theta_\ell v_1^\ell) + (\sum_{0 \le \ell \le p} \theta_\ell) \Gamma),$$

with the convention that η_k is everywhere replaced by t. The bracket represents an integral over Σ_N^{p+3}; the variables $\boldsymbol{v}^\ell, 0 \le \ell \le p$ are (thermally) independent copies of the overlap vector $\boldsymbol{u}$; and, for $\ell = 1, \cdots, p, x_\ell$ is one of the quantities

$$\xi \tilde{u}_k + \theta_0 v_k^0 + \overline{\gamma} g_k ; \theta_\ell v_k^\ell + \overline{\gamma} g_k .$$

We expend the product $x_1 \cdots x_p$. We use the inequality $|\prod_{\ell \le p+1} y_\ell| \le L(p) \sum_{\ell \le p+1} |y_\ell^{p+1}|$ to get that each term (6.12) is bounded by a sum of terms

$$\langle |\tilde{\boldsymbol{u}} \cdot \boldsymbol{v}| |x_k^{p+1}| \mathcal{E} \rangle_0 \tag{6.13}$$

where x_k is one of $v_k, \tilde{u}_k, v_k^\ell(\ell = 0, \cdots, p), \overline{\gamma} g_k$.

We take expections, integrate for $-1 \leq t \leq 1$; keeping in mind that, since $\varphi(N) \leq \sqrt{N}$, we have $\overline{\gamma}^4 \leq N^{-2}$, we then see that the right hand side of (6.12) is bounded by

$$KE(\langle|\tilde{\boldsymbol{u}} \cdot \boldsymbol{v}|(\sum_{k \leq M} u_k^4 + {u'_k}^4 + v_k^4 + w_k^4)\rangle) + \frac{K}{\sqrt{N}} \tag{6.14}$$

where (w_k) is an independent copy of (v_k) (we could in fact put K/N rather than $K/\sqrt{N}$).

Proposition 6.2. (Integration by parts) *Within an error at most (6.14), we can write*

$$C_{N+1}^{(1)} \approx C_{N=1}^{(9)} + C_{N+1}^{(10)} + C_{N+2}^{(11)} \tag{6.15}$$

where

$$C_{N+1}^{(9)} = \frac{1}{4}\beta' E(Z^{-3}\langle \sum_{\xi,\theta=\pm 1} (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})^2 \mathcal{E}(\xi,\theta)\rangle_0)$$

$$C_{N+1}^{(10)} = \frac{1}{4}\beta' E(Z^{-3}\langle \sum_{\xi,\theta=\pm 1} \xi\theta(\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})\|\boldsymbol{v}\|^2 \mathcal{E}(\xi,\theta)\rangle_0)$$

$$C_{N+1}^{(11)} = -\frac{3}{4}\beta' E(Z^{-4}\langle \sum_{\xi,\theta,\delta=\pm 1} \xi\delta(\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})(\boldsymbol{v} \cdot \boldsymbol{w})\mathcal{E}(\xi,\theta)\mathcal{E}'(\delta)\rangle_0)$$

for $\mathcal{E}(\xi,\theta)$ given by (6.9) and

$$\mathcal{E}'(\delta) = \exp \beta'(\delta\boldsymbol{\eta} \cdot \boldsymbol{w} + \delta\eta_1 w_1 + \delta\Gamma).$$

Proof. We use (6.12); we compute $f'_k(\eta_k)$ and regroup the terms, using formulas such as $\tilde{\boldsymbol{u}} \cdot \boldsymbol{v} = \sum_{2 \leq k \leq M} \tilde{u}_k v_k$. We then find the terms described in the statement of the Proposition, plus some other terms, that arise from the fact that Γ depends upon $\boldsymbol{\eta}$. Viewing $f_k(t)$ as a quotient, there are two such terms. The one occurring when taking the derivative of the numerator is

$$\frac{1}{4}\overline{\gamma} g_k Z^{-3}\langle\sum_{\xi,\theta} \xi\theta v_k(\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})\mathcal{E}\rangle_0.$$

To compute the expectation we integrate by parts in g_k, at all the other variables fixed. We then take absolute values, and the expectation in the variable $\boldsymbol{\eta}$, $(g_\ell)_{\ell \neq k}$. We then get a bound $\overline{\gamma}^2 KE\langle|v_k|\rangle_0$. To evaluate the sum over k, we use that

$$\sum_{k \leq M} |v_k| \leq \sqrt{M}(\sum_{k \leq M} v_k^2)^{1/2}.$$

Since $\overline{\gamma}^2 \leq K\varphi(N)^2/N^2 \leq K/N$, the result follows. (The term coming from the denominator is handled similarly.) $\square$

So, we now have to learn how to compute $C_{N+1}^{(\ell)}, 9 \leq \ell \leq 11$. An essential ingredient of the main computation will be a conditioning argument with respect to a variable of the type $\sum\limits_{2\leq k\leq M} b_k\eta_k$. To make this argument possible, the following is an essential step.

Proposition 6.3. (Gaussian smoothing). *In equation (6.15) we still make an error at most of type (6.14) when we assume now, in the definition of $C_{N+1}^{(\ell)}(\ell = 9, 10, 11)$ that the variables $(\eta_k)_{k\geq 2}$ (but NOT η_1) are standard normal rather than Bernoulli.*

Proof. The method we use was invented by Trotter to prove the CLT without using characteristic functions. Consider a function S from $\mathbb{R}^{M-1}$ to $\mathbb{R}$. Given independent $N(0,1)$ variables $(g_k)_{2\leq k\leq M}$ and independent Bernoulli variables $(\eta_k)_{2\leq k\leq M}$ we want to compare $L_1 = ES(\eta_2, \cdots, \eta_M)$ and $L_M = ES(g_2, \cdots, g_M)$. To do this we consider $L_k = ES(g_2, \cdots, g_k, \eta_{k+1}, \cdots, \eta_M)$ and we write

$$|L_1 - L_M| \leq \sum_k |L_{k+1} - L_k|.$$

In other words, we replace the η_k's by the g_k's one at a time. Consider, for a given k, the function

$$f_k(t) = S(g_2, \cdots, g_k, t, \eta_{k+2}, \cdots, \eta_M).$$

Thus

$$f_k(t) = \sum_{j=0}^{3} \frac{t^j}{j!} f_k^{(j)}(0) - \int_0^t \frac{(t-u)^3}{3!} f_k^{(4)}(u)du.$$

Using the fact that the first three moments of η_k and g_k coincide, we have

$$\begin{aligned}|L_{k+1} - L_k| &= |E(f_k(g_k) - f_k(\eta_k))| \\ &\leq E|g_k|^4 \sup_{|u|\leq|g_k|} |f_k^{(4)}(u)| + E \sup_{|u|\leq 1} |f_k^{(4)}(u)|.\end{aligned}$$

The function f_k is the same as before. Very much the same estimates as those used in Proposition 6.2 finish the proof. $\square$

In the previous propositions, we have conducted the calculation in the case of $C_{N+1}^{(1)}$ because it should be obvious how to conduct this calculation in the other (simpler) cases we need, so that it was useless to state a general result. On the other hand, the main calculation is more delicate, and it helps to formulate a general principle that will cover all the further needs. The framework is as follows. We consider a small integer p (say, $p \leq 10$). For $\ell \leq p$ consider thermally independent vectors $\boldsymbol{w}_\ell$ distributed like $\boldsymbol{u}$. We consider $\delta_\ell \in \{-1,1\}$, and we set $\Delta = \sum_{\ell\le p} \delta_\ell$. We consider a real-valued function $f = f(\eta_1, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_p)$, and the bracket

$$U = \langle f \exp \beta'(\boldsymbol{\eta}\cdot(\sum_{\ell\le p}\delta_\ell \boldsymbol{w}_\ell) + \eta_1 \sum_{\ell\le p}\delta_\ell(w_{1,\ell}+h') + \Delta\overline{\gamma}\sum_{2\le k\le M} g_k\eta_k)\rangle_0. \tag{6.16}$$

We consider

$$Z = \frac{1}{2}\sum_{\xi=\pm1}\langle \exp\beta'(\xi\boldsymbol{\eta}\cdot\boldsymbol{u} + \eta_1(u_1+h') + \xi\overline{\gamma}\sum_{2\le k\le M} g_k\eta_k)\rangle_0.$$

Consider now a standard normal r.v. g that is independent of all the other variables. We consider the r.v.

$$Y = \beta'(g\|\langle\boldsymbol{u}\rangle_0\| + \eta_1(\langle u_1\rangle_0 + h')). \tag{6.17}$$

Proposition 6.4. *In order to evaluate $E(U/Z^p)$, we can use the approximate equality*

$$E(\frac{U}{Z^p}) \approx \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} \tag{6.18}$$

where

$$\mathrm{I} = E(\frac{\exp\Delta Y}{\mathrm{ch}^p Y}\langle f\rangle_0)$$
$$\mathrm{II} = \beta'^2 E(\frac{\exp\Delta Y}{\mathrm{ch}^p Y}\langle f\sum_{\ell<\ell'}\delta_\ell\delta_{\ell'}\dot{\boldsymbol{w}}_\ell\cdot\dot{\boldsymbol{w}}_{\ell'}\rangle_0)$$
$$\mathrm{III} = -\beta'^2 E(\frac{\exp\Delta Y}{\mathrm{ch}^p Y}\langle f\sum_{\ell\le p}\dot{\boldsymbol{w}}_\ell\cdot\boldsymbol{b}\rangle_0)$$
$$\mathrm{IV} = \beta'^2 \Delta E(\frac{\exp\Delta Y}{\mathrm{ch}^p Y}\langle f\sum_{\ell\le p}\delta_\ell\dot{\boldsymbol{w}}_\ell\cdot\boldsymbol{b}\rangle_0)$$
$$\mathrm{V} = -\beta'^2 p E(\exp\Delta Y\frac{\mathrm{th}Y}{\mathrm{ch}^p Y}\langle f\sum_{\ell\le p}\delta_\ell\dot{\boldsymbol{w}}_\ell\cdot\boldsymbol{b}\rangle_0).$$

The error made while using (6.18) is at most $\sum_{j\le 4} E(j)$ where

$$E(1) = KE\langle|f|\sum_{\ell\le p+2}(\dot{\boldsymbol{w}}_\ell\cdot\dot{\boldsymbol{w}}_{\ell'})^2\rangle_0$$
$$E(2) = KE\langle|f|\sum_{\ell\le p+2}(\dot{\boldsymbol{w}}_\ell\cdot\boldsymbol{b})^2\rangle_0$$
$$E(3) = K(Ef^2)^{1/2}(E\langle(\|\boldsymbol{u}\|^2 - \langle\|\boldsymbol{u}\|^2\rangle_0)^2\rangle_0)^{1/2}$$
$$E(4) = K(Ef^2)^{1/2}(E\langle(u_1 - \langle u_1\rangle_0)^2\rangle_0)^{1/2}.$$

Proof. The first task is to show that the term containing $\overline{\gamma}$ creates only lower order effects. Let us denote by V the quantity defined as U, but with $\overline{\gamma}=0$. We simply write

$$\begin{aligned} E\Big|\frac{V}{Z}-\frac{U}{Z}\Big| &\le E|V||\exp\Delta\overline{\gamma}\sum_{2\le k\le M} g_k\eta_k - 1| \\ &\le (EV^2)^{1/2}(E(\exp\Delta\overline{\gamma}\sum_{2\le k\le M} g_k\eta_k - 1)^2)^{1/2} \end{aligned}$$

Since $\overline{\gamma}M \le \alpha\varphi^2(N)/N \le \alpha$, a straight forward computation yields a bound $K\varphi(N)^2/N$, that goes to zero if, say, $\varphi(N)=N^{1/3}$.

At this stage we have seen an example of each of the arguments needed to show that the perturbation term in the Hamiltonian creates only errors that go to zero with $\varphi(N)^2/N$. Before the real work starts, it is better, for the clarity of the exposition to decide once and for all that all terms containing $\overline{\gamma}$ will be ignored, and that we can pretend we work with the original Hamiltonian rather than with the perturbed Hamiltonian.

Throughout the end of Section 8, we will use the notation $\boldsymbol{b}=\langle \boldsymbol{u}\rangle_0$. The key idea, also central in [T4] is that a bracket $\langle \exp\xi\beta'\boldsymbol{\eta}\cdot\boldsymbol{u}\rangle_0$ depends upon $\boldsymbol{\eta}$ essentially through $\boldsymbol{\eta}\cdot\boldsymbol{b}$. Thus, if we write

$$\langle \exp\beta'(\xi\boldsymbol{\eta}\cdot\boldsymbol{u}+\eta_1(u_1+h'))\rangle_0 = \exp\beta'\xi\boldsymbol{\eta}\cdot\boldsymbol{b}\langle\exp\beta'\xi\boldsymbol{\eta}\cdot\dot{\boldsymbol{u}}+\eta_1(u_1+h')\rangle_0$$

the last bracket should be essentially independent of $\boldsymbol{\eta}$, so almost equal to its expectation in $\boldsymbol{\eta}$. Approximating $\|\dot{\boldsymbol{u}}\|^2$ and u_1 by their averages for $\langle\cdot\rangle_0$, we are led to set

$$\begin{aligned} \hat{Z} &= \mathrm{ch}\beta'(\boldsymbol{\eta}\cdot\boldsymbol{b}+\eta_1(\langle u_1\rangle_0+h'))\exp\frac{\beta'^2}{2}\langle\|\dot{\boldsymbol{u}}\|^2\rangle_0 \\ &= A\mathrm{ch}Y \end{aligned} \tag{6.18}$$

where $A=\exp\frac{\beta'^2}{2}\langle\|\dot{\boldsymbol{u}}\|^2\rangle_0, Y=\beta'(\boldsymbol{\eta}\cdot\boldsymbol{b}+\eta_1(\langle u_1\rangle_0+h'))$.

We will write

$$\frac{U}{Z^p}=\frac{U}{\hat{Z}^p}+\frac{U(\hat{Z}^p-Z^p)}{\hat{Z}^{2p}}+\frac{U(\hat{Z}^p-Z^p)^2}{\hat{Z}^{2p}Z^p}. \tag{6.19}$$

The expectation of the terms on the right will first be computed conditionally on $\boldsymbol{\eta}\cdot\boldsymbol{b}$. Thus, we first must learn how to compute such expectations.

For a vector $\boldsymbol{x}$ of $\mathbb{R}^{M-1}$, we write

$$S(\boldsymbol{x})=\frac{\boldsymbol{b}(\boldsymbol{x}\cdot\boldsymbol{b})}{\|\boldsymbol{b}\|^2} \qquad\qquad R(\boldsymbol{x})=\boldsymbol{x}-S(\boldsymbol{x}).$$

Thus $R(\boldsymbol{x})\cdot\boldsymbol{b}=0$ so that $R(\boldsymbol{x})\cdot\boldsymbol{\eta}$ is independent of $\boldsymbol{\eta}\cdot\boldsymbol{b}$, while $S(\boldsymbol{x})\cdot\boldsymbol{\eta}$ is proportional to $\boldsymbol{\eta}\cdot\boldsymbol{b}$.

We denote E_0 conditional expectation given $\boldsymbol{\eta}\cdot\boldsymbol{b}$. The typical computation is that, for a vector $\boldsymbol{x}$ of $\mathbb{R}^{M-1}$, we have

$$\begin{aligned} E_0 \exp \boldsymbol{\eta}\cdot\boldsymbol{x} &= E_0 \exp(\boldsymbol{\eta}\cdot R(\boldsymbol{x}) + \boldsymbol{\eta}\cdot S(\boldsymbol{x})) \\ &= \exp(\frac{\|R(\boldsymbol{x})\|^2}{2} + \boldsymbol{\eta}\cdot S(\boldsymbol{x})) \end{aligned}$$

Thus

$$\begin{aligned} E_0 U = \langle f \exp \frac{\beta'^2}{2} \|R(\sum_{\ell\le p}\delta_\ell \boldsymbol{w}_\ell)\|^2 + \beta'\boldsymbol{\eta}\cdot S(\sum_{\ell\le p}\delta_\ell\boldsymbol{w}_\ell) \\ + \beta'\eta_1(\sum_{\ell\le p}\delta_\ell(w_{1,\ell}+h'))\rangle_0. \end{aligned}$$

The game now is to extract from this a manageable expression, allowing small errors.

We observe that (by construction) $\langle \boldsymbol{w}_\ell\rangle_0=\boldsymbol{b}$, so that $\dot{\boldsymbol{w}}_\ell=\boldsymbol{w}_\ell-\boldsymbol{b}$. Since $R(\boldsymbol{b})=\boldsymbol{0}$, we have $R(\boldsymbol{w}_\ell)=R(\dot{\boldsymbol{w}}_\ell)$. Since $\|\boldsymbol{x}\|^2=\|R(\boldsymbol{x})\|^2+\|S(\boldsymbol{x})\|^2$, we then have

$$E_0 U = \langle f T \exp \frac{\beta'^2}{2} \|\sum_{\ell\le p}\delta_\ell\dot{\boldsymbol{w}}_\ell\|^2\rangle_0 \tag{6.20}$$

where

$$T=\exp\Big(\beta'\boldsymbol{\eta}\cdot S(\sum_{\ell\le p}\delta_\ell\boldsymbol{w}_\ell)+\beta'\eta_1(\sum_{\ell\le p}\delta_\ell(w_{1,\ell}+h'))-\frac{\beta'^2}{2}\|S(\sum_{\ell\le p}\delta_\ell\dot{\boldsymbol{w}}_\ell)\|^2\Big). \tag{6.21}$$

We write

$$\|\sum_{\ell\le p}\delta_\ell\dot{\boldsymbol{w}}_\ell\| = \sum_{\ell\le p}\|\dot{\boldsymbol{w}}_\ell\|^2 + 2\sum_{\ell<\ell'}\delta_\ell\delta_{\ell'}\dot{\boldsymbol{w}}_\ell\cdot\dot{\boldsymbol{w}}_{\ell'}.$$

We use the inequality

$$|e^x - x - 1| \le x^2 e^{|x|} \tag{6.22}$$

to see that we can write

$$E_0 U = X_1 + \langle fT(1+\beta'^2\sum_{\ell<\ell'}\dot{\boldsymbol{w}}_\ell\cdot\dot{\boldsymbol{w}}_{\ell'})\exp\frac{\beta'^2}{2}\sum_{\ell\le p}\|\dot{\boldsymbol{w}}_\ell\|^2\rangle_0 \tag{6.23}$$

where $E|X_1|\le E(1)$.

We write

$$\|\dot{\boldsymbol{w}}_\ell\|^2 = \|\boldsymbol{w}_\ell\|^2 - \|\boldsymbol{b}\|^2 - 2\dot{\boldsymbol{w}}_\ell \cdot \boldsymbol{b}$$

and we use again (6.22) to obtain

(6.24) $$E_0 U = X_1 + X_2 + \langle fT(1 + \beta'^2 \sum_{\ell<\ell'} \delta_\ell \delta_{\ell'} \dot{\boldsymbol{w}}_\ell \cdot \dot{\boldsymbol{w}}_{\ell'} - \beta'^2 \sum_{\ell \le p} \dot{\boldsymbol{w}}_\ell \cdot \boldsymbol{b}) \exp \frac{\beta'^2}{2} \sum_{\ell \le p} (\|\boldsymbol{w}_\ell\|^2 - \|\boldsymbol{b}\|^2)\rangle_0$$

and $|X_2| \le E(2)$.

