EECS 723-Microwave Engineering

Teacher: *"Bart, do you even* ***know*** *your multiplication tables?"*

Bart: *" Well, I know* ***of*** *them".*

Like Bart and his multiplication tables, many electrical engineers know **of** the concepts of microwave engineering.

Concepts such as characteristic impedance, scattering parameters, Smith Charts and the like are familiar, but often we find that a **complete**, **thorough** and **unambiguous** understanding of these concepts can be somewhat lacking.

Thus, the goals of this class are for **you** to:

1. Obtain a complete, thorough, and unambiguous understanding of the fundamental concepts on microwave engineering.

2. Apply these concepts to the **design** and **analysis** of useful microwave devices.

2.1 -The Lumped Element Circuit Model for Transmission Lines

Reading Assignment: *pp. 1-5, 49-52*

The most important fact about microwave devices is that they are connected together using transmission lines.

Q: *So just what is a **transmission line**?*

A: A passive, linear, two port device that allows bounded E. M. energy to flow from one device to another.

→ Sort of an "electromagnetic pipe" !

Q: *Oh, so it's simply a conducting **wire**, right?*

A: NO! At high frequencies, things get much more complicated!

HO: The Telegraphers Equations

HO: Time-Harmonic Solutions for Linear Circuits

Q: *So, what complex functions $I(z)$ and $V(z)$ **do** satisfy both telegrapher equations?*

A: The solutions to the transmission line **wave equations**!

HO: The Transmission Line Wave Equations

Q: *Are the solutions for I(z) and V(z) completely **independent**, or are they **related** in any way ?*

A: The two solutions are related by the transmission line **characteristic impedance**.

HO: The Transmission Line Characteristic Impedance

Q: *So what is the significance of the complex constant γ? What does it tell us?*

A: It describes the **propagation** of each **wave** along the transmission line.

HO: THE COMPLEX PROPAGATION CONSTANT

Q: *Now, **you** said earlier that **characteristic** impedance Z_0 is a **complex** value. But I recall engineers referring to a transmission line as simply a "50 Ohm line", or a "300 Ohm line". But these are **real** values; are they **not** referring to characteristic impedance Z_0 ??*

A: These real values are in fact some **standard** Z_0 values. They are **real** values because the transmission line is **lossless** (or nearly so!).

HO: THE LOSSLESS TRANSMISSION LINE

Q: *Is **characteristic** impedance Z_0 the same as the concept of impedance I learned about in **circuits** class?*

A: **NO!** The Z_0 is a **wave** impedance. However, we can also define **line impedance**, which **is** the same as that used in circuits.

HO: Line Impedance

Q: *These **wave** functions $V^+(z)$ and $V^-(z)$ seem to be important. How are **they** related?*

A: They are in fact **very** important! They are related by a function called the **reflection coefficient**.

HO: The Reflection Coefficient

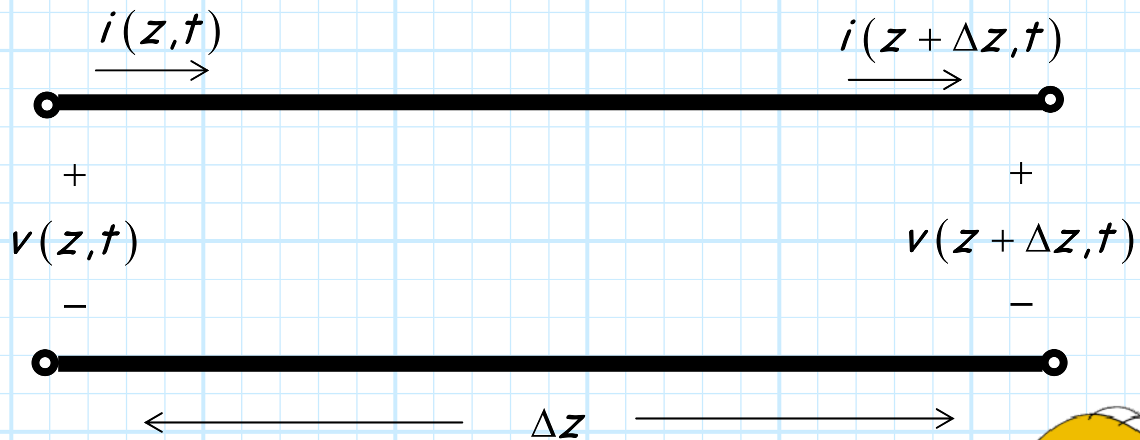
Q: *Does this mean I can describe transmission line activity in terms of (complex) voltage, current, and impedance, **or alternatively** in terms of an incident wave, reflected wave, and reflection coefficient?*

A: Absolutely! A microwave engineer has a **choice** to make when describing transmission line activity.