Now

$$\|\boldsymbol{w}_\ell\|^2 - \|\boldsymbol{b}\|^2 = \|\boldsymbol{w}_\ell\|^2 - \langle \|\boldsymbol{w}_\ell\|^2 \rangle_0 + \langle \|\dot{\boldsymbol{w}}_\ell\|^2 \rangle_0$$

so that, recalling $A = \exp \frac{\beta'^2}{2} \langle \|\dot{\boldsymbol{u}}\|^2 \rangle_0$, we can use $|e^x - 1| \le |x| e^x$ to write

(6.25) $$E_0 U = X_1 + X_2 + X_3 + A^p \langle fT(1 + \beta'^2 \sum_{\ell<\ell'} \delta_\ell \delta_{\ell'} \dot{\boldsymbol{w}}_\ell \cdot \dot{\boldsymbol{w}}_{\ell'} - \beta'^2 \sum_{\ell \le p} \dot{\boldsymbol{w}}_\ell \cdot \boldsymbol{b})\rangle_0$$

where (using Cauchy Schwarz) we see that

$$E|X_3| \le K(E\langle f^2 \rangle_0)^{1/2} (E\langle (\|\boldsymbol{u}\|^2 - \langle \|\boldsymbol{u}\|^2 \rangle_0)^2 \rangle_0)^{1/2}.$$

A similar argument shows that we can write

(6.26) $$E_0 U = X_1 + X_2 + X_3 + X_4 + U_1$$

for

$$U_1 = A^p \langle fT'(1 + \beta'^2 \sum_{\ell<\ell'} \delta_\ell \delta_{\ell'} \dot{\boldsymbol{w}}_\ell \cdot \dot{\boldsymbol{w}}_{\ell'} - \beta'^2 \sum_{\ell \le p} \ dot\boldsymbol{w}_{\ell'} \cdot \boldsymbol{b})\rangle_0$$

where T' is defined like T, except that we replace $w_{1,\ell}$ by $\langle w_{1,\ell} \rangle_0 = \langle u_1 \rangle_0$, and where

$$E_0 |X_4| \le K(E\langle f^2 \rangle_0)^{1/2} (E\langle (u_1 - \langle u_1 \rangle_0)^2 \rangle_0)^{1/2}.$$

Since $S(\boldsymbol{b}) = \boldsymbol{b}$, and $\sum_{\ell \le p} \delta_\ell = \Delta$, we have

$$T' = \exp(\beta' \Delta \boldsymbol{\eta} \cdot \boldsymbol{b} + \beta' \eta_1 \Delta(\langle u_1 \rangle_0 + h') + \boldsymbol{\eta} \cdot \boldsymbol{b} c - \|\boldsymbol{b}\|^2 c^2 / 2)$$

where

$$c\boldsymbol{b} = \beta' S(\sum_{\ell \le p} \delta_\ell \dot{\boldsymbol{w}}_\ell).$$

Now we are interested in evaluating $E(U/\hat{Z}^q)$, where $0 \le q \le p$. Since $\hat{Z} \ge 1$, writing

$$E(\frac{U}{\hat{Z}^q}) = E(\frac{E_0 U}{\hat{Z}^q}),$$

we see that we can replace U by U_1 of (6.23). We will first integrate in the gaussian variable $g = \boldsymbol{\eta} \cdot \boldsymbol{b}$ (then in the quenched variables). To integrate in g, (we denote this by E_g) we observe the following elementary fact. If W is a real valued function, and g is $N(0, \sigma^2)$, we have

$$E_g(W(g)\exp(cg - \frac{c^2\sigma^2}{2})) = EW(g + c\sigma^2). \tag{6.27}$$

Now,we can write

$$EW(g + c\sigma^2) = EW(g) + c\sigma^2 EW'(g) + \mathcal{R} \tag{6.28}$$

where

$$|\mathcal{R}| \le c^2\sigma^4 E(\sup_{|t|\le c\sigma^2} |W''(g+t)|).$$

In our case,

$$W(g) = \frac{\exp\beta'\Delta g}{(\mathrm{ch}\beta'(g + \eta_1(\langle u_1\rangle_0 + h')))^q}$$

so that

$$W'(g) = \beta'\Delta\frac{\exp\beta'\Delta g}{\mathrm{ch}^q Y} - q\beta'\frac{\exp\beta'\Delta g}{\mathrm{ch}^q Y}\mathrm{th}Y$$

and $|W''(t)| \le K\exp p\beta'|t|$. Since $\sigma^2 = \|\boldsymbol{b}\|^2$, we have

$$c\sigma^2 = \beta'(\sum_{\ell\le p}\delta_\ell \dot{\boldsymbol{w}}_\ell \cdot \boldsymbol{b}).$$

To evaluate $E(U_1/\mathrm{ch}^q Y)$, we write

$$E_g\frac{U_1}{\mathrm{ch}^q Y} = A^p\langle \tilde{f} E_g W(g + c\sigma^2)\exp\beta'\eta_1\Delta(\langle u_1\rangle_0 + h')\rangle_0$$

where

$$\tilde{f} = f(1 + \beta'^2\sum_{\ell<\ell'}\delta_\ell\delta_{\ell'}\dot{\boldsymbol{w}}_\ell\cdot\dot{\boldsymbol{w}}_{\ell'} - \beta'^2\sum_{\ell\le p}\boldsymbol{w}_\ell\cdot\boldsymbol{b})$$

and then we use (6.28). The expected value of the remainder term is bounded by $E(2)$. Thus, within errors $\sum_{j\le 4} E(j)$, we can approximate $E(U/\mathrm{ch}^q Y)$ by the sum of the following terms

$$A^{p-q}E(\frac{\exp\Delta Y}{\mathrm{ch}^q Y}\langle\tilde{f}\rangle_0) \tag{6.29}$$

$$A^{p-q}\beta'^2\Delta E(\frac{\exp\Delta Y}{\mathrm{ch}^q Y}\langle f\sum_{\ell\le p}\delta_\ell\dot{\boldsymbol{w}}_\ell\cdot\boldsymbol{b}\rangle_0) \tag{6.30}$$

$$-A^{p-q}\beta'^2 qE(\frac{\exp\Delta Y}{\mathrm{ch}^q Y}\mathrm{th}Y\langle f\sum_{\ell\le p}\delta_\ell\dot{\boldsymbol{w}}_\ell\cdot\boldsymbol{b}\rangle_0). \tag{6.31}$$

The reason we can put f rather than $\tilde{f}$ in (6.30), (6.31), is that doing this creates only further errors of the type $E(1), E(2)$.

Consider now the case $q = p$. Taking in account that $\hat{Z}^p = A^p \mathrm{ch}^p Y$, we see that (6.29) is the term I + II + III of Proposition 6.4, while (6.30) is IV and (6.31) is V. Thus, to finish the proof, we have to show that the last two terms of (6.19) produce a contribution controlled by the error term $\sum_{j \le 4} E(j)$.

First, we consider the case of the second term of (6.19). We write it as a sum of terms of the type

$$UZ^{q-p-1}(Z - \hat{Z})/\hat{Z}^q = \frac{U'Z}{\hat{Z}^q} - \frac{U'}{\hat{Z}^{q-1}} \tag{6.32}$$

for $p + 1 \le q \le 2p$, where $U' = UZ^{q-p-1}$, and we write

$$\frac{U'Z}{\hat{Z}^q} - \frac{U'}{\hat{Z}^{q-1}} = \frac{1}{2}(D^+ + D^-) - \frac{U'}{\hat{Z}^{q-1}}$$

where

$$D^+ = \frac{U'}{\hat{Z}^q} \langle \exp \beta'(\boldsymbol{\eta} \cdot \boldsymbol{u} + \eta_1(u_1 + h')) \rangle_0$$

and D^- is defined similarly with a minus sign in the exponent. We use the replica trick to make each numerator appear as of the type (6.16). Now we have q variables $w_1, \cdots, w_q$. Most importantly, f depends only upon $w_1, \cdots, w_p$, so that, by independence, $\langle f \dot{\boldsymbol{w}}_\ell \cdot \dot{\boldsymbol{w}}_{\ell'} \rangle_0 = 0 = \langle f \dot{\boldsymbol{w}}_\ell \cdot \boldsymbol{b} \rangle_0$ unless $\ell, \ell' \le p$. It should then be obvious that if one considers the terms (6.29) to (6.31) arising from $D^+, D^-, U'/\hat{Z}^{q-1}$, these terms cancel out when calculating $E((D^+ + D^-)/2 - U'/\hat{Z}^q)$. As for the error terms, they are controlled by $\sum_{j \le 4} E(j)$. (Observe that in the definition of $E(1)$ to $E(2)$, we have taken $\ell, \ell' \le p + 2$ to make sure that terms where $\dot{\boldsymbol{w}}_\ell, \dot{\boldsymbol{w}}'_\ell$ are independent of f occur, as these terms occur in the control of D^+, D^-.)

Now, to control the last term of (6.19), we write

$$E\left(\left|\frac{U(Z^p - \hat{Z}^p)^2}{\hat{Z}^{2p} Z^p}\right|\right) \le E(|U|(Z^p - \hat{Z}^p)^2).$$

We bound $|U|$ by the expression of the type (6.16), replacing f by $|f|$. We then proceed as in the case of the previous term. □

The error terms $E(3), E(4)$ are not dangerous. While it cannot be guaranteed that these terms are small for a given value of the parameters, Proposition (10.8) shows that they are small once we average over the parameters. To simplify notation we will denote by $\mathcal{A}_N$ a nonnegative term (depending upon β, γ, h) that has the property that

$$\lim_{N \to \infty} \int_{\beta \le \beta_0, h \le h_0, |\gamma| \le 1} \mathcal{A}_N d\beta dh d\gamma = 0.$$

These terms are "automatically small" in the sense that we already know they are small, in contrast with other terms that will be proved small through iteration.

Controlling the error terms $E(1), E(2)$ is trickier. We will use two different techniques to do this; this is the object of the next two sections.

7. The small α region.

In this section we prove Theorem 1.6. We will first assume $\beta \leq 2$, which is the difficult case. We will then indicate the modification to make in the case $\beta \geq 2$. We will make use of the results of Section 3, and in particular Proposition 3.15. (The reader will check that this Proposition remains valid for the perturbated Hamiltonian.) We assume that for a certain number S, we have

$$E(G(\{\|\boldsymbol{m} - m^*\boldsymbol{e}_1\| \geq S\})) \leq K\exp(-N/K). \tag{7.1}$$

There, G is the same Gibbs measure as in the bracket $\langle \cdot \rangle_0$.

In particular

$$E(G(\{\|\boldsymbol{u}\| \geq S\})) \leq K\exp(-N/K)$$

since the vector $\boldsymbol{u}$ simply forgets the first coordinate of the overlap vector $\boldsymbol{m}$.

As usual, we will ignore the exponentially small terms arising from the fact that it is not true that $\|\boldsymbol{u}\| \leq S$ always.

To study $C^{(9)}_{N+1}$ we observe that it is the average of 4 terms (corresponding to the choices of ξ, θ), that are all of the type of Proposition 6.4 for $p = 3$ and

$$\boldsymbol{w}_1 = \boldsymbol{u}, \boldsymbol{w}_2 = \boldsymbol{u}', \boldsymbol{w}_3 = \boldsymbol{v}, f = (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})^2.$$

All terms such as $|\langle f(\dot{\boldsymbol{w}}_\ell \cdot \dot{\boldsymbol{w}}_\ell)\rangle_0|, |\langle f(\dot{\boldsymbol{w}}_\ell \cdot \boldsymbol{b})\rangle_0|$ are bounded by $\langle f\rangle_0 S^2$, so that we have

Lemma 7.1. *Under (7.1) we have*

$$C^{(9)}_{N+1} \leq \beta' E(\frac{1}{ch^2 Y}\langle(\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})^2\rangle_0) + KS^2\langle(\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})^2\rangle_0 + E(5) \tag{7.2}$$

where

$$E(5) \leq KE(\langle|\tilde{\boldsymbol{u}} \cdot \boldsymbol{v}| \sum_{2 \leq k \leq M} (u_k^4 + {u'_4}^4 + v_k^4 + w_4^k)\rangle_0).$$

The term $E(5)$ above arises from the error made while "integrating by parts", and as usual $Y = \beta'(g\|\boldsymbol{b}\| + \eta_1(\langle u_1\rangle_0 + h'))$.

Let us now study $C^{(10)}_{N+1}$. In that case, with $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ as before, we have $f(\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{v}) = \tilde{\boldsymbol{u}} \cdot \boldsymbol{v}\|\boldsymbol{v}\|^2$, so that $\langle f\rangle_0 = 0$ and there is no contribution from the term I of Proposition 6.4. We claim that the other contributions are bounded by $KS^2\langle(\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})^2\rangle_0$. The factor S^2 arises from $\|\boldsymbol{v}\|^2$. We also observe that

$$\langle|f\dot{\boldsymbol{w}}_\ell \cdot \boldsymbol{b}|\rangle_0 \leq \langle f^2\rangle_0^{1/2}\langle(\dot{\boldsymbol{w}}_\ell \cdot \boldsymbol{b})^2\rangle_0^{1/2}$$
$$\langle|f\dot{\boldsymbol{w}}_\ell \cdot \dot{\boldsymbol{w}}_{\ell'}|\rangle_0 \leq \langle f^2\rangle_0^{1/2}\langle(\dot{\boldsymbol{w}}_\ell \cdot \dot{\boldsymbol{w}}_{\ell'})^2\rangle_0^{1/2}$$

Now, by Jensen's inequality

$$\langle(\dot{\boldsymbol{w}}_\ell\cdot\boldsymbol{b})^2\rangle_0=\langle(\dot{\boldsymbol{u}}\cdot\boldsymbol{b})^2\rangle_0\le\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0 \tag{7.3}$$

$$\langle(\dot{\boldsymbol{w}}_\ell\cdot\dot{\boldsymbol{w}}_{\ell'})^2\rangle_0\le 2\langle(\dot{\boldsymbol{w}}\cdot\boldsymbol{v})^2\rangle_0\le 2\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0. \tag{7.4}$$

The contributions of $C^{(11)}_{N+1}$ are handled the same way, and (7.2) remains valid with $C^{(1)}_{N+1}$ rather than $C^{(9)}_{N+1}$.

To estimate the contribution of $C^{(2)}_{N+1}$, integration by parts is not needed, and we have

$$C^{(2)}_{N+1}=\frac{1}{4}\frac{M-1}{N}E\frac{1}{Z^3}\langle\sum_{\xi,\theta=\pm1}(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})\exp\beta'(\xi\boldsymbol{\eta}\cdot\tilde{\boldsymbol{u}}+\theta\boldsymbol{\eta}\cdot\boldsymbol{v}+\xi\eta_1\tilde{u}_1+\theta\eta_1(v_1+h'))\rangle_0. \tag{7.5}$$

We use Proposition 6.4 with $f=\tilde{\boldsymbol{u}}\cdot\boldsymbol{v}$, so $\langle\tilde{\boldsymbol{u}}\cdot\boldsymbol{v}\rangle_0=0$. The terms II to IV are bounded by $KE\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0$ using (7.3), (7.4). No subtlety is needed because the factor α (or rather $\frac{M-1}{N}$) in (7.5) will be very small. Collecting all estimates, we now have

Lemma 7.2. *Under (7.1) we have*

$$C_{N+1}\le\beta'E(\frac{1}{ch^2Y}\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0)+K(S^2+\alpha)E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0+E(5)+\mathcal{A}_N. \tag{7.6}$$

In order to use a relation such as (7.6), we need information on $E_g\mathrm{ch}^{-2}Y$. We will use some notation from Section 3, (except that all parameters have now a "prime"), i.e.

$$m^*=\mathrm{th}\beta'(m^*+h')\text{ and }a^*=1-\beta'(1-m^*)^2.$$

Lemma 7.3. *Under (7.1), with probability at least* $1-K\exp(-N/K)$*, we have*

$$\beta'E_g\frac{1}{ch^2Y}\le 1-a^*+L\beta^3(S^2+m^*S).$$

Proof. Consider the function

$$\varphi(x,y)=E_g\frac{1}{\mathrm{ch}^2(xg+y)}.$$

Then $\varphi(0,\beta'(m^*+h'))=1-m^{*2}$, so that

$$\beta'\varphi(0,\beta'(m^*+h'))=1-a^*.$$

Next,

$$|\frac{\partial \varphi}{\partial y}(0,y)| = |-\frac{2\mathrm{th}y}{\mathrm{ch}^2 y}| \le 2y,$$

so that

$$\beta'\varphi(0,y) \le 1 - a^* + 2\max(|y|, \beta'(m^* + h')|y - \beta'(m^* + h')|.$$

Finally, using integration by parts

$$|\frac{\partial}{\partial x}\varphi(x,y)| = |-2E_g g\frac{\mathrm{th}}{\mathrm{ch}^2}(xg+y)| \le 4|x|$$

since the derivative of the function $\mathrm{th}(z)\mathrm{ch}^{-2}(z)$ is bounded by 2, and thus

$$\varphi(x,y) \le 2x^2 + \varphi(0,y).$$

We use these estimates for $x = \beta'\|\boldsymbol{b}\| \le \beta' S$, $y = \beta'(\langle m_1\rangle_0 + h')$ so that $|y - \beta'(m^* + h')| \le \beta' S$ (and we note that $m^* \ge h'$). □

Thus, we now have

$$C_{N+1} \le (1 - a^* + K(Sm^* + S^2 + \alpha))C_N + E(5) + \mathcal{A}_N. \tag{7.7}$$

There, $C_N = \langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0 = C_N(\alpha', \beta', h', \gamma')$.

We first consider the case $\beta < 1$, which is easier because Lemma 2.9 shows that the term $E(5)$ is of the type $\mathcal{A}_N$.

Lemma 7.4. *There is a number L such that if $0 < h \le 1$, $\frac{1}{2} \le \beta < 1$, we have*

$$\alpha \le \frac{1}{L}((\beta-1)^2 + m^{*4}) \Rightarrow C_{N+1} \le (\frac{1+\beta}{2})C_N + \mathcal{A}_N.$$

Proof. We recall that $a^* = 1 - \beta + \beta m^{*2} \ge 1 - \beta$. (For simplicity we will not distinguish between β and β'.) Thus it suffices to achieve

$$Sm^* + S^2 + \alpha \le a^*/L_0.$$

(The constant K of (7.7) is now universal as β, h are bounded.)

Since $m^{*2} \le 1$, we have $a^* \ge (1 - \beta + \beta)m^{*2} \ge m^{*2}$, and thus $(\beta-1)^2 + m^{*2} \le 2a^{*2}$.

If $\alpha \le m^{*2}/L_1$, Proposition 3.15 shows that we can take

$$S \le (L_1\alpha/m^{*2})^{1/2} \le L\alpha^{1/4}.$$

On the other hand, if $m^{*2} \leq L_1\alpha$, (and $\alpha \leq a^{*2}/2$) Lemma 3.7 shows that we can take

$$S \leq 2m^* + L(\alpha \log \frac{a^{*2}}{\alpha})^{1/4} \leq 2L(\alpha \log \frac{a^{*2}}{\alpha})^{1/4}.$$

The result follows easily. □

Thus, if $\beta_0 < 1$, and

$$\alpha_0 \leq \frac{1}{L} \max(|1-\beta_0|^2, m^{*4}(\beta_0, h_0))$$

we can find a neighbourhood D_0 of (β, h), and $\theta < 1$ such that if we set

$$I_N(\alpha) = \int_{(\beta,h)\in D_0, |\gamma|\leq 1} C_N(\alpha,\beta,h,\gamma)d\beta d\gamma dh$$

then, for all α in a neighbourhood of α_0, we have

$$I_{N+1}(\alpha) \leq \theta I_N(\alpha') + a_N$$

where $a_N \to 0$, so that $\lim I_N(\alpha) = 0$ for all α in this neighborhood of α_0. This is the main step in proving the validity of the RS solution. The other steps will be delayed until the end of the section, and detailed in the more delicate case $1 \leq \beta \leq 2$, to which we turn now. We assume that $\alpha \leq m^{*4}/L$; Proposition 3.13 shows that we can choose L large enough that (with overwhelming probability)

$$E_g \frac{1}{\mathrm{ch}^2 Y} \leq 1 - \frac{m^{*2}}{L} \tag{7.8}$$

The problem with (7.5) is that I do not see how to control $E(5)$ unless I can show that

$$A_N = E\langle \sum_{2\leq k\leq M} m_k^4 \rangle$$

is small. Thereby, it seems that this term should be studied prior to a study of C_N. Fortunately, the techniques we have developed do bear on A_N.

Lemma 7.5. (Expansion) *We have*

$$A_{N+1}(\alpha,\beta,h,\gamma) \leq \frac{K}{N} + (\frac{N}{N+1})^2 E \sum_{2\leq k\leq M} \eta_k \langle \epsilon_{N+1} u_k^3 \rangle.$$

Proof. We write

$$m_k^4 = \frac{1}{N+1} \sum_{i\leq N} \epsilon_i \eta_{i,k} m_k^3$$

so that by symmetry $A_{N+1} = E \sum_{2\leq k\leq M} \eta_k \langle \epsilon_{N+1} m_k^3 \rangle$. We then replace m_k by its value (6.4). □

Lemma 7.6. (Integration by parts) *We have*

$$E\Big(\sum_{2\le k\le M}\eta_k\langle\epsilon_{N+1}u_k^3\rangle\Big)\le\beta' E\Big(\sum_{2\le k\le M}\langle u_k^4\rangle-\sum_{2\le k\le M}\langle\epsilon_{N+1}u_k^3\rangle\langle\epsilon_{N+1}v_k\rangle\Big)\qquad(7.9)$$
$$+KE\langle\sum_{2\le k\le M}u_k^6\rangle.$$

Proof. This should be obvious once understood the proof of Proposition 6.2, after we observe that $\langle\sum u_k^3v_k^3\rangle\le\langle\sum u_k^6\rangle$.

Next, we transform the brackets $\langle\ \cdot\ \rangle$ into brackets $\langle\ \cdot\ \rangle_0$ and we appeal to Proposition 6.4. It should be apparent that under (7.1) we have

$$E\langle\sum_{2\le k\le M}u_k^4\rangle\le(1+KS^2)E\langle\sum_{2\le k\le M}u_k^4\rangle_0.$$

Concerning the term $\sum_{2\le k\le M}\langle\epsilon_{N+1}u_k^3\rangle\langle\epsilon_{N+1}v_k\rangle$, we transform it in a single bracket, and we use Proposition 6.4 for $f=\sum_{2\le k\le M}u_k^3v_k$. The terms I contribute as $E(\mathrm{th}^2Y\langle\sum_{2\le k\le M}u_k^4v_k\rangle_0)$. The contribution of all the other terms can be bounded by $KS^2E\langle\sum_{2\le k\le M}u_k^4\rangle_0$, using that $\langle\sum u_k^3v_k\rangle_0\le\langle\sum u_k^4\rangle_0$.

Thus we have proved the following.

Lemma 7.7. *Under (7.1) we have*

$$A_{N+1}\le(\beta'+KS^2)E(\langle\sum_{2\le k\le M}u_k^4\rangle_0)-\beta' E(th^2Y\langle\sum_{2\le k\le M}u_k^3v_k\rangle_0)+\mathcal{A}_N.$$

We rewrite this as

$$A_{N+1}\le KS^2A_N+\beta' E\left(\frac{1}{\mathrm{ch}^2Y}\langle\sum_{2\le k\le M}u_k^4\rangle_0\right)\qquad(7.10)$$
$$+\beta' E\left(\left(\langle\sum_{2\le k\le M}u_k^4\rangle_0-\langle\sum_{2\le k\le M}u_k^3v_k\rangle_0\right)\mathrm{th}^2Y\right)+\mathcal{A}_N.$$

If we compare with what we did in Corollary 7.5, the first two terms are promising. But to handle the third term, there seems to be no other way than to consider the new quantity

$$B_N=E(\langle\sum_{2\le k\le M}m_k^4\rangle-\sum_{2\le k\le M}\langle m_k^3\rangle\langle m_k\rangle).\qquad(7.11)$$

At this point appears the drawback (or, if one prefers, the charm) of our method; it tends to take us further and further from the original problem. On the other hand, there is only a small number of expressions such as A_N, B_N that one can write down, so at some stage we are bound to succeed in getting real information.

To study B_{N+1}, we go through expansion, integration by parts, use of Proposition 6.4. The term $\langle \sum m_k^4 \rangle$ is handled as for A_N. For the term $\sum \langle m_k^3 \rangle \langle m_k \rangle$, we replace m_k by $(N+1)^{-1} \sum\limits_{i \leq N+1} \eta_{i,k}\epsilon_i$ in the last bracket only, and after use of (6.4) we get

$$B_{N+1} \leq (\frac{N}{N+1})^3 E(\sum_{2 \leq k \leq M} \eta_k (\langle \epsilon_{N+1} u_k^3 \rangle - \langle u_k^3 \rangle \langle \epsilon_{N+1} \rangle)) + K/N.$$

Using integration by parts, we then have

$$\begin{aligned} B_{N+1} \leq \beta' E & \sum_{2 \leq k \leq M} (\langle u_k^4 \rangle - \langle \epsilon_{N+1} u_k^3 \rangle \langle \epsilon_{N+1} v_k \rangle \\ & - \langle \epsilon_{N+1} u_k^4 \rangle \langle \epsilon_{N+1} \rangle - \langle u_k^3 \rangle \langle u_k \rangle + 2 \langle u_k^3 \rangle \langle \epsilon_{N+1} \rangle \langle \epsilon_{N+1} v_k \rangle) \\ & + KE \langle \sum_{2 \leq k \leq M} u_k^6 \rangle . \end{aligned}$$

Each term of the first summation is transformed in a bracket $\langle \cdot \rangle_0$, to which we apply Proposition 6.4. It is quite fortunate that the contributions of the terms II to V can all be bounded by $KS^2 E \langle \sum\limits_{2 \leq k \leq M} u_k^4 \rangle_0$; such is also the case of the error terms $E(1), E(2)$. As for the contributions of the terms I, the situation is saved by the fact that each term ϵ_{N+1} simply creates a factor $\mathrm{th} Y$, independently of where it is located. Thus cancellation occurs. Using that $\mathrm{th}^2 Y = 1 - 1/\mathrm{ch}^2 Y$, and that

$$\langle \sum u_k^6 \rangle_0 \leq \langle (\sum u_k^2)(\sum u_k^4) \rangle_0 \leq S^2 \langle \sum u_k^4 \rangle_0$$

we have proved the following.

Lemma 7.8. *Under (7.1) we have*

$$B_{N+1} \leq \beta' E \bigg(\frac{1}{ch^2 Y} \bigg(\langle \sum_{2 \leq k \leq M} u_k^4 \rangle_0 - \sum_{2 \leq k \leq M} \langle u_k^3 v_k \rangle_0 \bigg) \bigg) + KS^2 A_N + \mathcal{A}_N. \tag{7.12}$$

The only remaining obstacle is that the small term $KS^2 A_N$ contains A_N rather than B_N; and, conceivably, $A_N >> B_N$. On the other hand, should $B_N << A_N$, then (7.10) is the inductive relation we want; so we should try to combine (7.10) and (7.12). Given a number $\rho > 0$, it follows from (7.10), (7.12) that

$$\begin{aligned} & \rho A_{N+1} + B_{N+1} \\ & \leq \beta E \bigg(\Big(\frac{1}{\mathrm{ch}^2 Y} + \rho \mathrm{th}^2 Y \Big) \Big(\rho \langle \sum_{2 \leq k \leq M} u_k^4 \rangle_0 + \langle \sum_{2 \leq k \leq M} u_k^4 \rangle_0 - \langle \sum_{2 \leq k \leq M} u_k^3 v_k \rangle_0 \Big) \bigg) \\ & + KS^2 (\rho A_N + B_N) + \mathcal{A}_N. \end{aligned} \tag{7.13}$$

Now, using (7.7), with overwhelming probability

$$\beta E_g\left(\frac{1}{\mathrm{ch}^2 Y}+\rho\mathrm{th}^2 Y\right)\le\beta\rho+(1-\rho)\beta E_g\frac{1}{\mathrm{ch}^2 Y}$$
$$\le(\beta-1)\rho+1-(1-\rho)\frac{m^{*2}}{L_0}.$$

As $m^{*2}\ge(\beta-1)/L_1$, we can choose ρ universal constant such that, setting $A'_N=\rho A_N+B_N$ we have

$$A'_{N+1}\le\left(1-\frac{m^{*2}}{L_1}+KS^2\right)A'_N+\mathcal{A}_N.$$

Consider now β_0,h_0, and $\alpha_0\le m^{*4}(\beta_0,h_0)/L$. We can find a neighborhood J of α_0, a neighborhood D_0 of (β,h), such that if $D=D_0\times[0,1]$, then, for α in J, we have

$$J_{N+1}(\alpha)\le\theta J_N(\alpha')+a_N$$

where $a_N\to 0,\theta<1$, and

$$J_N(\alpha)=\int_D E(A'_N(\alpha,\beta,h,\gamma))d\beta dhd\gamma. \tag{7.14}$$

Consequently, $\lim_{N\to\infty}J_N(\alpha)=0$ for all α in J.

As we know now how to control the term $E(5)$, we see that if $I_N(\alpha)=\int_D C_N(\alpha,\beta,h,\gamma)d\beta dhd\gamma$, then we have

$$\lim_{N\to\infty}I_N(\alpha)=0.$$

We now turn to the proof of the validity of the RS solution. We introduce the parameters

$$R_N=E\sum_{2\le k\le M}\langle m_k\rangle^2$$
$$W_N=E\sum_{2\le k\le M}\langle m_k^2\rangle$$
$$U_N=E\sum_{2\le k\le M}\langle\tilde{m}_k^2\rangle=2(W_N-R_N)$$
$$\mu_N=E\langle m_1\rangle$$
$$Q_N=E\langle\epsilon_1\rangle^2.$$

Let us first consider the case of W_N. Then, with our usual notation

$$\begin{aligned} W_{N+1} &= E\sum_{2\le k\le M}\langle m_k^2\rangle \\ &= E\sum_{2\le k\le M}\langle\frac{1}{N+1}\sum_{i\le N+1}\epsilon_i\eta_{i,k}m_k\rangle \\ &= E\sum_{2\le k\le M}\eta_k\langle\epsilon_{N+1}m_k\rangle \\ &= \frac{M-1}{N+1}+E\sum_{2\le k\le M}\eta_k\langle\epsilon_{N+1}u_k\rangle. \end{aligned}$$

We now use integration by parts, and Proposition 6.4 to transform this expression. What is nice, is that now that we have done the hard work, we know that the error terms are automatically small (and still denoted by $\mathcal{A}_N$), in the sense that their integral over the domain D considered in (7.14) goes to zero. Integration by parts yield

$$W_{N+1}=\alpha+\beta' E(\sum_{2\le k\le M}\langle u_k^2\rangle-\langle u_k\epsilon_{N+1}\rangle^2)+\mathcal{A}_N. \tag{7.15}$$

Use of Proposition 6.4 yields

$$W_{N+1}=\alpha+\beta W_N-\beta E(\mathrm{th}^2 Y\sum_{2\le k\le M}\langle u_k\rangle_0^2)+\mathcal{A}_N, \tag{7.16}$$

where as usual $Y=\beta'(g\|\boldsymbol{b}\|+\eta_1(\langle u_1\rangle_0+h'))$,

$$\|\boldsymbol{b}\|=\sum_{2\le k\le M}\langle u_k\rangle_0^2,\ \text{and}\ W_N=W_N(\alpha',\beta',\gamma',h').$$

(Since we now know that $E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0$ is small - after averaging over β,h,γ - only the terms I have to be considered, and every factor ϵ_{N+1} gives rise to a term th$Y$).

Proceeding in a similar fashion, for U_{N+1} we get

$$U_{N+1}=2\alpha(1-Q_{N+1})+\beta E(\frac{1}{\mathrm{ch}^2 Y}\langle\|\tilde{\boldsymbol{u}}\|^2\rangle_0)+\mathcal{A}_N. \tag{7.17}$$

Computing Q_{N+1} is a straightforward use of Proposition 6.4, and

$$Q_{N+1}=E\mathrm{th}^2 Y+\mathcal{A}_N. \tag{7.18}$$

To compute μ_{N+1}, we observe first that

$$\mu_{N+1}=E\eta_1\langle\epsilon_{N+1}\rangle+\mathcal{A}_N$$

and we use Proposition 6.4 (and the fact that f there may depend upon η_1) to get

$$\mu_{N+1} = E(\eta_1 \mathrm{th} Y) + \mathcal{A}_N. \tag{7.19}$$

We now substitute (7.18) into (7.17) to obtain

$$U_{N+1} = 2\alpha E \frac{1}{\mathrm{ch}^2 Y} + \beta E\left(\frac{1}{\mathrm{ch}^2 Y} \langle \|\tilde{\boldsymbol{u}}\|^2 \rangle_0\right) + \mathcal{A}_N. \tag{7.20}$$

We replace W_{N+1} by $R_{N+1} + U_{N+1}/2$ in (7.16), and we combine with (7.18) to obtain

$$R_{N+1} = \alpha E \mathrm{th}^2 Y + \beta E\left(\frac{1}{\mathrm{ch}^2 Y} \langle \|\boldsymbol{u}\|^2 \rangle_0\right) + \frac{\beta}{2} E(\mathrm{th}^2 Y \langle \|\tilde{\boldsymbol{u}}\|^2 \rangle_0) + \mathcal{A}_N. \tag{7.21}$$

At this point we observe that

$$\|\boldsymbol{b}\|^2 = \|\langle \boldsymbol{u} \rangle_0\|^2 = \langle \|\boldsymbol{u}\|^2 \rangle_0 - \langle \|\boldsymbol{u} - \boldsymbol{b}\|^2 \rangle_0$$

so that

$$\mathrm{Var}\|\boldsymbol{b}\|^2 \leq 2\mathrm{Var}\langle \|\boldsymbol{u}\|^2 \rangle_0 + 2\mathrm{Var}\langle \|\boldsymbol{u} - \boldsymbol{b}\|^2 \rangle_0.$$

We appeal to Proposition 10.5 to see that the first term is of the type $\mathcal{A}_N$, and to Proposition 10.7 to see (now that we control C_N!) that the second term is also of the type $\mathcal{A}_N$. It is at this point that the perturbed Hamiltonian is needed. Consider the function $\Phi_{\alpha,\beta,h}$ from $\mathbb{R}^3$ to itself that transforms the point (x, y, z) into the point (x', y', z') given by

$$\begin{aligned} x' &= E \mathrm{th} Y \\ y' &= (2\alpha + \beta y) E \frac{1}{\mathrm{ch}^2 Y} \\ z' &= \alpha E \mathrm{th}^2 Y + \beta z E \frac{1}{\mathrm{ch}^2 Y} + \frac{\beta}{2} y E \mathrm{th}^2 Y \end{aligned}$$

where $Y = \beta(g\sqrt{z} + x + h)$. Then, using the symmetry of g to get rid of the terms η_1, (7.19) to (7.21) become

$$(\mu_{N+1}, U_{N+1}, R_{N+1}) = \Phi_{\alpha,\beta,h}(\mu_N, U_N, R_N) + \mathcal{A}_N \tag{7.22}$$

where $\mathcal{A}_N$ has the obvious meaning, and where μ_N, U_N, R_N are as usual for $\alpha', \beta', h', \gamma'$. Since (α, β, h) stays in an arbitrarily small neighborhood of (α_0, β_0, h_0), using Theorem 1.4, we see that for all N, setting $m_0 = m_0(\beta_0, h_0)$, we have

$$(\mu_N, U_N, R_N) \in \Delta = \{(x, y, z);\, |x - m^*| \leq L_0 \sqrt{\frac{\alpha_0}{m_0^{*2}}};\, |y|, |z| \leq L_0 \frac{\alpha_0}{m_0^{*2}}\}.$$

We define by induction $\Delta^0_{\alpha,\beta,h} = \Delta$ and $\Delta^{p+1}_{\alpha,\beta,h} = \Phi_{\alpha,\beta,h}(\Delta^p_{\alpha,\beta,h})$. It is now simple to show by induction over p that for each p, we have

$$\sup_{\alpha \in J} \lim_{N \to \infty} \int_D \mathrm{dist}\,((\mu_{N+1}, U_{N+1}, R_{N+1}), \Delta^p_{\alpha,\beta,h}) d\beta dh d\gamma = 0. \tag{7.23}$$

Lemma 7.9. *The set* $\bigcap_p \Delta^p_{\alpha,\beta,h}$ *consists of a single point.*

Proof. This very tedious proof should be omitted at first reading. We would be done if $\Phi = \Phi_{\alpha,\beta,h}$ were a contraction. The differential of Φ is the matrix

$$V = \begin{pmatrix} \beta E\frac{1}{\mathrm{ch}^2 Y} & 0 & A \\ (2\alpha+\beta y)C & \beta E\frac{1}{\mathrm{ch}^2 Y} & (2\alpha+\beta y)B \\ tC & \frac{\beta}{2}E\mathrm{th}^2 Y & \beta E\frac{1}{\mathrm{ch}^2 Y} + tB \end{pmatrix}$$

where $t = \beta z - \alpha - \beta y/2$,

$$C = \frac{\partial}{\partial x}E\frac{1}{\mathrm{ch}^2 Y} \quad B = \frac{\partial}{\partial z}E\frac{1}{\mathrm{ch}^2 Y} \quad A = \frac{\partial}{\partial z}E\mathrm{th}Y.$$

Thus, setting $Q = E\mathrm{th}^2 Y$, we have

$$C = -2\beta E\frac{\mathrm{th}Y}{\mathrm{ch}^2 Y}.$$

so that, by Cauchy-Schwarz,

$$|C| \le 2\beta\sqrt{Q}. \tag{7.24}$$

Using integration by parts, we have

$$B = \frac{\beta}{\sqrt{z}}E(g\frac{\mathrm{th}Y}{\mathrm{ch}^2 Y}) = \beta^2 E(\frac{1-2\mathrm{sh}^2 Y}{\mathrm{ch}^4 Y})$$

so that

$$|B| \le 2\beta^2. \tag{7.25}$$

Similarly,

$$A = \frac{\beta}{2\sqrt{z}}E\frac{g}{\mathrm{ch}^2 Y} = -\beta^2 E\frac{\mathrm{th}Y}{\mathrm{ch}^2 Y}$$

so that

$$|A| \le \beta^2\sqrt{Q}. \tag{7.26}$$

Despite these estimates, and the fact that

$$|t| \le L\beta\frac{\alpha_0}{m_0^{*2}} \tag{7.27}$$

it does not seem that V is a contraction (for the euclidean norm). On the other hand, if (e_1, e_2, e_3) denotes the canonical basis of $\mathbb{R}^3$, in the basis $(\lambda_1 e_1, \lambda_2 e_2, e_3)$, the matrix of V becomes

$$V' = \begin{pmatrix} \beta(1-Q) & 0 & \lambda_1 a_{13} \\ \frac{\lambda_2}{\lambda_1} a_{21} & \beta(1-Q) & \lambda_2 a_{23} \\ a_{31}/\lambda_1 & a_{32}/\lambda_2 & \beta(1-Q) + t\beta \end{pmatrix} \tag{7.28}$$

where

$$\begin{aligned} |a_{13}| &= |A| \le L\sqrt{Q} \\ |a_{21}| &= |(2\alpha + \beta y)C| \le L\frac{\alpha_0}{m_0^{*2}}\sqrt{Q} \\ |a_{23}| &= |(2\alpha + \beta y)C| \le L\frac{\alpha_0}{m_0^{*2}}\sqrt{Q} \\ |a_{31}| &= |tC| \le L\frac{\alpha_0}{m_0^{*2}}\sqrt{Q} \\ |a_{32}| &= \frac{\beta}{2}Q \le LQ. \end{aligned}$$

(We recall that $\beta \le 2$).

If we can find λ_2, λ_1 (independent of $(\beta, h) \in D_0, (x, y, z) \in \Delta$) such that V' is always a contraction, the proof is finished. Assuming $\alpha_0/m_0^{*2} \le Q$, in an effort to minimize the largest off diagonal term of V', we take

$$\lambda_1 = (\frac{\alpha_0}{m_0^{*2}})^{1/3} Q_0^{1/6} \qquad \lambda_2 = (\frac{\alpha_0}{m_0^{*2}})^{-1/3} Q_0^{1/3}$$

where Q_0 is the maximum value of Q over $\alpha \in J, (\beta, h) \in D_0, (x, y, z) \in \Delta$. The maximum off diagonal term of V' is then at most $L(\frac{\alpha_0}{m_0^{*2}})^{1/3} Q_0^{2/3}$. To show that V' is a contraction under the condition $\alpha_0 \le m_0^{*4}/L'$, it then suffices to check that

$$\frac{1}{L}\max(m_0^{*2}, Q_0) + \beta(1-Q) < 1$$

which follows easily from the method of Lemma 7.3. □

We denote by (μ, u, r) the unique point of $\bigcap_p \Delta^p_{\alpha,\beta,h}$, the dependence in the parameters being kept implicit. Thus (7.23) implies

$$\lim_{N\to\infty} \int_D (|\mu_N - \mu| + |U_N - u| + |R_N - r|) d\beta dh d\gamma = 0. \tag{7.29}$$

Since (μ, u, r) is a fixed point of Φ, setting $Y = \beta(g\sqrt{r} + \mu + h)$ and $q = E\mathrm{th}^2 Y$, we get the equations

$$\mu = E\mathrm{th}Y \tag{7.30}$$

$$u = (2\alpha + \beta u)(1 - q) \tag{7.31}$$

$$r = \alpha q + \beta r(1-q) + \frac{\beta u}{2}(1-q). \tag{7.32}$$

The second equation yields

$$u = \frac{2\alpha(1-q)}{1 - \beta(1-q)} \tag{7.33}$$

and the third then yields

$$r = \frac{\alpha q}{(1 - \beta(1-q))^2}. \tag{7.34}$$

We now turn to the proof of (1.13). When $\alpha = 1/N (M = 1)$ the Hopfield model reduces to the simple Curie Wiess model, so that $N^{-1} EF_N \simeq RS$ is easy in that case. Thus it suffices to prove that

$$N^{-1} E \frac{\partial F_N}{\partial \alpha} \simeq \frac{\partial RS}{\partial \alpha}. \tag{7.35}$$

Since F_N is defined only for values of α of the type $M/N (M \geq 1)$, the left-hand side of (7.35) makes no sense apriori. So we extend the definition of $F_N(\alpha, \beta, h, \gamma)$ to all values of $\alpha \geq 1/M$ by linear interpolation in α (at β, h, γ fixed). In that way, we have

$$\frac{\partial F_N}{\partial \alpha}(\alpha, \beta, h, \gamma) = N\left(F_N\left(\alpha_1 + \frac{1}{N}, \beta, h, \gamma\right) - F_N(\alpha_1, \beta, h, \gamma)\right) \tag{7.36}$$

where $\alpha_1 = M/N$, for the integer M with $M/N \leq \alpha < (M+1)/N$ and where the derivative is understood as a right derivative.