HO: V, I, Z OR V^+, V^-, Γ ?

The Telegrapher Equations

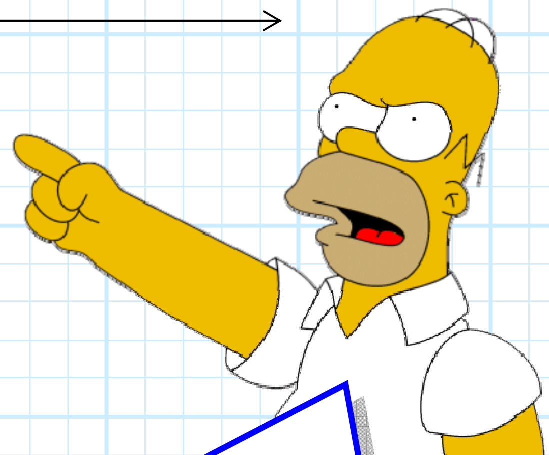
Consider a section of "wire":



Where:

$$i(z,t) \neq i(z+\Delta z,t)$$

$$v(z,t) \neq v(z+\Delta z,t)$$

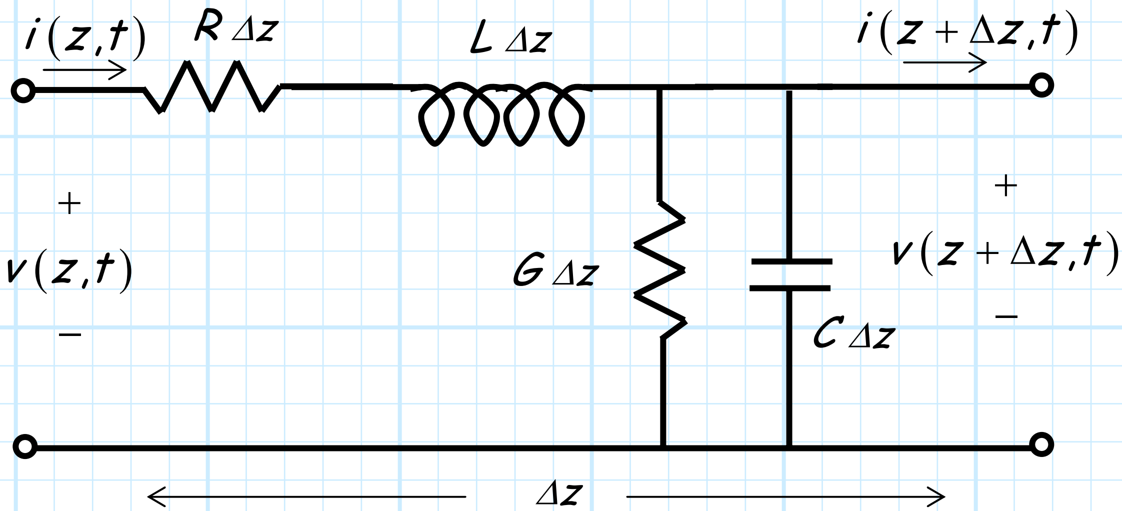


Q: *No way! Kirchoff's Laws tells me that:*

$$i(z,t) = i(z+\Delta z,t)$$
$$v(z,t) = v(z+\Delta z,t)$$

*How can the voltage/current at the **end** of the line (at $z+\Delta z$) be **different** than the voltage/current at the **beginning** of the line (at z)??*

A: Way. The structure above actually exhibits some non-zero value of **inductance**, **capacitance**, **conductance**, and **admittance**! A more accurate transmission line model is therefore:



Where:

R = resistance/unit length
L = inductance/unit length
C = capacitance/unit length
G = conductance/unit length

∴ resistance of wire length Δz is $R\Delta z$

Now evaluating **KVL**, we find:

$$v(z+\Delta z,t)-v(z,t)=-R\Delta z\, i(z,t)-L\Delta z\frac{\partial i(z,t)}{\partial t}\neq 0$$

and from **KCL**:

$$i(z+\Delta z,t)-i(z,t)=-G\Delta z\, v(z,t)-C\Delta z\frac{\partial v(z,t)}{\partial t}\neq 0$$

Dividing the first equation by Δz, and then taking the limit as $\Delta z \rightarrow 0$:

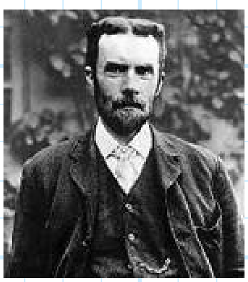
$$\lim_{\Delta z \rightarrow 0} \frac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

which, by definition of the derivative, becomes:

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

Similarly, the KCL equation becomes:

$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t}$$



These coupled differential equations are quite famous! Derived by **Oliver Heavyside**, they are known as the **telegrapher's equations,** and are essentially the Maxwell's equations of transmission lines.