Consider a fresh Bernoulli sequence $(\eta_i)_{i \leq N}$. Then (7.36) implies

$$EN^{-1} \frac{\partial F_N}{\partial \alpha}(\alpha, \beta, h, \gamma) = E \log\langle \exp \frac{\beta}{2N} \Big(\sum_{i \leq N} \epsilon_i \eta_i \Big)^2 \rangle \tag{7.37}$$

where the bracket is for the values $\alpha_1, \beta, h, \gamma$ of the parameters. To compute the right-hand side of (7.35), we observe that if in (1.14) we think of the right-hand side as a function of independent variables $\alpha, \beta, h, \mu, q, r$, equations (1.11) , (1.12), (7.34) mean that the partial derivatives of this function in μ, q, r, are zero. This makes it obvious that

$$\frac{\partial RS}{\partial \alpha} = \frac{1}{2}\left(\frac{\beta q}{1 - \beta(1-q)} - \log(1 - \beta(1-q)) \right). \tag{7.38}$$

The key to (1.13) is the following precise version of (7.35).

Proposition 7.10. *If D_0 is a small enough neighborhood of $(\beta_0, h_0)(\beta_0 < 2, h_0 > 0)$ we have*

$$\lim_{N\to\infty}\int_D |E\log\langle\exp\frac{\beta}{2N}\Big(\sum_{i\le N}\epsilon_i\eta_i\Big)^2\rangle - \frac{1}{2}\Big(\frac{\beta q}{1-\beta(1-q)} - \log(1-\beta(1-q))\Big)|d\beta dh d\gamma = 0$$

uniformly over $\alpha \le \frac{1}{L}((\beta_0-1)^2 + m^{*2}(\beta_0, h_0))$.

Indeed, once this is proved, we integrate over α (using (7.37), (7.38)) to get

$$\lim_{N\to\infty}\int_D |\frac{1}{N}EF_N(\alpha,\beta,h,\gamma) - RS(\alpha,\beta,h)|d\beta dh d\gamma = 0$$

Since D_0 is arbitrarily small, and since F_N and RS are convex functions of β, h, γ, (1.13) follows.

The basic fact is the elementary formula (left to the reader). If $c < 1$,

$$E_g\exp(ag + \frac{c}{2}g^2) = \frac{1}{\sqrt{1-c^2}}\exp\frac{a^2}{2(1-c)}. \tag{7.39}$$

The heuristic argument goes as follows. We write

$$\begin{aligned}\langle\exp\frac{\beta}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\eta})^2\rangle &= E_g\langle\exp\sqrt{\frac{\beta}{N}}g\boldsymbol{\epsilon}\cdot\boldsymbol{\eta}\rangle \\ &= E_g\exp\sqrt{\frac{\beta}{N}}g\boldsymbol{a}\cdot\boldsymbol{\eta}\langle\exp\sqrt{\frac{\beta}{N}}g\dot{\boldsymbol{\epsilon}}\cdot\boldsymbol{\eta}\rangle\end{aligned} \tag{7.40}$$

where $\boldsymbol{a} = \langle\boldsymbol{\epsilon}\rangle$. We know that (4.5), and hence (4.3) hold. The argument of Lemma 4.1 (used at many other places) then shows that

$$\langle\exp\sqrt{\frac{\beta}{N}}g\dot{\boldsymbol{\epsilon}}\cdot\boldsymbol{\eta}\rangle \simeq \langle\exp\frac{\beta g^2}{2N}\sum_{i\le N}\dot{\epsilon}_i^2\rangle. \tag{7.41}$$

We certainly hope that $\sum_{i\le N}\dot{\epsilon}_i^2$ to be nearly constant, so equal to

$$E\langle\sum_{i\le N}\dot{\epsilon}_i^2\rangle = N(1 - E(\frac{1}{N}\sum_{i\le N}\langle\epsilon_i^2\rangle)) \simeq 1-q$$

so that (7.41) should yield

$$\langle\exp\sqrt{\frac{\beta}{N}}g\dot{\boldsymbol{\epsilon}}\cdot\boldsymbol{\eta}\rangle \simeq \exp\frac{\beta}{2}(1-q)g^2.$$

Using (7.39),(7.40), we get

$$\langle \exp \frac{\beta}{2N} (\boldsymbol{\epsilon} \cdot \boldsymbol{\eta})^2 \rangle \simeq \frac{1}{\sqrt{1-\beta(1-q)}} \exp \frac{\beta}{2N} \frac{(\boldsymbol{a} \cdot \boldsymbol{\eta})^2}{1-\beta(1-q)}.$$

Taking logarithm and expectation conclude this scheme of proof.

It is unfortunately a nontrivial task to justify rigorously the previous approximations.

Proof of Proposition 7.10. A first observation is that

$$\log \langle \exp \frac{\beta}{2N} (\boldsymbol{\epsilon} \cdot \boldsymbol{\eta})^2 \rangle \leq \frac{\beta N}{2}$$

so that the influence of exponentially small events (in the quenched variables imlicit in $\langle \cdot \rangle$ and $\boldsymbol{\eta}$) is negligible. Appealing to Proposition 3.14, we can pretend that if

$$C = \{\boldsymbol{\epsilon} \in \Sigma_N, |m_1(\boldsymbol{\epsilon}) - m^*| \leq L \left(\frac{\alpha}{m^*} \log \frac{La^*}{m^{*2}} \right)^{1/2} \}$$

where $m^* = m^*(\beta, h)$, then $G(C) \geq 1 - \exp(-N/K) \geq 1/2$, and, appealing again to Proposition 3.14, this time for $M+1$ rather than for M, we can pretend also that

$$\langle \exp \frac{\beta}{2N} (\boldsymbol{\epsilon} \cdot \boldsymbol{\eta})^2 \rangle \leq (1 + e^{-N/K}) \langle 1_C \exp \frac{\beta}{2N} (\boldsymbol{\epsilon} \cdot \boldsymbol{\eta})^2 \rangle.$$

Thus, setting

$$T_N = \log \langle 1_C \exp \frac{\beta}{2N} (\boldsymbol{\epsilon} \cdot \boldsymbol{\eta})^2 \rangle$$

we have to show that

$$\lim_{N \to \infty} \int_D |T_N - \frac{\partial RS}{\partial \alpha}| d\beta dh d\gamma = 0. \tag{7.42}$$

Lemma 7.11. *Consider an event $\Omega_N = \Omega_N(\alpha, \beta, \gamma, h)$ (depending upon α, β, γ, h in a measurable way). If*

$$\lim_{N \to \infty} \int_D P(\Omega_N) d\beta dh d\gamma = 0 \tag{7.43}$$

then

$$\lim_{N \to \infty} \int_D E(1_{\Omega_N} T_N) d\beta dh d\gamma = 0. \tag{7.44}$$

In this statement, and the rest of the proof, we make the convention that it is understood that all limits are uniform on α satisfying condition (1.17).

Proof. If δ is a positive number (to be specified below) we have the inequality

$$(x+y)^2 \leq (1+\frac{1}{\delta})x^2 + (1+\delta)y^2. \tag{7.45}$$

This implies

$$T_N \leq \log U + \log V \tag{7.46}$$

for

$$U = \exp \frac{\beta}{2N}(1+\frac{1}{\delta})\Big(\sum_{i\leq N} \eta_i \eta_{i,1} m^*\Big)^2 \tag{7.47}$$

$$V = \langle 1_C \exp \frac{\beta}{2N}(1+\delta)\Big(\sum_{i\leq N} \eta_i(\epsilon_i - \eta_{i,1}m^*)\Big)^2 \rangle. \tag{7.48}$$

Using Cauchy Schwarz to write $E(1_{\Omega_N} T_N)^2 \leq P(1_{\Omega_N})E(T_N^2)$, it suffices from (7.46) to show that $E(\log U)^2 \leq K, E(\log V)^2 \leq K$. The first statement is obvious. For the second, we write, for $\boldsymbol{\epsilon} \in C$

$$\begin{aligned} N^{-1}\sum_{i\leq N}(\epsilon_i - \eta_{i,1}m^*)^2 &= 1 + m^{*2} - 2N^{-1}\sum_{i\leq N}\epsilon_i\eta_{i,1}m^* \\ &= 1 + m^{*2} - 2m^* m_1(\boldsymbol{\epsilon}) \\ &= 1 - m^{*2} + 2m^*(m^* - m_1(\boldsymbol{\epsilon})) \\ &\leq 1 - m^{*2} + Lm^*\left(\frac{\alpha}{m^*}\log\frac{La^*}{m^{*2}}\right)^{1/2} \end{aligned} \tag{7.49}$$

since $\boldsymbol{\epsilon} \in C$. By arguments already used, we see that if the constant of (1.17) is small enough, (and if D_0 is small enough), we can find $\delta > 0$ such that for (β, h) in D_0, and $\boldsymbol{\epsilon}$ in C we have

$$\beta(1+\delta)^3 N^{-1}\sum_{i\leq N}(\epsilon_i - \eta_{i,1}m^*)^2 \leq 1-\delta. \tag{7.50}$$

(The term $(1+\delta)^3$ rather than $(1+\delta)$ is required for further purposes). A straightforward extension of Lemma 2.1 show that

$$E\exp\frac{1}{2}\Big(\sum_{i\leq N}\alpha_i\eta_i\Big)^2 \leq \Big(1 - \sum_{i\leq N}\alpha_i^2\Big)^{-1/2}$$

and thus from (7.50) we see that $EV \leq K$. Since $V \geq 1/2$ (as $G(C) \geq 1/2$) it follows that $E(\log V)^2 \leq C$. □

We now construct the events Ω_N to which we will apply Lemma 7.11.

Lemma 7.12. *We can find a sequence $\delta_N \to 0$ such that the event*

$$\Omega_N = \{G(B) \le 1 - \delta_N \text{ or } |N^{-1} \sum_{i \le N} \langle \epsilon_i \rangle^2 - q| \ge \delta_N\} \cap C \tag{7.52}$$

satisfies (7.43).

Proof. We first show how to control the event

$$\{|\frac{1}{N} \sum_{i \le N} \langle \epsilon_i \rangle^2 - q| \ge \delta_N\}.$$

¿From (7.18) and (7.28), we have

$$\lim_{N \to \infty} \int_D |\frac{1}{N} E \sum_{i \le N} \langle \epsilon_i \rangle^2 - q| d\beta dh d\gamma = 0.$$

Thus it suffices to show that

$$\lim_{N \to \infty} \int_D |\frac{1}{N^2} E\Big(\sum_{i \le N} \langle \epsilon_i \rangle^2 - E \sum_{j \le N} \langle \epsilon_j \rangle^2 \Big)^2 | d\beta dh d\gamma = 0$$

or, equivalently, that if $i \ne j$

$$\lim_{N \to \infty} \int_D |E \langle \epsilon_i \rangle^2 \langle \epsilon_j \rangle^2 - q^2 | d\beta dh d\gamma = 0.$$

The reader should feel that to prove this is standard (reduction from N to $N-2$ spins) now that we know (7.29) and we have the tools of Section 6.

Since $\langle \dot\epsilon_i \rangle^2 = 1 - \langle \epsilon_i \rangle^2$, it suffices now to show that

$$\lim_{N \to \infty} \int_D \frac{1}{N^2} E \langle \Big(\sum_{i \le N} \dot\epsilon_i^2 - \sum_{i \le N} \langle \dot\epsilon_i \rangle^2 \Big)^2 \rangle d\beta dh d\gamma = 0$$

or, equivalently, that, for $i \ne j$

$$\lim_{N \to \infty} \int_D \frac{1}{N^2} E(\langle \dot\epsilon_i^2 \dot\epsilon_j^2 \rangle - \langle \dot\epsilon_i \rangle^2 \langle \dot\epsilon_j \rangle^2) d\beta dh d\gamma = 0.$$

Straightforward algebra shows that

$$\langle \dot\epsilon_i^2 \dot\epsilon_j^2 \rangle - \langle \dot\epsilon_i \rangle^2 \langle \dot\epsilon_j \rangle^2 = 4 \langle \dot\epsilon_i \dot\epsilon_j \rangle \langle \epsilon_i \rangle \langle \epsilon_j \rangle.$$

Thus is suffices to show that

$$\lim_{N \to \infty} \int_D \frac{1}{N^2} E \langle \dot\epsilon_i \dot\epsilon_j \rangle^2 d\beta dh d\gamma = 0. \tag{7.53}$$

The computation of $E\langle\dot{\epsilon}_i\dot{\epsilon}_j\rangle^2$, or, more conveniently, of $E\langle\tilde{\epsilon}_i\tilde{\epsilon}_j\rangle^2$ should again be felt as easy. □

Combining Lemmas 7.11 and 7.12, to prove (7.41) it then suffices to prove that (denoting by Ω_N^c the complement of Ω_N)

$$\lim_{N\to\infty}\int_D E|1_{\Omega_N^c}(T_N-\frac{\partial RS}{\partial\alpha})|d\beta dh d\gamma=0. \tag{7.54}$$

We start by a preparatory lemma.

Lemma 7.13. *a) If $x,y\geq 1/2$, we have*

$$|\log x-\log y|\leq\log(1+2|x-y|) \tag{7.55}$$

b) If $U,V\geq 0,t>1$, we have

$$\log(1+UV)\leq tV+\frac{(\log U)^2}{\log t}. \tag{7.56}$$

Proof. Since (a) is obvious, we prove only (b). If $U\leq t$, we have

$$\log(1+UV)\leq\log(1+tV)\leq tV.$$

If $U\geq t$, we have $U\geq 1$, and

$$\log(1+UV)\leq\log(U(1+V))\leq\log U+V\leq\frac{(\log U)^2}{\log t}+V.$$

Comment. The use of (b) is that if $E(\log U)^2\leq K, EV\to 0$, then $E\log(1+UV)\to 0$.

Lemma 7.14. *Consider the set B of (7.52) and*

$$T'_N=\log\langle 1_B\exp\frac{\beta}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\eta})^2\rangle.$$

Then

$$\lim_{N\to\infty}\int_D E1_{\Omega_N^c}|T'_N-T_N|d\beta dh d\gamma=0.$$

Proof. Using (7.55), we have

$$|T_N-T'_N|\leq\log(1+2\langle 1_{C\setminus B}\exp\frac{\beta}{2N}(\boldsymbol{\epsilon}\cdot\boldsymbol{\eta})^2\rangle).$$

Proceeding as in Lemma 7.11, we have

$$1_{\Omega_N^c}|T_N - T'_N| \le \log(1 + 2UV)$$

where U is given by (7.47) and V by (7.48), except that we have replaced 1_C by $1_{C\backslash B}$. Using Holder's inequality for Gibbs measure, we have

$$V \le G(C\backslash B)^{\frac{\delta}{1+\delta}} \langle 1_C \exp \frac{\beta}{2N}(1+\delta)^2 \Big(\sum_{i\le N} \eta_i(\epsilon_i - \eta_{i,1}m^*) \Big)^2 \rangle^{\frac{1}{1+\delta}}$$

Appealing to (7.50), we then see that $EV \le K\delta_N^{\delta/(1+\delta)}$. The result then follows from Lemma 7.13. □

Lemma 7.15. *Consider*

$$\varphi_2 = \exp \sqrt{\frac{\beta}{N}} g\boldsymbol{a}\cdot\boldsymbol{\eta} \langle 1_B \exp \frac{\beta}{2N} g^2 \sum_{i\le N} \dot{\epsilon}_i^2 \rangle$$

and $T''_N = \log E_g \varphi_2$, *where* E_g *denotes expectation in* g *only. Then*

$$\lim_{N\to\infty} \int_D E 1_{\Omega_N^c} |T''_N - T_N| d\beta dh d\gamma = 0. \tag{7.57}$$

To provide motivation for this last effort, we show why this proves (7.54), and finishes the proof of Proposition 7.10. Using (7.39), we have

$$T''_N = \log \langle 1_B \frac{1}{\sqrt{1 - \beta(N^{-1} \sum_{i\le N} \dot{\epsilon}_i^2)}} \exp \frac{\beta(\boldsymbol{a}\cdot\boldsymbol{\eta})^2}{2(1 - \beta N^{-1} \sum_{i\le N} \dot{\epsilon}_i^2)} \rangle.$$

We use the definition of B to control $\sum_{i\le N} \dot{\epsilon}_i^2$ from above and from below; we then take expectation in $\boldsymbol{\eta}$, and use the definition of Ω_N to control $\|\boldsymbol{a}\|$ and $G(B)$. The result follows.

Proof of Lemma 7.15. We set

$$\varphi_1 = \exp g \sqrt{\frac{\beta}{N}} \boldsymbol{a}\cdot\boldsymbol{\eta} \langle 1_B \exp g \sqrt{\frac{\beta}{N}} \dot{\boldsymbol{\epsilon}}\cdot\boldsymbol{\eta} \rangle$$

so that $T'_N = \log E_g \varphi_1$. In view of Lemma 7.14, we can replace T_N by T'_N in (7.57). Appealing to (7.55), we write (since $G(B) \ge 1/2$)

$$|T''_N - T'_N| \le \log(1 + 2E_g|\varphi_1 - \varphi_2|).$$

For $\xi > 0$, we have, with obvious notation

$$E_g|\varphi_1 - \varphi_2| \leq \xi + UV$$

where

$$U(E_g|\varphi_1 - \varphi_2|^{1+\delta})^{\frac{1}{1+\delta}}$$
$$V = (P_g(|\varphi_1 - \varphi_2| \geq \xi))^{\frac{\delta}{1+\delta}}$$

As $\log(x+\xi) \leq \xi + \log x$ for $\xi \geq 0, x \geq 1$, it suffices to show by Lemma 7.13 that for any ξ we have

$$E(\log U)^2 \leq K \tag{7.58}$$

$$\lim_{N\to\infty} \int_D EV d\beta dh d\gamma = 0. \tag{7.59}$$

We start by writing $|\varphi_1 - \varphi_2| = \varphi_3\varphi_4$ for

$$\varphi_3 = \exp g\sqrt{\frac{\beta}{N}}\boldsymbol{a}\cdot\boldsymbol{\eta}$$
$$\varphi_4 = |\langle 1_B(\exp\sqrt{\frac{\beta}{N}}g\dot{\boldsymbol{\epsilon}}\cdot\boldsymbol{\eta} - \exp\sqrt{\frac{\beta}{N}}g^2\sum_{i\leq N}\dot{\epsilon}_i^2)\rangle|.$$

Thus, from Hölder's inequality

$$U = E_g\left(\varphi_3^{1+\delta}\varphi_4^{1+\delta}\right)^{\frac{1}{1+\delta}} \leq \left(E_g\varphi_3^{\frac{(1+\delta)^2}{\delta}}\right)^{\frac{\delta}{(1+\delta)^2}}\left(E_g\varphi_4^{(1+\delta)^2}\right)^{\frac{1}{(1+\delta)^2}}.$$

Using (7.50), one sees that the expected value of the last term remains bounded; then (7.58) follows easily. To prove (7.59), it suffices to prove that

$$\lim_{N\to\infty}\int_D P(|\varphi_1 - \varphi_2| \geq \xi) d\beta dh d\gamma = 0.$$

Since for each $t > 0, P(\varphi_3 \geq e^t) \leq K/t$, (because $E\log\varphi_3 \leq K$) it suffices to show that

$$\lim_{N\to\infty}\int_D P(|\varphi_4| \geq \xi) d\beta dh d\gamma = 0.$$

Now, it suffices to show that for each value of g

$$\lim_{N\to\infty}\int_D E'(\varphi_4) d\beta dh d\gamma = 0$$

where E' denotes conditional expectation at g given. But this follows from (7.53) and the fact that, as shown by the argument of Lemma 4.1, for each value of g we have

$$E'(\varphi_4) \leq K(g)\left(\langle(\dot{\boldsymbol{\epsilon}} \cdot \dot{\boldsymbol{\epsilon}}')^2\rangle^{1/2} + \frac{1}{N}\right).$$

(One of the difficulties that make the present proof delicate is that it is not true that $E_g K(g) < \infty$). The proof is finished. □

Finally, we explain how to handle the (much easier) case where $\beta \geq 2$. (More precisely, we will consider only the case $\beta \geq \beta_0$, where β_0 is a large enough constant. This is sufficient because the analysis done for $\beta \leq 2$ also holds for $\beta \leq \beta_0$, possibly with different constants). The main difficulty in the case $\beta \leq 2$ was the crucial coefficient $\beta E \frac{1}{\mathrm{ch}^2 Y}$ was possibly dangerously close to one.

We will show that this is not possible for$\beta \geq \beta_0, \alpha \leq K^{-1} \log \beta$. We now have $m^*, \geq 1/K$, and (7.1) holds $S = \sqrt{\alpha}/K$, so that $\langle u_1 \rangle_0 \geq 1/K$. We have

$$E \frac{1}{\mathrm{ch}^2 Y} \leq \frac{1}{\mathrm{ch}(\beta/2)} + P(|g|\|b\| \geq \langle u_1 \rangle_0).$$

Since (7.1) implies $\|b\| \leq K\sqrt{\alpha}$, we thus have

$$E \frac{1}{\mathrm{ch}^2 Y} \leq \frac{1}{\mathrm{ch}(\beta/2)} + \exp(-\frac{1}{K\alpha})$$

so that $\beta E 1/\mathrm{ch}^2 Y$ is small for $\beta \geq \beta_0$ and $\alpha \leq K^{-1} \log \beta$. We leave to the reader the easy to perform necessary modifications of the previous arguments. These results actually in great simplifications (e.g. the matrix V of Lemma 7.9 can now be assumed to be a contraction).

8. Stability.

In this section we study the properties of the system when the parameters are close to a given value α_0, β_0, h_0. We fix a small interval J around α_0, a small cube D_0 around (β_0, h_0). We set $D = D_0 \times [0,1]$. The sentence "for the parameters in $J \times D$" means "for $(\alpha, \beta, h, \gamma) \in J \times D$".

Considering a function U of β, h, γ, we will write

$$I(U) = \int_D U d\beta dh d\gamma$$

By the expression "A random function g_N from $\Sigma_N^p \to \mathbb{R}$ defined on D" we mean a (measurable) function g_N from $D \times \{-1,1\}^{N\times M} \times \mathbb{R}^{M-1} \times \Sigma_N^p$ to $\mathbb{R}$; that is, g_N depends upon the parameters, the quenched variables, the coefficients $(g_k)_{2\le k\le M}$ of the perturbed Hamiltonian, the spins.

Fixing all these quantities except the spins, we can integrate g with respect to the Gibbs measure, take expectations, and integrate over D to define $I(E\langle g\rangle)$ (that depends upon α only).

Definition 8.1 (Condition (EI)). We say that a family (f_N) of positive random functions defined on D satisfies condition (EI) (for "equintegrable") if, given $\rho > 0$, there exists $\rho_1 > 0$ and $N_0 > 0$, such that for each $N \ge N_0$, for each random function g defined on D, valued in $[0,1]$, we have, for each α in J,

$$I(E\langle g\rangle) \le \rho_1 \Rightarrow I(E\langle f_N g\rangle) \le \rho I((E\langle f_N\rangle). \tag{8.1}$$

What this means is simply that, when computing $I(E\langle f_N\rangle)$, the contribution of sets that are very small for the Gibbs measure, of rare events, and of exceptional sets of parameters, is vanishingly small compared to $I(E\langle f_N\rangle)$.

A much stronger (but easier to understand) property would be an inequality

$$I(E\langle f_N^2\rangle) \le K(I(E\langle f_N\rangle))^2$$

as follows from Cauchy Schwarz inequality

$$I(E\langle f_N\rangle) \le (I(E\langle g^2\rangle))^{1/2}(I(E\langle f_N^2\rangle))^{1/2}.$$

It seems very likely that this stronger property holds over the entire range of parameters for the two functions considered in the next conjecture.

Conjecture 8.2. *When α, β, h satisfy the condition of Theorem 1.7, if J and D_0 are small enough, the families of random functions $(\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})^2$ and $\sum_{2\le k\le M} m_k^4$ (from Σ_N^3 and Σ_N respectively) to $\mathbb{R}^+$ satisfy condition (EI).*

We now assume that J and D_0 have been chosen so that the functions of Conjecture 8.2 satisfy condition (EI). *The subsequent results of this chapter depend on* *this unproven fact.* The reader can check that in fact quite less would be necessary. But, at this stage of our ignorance, it seems simpler to make a convenient blanket assumption. To simplify notation, we will denote by $\mathcal{N}(A_N)$ a quantity (depending upon α, β, h, γ only) with the following property: Given $\rho > 0$, there is $\rho_1 > 0$ such that for each α in J,

$$I(C_N), I(A_N) \le \rho_1 \Rightarrow I(\mathcal{N}(A_N)) \le \rho I(A_N). \tag{8.2}$$

Here A_N, C_N are as in Sections 6 and 7. Quantities $\mathcal{N}(C_N)$ are defined similarly. The idea is simply that when trying to establish a relation $I(A_{N+1}) \le \theta I(A_N)$, for $\theta < 1$, if we know that $I(A_N)$ and $I(C_N)$ are small, terms $\mathcal{N}(A_N)$ are irrelevant.

The following explains one way to use condition (EI). The notation is as in Proposition 6.4.

Lemma 8.3. *We have*

$$E\langle \sum_{2\le k\le M} m_k^6\rangle = \mathcal{N}(A_N) \tag{8.3}$$

$$E\langle UV\rangle = \mathcal{N}(A_N) \tag{8.4}$$

for $U = \sum_{2\le k\le M} u_k^3 v_k$ *or* $U = \sum_{2\le k\le M} u_k^4$, $V = |\dot{\boldsymbol{w}}_\ell \cdot \dot{\boldsymbol{w}}_{\ell'}|$ *or* $V = |\boldsymbol{w}_\ell \cdot \boldsymbol{b}|$.

Proof. Inequality (8.3) should be obvious if one observes that

$$\langle \sum_{2\le k\le M} m_k^6\rangle \le \langle (\sum_{2\le k\le M} m_k^4) w\rangle$$

for $w = (\sum_{2\le k\le M} m_k^4)^{1/2}$, and thus $E\langle w\rangle \le A_N^{1/2}$. To prove (8.4), we first reduce to the case $U = \sum_{2\le k\le M} u_k^4$ by writing

$$E\langle |\sum_{2\le k\le M} u_k^3 v_k V|\rangle \le (E\langle \sum_{2\le k\le M} u_k^4 V\rangle)^{3/4} (E\langle \sum_{2\le k\le M} v_k^4 V\rangle)^{1/4}.$$

We then observe that $E\langle V\rangle \le LC_N$ by (7.3), (7.4). □

Using Lemma 8.3 to control the error terms it should be obvious, following the computations of Lemmas 7.5 to 7.7 that

Lemma 8.4. *We have*

$$\begin{aligned} A_{N+1} \le\ & \beta' E\Big(\frac{1}{ch^2 Y}\langle \sum_{2\le k\le M} u_k^4\rangle_0\Big) \\ & + \beta' E\big(th^2 Y(\langle \sum_{2\le k\le M} u_k^4\rangle_0 - \langle \sum_{2\le k\le M} u_k^3 v_k\rangle_0)\big) + \mathcal{A}_N + \mathcal{N}(A_N) \end{aligned}$$

$$B_{N+1} \le \beta' E\Big(\frac{1}{ch^2 Y}(\langle \sum_{2\le k\le M} u_k^4\rangle_0 - \langle \sum_{2\le k\le M} u_k^3 v_k\rangle_0)\Big) + \mathcal{A}_N + \mathcal{N}(A_N).$$

Combining these relations, we will be able to take care of A_N, and we go back to the main point, the study of C_{N+1}, that is of $C_{N+1}^{(1)}$ and $C_{N+1}^{(2)}$. First we study $C_{N+1}^{(1)}$. The error (6.14) is now $\mathcal{N}(A_N)$, so we study $C_{N+1}^{(\ell)}, \ell = 9, 10, 11$ through Proposition 6.4. The error terms $E(1), E(2)$ are $\mathcal{N}(C_N)$ (appealing again to (7.3), (7.4)). Thus, we have to account for the contributions of the terms I to V. This is easy for $C_{N+1}^{(9)}$. The contribution of I is

$$\beta' E(\frac{1}{\operatorname{ch}^2 Y}\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})\rangle_0) \tag{8.5}$$

and the contributions of II to V are $\mathcal{N}(A_N)$. Concerning $C_{N+1}^{(10)}$ and $C_{N+1}^{(11)}$, a bit of patience is needed. We regroup the contributions of each term after averaging over the signs ξ, θ, δ.

Study of $C_{N+1}^{(10)}$. To use Proposition 6.4, we take $\boldsymbol{w}_1 = \boldsymbol{u}, \boldsymbol{w}_2 = \boldsymbol{u}', \boldsymbol{w}_3 = \boldsymbol{v}, \delta_1 = \xi, \delta_2 = -\xi, \delta_3 = \theta$.

Contribution of the terms I and III. These are zero. This is obvious for I, and for III this follows from cancellation when summing over ξ.

Contribution of the term II. This is

$$\begin{aligned}&\frac{\beta'^3}{4}\sum_{\xi,\theta=\pm 1}\xi\theta E(\frac{\exp\theta Y}{\operatorname{ch}^3 Y}\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})\|\boldsymbol{v}\|^2(-\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}}' + \xi\theta\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{v}} - \xi\theta\dot{\boldsymbol{u}}'\cdot\dot{\boldsymbol{v}})\rangle_0)\\&\quad = \beta'^3 E(\frac{1}{\operatorname{ch}^2 Y}\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})(\tilde{\boldsymbol{u}}\cdot\dot{\boldsymbol{v}})\|\boldsymbol{v}\|^2\rangle_0)\end{aligned}$$

Contribution of the term IV. This is

$$\begin{aligned}&\frac{\beta'^3}{4}\sum_{\xi,\theta\pm 1}\xi\theta E(\theta\frac{\exp\theta Y}{\operatorname{ch}^3 Y}\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})\|\boldsymbol{v}\|^2(\xi\dot{\boldsymbol{u}}\cdot\boldsymbol{b} - \xi\dot{\boldsymbol{u}}'\cdot\boldsymbol{b} + \theta\dot{\boldsymbol{v}}\cdot\boldsymbol{b})\rangle_0\\&\quad = \beta'^3 E(\frac{1}{\operatorname{ch}^2 Y}\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})(\tilde{\boldsymbol{u}}\cdot\boldsymbol{b})\|\boldsymbol{v}\|^2\rangle_0)\end{aligned}$$

Contributions of the term V. This is

$$\begin{aligned}&-\frac{3\beta'^2}{4}\sum_{\xi,\theta=\pm 1}\xi\theta E(\exp\theta Y\frac{\operatorname{th}Y}{\operatorname{ch}^4 Y}\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})\|\boldsymbol{v}\|^2(\xi\dot{\boldsymbol{u}}\cdot\boldsymbol{b} - \xi\dot{\boldsymbol{u}}'\cdot\boldsymbol{b} + \theta\dot{\boldsymbol{v}}\cdot\boldsymbol{b}\rangle_0)\\&= -3\beta'^3 E(\frac{\operatorname{th}^2 Y}{\operatorname{ch}^2 Y}\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})(\tilde{\boldsymbol{u}}\cdot\boldsymbol{b})\|\boldsymbol{v}\|^2\rangle_0).\end{aligned}$$

Study of $C_{N+1}^{(11)}$. There we have one more variable $\boldsymbol{w}_4 = \boldsymbol{w}$, and $\delta_4 = \delta$.

It should be obvious that if one applies Proposition 6.4 to the case of $f = (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})(\boldsymbol{v} \cdot \dot{\boldsymbol{w}})$, or $f = (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})(\dot{\boldsymbol{v}} \cdot \boldsymbol{b})$, all the contributions are $\mathcal{N}(C_N)$. Thus, writing

$$\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{b}\|^2 + \dot{\boldsymbol{v}} \cdot \boldsymbol{b} + \boldsymbol{v} \cdot \dot{\boldsymbol{w}}$$

we can replace $f = (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})(\boldsymbol{v} \cdot \dot{\boldsymbol{w}})$ by $f = \|\boldsymbol{b}\|^2(\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})$, up to terms that are $\mathcal{N}(C_N)$. The contribution of the terms I and III is zero as in the case of $C_{N+1}^{(10)}$.

Contribution of the terms II. This is

$$\begin{aligned}
&\frac{-3\beta'^3}{8} \sum_{\xi,\theta,\delta=\pm 1} \xi\delta E\Big(\frac{\exp(\delta+\theta)Y}{\mathrm{ch}^4 Y} \langle (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v}) \|\boldsymbol{b}\|^2 (-\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}' + \xi\theta(\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{v}} - \dot{\boldsymbol{u}}' \cdot \boldsymbol{v}) \\
&+ \xi\delta(\dot{\boldsymbol{u}} \cdot \boldsymbol{w} - \dot{\boldsymbol{u}}' \cdot \dot{\boldsymbol{w}}) + \theta\delta(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{w}}))\rangle_0\Big) \\
&= -3\beta'^3 E\Big(\frac{\mathrm{th}^2 Y}{\mathrm{ch}^2 Y} \langle (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})(\tilde{\boldsymbol{u}} \cdot \dot{\boldsymbol{v}}) \|\boldsymbol{b}\|^2 \rangle_0\Big)
\end{aligned}$$

Contribution of the terms IV. This is

$$\begin{aligned}
&\frac{-3\beta'^3}{8} \sum_{\xi,\theta,\delta=\pm 1} \xi\delta E\Big((\delta+\theta)\frac{\exp(\delta+\theta)Y}{\mathrm{ch}^4 Y} \langle (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v}) \|\boldsymbol{b}\|^2 (\xi\dot{\boldsymbol{u}} \cdot \boldsymbol{b} - \xi\dot{\boldsymbol{u}}' \cdot \boldsymbol{b} \\
&+ \theta\dot{\boldsymbol{v}} \cdot \boldsymbol{b} + \delta\dot{\boldsymbol{w}} \cdot \boldsymbol{b})\rangle_0\Big) \\
&= -3\beta'^3 E\Big(\Big(\frac{1}{\mathrm{ch}^2 Y} + \frac{\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}\Big) \langle (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})(\tilde{\boldsymbol{u}} \cdot \boldsymbol{b}) \|\boldsymbol{b}\|^2 \rangle_0\Big)
\end{aligned}$$

Contribution of the term V. This is

$$\begin{aligned}
&\frac{12\beta'^3}{8} \sum_{\xi,\theta,\delta=\pm 1} \xi\delta E\Big(\frac{\exp(\delta+\theta)Y \,\mathrm{th} Y}{\mathrm{ch}^5 Y} \langle (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v}) \|\boldsymbol{b}\|^2 (\xi(\dot{\boldsymbol{u}} \cdot \boldsymbol{b} - \dot{\boldsymbol{u}}' \cdot \boldsymbol{b}) \\
&+ \theta\dot{\boldsymbol{v}} \cdot \boldsymbol{b} + \delta\dot{\boldsymbol{w}} \cdot \boldsymbol{b})\rangle \\
&= 12\beta'^3 E\Big(\frac{\mathrm{th}^2 Y}{\mathrm{ch}^4 Y} \langle (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})(\tilde{\boldsymbol{u}} \cdot \boldsymbol{b}) \|\boldsymbol{b}\|^2 \rangle_0\Big).
\end{aligned}$$

We leave to the reader to perform a similar (but simpler) computation for $C_{N+1}^{(2)}$ to find a total contribution for $C_{N+1}^{(2)}$ of

$$\alpha\beta'^2 E\Big(\frac{1}{\mathrm{ch}^2 Y} \langle (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})^2 \rangle_0 - 3\frac{\mathrm{th}^2 Y}{\mathrm{ch}^4 Y} \langle (\tilde{\boldsymbol{u}} \cdot \boldsymbol{b})^2 \rangle_0\Big).$$

We regroup the terms, observing that $\langle (\tilde{\boldsymbol{u}} \cdot \boldsymbol{v})(\tilde{\boldsymbol{u}} \cdot \boldsymbol{b}) \rangle_0 = \langle (\tilde{\boldsymbol{u}} \cdot \boldsymbol{b})^2 \rangle_0$, and observing that, (using Lemma 10.11) we can replace $\|\boldsymbol{v}\|^2$ by $\langle \|\boldsymbol{u}\|^2 \rangle_0$ everywhere. We thus have proved the following.

Proposition 8.5. *We have*

$$(8.6) C_{N+1} \le \beta' E\Big(\Big(\frac{1}{\mathrm{ch}^2 Y} + \beta'^2 \frac{1}{\mathrm{ch}^2 Y}\langle \|\boldsymbol{u}\|^2\rangle_0 - 3\beta'^2 \frac{\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}\|\boldsymbol{b}\|^2 + \alpha\beta' \frac{1}{\mathrm{ch}^2 Y}\Big)\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0\Big)$$
$$+ 3\beta'^2 E\Big(\Big(\frac{4\beta\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}\|\boldsymbol{b}\|^2 - \beta' \frac{\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}\langle\|\boldsymbol{u}\|^2\rangle_0 - \beta' \frac{1}{\mathrm{ch}^2 Y}\|\boldsymbol{b}\|^2 - \frac{\alpha\mathrm{th}^2 Y}{\mathrm{ch}^2 Y}\Big)\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{b})^2\rangle_0\Big)$$
$$+ \mathcal{A}_N + \mathcal{N}(A_N) + \mathcal{N}(C_N)$$

Keeping the notation of Section 7, consider now $Y_N = \beta'(g\sqrt{R_N} + \mu_N + h')$. Using Propositions 10.5, 10.8, it should be clear that (8.6) implies

$$(8.7)\ C_{N+1} \le \beta E\Big(\frac{1}{\mathrm{ch}^2 Y_N} + \frac{\beta^2}{\mathrm{ch}^2 Y_N} W_N - 3\beta^2 \frac{\mathrm{th}^2 Y_N}{\mathrm{ch}^2 Y_N} R_N + \frac{\alpha\beta}{\mathrm{ch}^2 Y_N}\Big) E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0$$
$$+ 3\beta^2 E\Big(4\beta \frac{\mathrm{th}^2 Y_N}{\mathrm{ch}^2 Y_N} R_N - \beta \frac{th^2 Y_N}{\mathrm{ch}^2 Y_N} W_N - \beta \frac{R_N}{\mathrm{ch}^2 Y_N} R_N - \alpha \frac{\mathrm{th}^2 Y_N}{\mathrm{ch}^2 Y_N}\Big) E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{b})^2\rangle_0$$
$$+ \mathcal{A}_N + \mathcal{N}(A_N) + \mathcal{N}(C_N) + KE(\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0 \varphi(\|\boldsymbol{b}\|^2 - R_N)$$

where φ is a function valued in $[0,1]$ such that $\lim_{x\to 0}\varphi(x) = 0$. We now show that the last term of (8.7) is $\mathcal{N}(C_N)$. Applying Definition 8.1 to the random function $g = \varphi(\|\boldsymbol{b}\|^2 - E\|\boldsymbol{b}\|^2)$, it then suffices to observe that $I(g)$ goes to zero as $N \to 0$ and $I(C_N)$ goes to zero by Propositions 10.5 and 10.7 (observe that $\|\boldsymbol{b}\|^2 = \langle\|\boldsymbol{u}\|^2 - \|\dot{\boldsymbol{u}}\|^2\rangle_0$).

Now we appeal to Guerra's identity (Proposition 10.9), that, with our notation can be written as

$$(8.8) \qquad 4E\langle(\boldsymbol{u}\cdot\boldsymbol{b})^2\rangle_0 = 3E\|\boldsymbol{b}\|^2 + E\langle(\boldsymbol{u}\cdot\boldsymbol{v})^2\rangle_0 + \mathcal{A}_N$$

and we note that

$$\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0 = 2\langle(\boldsymbol{u}\cdot\boldsymbol{v})^2\rangle_0 - 2\langle(\boldsymbol{u}\cdot\boldsymbol{b})^2\rangle_0$$
$$\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{b})^2\rangle_0 = 2\langle(\boldsymbol{u}\cdot\boldsymbol{b})^2\rangle_0 - 2\|\boldsymbol{b}\|^2$$

so that (8.7) implies

$$(8.9) \qquad E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{v})^2\rangle_0 = 3E\langle(\tilde{\boldsymbol{u}}\cdot\boldsymbol{b})^2\rangle_0 + \mathcal{A}_N.$$

Combining with (8.6), we now have

Proposition 8.6. *We have*

$$(8.10) \qquad C_{N+1} \le \Big(\beta E\Big(\frac{1}{ch^2 Y_N}\Big) + \Big(\beta^3 \frac{U_N}{2} + \alpha\beta^2\Big) E \frac{1}{ch^4 Y_N}\Big) C_N$$
$$+ \mathcal{A}_N + \mathcal{N}(C_N) + \mathcal{N}(A_N).$$

We can now prove the following precise version of Theorem 4.2.

Theorem 8.7. *Consider α, β, h such that*

$$\alpha\beta^2 E\frac{1}{ch^4\beta(g\sqrt{r}+\mu+h)} < (1-\beta(1-q))^2$$

where μ, r, q are as in (7.30) to (7.32). Under Conjecture 8.2, we can find J, D_0 small enough, we can find $\theta < 1$, we can find $\rho > 0, \epsilon_0 > 0$, and a sequence $a_N \to 0$ such that, if one sets $D_N = C_N + \rho A_N + B_N$, then for all α in J we have

$$\begin{aligned} &I(D_N) < \epsilon_0, I(|R_N - r|) \leq \epsilon_0, I(|U_n - u|) \leq \epsilon_0 \\ &\Rightarrow I(D_{N+1}) \leq \theta I(D_N) + a_N. \end{aligned}$$

Here $u = \alpha(1-q)/(1-\beta(1-q))$.