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t}$$

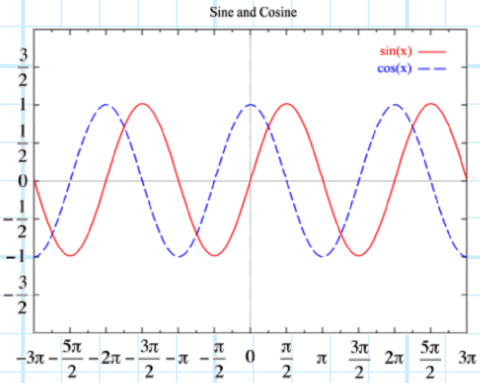
Although **mathematically** the functions $v(z,t)$ and current $i(z,t)$ can take any form, they can **physically** exist **only if** they satisfy the both of the differential equations shown above!

Time-Harmonic Solutions for Linear Circuits

There are an unaccountably **infinite** number of solutions $v(z,t)$ and $i(z,t)$ for the telegrapher's equations! However, we can simplify the problem by assuming that the function of time is **time harmonic** (i.e., sinusoidal), oscillating at some radial **frequency** ω (e.g., $\cos \omega t$).

Q: *Why on earth would we assume a **sinusoidal** function of time? Why not a **square wave**, or **triangle wave**, or a "sawtooth" function?*

A: We assume **sinusoids** because they have a **very** special property!



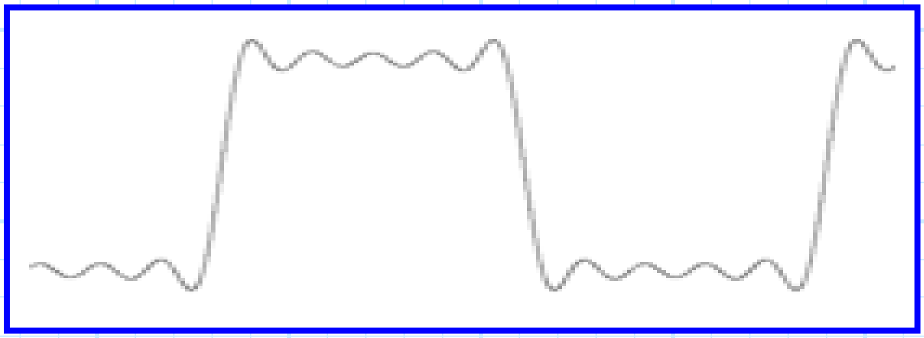
Sinusoidal time functions—and **only** a sinusoidal time functions—are the **eigen functions** of **linear, time-invariant** systems.

Q: *???*

A: If a sinusoidal voltage source with frequency ω is used to excite a linear, time-invariant circuit (and a transmission line is **both** linear **and** time invariant!), then the voltage at each and **every** point with the circuit will likewise vary sinusoidally—at the same frequency ω!

Q: *So what? Isn't that **obvious**?*

A: Not at all! If you were to excite a linear circuit with a **square wave**, or **triangle wave**, or **sawtooth**, you would find that—generally speaking—**nowhere** else in the circuit is the voltage a perfect square wave, triangle wave, or sawtooth. The linear circuit will effectively **distort** the input signal into something **else**!



Q: *Into what function will the input signal be distorted?*

A: It depends—both on the original form of the **input signal**, and the parameters of the **linear circuit**. At **different** points within the circuit we will discover **different** functions of time—**unless**, of course, we use a **sinusoidal** input. Again, for a sinusoidal excitation, we find at **every** point within circuit an **undistorted** sinusoidal function!

Q: *So, the sinusoidal function at every point in the circuit is **exactly** the same as the input sinusoid?*

A: Not quite **exactly** the same. Although at every point within the circuit the voltage will be precisely sinusoidal (with frequency ω), the **magnitude** and **relative phase** of the sinusoid will generally be different at each and every point within the circuit.

Thus, the voltage along a transmission line—when excited by a sinusoidal source—**must** have the form:

$$v(z,t) = v(z)\cos\left(\omega t + \varphi(z)\right)$$

Thus, at some arbitrary location z along the transmission line, we **must** find a time-harmonic oscillation of **magnitude** $v(z)$ and **relative phase** $\varphi(z)$.