Proof. First, we pick J and D_0 small enough that for values of the parameters in $J \times D_0$ we have

$$\beta(1-q) + \frac{\alpha\beta^2}{1-\beta(1-q)} E\frac{1}{\mathrm{ch}^4\beta(g\sqrt{r}+\mu+h)} < \theta_1 < 1. \tag{8.11}$$

Next, we consider $\theta_1 < \theta_2 < \theta_3 < \theta < 1$, and the (random) subset D_N of D given by

$$D_N = \{(\beta, h); \beta E\frac{1}{\mathrm{ch}^2 Y_N} + (\beta^2\frac{U_N}{2} + \alpha\beta^2)E\frac{1}{\mathrm{ch}^4 Y_N} > \theta_2\}.$$

Thus (8.10) implies

$$I(C_{N+1}) \leq \theta_2 I(C_N) + KI(C_N 1_{D_N}) + I(\mathcal{A}_N) + I(\mathcal{N}(C_N)) + I(\mathcal{N}(A_N)). \tag{8.12}$$

Next, we observe that if in the expression

$$\beta E\frac{1}{\mathrm{ch}^2 Y_N} + (\beta^2\frac{U_N}{2} + \alpha\beta^2)E\frac{1}{\mathrm{ch}^4 Y_N}$$

we substitute $R_N = r$ and $U_N = u$, we find the left-hand side of (8.11). This implies that, given $\rho_1 > 0$, we can find ϵ_0 such that if

$$I(|R_N - r|) < \epsilon_0, I(|U_N - u|) < \epsilon_0 \tag{8.13}$$

then $I(1_{D_N}) < \rho_1$. Using Definition 8.1, we see that ϵ_0 can be found such that (8.13) implies

$$I(C_{N+1}) \leq \theta_3 I(C_N) + I(\mathcal{A}_N) + I(\mathcal{N}(C_N)) + I(\mathcal{N}(A_N)). \tag{8.14}$$

Next, using the argument of Proposition 8.6, it follows from Lemma 8.4 that

$$\begin{aligned} A_{N+1} &\leq \beta A_N E \frac{1}{\mathrm{ch}^2 Y_N} + \beta B_N E \mathrm{th}^2 Y_N + \mathcal{A}_N + \mathcal{N}(A_N) \\ B_{N+1} &\leq \beta B_N E \frac{1}{\mathrm{ch}^2 Y_N} + \mathcal{A}_N + \mathcal{N}(A_N). \end{aligned}$$

so that

$$\rho A_{N+1} + B_{N+1} \leq \beta (E \frac{1}{\mathrm{ch}^2 Y_N} + \rho E \mathrm{th}^2 Y_N)(\rho A_N + B_N) + \mathcal{A}_N + \mathcal{N}(A_N).$$

We fix ρ small enough that $\beta(1-q) + \rho q < \theta_1$ for all values of the parameters in $J \times D_0$. The argument that led to (8.14) (together with the fact that $B_N \leq A_N$) show that if ϵ_0 is small enough, (8.13) implies

$$I(\rho A_{N+1} + B_{N+1}) \leq \theta_3 I(\rho A_N + B_N) + \mathcal{A}_N + \mathcal{N}(A_N). \tag{8.15}$$

Combining with (8.14) we get

$$I(D_{N+1}) \leq \theta_3 I(D_N) + \mathcal{A}_N + I(\mathcal{N}(A_N)) + I(\mathcal{N}(C_N)). \tag{8.16}$$

Thus, if ϵ_0 is small enough

$$I(C_N) < \epsilon_0, I(A_N) < \epsilon_0 \Rightarrow I(D_{N+1}) \leq \theta I(D_N) + a_N.$$ □

9. Zero temperature.

In this section we study the random function

$$H(\boldsymbol{\epsilon}) = -\frac{N}{2} \sum_{k \le M} m_k^2(\boldsymbol{\epsilon}).$$

Thus, (except in Proposition 9.2), we assume $h = 0$.

First, we prove the simple fact that, for α small, the minima are located close to the prototypes.

Proposition 9.1. *Given $t < 1$, there is $\gamma(t) > 0$ such that, if $\alpha < \gamma(t)$, with overwhelming probability we have*

$$\sup\{\sum_{k \le M} m_k^2(\boldsymbol{\epsilon}); \forall k \le M, |m_k(\boldsymbol{\epsilon})| \le t\} \le 1 - \gamma(t).$$

Comment. Since, with overwhelming probability, the energy of each prototype is about $-N(\alpha+1)/2$, $(m_k(\boldsymbol{\eta}_k) = 1, \sum_{\ell \ne k} m_\ell(\boldsymbol{\eta}_k)^2 \sim \alpha)$, the minimum of H is located near a prototype.

Proof. First, we observe that if $\sum\limits_{k \le M} x_k^2 \ge b^2$ and $|x_k| \le t$ for each $k \le M$, we can find $(a_k)_{k \le M}$ with $\sum\limits_{k \le M} a_k^2 = 1, |a_k| \le t/b$, such that $\sum\limits_{k \le M} x_k a_k \ge 1$. (Indeed, $a_k = x_k(\sum\limits_{\ell \le M} x_\ell^2)^{-1/2}$ works). Thus, if $b = 1 - \gamma(t)$ and $t' = t/(1-\gamma(t))$, it suffices to show that

$$\sup\{\sum_{k \le M} a_k m_k(\boldsymbol{\epsilon}); \boldsymbol{\epsilon} \in \Sigma_N, |a_k| \le t', \sum_{k \le M} a_k^2 \le 1\} \le 1 - \gamma(t) \tag{9.1}$$

with overwhelming probability. Using concentration of measure, and more specifically Theorem 6.6 of [T2], it suffices to prove that

$$E \sup\{\sum_{k \le M} a_k m_k(\boldsymbol{\epsilon}) : \boldsymbol{\epsilon} \in \Sigma_N, |a_k| \le t', \sum_{k \le M} a_k^2 \le 1\} \le 1 - 2\gamma(t). \tag{9.2}$$

We observe that

$$\begin{aligned}\sup_{\boldsymbol{\epsilon} \in \Sigma_N} \sum_{k \le M} a_k m_k(\boldsymbol{\epsilon}) &= \sup_{\boldsymbol{\epsilon}} \frac{1}{N} \sum_{i \le N} \epsilon_i (\sum_{k \le M} \eta_{i,k} a_k) \\ &= \frac{1}{N} \sum_{i \le N} |\sum_{k \le M} \eta_{i,k} a_k|\end{aligned}$$

The key point, that is easy and left to the reader, is to show that there is a number $b(t) < 1$ such that

$$\sum_{k\le M} a_k^2 \le 1, \forall k \le M, |a_k| \le \frac{1+t}{2} \Rightarrow E|\sum_{k\le M} \eta_{i,k} a_k| \le b(t).$$

Thus, assuming as we may, $\gamma(t)$ small enough that $t' \le (1+t)/2$, we see that it suffices to prove that

$$E = E\sup\{|\frac{1}{N}\sum_{i\le N}(|Y_{i,a}| - E|Y_{i,a}|)|; a = (a_k), \sum_{k\le M} a_k^2 \le 1\} \le 4\sqrt{\alpha}.$$

where $Y_{i,a} = \sum_{k\le M} \eta_{i,k} a_k$. To do this we introduce independent copies $Y'_{i,a}$ of $Y_{i,a}$, so that we can replace $|Y_{i,a}| - E|Y_{i,a}|$ first by $|Y_{i,a}| - |Y'_{i,a}|$, then by $\eta_i|Y_{i,a}|$ where $(\eta_k)_{i\le N}$ is a fresh Bernoulli sequence. The comparison theorem for Bernoulli sequences [T1, Theorem 2.1] then allows to get rid of the absolute values, and one is reduced to

$$\begin{aligned} E\sup\{\frac{1}{N}|\sum_{i\le N, k\le M} \eta_i \eta_{i,k} a_k|; \sum_{k\le M} a_k^2 \le 1\} \\ = \frac{1}{N} E(\sum_{k\le M}(\sum_{i\le N}\eta_{i,k})^2)^{1/2} \le \sqrt{\alpha} \end{aligned}$$

using Cauchy-Schwarz. □

Here is a simple corollary, that was needed in Section 3. We use the notation of this Section.

Proposition 9.2. *Given $\delta > 0$ there is $\alpha(\delta) > 0, \beta(\delta)$ such that if $\beta > \beta(\delta), \alpha < \alpha(\delta)$, we have $EG(C) \le K\exp(-N/K)$, where*

$$C = \{\boldsymbol{\epsilon}; \forall k \le M, \|\boldsymbol{m}(\boldsymbol{\epsilon}) \pm m^* \boldsymbol{e}_k\| \ge \delta\}.$$

Proof. Since $\lim_{\beta\to\infty} m^* = 1$, we can replace $m^*\boldsymbol{e}_k$ by $\boldsymbol{e}_k$. We can then replace C by

$$C' = \{\boldsymbol{\epsilon}; \forall k \le M, |m_k(\boldsymbol{\epsilon})| \le t\}$$

for a certain t depending on δ only. Indeed, if, say, $m_k(\boldsymbol{\epsilon}) \ge t$, then all but a proportion of $(1-t)/2$ of components of $\boldsymbol{\epsilon}$ differ from the corresponding components of $\boldsymbol{\eta}_k$; It is then easily seen that, with overwhelming probability, for each such $\boldsymbol{\epsilon}$, $\sum_{\ell\ne k} m_\ell^2(\boldsymbol{\epsilon}) \le a(t)$, where $a(t)$ goes to zero with t.

Now, by Proposition 9.1, if α is small enough,

$$\sum_{\boldsymbol{\epsilon}\in C'} \exp(-\beta H(\boldsymbol{\epsilon})) \leq 2^N \exp(\frac{\beta N}{2}(1-\gamma(t)) + \beta h N)$$

while

$$\exp(-\beta H(\boldsymbol{\eta}_1)) \geq \exp(\frac{\beta N}{2} + \beta h N)$$

so that it suffices to ensure that $\beta\gamma(t) > \log 4$. □

Much of the rest of the present section is devoted to the study of the energy function $H(\boldsymbol{\epsilon})$ in the neighborhood of the prototype $\boldsymbol{\eta}_1$ (by symmetry, all prototypes play the same role). It is very useful to think about a point $\boldsymbol{\epsilon}$ close to $\boldsymbol{\eta}_1$ as a small perturbation of $\boldsymbol{\eta}_1$. As $\boldsymbol{\eta}_1$ plays now a special role, it is convenient to assume that $\eta_{1,k} = 1$ for all k. This does not change the distribution of H, as is seen by the transformation $(\epsilon_i)_{i\leq N} \to (\epsilon_i \eta_{1,k})_{i\leq N}$. Given a subset $I \subset \{1,\cdots,N\}$, we denote by $\boldsymbol{\eta}_I$ the point obtained from $\boldsymbol{\eta}_1$ by reversing the sign of all coordinates in I. Thus $\eta_{I,i} = -1$ if $i \in I$ and $= 1$ otherwise. By elementary algebra, we have the following.

Lemma 9.3. *If $n = \mathrm{card} I$, we have*

$$H(\boldsymbol{\eta}_I) = H(\boldsymbol{\eta}_1) + 2n - \frac{2n^2}{N} - \frac{2}{N}\sum_{k\geq 2}(\sum_{i\in I}\eta_{i,k})^2 + 2\sum_{i\in I}\sum_{k\geq 2}\eta_{i,k} m_k(\boldsymbol{\eta}_1) \tag{9.3}$$

Looking at this formula, we think of the term before the last as a perturbation term; To understand the last term we write

$$\sum_{k\geq 2}\eta_{i,k} m_k(\boldsymbol{\eta}_1) = \frac{M-1}{N} + \sum_{k\geq 2}\eta_{i,k} S_{i,k}$$

where

$$S_{i,k} = \frac{1}{N}\sum_{j\neq i}\eta_{j,k}.$$

This provides motivation to study the sequence $T_i = \sum_{k\geq 2}\eta_{i,k}S_{i,k}$.

Proposition 9.4. *The random distribution $\sum_{i\leq N} N^{-1}\delta_{T_i}$ converges weakly to $N(0,\alpha)$ with probability one as $N\to\infty$.*

Comment. This result was obtained independently by the author [T3] and by Bovier and Gayrard [B-G2].

Proof. Throughout the section, we set $\Phi(t) = P(g \geq t)$, where g is standard normal. We fix t in $\mathbb{R}$, and consider the set

$$A_i = \{(\eta_{j,k}); T_i \geq t\}.$$

We will prove that

$$E\left(\frac{1}{N}\sum_{i\le N} 1_{A_i} - \Phi\left(\frac{t}{\sqrt{\alpha}}\right)\right)^2 \le K\frac{\log N}{\sqrt{N}}. \tag{9.4}$$

This is a quantitative version of what we want. The proof relies upon the Berry-Essen theorem.

Lemma 9.5. [F, p. 542] *If $(X_i)_{i\le N}$ are centered independent r.v. then*

$$\sup_t \left|P\left(\sum_{i\le N} X_i \ge t\right) - \Phi\left(\frac{t}{\sigma}\right)\right| \le 3\frac{\rho^3}{\sigma^3} \tag{9.5}$$

where $\sigma^2 = \sum_{i\le N} EX_i^2, \rho^3 = \sum_{i\le N} E|X_i|^3.$

Since the variables $(\eta_{i,k}S_{i,k})_{k\ge 2}$ are independent and centered, we first deduce from the Berry-Essen theorem that

$$\left|P(A_i) - \Phi\left(\frac{t}{\sqrt{\alpha}}\right)\right| \le \frac{L}{\sqrt{N}}.$$

Thus, to prove (9.5) it suffices to prove that

$$i \ne j \Rightarrow P(A_i \cap A_j) \le \Phi^2\left(\frac{t}{\sqrt{\alpha}}\right) + \frac{K\log N}{\sqrt{N}}. \tag{9.6}$$

The beautiful idea there, that I learned from [Lou] is to use the theory of negatively associated r.v. [J-P]. Let us denote by P_m the conditional probability given $\boldsymbol{m} = (m_k)_{k\le M}$, where $m_k = m_k(\boldsymbol{\eta}_1) = N^{-1}\sum_{i\le N}\eta_{i,k}$, and by E_m the corresponding expectation. The theory of [J-P] implies that

$$P_m(A_i \cap A_j) \le P_m(A_i)P_m(A_j).$$

Thus, for any event Ω_1, using Cauchy-Schwarz, we have

$$\begin{aligned} P(A_i \cap A_j) &\le P(\Omega_1^c) + E(1_{\Omega_1} P_m(A_i \cap A_j)) \\ &\le P(\Omega_1^c) + E(P_m(A_i)^2 1_{\Omega_1}) \end{aligned}$$

Thus it suffices to show that we can choose Ω_1 with $P(\Omega_1^c) \le 1/N$ and

$$P_m(A_i)1_{\Omega_1} \le \Phi\left(\frac{t}{\sqrt{\alpha}}\right) + K\frac{\log N}{\sqrt{N}}.$$

We rewrite T_i as

$$T_i = \sum_{k\geq 2} \eta_{i,k} m_k - \frac{M-1}{N}.$$

Given $\boldsymbol{m}$, the variables $(\eta_{i,k})_{k\geq 2}$ are independent with respect to P_m; moreover $E_m(\eta_{i,k}) = m_k$ by symmetry. The variables

$$Y_k = \eta_{i,k} m_k - m_k^2$$

for $k \geq 2$ are thus independent centered with respect to P_m, and

$$T_i = \sum_{k\geq 2} Y_k + \sum_{k\geq 2} m_k^2 - \frac{M-1}{N}.$$

Moreover, we have

$$\sigma^2 := \sum_{k\geq 2} E_m Y_k^2 = \sum_{2\leq k\leq M} (m_k^2 - m_k^4) \tag{9.7}$$

$$\rho^3 := \sum_{k\geq 2} E_m |Y_k^3| \leq \sup_{k\leq M} |m_k| (\sum_{k\leq M} m_k^2). \tag{9.8}$$

Using the Berry-Essen theorem (conditionally in $\boldsymbol{m}$) we then have

$$P_m(A_i) \leq \Phi(\frac{t - (\sum_{k\geq 2} m_k^2 - \frac{M-1}{N})}{\sigma}) + 3\frac{\rho^3}{\sigma^2}.$$

Thus it seems a good idea to define Ω_1 by

$$|\sum_{k\geq 2} m_k^2 - \frac{M-1}{N}| \leq C_1; \sup_{k\leq M} |m_k| \leq C_2.$$

Elementary exponential estimates show that we can achieve $P(\Omega_1) \geq 1 - 1/N$ with $C_2 = L\sqrt{\log N}/\sqrt{N}$ and $C_1 = L\sqrt{\alpha \log N}/\sqrt{N}$. The result follows by more elementary estimates. □

Let us now try to explain the importance of Proposition 9.4. We rewrite (9.3) as

$$H(\boldsymbol{\eta}_I) = H(\boldsymbol{\eta}_1) + 2n - \frac{2n^2}{N} + 2\sum_{i\in I} T_i - R_I =: \overline{H}(\boldsymbol{\eta}_I) - R_I \tag{9.9}$$

where $R_I = \frac{2}{N}(\sum_{k\geq 2}(\sum_{i\in I} \eta_{i,k})^2 - (M-1)\mathrm{card} I)$. Let us now think conditionally upon the sequence T_i, the distribution of which is more or less known by Proposition 9.4. Then (9.4) gives an explicit expression for $H(\boldsymbol{\eta}_I)$, up to the error term R_I. It should be apparent that, when $\mathrm{card} I$ is small, R_I is small.

Throughout this section, we denote, for $0 < \delta < 1$

$$I(\delta) = \delta \log \frac{1}{\delta} + (1-\delta) \log \frac{1}{1-\delta}$$

so that, by the Chernov bounds, as $N \to \infty$,

$$\mathrm{card}\{I \subset \{1, \cdots, N\}; \mathrm{card} I \leq \delta N\} \leq \exp N I(\delta). \tag{9.10}$$

Lemma 9.6. *If we have*

$$x - \log(1+x) > \frac{2I(\delta)}{\alpha}$$

then, as $N \to \infty$, with overwhelming probability we have

$$\forall I \subset \{1, \cdots, N\}, cardI \leq \delta N \Rightarrow R_I \leq 2N\alpha\delta x.$$

Proof. Lemma 2.1 shows that

$$E \exp \frac{\beta}{2\text{card}I} \sum_{k\geq 2} \Big(\sum_{i\in I} \eta_{i,k}\Big)^2 \leq (\frac{1}{1-\beta})^{(M-1)/2}$$

and optimization over u in the inequality

$$P(Y \geq t) \leq \exp(-ut) E \exp uY$$

yields

$$P\bigg(\frac{1}{\text{card}I} \sum_{k\geq 2} \Big(\sum_{i\in I} \eta_{i,k}\Big)^2 \geq (M-1)(1+x)\bigg) \leq \exp(-\frac{M-1}{2}(x - \log(1+x)).$$

The result follows easily. □

One popular topic about the Hopfield model at zero temperature is the study of dynamics. A dynamic is a rule to construct, given a configuration $\boldsymbol{\epsilon}$, a new configuration $\boldsymbol{\epsilon}$. The two most popular dynamics are as follows:

Rule 1. Change the sign of the spin for which the change creates the greatest decrease for energy.

Rule 2. Select a spin at random. Flip the spin if this decreases the energy; do nothing otherwise.

The idea is that the dynamics describes the spontaneous evaluation of the "memory". The topic of interest is the evaluation of the dynamics upon starting with $\boldsymbol{\eta}_1$. (Subsequent deviations from $\boldsymbol{\eta}_1$ are then errors made by the memory.)

Whichever of the previous rules we choose, the dynamics decreases the energy. This motivates the notion of energy barriers, that are a way to insure that a dynamic never strays far from $\boldsymbol{\eta}_1$.

Definition 9.7. We say there is an energy barrier at level n if

$$\forall I \subset \{1, \cdots, N\}, \text{card}I = n \Rightarrow H(\boldsymbol{\eta}_I) > H(\boldsymbol{\eta}_1). \tag{9.11}$$

Thus, the dynamics cannot cross the energy barrier, and hence can never reverse more than n spins.

Proposition 9.8. *Consider $s > 0$, and $\delta = \Phi(s)$. Let*

$$x = \frac{1-\delta}{\alpha} - \frac{1}{\delta\sqrt{2\pi\alpha}} e^{-s^2/2}.$$

Assume that

$$\frac{\alpha}{2}(x - \log(1+x)) > I(\delta). \tag{9.12}$$

Then, for N large enough, with overwhelming probability there is an energy barrier at level $n = [\delta N]$.

Comment. Numerical computations (that carry absolutely no warranty) seem to indicate that the Proposition proves that energy barriers exist up to values of α larger than those of [Lou], [N].

Proof. The nicest feature of the proof is that it suffices to show that for some $\theta > 0$, we have

$$\lim_{N\to\infty} P(\inf_{\mathrm{card} I = m} (H(\boldsymbol{\eta}_I) - H(\boldsymbol{\eta}_1)) > \theta N) = 0$$

and then the statement "with overwhelming probability" follows from concentration of measure (This statement does not follow directly from the proof).

Certainly we can pretend that $\delta = n/N$. Using (9.9), we see that if $\mathrm{card} I = n$,

$$\frac{1}{2}\Big(H(\boldsymbol{\eta}_I) - H(\boldsymbol{\eta}_1)\Big) \geq n - \frac{n^2}{N} + \sum_{i\in I} T_i - \frac{1}{2}R_I \tag{9.13}$$

so that

$$\frac{1}{2N}(H(\boldsymbol{\eta}_I) - H(\boldsymbol{\eta}_1)) \geq \delta - \delta^2 + \frac{1}{N}\sum_{i\in I} T_i - \frac{1}{2N}R_I.$$

Consider $s' < s$, and $\delta' = \Phi(s')$, so $\delta' > \delta$. Consider the set I^* that consists of the indexes i for which T_i takes its smallest n values, and $J = \{i \leq N; T_i \leq -\sqrt{\alpha}s'\}$. Proposition 9.4 shows that for large N we have $\mathrm{card} J \geq n = \delta N$, so that $I^* \subset J$. Since $T_i \leq 0$ for $i \in J$, we have

$$\frac{1}{N}\sum_{i\in I} T_i \geq \frac{1}{N}\sum_{i\in I^*} T_i \geq \frac{1}{N}\sum_{i\leq N} T_i 1_{\{T_i \leq -\sqrt{\alpha}s'\}}.$$

We leave the reader to deduce from Proposition 9.4 that, for any $\gamma > 0$,

$$\begin{aligned}\frac{1}{N\sqrt{\alpha}}\sum_{i\leq N} T_i 1_{\{T_i\leq -\sqrt{\alpha}s'\}} &\geq -\left[\int_{s'}^{\infty} \frac{1}{\sqrt{2\pi}} u e^{-u^2/2} du + \gamma\right] \\ &= -\left[\frac{1}{\sqrt{2\pi}} e^{-s'^2/2} + \gamma\right]\end{aligned}$$

with probability going to one as $N \to \infty$.

The result then follows easily from Lemma 9.6. $\square$

Thus, the dynamics stays close to $\boldsymbol{\eta}_1$. As it can only decrease the energy, the dynamic can stop only at a local minimum, that is a configuration $\boldsymbol{\epsilon}$ such that one cannot decrease the energy by changing the sign of a single coordinate. Since H is (in the vicinity of $\boldsymbol{\eta}_1$) a small perturbation of the function $\overline{H}$ of (9.19), one should expect that a local minima of H there will be close to the global minimum of $\overline{H}$ (which is easy to identify). In order to prove this, it is not sufficient that the remainder R_I be small, it is also necessary that it should be smooth. What is precisely needed that, given I, there are not too many i such that $R_{I\cup\{i\}}$ or $R_{I\setminus\{i\}}$ is rather different from R_I. For example

$$R_{I\cup\{i\}} - R_I = \frac{4}{N} \sum_{2\leq k\leq M} \eta_{i,k} \left(\sum_{j\in I} \eta_{j,k} \right)$$

To control this difference, one works conditionally on $S_{I,k} = \sum\limits_{j\in I} \eta_{j,k}$. The independence of the sums $\sum\limits_{2\leq k\leq M} \eta_{i,k} S_{I,k}$ as i varies the exponential inequality they satisfy, and the control of $\sum\limits_{2\leq k\leq M} S^2_{I,k}$ allow to show that for all I, only few of these differences are not small. Optimization over the parameters, and a few pages of tedious and totally standard estimates yield results such as [T3, Théorème 9], the most remarkable fact (to be traced back to Proposition 9.4) being that the accuracy of the approximations become excellent as $\alpha \to 0$.

On the other hand, for large α, there are no local minima very close to the prototypes (although simulation [A-G-S] indicates that there does exist local minima $\boldsymbol{\epsilon}$ with $m_1(\boldsymbol{\epsilon}) > .1$).

The best numerical results currently available seem to be given by the following.

Proposition 9.9. *Consider* $0 < \delta < 1$, *and assume that*

$$\forall u \in \mathbb{R}, \Phi(-8\sqrt{I(\delta)} - 4u) \exp(I(\delta) + f(u)) < 1$$

where $f(u) = \inf\limits_{\lambda>0} \exp(-\lambda u + E \log(1 - \delta + \delta e^{\lambda g}))$, g *standard normal.*

Then, if N *and* α *are large enough, with probability going to* 1 *as* $N \to \infty$ *there is no local minimum* $\boldsymbol{\epsilon}$ *of* H *such that* $|m_1(\boldsymbol{\epsilon})| \geq 1 - 2\delta$.

Comment. Numerical computation (based on a previous, less elaborate version of this result) indicates that one can take δ of values up to .16.

Proof. This proof is based on the observation that the arguments of the first version of Loukianova's work [Lou] greatly simplify if one lets $\alpha \to \infty$ rather than trying to study what happens at given α, a fact that was also used by this author in the final version of her work.

Given $\boldsymbol{\epsilon}$ in Σ_N, denote by $\boldsymbol{\epsilon}^j$ the point obtained from $\boldsymbol{\epsilon}$ by changing the sign of ϵ_j. By algebra

$$H(\boldsymbol{\epsilon}^j) - H(\boldsymbol{\epsilon}) = 2N \sum_{k \le M} \epsilon_j \eta_{j,k} m_k(\boldsymbol{\epsilon}) - 2M.$$

Thus, $\boldsymbol{\epsilon}$ is a local minimum if and only if

$$\forall j \le N, \sum_{k \le M} \epsilon_j \eta_{j,k} m_k(\boldsymbol{\epsilon}) \ge \alpha.$$

We denote by P_m the probability given the sequence $(m_k(\boldsymbol{\epsilon}))_{k \le M}$. The key is again negative association to get

$$P_m(\boldsymbol{\epsilon} \text{ local minimum}) \le \prod_{j \le N} P_m(\sum_{k \le M} \epsilon_j \eta_{j,k} m_k(\boldsymbol{\epsilon}) \ge \alpha) \tag{9.14}$$

To evaluate the last term, we proceed as in Proposition 9.4. Given $j \le N$, for P_m, the variables $(\epsilon_j \eta_{j,k})_{k \le M}$ are independent, the expectation of $\epsilon_j \eta_{j,k}$ is $m_k(\boldsymbol{\epsilon})$ so that, setting

$$\sigma^2 = \sigma^2(\boldsymbol{\epsilon}) = \sum_{k \le M} (m_k^2(\boldsymbol{\epsilon}) - m_k^4(\boldsymbol{\epsilon}))$$

we have

$$P_m(\sum_{k \le M} \epsilon_j \eta_{j,k} m_k(\boldsymbol{\epsilon}) \ge \alpha) \le \Phi(\frac{\alpha - \sum_{k \le M} m_k^2(\boldsymbol{\epsilon})}{\sigma}) + \frac{6}{\sigma}$$

where we have used the fact that, for $|Y| \le 2, E|Y|^3 \le 2EY^2$.

We leave the reader to check that for large α, the event

$$\Omega_1 := \{(\eta_{i,k}); \forall \boldsymbol{\epsilon} \in \Sigma_N, \sum_{k \le M} m_k^2(\boldsymbol{\epsilon}) \ge \alpha/2\}$$

occurs with overwhelming probability. Now, for $k \ge 2$,

$$P(|m_k(\boldsymbol{\epsilon})| \ge t) \le \exp(-\frac{Nt^2}{2})$$

so that for any set $J \subset \{2, \cdots, M\}$

$$P(\forall k \in J, |m_k(\boldsymbol{\epsilon})| \ge t) \le \exp(-\frac{Nt^2 \mathrm{card} J}{2}).$$

Thus given an integer p

$$P(\exists \boldsymbol{\epsilon} \in \Sigma_N; \mathrm{card}\{k \in \{2, \cdots, M\}; |m_k(\boldsymbol{\epsilon})| \ge t\} \ge p\} \le M^p 2^N \exp(-\frac{Npt^2}{2})$$

so that the set

$$\Omega_2 = \{(\eta_{i,k}) \in \Omega_1; \forall \boldsymbol{\epsilon} \in \Sigma_N, \operatorname{card}\{k \leq M; |m_k(\boldsymbol{\epsilon})| \geq 2/\sqrt{p}\} \leq p\}$$

has overwhelming probability for N large enough.

Now, on Ω_2 we have

$$\sum_{k \leq M} m_k^4(\boldsymbol{\epsilon}) \leq p + \frac{4}{p} \sum_{k \leq M} m_k^2(\boldsymbol{\epsilon}) \tag{9.15}$$

so that we have

$$\sigma^2 \geq (1 - \frac{4}{p}) \sum_{k \leq M} m_k^2(\boldsymbol{\epsilon}) - p. \tag{9.16}$$

Concerning the term $\Phi((\alpha - \sum_{k \leq M} m_k^2(\boldsymbol{\epsilon})/\sigma)$, we bound it by $1/2$ if $\sum_{k \leq M} m_k^2(\boldsymbol{\epsilon}) \leq \alpha$. If $\sum_{k \leq M} m_k^2(\boldsymbol{\epsilon}) \geq \alpha$, we bound it by

$$B = \Phi\left(\frac{\alpha - \sum_{k \leq M} m_k^2(\boldsymbol{\epsilon})}{\sqrt{\alpha(1 - \frac{4}{p}) - p}} \right)$$

using (9.15) (Observe that in the denominator α could be replaced by $\sum_{k \leq M} m_k^2(\boldsymbol{\epsilon})^N$; we have not tried to refine the argument along this line). Finally, observing by (9.16) that on Ω_1, for $p \geq 8$ and α large, we have $\sigma^2 \geq \alpha/16$, we get from (9.14)

$$P(\Omega_2 \cap \{\boldsymbol{\epsilon} \text{ local minimum}\}) \leq \left(\frac{L}{\sqrt{\alpha}} + \frac{1}{2} \right)^N + \left(B + \frac{L}{\sqrt{\alpha}} \right)^N.$$

The dangerous situation is that $\sum_{k \leq M} m_k^2(\boldsymbol{\epsilon})$ can be large (so that B is close to one).

We now specialize to the case where $\boldsymbol{\epsilon}$ is close to $\boldsymbol{\eta}_1$, i.e. $\boldsymbol{\epsilon} = \boldsymbol{\eta}_I$, I small. To control $\sum_{k \leq M} m_k^2(\boldsymbol{\eta}_I)$, we rewrite (9.3) as

$$\sum_{k \leq M} m_k(\boldsymbol{\eta}_I)^2 = \sum_{k \leq M} m_k(\boldsymbol{\eta}_1)^2 - \frac{4n}{N} + 4(\frac{n}{N})^2 - \frac{4}{N} \sum_{ii^{th} I} T_i + \frac{2}{N} R_I.$$

In the proof of Lemma 9.6, we have seen that if $y(\alpha)$ is such that

$$\frac{\alpha}{2}(y(\alpha) - \log(1 + y(\alpha))) > I(\delta)$$

then with overwhelming probability, when $\mathrm{card} I \leq [\delta N]$, we have

$$\frac{2}{N} R_I \leq 4\delta\alpha y(\alpha)$$

and thus, since $(\frac{n}{N})^2 \leq \frac{n}{N}$,

$$\sum_{k \leq M} m_k^2(\boldsymbol{\eta}_I) \leq (1+\alpha) + 4\delta\alpha y(\alpha) - \frac{4}{n}\sum_{i \leq I} T_i$$

so that we get

$$\begin{aligned} &P(\exists I, \mathrm{card} I \leq n, \boldsymbol{\eta}_I \text{ local minimum}) \\ &\leq \sum_I \left(\left(\frac{1}{2} + \frac{L}{\sqrt{\alpha}}\right)^N + \left(\Phi\left(\frac{-1 - 4\delta\alpha y(\alpha) + \frac{4}{p}\sum_{i \in I} T_i}{\sqrt{\alpha(1-\frac{p}{4}) - p}}\right) + \frac{L}{\sqrt{\alpha}}\right)^N\right). \end{aligned}$$

We want to show that this sum is $< \gamma^N$, for some $\gamma < 1$.

Suppose now that we have a sequence $u_1 \leq \cdots \leq u_q = 4\sqrt{I(\delta)}$, and that we know that if $U_i = T_i/\sqrt{\alpha}$,

$$\frac{1}{N}\sum_{i \in I} U_i \geq u_1 \quad \forall I$$

and that

$$\forall \ell \leq q, \mathrm{card}\{I : \mathrm{card} I \leq n, \frac{1}{N}\sum_{i \in I} U_i \geq u_{\ell+1}\}\Phi(-8\sqrt{I(\delta)} - 4u_\ell)^N < \gamma^N.$$

Then, taking p large enough, and then α large enough, and observing that we can take $y(\alpha) \simeq 2\sqrt{I(\delta)/\alpha}$, the result will follow.

Now consider independent r.v. $\xi_i \in \{0,1\}$ with $P(\xi_i) = 1 = n/N$. Using the fact that the set $\{i; \xi_i = 1\}$ has cardinality n with probability of order $1/\sqrt{N}$, it suffices to show that

$$\exp NI(\delta) P(\frac{1}{N}\sum_{i \in I} \xi_i U_i \geq u_{\ell+1})\Phi(-8\sqrt{I(\delta)} - 4u_\ell)^N < \gamma^N.$$

Now, for each λ

$$P(\frac{1}{N}\sum_{i \in I} \xi_i U_i \geq u_{\ell+1}) \leq \exp -N\left(\lambda u_{\ell+1} + \frac{1}{N}\sum_{i \leq N} \log(1 - \delta + \delta e^{\lambda U_i})\right).$$

Given any value of λ, the last term, with large probability, is close to $E\log((1-\delta) + \delta e^{\lambda g}))$. The result follows easily. $\square$

Remark. One can use large deviation estimates as in [Lou] to control the large values of $\sum m_k^2(\boldsymbol{\eta}_I)$. This gives a smaller value of δ, but the result then holds with overwhelming probability.

As a last topic, we will consider the *dilute* Hopfield model. Consider a number $0 < p \leq 1$, the "dilution parameter". Consider independent r.v. $\delta_{ij} \in \{0,1\}, E\delta_{ij} = p$ (that are independent of the quenched variables). The idea is that given spins (or neurons...) i and j interact directly if and only if $\delta_{ij} = 1$. The point is that no realistic model for the brain can assume that every pair of neurons interacts, so one tries to show that the essential properties of the model remain valid when only a small proportion of the connections do exist. The Hamiltonian is given by

$$H_d(\boldsymbol{\epsilon}) = -\frac{1}{2Np} \sum_{i,j \leq N} \epsilon_i \epsilon_j J_{ij} \delta_{ij}$$

where $J_{ij} = \sum_{k \leq M} \eta_{i,k} \eta_{j,k}$. The factor $1/p$ at the denominator is to ensure that the expected value over the δ_{ij} is $H(\epsilon)$. The key to the study of the dilute Hopfield model is the following elementary fact.

Proposition 9.10. *For every $\boldsymbol{\epsilon}$ in Σ_N, we have*

$$0 \leq t \leq N\sqrt{M} \Rightarrow P(|H_d(\boldsymbol{\epsilon}) - H(\boldsymbol{\epsilon})| \geq t) \leq \exp\left(-\frac{pt^2}{LN\alpha(1+\alpha)}\right).$$

As a consequence, if $\Sigma_0 \subset \Sigma_N$ has a cardinal $\leq 2^{\gamma N}$, with overwhelming probability, we have

$$\sup_{\boldsymbol{\epsilon} \in \Sigma_0} |H_d(\boldsymbol{\epsilon}) - H(\boldsymbol{\epsilon})| \leq LN\sqrt{\frac{\gamma}{p}\alpha(1+\alpha)}$$

with overwhelming probability. Taking $\gamma = 1$, it is then simple to see that there is L_0 such that if $p \geq L_0\alpha$ and $\alpha \leq 1/L_0$, the dilute Hopfield model has an energy barrier around each prototype, a result that was proved in [B-G1] by more complicated estimates.

The proof of Proposition 9.9 relies upon the following observation, that is of independent interest. Assume that $(\delta_\ell)_{\ell \leq R}$ are independent, $\delta_\ell \in \{0,1\}, E\delta_\ell = p$, and consider numbers $(a_\ell)_{\ell \in L}$. To bound

$$P\Big(\sum_{\ell \leq R} (\delta_\ell - p) a_\ell \geq t\Big); P\Big(\sum_{\ell \leq R} (\delta_\ell - p) a_\ell \leq -t\Big)$$

one can use the Chernov bounds, replacing each a_ℓ by $a = (R^{-1} \sum_{\ell \leq R} a_\ell^2)^{1/2}$. To see this, we write

$$E \exp \lambda \sum_{\ell \leq R} (\delta_\ell - p) a_\ell = \exp \sum_{\ell \leq R} f_p(\lambda a_\ell)$$

where

$$f_p(x) = \log((1-p)e^{-px} + pe^{(1-p)x})$$

and we observe by calculus that the function $f_p(\sqrt{x})$ is concave, so that

$$\sum_{\ell \le R} f_p(\lambda a_\ell) \le R f_p(\lambda a)$$

so that

$$P\Big(\sum_{\ell \le R} (\delta_\ell - p) a_\ell \ge t\Big) \le \Big(\inf_\lambda \exp(-\lambda t + f_p(\lambda a))\Big)^R. \tag{9.18}$$

Using the elementary fact that $f_p(x) \le Lpx^2$ for $x \le 1$, we then see that if $\sum_{\ell \le R} a_\ell^2 \le A^2$, then for $t \le pA^2 R$, we have

$$P\Big(\sum_{\ell \le R} (\delta_\ell - p) a_\ell \ge t\Big) \le \exp\Big(-\frac{t^2}{LpA^2}\Big),$$

and similarly

$$P(|\sum_{\ell \le R} (\delta_\ell - p) a_\ell| \ge t) \le 2\exp\Big(-\frac{t^2}{LpA^2}\Big).$$

In the situation of Proposition 9.9, we have $a_{ij} = \epsilon_i \epsilon_j J_{ij}/2Np$. In Lemma 11.3, it is shown that with probability $\ge 1 - \exp -M$, the operator norm of the matrix $(\eta_{i,k})$ from $\mathbb{R}^M$ to $\mathbb{R}^N$ is at most $L\sqrt{N+M}$. Thus for any numbers $(b_k)_{k \le M}$

$$\sum_{i \le N} \Big(\sum_{k \le M} \eta_{i,k} b_k\Big)^2 \le L(N+M)\Big(\sum_{k \le M} b_k^2\Big).$$

Taking $b_k = \eta_{j,k}$, and summing over j gives

$$\sum_{i,j \le N} J_{ij}^2 \le LNM(N+M) = LN^3 \alpha(1+\alpha).$$

The proof is finished. $\square$

10. Appendix 1: Variance estimates.

A basic tool for the this section is that "the derivative of a random convex function does not fluctuate more than the function itself". This principle, when applied to the derivatives of the free energy with respect to the various parameters is very powerful. However, as simple examples show, this principle is not true at each point, but only "in average". A possible rigorous formulation is as follows.

Proposition 10.1. *Consider a random convex function U defined on $\mathbb{R}$. Then for $0 < v < x_0$, we have*

$$(10.1)\int_{-x_0}^{x_0} (\mathit{Var}\frac{dU}{dx})dx \le 12\frac{x_0}{v} \sup_{|x|\le x_0} \mathit{Var}U(x) + 12\frac{v}{x_0^2}(EU(-3x_0) + EU(3x_0) - 2EU(0))^2$$

Proof. Replace $U(x)$ by $U(x) - U(0) - xEU'(0)$ to reduce to [T4, Proposition 4.3].

In order to use this result for $U = F$, the free energy for the Hamiltonian (4.16), we need to control the variance of F.

Proposition 10.2. [S-T]. *We have $\mathit{Var}F \le KN$.*

Comment. Thus, the free energy per site F/N is of order 1 but has a variance of order $1/N$.

Proof. We fix N, M, β, h, γ. We indicate the dependence of F in the random variables $\eta_{i,k}, g_k$ by writing $F = F(\boldsymbol{\eta}, \boldsymbol{g})$, where $\boldsymbol{\eta} = (\eta_{i,k})_{i\le N, k\le M}$ and $\boldsymbol{g} = (g_k)_{2\le k\le M}$.

Considering independent copies $\boldsymbol{\eta}', \boldsymbol{g}'$ of $\boldsymbol{\eta}, \boldsymbol{g}$, we have

$$(10.2)\qquad \mathrm{Var}F(\boldsymbol{\eta},\boldsymbol{g}) = \frac{1}{2}E(F(\boldsymbol{\eta},\boldsymbol{g}) - F(\boldsymbol{\eta}',\boldsymbol{g}'))^2$$
$$\le E(F(\boldsymbol{\eta},\boldsymbol{g}) - F(\boldsymbol{\eta}',\boldsymbol{g}))^2 + E(F(\boldsymbol{\eta}',\boldsymbol{g}) - F(\boldsymbol{\eta}',\boldsymbol{g}'))^2.$$

Fixing $\boldsymbol{\eta}'$, the function $\boldsymbol{g} \to F(\boldsymbol{\eta}',\boldsymbol{g})$, as a function on $\mathbb{R}^{M-1}$, has a Lipschitz constant at most $\beta\gamma\varphi(N)\sup_{\boldsymbol{\epsilon}}\|\boldsymbol{m}(\boldsymbol{\epsilon})\|^2$, as follows from Cauchy Schwarz. A general property of $\mathbb{R}^M$ provided with Gaussian measure [I-S-T], [L-T] show that the last term of (10.2) is at most

$$L\beta^2\gamma^2\varphi(N)^2 E\sup_{\boldsymbol{\epsilon}}\|\boldsymbol{m}(\boldsymbol{\epsilon})\|^2 \le L\beta^2\gamma^2\varphi(N)^2 \le L\beta^2\gamma^2 N.$$

To study the first term of (10.2), it is shown in [T2] that when $h = \gamma = 0$, this term is at most $L\beta^2 N$. Inspection of the proof shows however that the influence of the terms containing h, γ is at most the square of the Lipschitz constant of the real valued function on $\mathbb{R}^{N\times M}$ given by

$$\boldsymbol{z} \to \beta\gamma\varphi(N) \sum_{2\leq k\leq M} g_k \frac{1}{N} \sum_{i\leq N} z_{i,k}\epsilon_i + \beta h \sum_{i\leq N} z_{i,1}\epsilon_i. \tag{10.3}$$

Use of Cauchy Schwartz show that this Lipschitz constant is at most $\beta\gamma\frac{\varphi(N)}{\sqrt{N}}(\sum_{2\leq k\leq M} g_k^2)^{1/2} + \beta h\sqrt{N}$. Thus, finally using again that $\varphi^2(N) \leq N$, we see that $\mathrm{Var}F \leq KN$. □

A typical application is as follows.

Proposition 10.3. *For all β, γ, we have*

$$\int_0^{h_0} Var\langle m_1\rangle dh \leq \frac{K}{\sqrt{N}}.$$

Proof. We fix β, γ, and we apply Proposition 10.1 to the function $U(h) = F(\beta, h, \gamma)$, so that $\frac{dU}{dh} = \beta N\langle m_1\rangle$. To control the last term of (4.11), we use that $|\langle m_1\rangle| \leq 1$, and that (easily) $|U(h) - U(0)| \leq \beta|h|N$. □

Proposition 10.4. *For each β_0, h_0,*

$$\int_{\beta\leq\beta_0, h\leq h_0|\gamma|\leq 1} Var\langle\|\boldsymbol{m}\|^2\rangle d\beta dh d\gamma \leq K/\sqrt{N}.$$

Proof. After one sees the proof of Proposition 10.3, one would like to consider $\frac{\partial F}{\partial\beta}$. It is however more convenient to consider

$$U(\beta) = F(\beta, \frac{h}{\beta}, \frac{\gamma}{\beta}) = \log E_\epsilon \exp(\beta N\|\boldsymbol{m}(\boldsymbol{\epsilon})\|^2 + hNm_1(\boldsymbol{\epsilon}) + \gamma\varphi(N) \sum_{2\leq k\leq M} g_k m_k(\boldsymbol{\epsilon})),$$

which is a convex function of β. Thus

$$\frac{dU}{d\beta} = N\langle\|\boldsymbol{m}(\boldsymbol{\epsilon})\|^2\rangle,$$

the bracket being for the parameters $(\beta, h/\beta, \gamma/\beta)$.

To control the last term of (10.1) one uses simply that

$$|U(\beta) - U(0)| \leq \beta N \sup_{\boldsymbol{\epsilon}} \|\boldsymbol{m}(\boldsymbol{\epsilon})\|^2.$$

The result follows easily. □

Since $\|\boldsymbol{m}\|^2 = \|\boldsymbol{u}\|^2 + m_1^2$, we have

Proposition 10.5. *For each β_0, h_0,*

$$\int_{\beta\le\beta_0, h\le h_0, |\gamma|\le 1} Var\langle\|\boldsymbol{u}\|^2\rangle d\beta dh d\gamma \le \frac{K}{N}.$$

We have applied Proposition 10.1 to $\frac{\partial F}{\partial h}$ and (essentially) to $\frac{\partial F}{\partial \beta}$. To handle the case of $\frac{\partial F}{\partial \gamma}$, we need the following

Lemma 10.6. *We have*

$$EF(\beta,h,0) \le EF(\beta,h,\gamma) \le EF(\beta,h,0) + K(\alpha)\beta^2\gamma^2\varphi(N)^2. \tag{10.4}$$

Proof. We integrate first in the variables g_k, using Jensen's inequality to integrate inside the log rather than outside, to get

$$EF(\beta,h,\gamma) \le E(F(\beta,h,0) + \beta^2\gamma^2\varphi(N)^2 \sum_{2\le k\le M} |m_k(\boldsymbol{\epsilon})|^2). \qquad \square$$

A noteworthy consequence of (10.4) is the fact, already mentioned, that, as $N\to\infty$, when $\varphi(N)^2/N$ goes to zero, the perturbation term of the Hamiltonian has a vanishing influence on the free energy per site.

We now consider a thermally independent copy $\boldsymbol{v}$ of $\boldsymbol{u}$, and the parameter $\langle(\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{v}})^2\rangle$ (closely related to the parameter of (4.5)). The following is a rigorous version of the claim "(4.4)⇒(4.5)".

Proposition 10.7. *If $N^{1/4}\le\varphi(N)\le N^{1/2}$, we have, for all β, h,*

$$\int_{-1}^{1}\gamma^2\, Var\langle\|\dot{\boldsymbol{u}}\|^2\rangle d\gamma \le \frac{KN}{\varphi^4(N)} + 2\int_{-1}^{1}\gamma^2 E\langle(\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{v}})^2\rangle d\gamma.$$

Proof. We start with the formula

$$\frac{\partial F}{\partial\gamma} = \sum_{2\le k\le M}\beta\varphi(N)g_k\langle m_k\rangle \tag{10.5}$$

so that

$$\beta^2\gamma\varphi(N)^2\langle\|\dot{\boldsymbol{u}}\|^2\rangle = \frac{\partial F}{\partial\gamma} + \beta\varphi(N)R,$$

where

$$R = \sum_{2\le k\le M}\gamma\beta\varphi(N)\langle\dot{m}_k^2\rangle - g_k\langle m_k\rangle.$$

Using the formula

$$\mathrm{Var}(X+Y) \le 2\mathrm{Var}X + 2EY^2,$$

we get

$$\beta^4\gamma^2\varphi(N)^4\mathrm{Var}\langle\|\dot{\boldsymbol{u}}\|^2\rangle \le 2\mathrm{Var}\frac{\partial F}{\partial\gamma} + 2\beta^2\varphi(N)^2 ER^2. \tag{10.6}$$

To control the integral of the first term, we appeal to Proposition 10.1, using Lemma 10.6 to control the last term of (10.1).

To control the second term, one expends the square and eliminate all terms g_k, by using the integration by part formula

$$E(\langle g_k A\rangle) = \gamma\beta\varphi(N)E(\langle A m_k\rangle - \langle A\rangle\langle m_k\rangle) \tag{10.7}$$

that hold for any smooth function A from Σ_N to $\mathbb{R}$. After a few lines of straight forward algebra, we find that

$$ER^2 = \sum_{2\le k\le M}\langle m_k\rangle^2 + \beta^2\gamma^2\varphi(N)^2 E\sum_{2\le k,\ell\le M}\langle\dot{m}_k\dot{m}_\ell\rangle^2. \tag{10.8}$$

Now, expending $\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}}' = \sum_{2\le k\le M}\dot{m}_k\dot{m}'_k$, squaring and using the replica trick show that the last expectation is $E\langle(\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{v}})^2\rangle$. □

The previous arguments, based upon the control of the (average of) the variance of partial derivatives of F have brought us precious information. There are other averages that can be controlled, this time in a trivial fashion; the averages of second partial derivatives of F. It is quite amazing that this brings equally interesting information.

Proposition 10.8. *For all β_0, h_0, we have*

$$\int E\langle(m_1 - \langle m_1\rangle)^2\rangle d\beta dh d\gamma \le \frac{K}{N} \tag{10.9}$$

$$\int E\langle(\|\boldsymbol{m}\|^2 - \langle\|\boldsymbol{m}\|^2\rangle)^2\rangle d\beta dh d\gamma \le \frac{K}{N} \tag{10.10}$$

$$\int E\langle(\|\boldsymbol{u}\|^2 - \langle\|\boldsymbol{u}\|^2\rangle)^2\rangle d\beta dh d\gamma \le \frac{K}{N} \tag{10.11}$$

where the integrals are over $0\le\beta\le\beta_0, 0\le h\le h_0, -1\le\gamma\le 1$.

Proof. We have, by a simple calculation

$$\frac{\partial F}{\partial h} = N\beta\langle m_1\rangle, \frac{\partial^2 F}{\partial h^2} = N^2\beta^2(\langle m_1^2\rangle - \langle m_1\rangle^2) = N^2\beta^2\langle(m_1 - \langle m_1\rangle)^2\rangle.$$

This implies (10.9), since $\frac{\partial F}{\partial h} \le KN$. To prove (10.10) one use similarly the function U of Proposition 10.4; (10.10) (and hence (10.11)) follows. □

Trying to use the same idea for $\frac{\partial^2 F}{\partial\gamma^2}$ yields a remarkable consequence of adding the perturbation term in (4.16). The following result is inspired by [G].

Proposition 10.9. (Guerra's identity). *For each β_0, h_0 we have*

$$\int_{\beta\leq\beta_0, h\leq h_0, -1\leq\gamma\leq 1} \gamma^2\beta^4 E|4\langle(\boldsymbol{u}\cdot\boldsymbol{b})^2\rangle - 3\|\boldsymbol{b}\|^2 - \langle(\boldsymbol{u}\cdot\boldsymbol{v})^2\rangle| d\beta d\gamma dh \leq \frac{K}{\sqrt{N}}.$$

Here, $\boldsymbol{b} = \langle\boldsymbol{u}\rangle = (\langle m_k\rangle)_{2\leq k\leq M}$, and $\boldsymbol{v}$ is an independent copy of $\boldsymbol{u}$.

Proof. We start again with (10.5), so that, by the integration by part formula (10.7), we have

$$E\frac{\partial F}{\partial\gamma} = \gamma(\beta\varphi(N))^2 E(\sum_{2\leq k\leq M}(\langle m_k^2\rangle - \langle m_k\rangle^2))$$

and thus

$$E\frac{\partial^2 F}{\partial\gamma^2} = (\beta\varphi(N))^2 E(\langle\|\boldsymbol{u}\|^2\rangle - \|\langle\boldsymbol{u}\rangle\|^2)$$
$$+\gamma(\beta\varphi(N))^3 E\sum_{2\leq k,\ell\leq M} g_\ell(\langle m_k^2 m_\ell\rangle - \langle m_k^2\rangle\langle m_\ell\rangle - 2\langle m_\ell\rangle(\langle m_k m_\ell\rangle - \langle m_k\rangle\langle m_\ell\rangle)).$$

Use of integration by part (10.7) to get rid of the factor g_ℓ yield after a straightforward but tedious computation

$$E\frac{\partial^2 F}{\partial\gamma^2} = (\beta\varphi(N))^2 E(\langle\|\boldsymbol{u}\|^2\rangle - \|\langle\boldsymbol{u}\rangle\|^2) + \gamma^2(\beta\varphi(N))^4 E(S + R_1 + R_2 + R_3) \tag{10.12}$$

where

$$S = \sum_{2\leq k,\ell\leq M}(8\langle m_k m_\ell\rangle\langle m_k\rangle\langle m_\ell\rangle - 6\langle m_k\rangle^2\langle m_\ell\rangle^2 - 2\sum_{2\leq k,\ell\leq M}\langle m_k m_\ell\rangle^2)$$
$$R_1 = 2\sum_{2\leq k,\ell\leq M}(\langle m_k\rangle^2\langle m_\ell^2\rangle - \langle m_k m_\ell^2\rangle\langle m_k\rangle)$$
$$R_2 = 2\sum_{2\leq k,\ell\leq M}(\langle m_\ell\rangle^2\langle m_k^2\rangle - \langle m_\ell m_k^2\rangle\langle m_\ell\rangle)$$
$$R_3 = \sum_{2\leq k,\ell\leq M}(\langle m_k^2 m_\ell^2\rangle - \langle m_k^2\rangle\langle m_\ell^2\rangle)$$

Use of the replica trick show that

$$S = 8\langle(\boldsymbol{u}\cdot\boldsymbol{b})^2\rangle - 6\|\boldsymbol{b}\|^2 - 2\langle(\boldsymbol{u}\cdot\boldsymbol{v})^2\rangle$$
$$R_1 = R_2 = -2\sum_{2\leq k\leq M}\langle m_k\rangle\langle m_k(\|\boldsymbol{u}\|^2 - \langle\|\boldsymbol{u}\|^2\rangle)\rangle$$
$$R_3 = \langle\|\boldsymbol{u}\|^4\rangle - \langle\|\boldsymbol{u}\|^2\rangle^2$$

Consider the function $V = \|\boldsymbol{u}\|^2 - \langle\|\boldsymbol{u}\|^2\rangle$ defined on Σ_N. Use of Cauchy-Schwarz show that $\langle m_k V\rangle \le \langle m_k^2\rangle^{1/2}\langle V^2\rangle^{1/2}$, so that

$$|R_1| \le 2\langle\|\boldsymbol{m}\|^2\rangle\langle V^2\rangle^{1/2}$$

and $R_3 = \langle V^2\rangle$. Thus (10.12) implies

$$\begin{aligned}(10.13)\int \gamma^2\beta^4|ES|d\beta d\gamma \le \frac{1}{\varphi(N)^4}\int E\frac{\partial^2 F}{\partial\gamma^2}d\beta d\gamma + \frac{1}{\varphi(N)^2}\int \beta^2\Big[E(\langle\|\boldsymbol{u}\|^2\rangle - \|\langle\boldsymbol{u}\rangle\|^2)\Big]d\beta d\gamma \\ +2\int E(\langle\|\boldsymbol{m}\|^2\rangle\langle V^2\rangle^{1/2} + \langle V^2\rangle)d\beta d\gamma\end{aligned}$$

where all the integrals are for $\beta \le \beta_0, h \le h_0, -1 \le \gamma \le 1$. Now

$$\int_{-1}^{1} E\frac{\partial^2 F}{\partial\gamma^2}d\gamma = E\frac{\partial F}{\partial\gamma}\Big|_{-1}^{1} = E\frac{\partial\tilde{F}}{\partial\gamma}\Big|_{-1}^{1}$$

where $\tilde{F}$ is given by (10.5). Thus, using (10.6), and the bound $\partial\tilde{F}/\partial\gamma(1) \le \tilde{F}(2)$, the first term on the right-hand side of (10.13) is bounded by $K\varphi(N)^{-2}$. This is also the case for the second term. Use of Cauchy Schwarz show that the last term is bounded by $K(I + I^{1/2})$, where $I = \int\langle V^2\rangle d\beta d\gamma$. The result then follows from (10.11). □

11. Appendix 2: Random Matrices.

We recall $\boldsymbol{\eta}_i = (\eta_{i,k})_{k \le M}$.

Lemma 11.1. *Consider $\boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{R}^M$, with $\|\boldsymbol{v}\|, \|\boldsymbol{w}\| \le 1$. Then we have*

$$P\Bigg(\sum_{i \le N} ((\boldsymbol{\eta}_i \cdot \boldsymbol{v})(\boldsymbol{\eta}_i \cdot \boldsymbol{w}) - \boldsymbol{v} \cdot \boldsymbol{w}) \ge t\Bigg) \le \left(\exp -\frac{1}{L} \min\left(t, \frac{t^2}{N}\right)\right)$$

for all $t \ge 0$.

Proof. We have

$$(\boldsymbol{\eta}_i \cdot \boldsymbol{v})(\boldsymbol{\eta}_i \cdot \boldsymbol{w}) - \boldsymbol{v} \cdot \boldsymbol{w} = X_i = \sum_{k \ne \ell} \eta_{i,k} \eta_{i,\ell} v_k w_\ell. \tag{11.1}$$

Now, $EX_i = 0$, $EX_i^2 = \sum_{k \ne \ell} v_k^2 w_\ell^2 \le 1$, so that, since X_i is an order 2 chaos, we have $E \exp(|X_i|/L) \le 2$ by [Bo]. Bernstein's inequality then implies the result.

Lemma 11.2. *If C is a bounded convex balanced set of $\mathbb{R}^M$, there is a subset R of $2C$ such that $C \subset convR$, $cardR \le 5^M$.*

Proof. It is easy to show that if R is maximal with respect to the property $x, y \in R \Rightarrow x - y \notin C$, then R works.

Lemma 11.3. *There exists an event Ω_0 in the quenched variables such*

$$P(\Omega_0^c) \le \exp(-M) \tag{11.2}$$

On Ω_0, for each $\boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{R}^M$

$$\sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{v})(\boldsymbol{\eta}_i \cdot \boldsymbol{w}) \le N\boldsymbol{v} \cdot \boldsymbol{w} + LN \max(\alpha, \sqrt{\alpha}) \|\boldsymbol{v}\| \|\boldsymbol{w}\|. \tag{11.3}$$

Comment. In particular, on Ω_0

$$\sum_{i \le N} (\boldsymbol{\eta}_i \cdot \boldsymbol{v})^2 \le N(1 + L \max(\alpha, \sqrt{\alpha})) \|\boldsymbol{v}\|^2 \tag{11.4}$$

which expresses that the operator norm of the matrix $(\eta_{i,k})$ from $\mathbb{R}^M$ to $\mathbb{R}^N$ is at most $\sqrt{N}(1 + L\sqrt{\alpha})$ if $\alpha \le 1$ and $L\sqrt{N\alpha}$ if $\alpha \ge 1$.

Proof. Since

$$\sum_{i \le N} ((\eta_i \cdot \boldsymbol{v})(\boldsymbol{\eta}_i \cdot \boldsymbol{w}) - \boldsymbol{v} \cdot \boldsymbol{w})$$

is bilinear in $\boldsymbol{v}, \boldsymbol{w}$, to ensure (11.3), it suffices that

$$\sum_{i \leq N} ((\boldsymbol{\eta}_i \cdot \boldsymbol{v})(\boldsymbol{\eta}_i \cdot \boldsymbol{w}) - \boldsymbol{v} \cdot \boldsymbol{w}) \leq L_1 N \max(\alpha, \sqrt{\alpha})$$

for $\boldsymbol{v}, \boldsymbol{w}$ in R, where R is the set constructed in Lemma 11.2 when C is the unit ball of $\mathbb{R}^M$. Then Lemma 11.1 shows that the probability that this fails is at most

$$\begin{aligned} &5^{2M} (\exp -\frac{1}{L} \min(\frac{L_1^2}{4} N\alpha, N L_1 \alpha)) \\ &\leq \exp(-M) \end{aligned}$$

if L_1 is large enough.

The following is less important, and will be used only once for a secondary result.

Lemma 11.4. *There is $\delta_0 > 0$, and α_0 such that, if $\alpha \leq \alpha_0$ there exists an event Ω_1, with $P(\Omega_1) \leq \exp(-N/L)$, such that, on Ω_1, for all $\boldsymbol{v}$ in $\mathbb{R}^M$, each subset J of $\{1, \cdots, N\}$ with card$J \leq \delta_0 N$ we have*

$$\sum_{i \in J} (\boldsymbol{\eta}_i \cdot \boldsymbol{v})^2 \leq \frac{N}{2} \|\boldsymbol{v}\|^2.$$

Proof. Using Lemma 11.1, and the method of Lemma 11.2, we see that, given J

$$\sum_{i \in J} ((\boldsymbol{\eta}_i \cdot \boldsymbol{v})(\boldsymbol{\eta}_i \cdot \boldsymbol{w}) - \boldsymbol{v} \cdot \boldsymbol{w}) \leq \frac{N}{4}$$

for all $\boldsymbol{w}, \boldsymbol{w}$ of norm of $\mathbb{R}^N \leq 1$, with probability at least $1 - \exp(5M - \frac{N}{L_2})$. It then suffices to take $\delta_0 \leq 1/4$ small enough that there are at most $\exp N/2L_2$ possible sets J. $\square$

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