Now, consider Euler's equation, which states:

$$e^{j\psi} = \cos\psi + j\sin\psi$$

Thus, it is apparent that:

$$Re\left\{e^{j\psi}\right\} = \cos\psi$$

and so we conclude that the voltage on a transmission line can be expressed as:

$$\begin{aligned} v(z,t) &= v(z)\cos\left(\omega t + \varphi(z)\right) \\ &= Re\left\{v(z)\,e^{j(\omega t + \varphi(z))}\right\} \\ &= Re\left\{v(z)\,e^{+j\varphi(z)}e^{j\omega t}\right\} \end{aligned}$$

Thus, we can specify the time-harmonic voltage at each an every location z along a transmission line with the **complex** function $V(z)$:

$$V(z) = v(z)\, e^{-j\varphi(z)}$$

where the **magnitude** of the complex function is the **magnitude** of the sinusoid:

$$v(z) = |V(z)|$$

and the phase of the complex function is the relative phase of the sinusoid :

$$\varphi(z) = arg\{V(z)\}$$

Q: *Hey wait a minute! What happened to the time-harmonic function $e^{j\omega t}$??*

A: There really is no reason to **explicitly** write the complex function $e^{j\omega t}$, since we know in fact (being the eigen function of linear systems and all) that if this is the time function at any **one** location (such as qt the excitation source) then this must be time function at **all** transmission line locations z!

The only **unknown** is the **complex** function $V(z)$. Once we determine $V(z)$, we can always (if we so desire) "recover" the **real** function $v(z,t)$ as:

$$v(z,t) = Re\{V(z)\, e^{j\omega t}\}$$

Thus, if we assume a **time-harmonic source**, finding the transmission line solution $v(z,t)$ reduces to solving for the **complex function** $V(z)$.

The Transmission Line Wave Equation

Let's assume that $v(z,t)$ and $i(z,t)$ each have the **time-harmonic** form:

$$v(z,t) = \text{Re}\left\{V(z)e^{j\omega t}\right\} \quad \text{and} \quad i(z,t) = \text{Re}\left\{I(z)e^{j\omega t}\right\}$$

The time-**derivative** of these functions are:

$$\frac{\partial v(z,t)}{\partial t} = \text{Re}\left\{V(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \text{Re}\left\{j\omega V(z)e^{j\omega t}\right\}$$

$$\frac{\partial i(z,t)}{\partial t} = \text{Re}\left\{I(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \text{Re}\left\{j\omega I(z)e^{j\omega t}\right\}$$

Inserting these results into the telegrapher's equations, we find:

$$\text{Re}\left\{\frac{\partial V(z)}{\partial z}e^{j\omega t}\right\} = \text{Re}\left\{-(R+j\omega L)I(z)\,e^{j\omega t}\right\}$$

$$\text{Re}\left\{\frac{\partial I(z)}{\partial z}e^{j\omega t}\right\} = \text{Re}\left\{-(G+j\omega C)V(z)\,e^{j\omega t}\right\}$$

Simplifying, we have the **complex** form of **telegrapher's equations**:

$$\frac{\partial V(z)}{\partial z} = -(R + j\omega L)\, I(z)$$

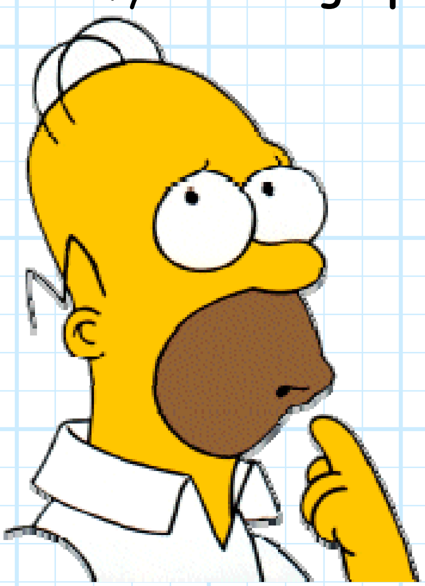
$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)\, V(z)$$

Note that these complex differential equations are **not** a function of **time** t !

* The functions $I(z)$ and $V(z)$ are **complex,** where the **magnitude** and **phase** of the complex functions describe the **magnitude** and **phase** of the sinusoidal time function $e^{j\omega t}$.

* Thus, $I(z)$ and $V(z)$ describe the current and voltage along the transmission line, as a function as position z.

* **Remember**, not just **any** function $I(z)$ and $V(z)$ can exist on a transmission line, but rather **only** those functions that satisfy the **telegraphers equations**.



Our task, therefore, is to **solve** the telegrapher equations and find **all** solutions $I(z)$ and $V(z)$!

Q: *So, what functions $I(z)$ and $V(z)$* ***do*** *satisfy both telegrapher's equations??*

A: To make this easier, we will combine the telegrapher equations to form **one** differential equation for $V(z)$ and **another** for $I(z)$.

First, take the **derivative** with respect to z of the **first** telegrapher equation:

$$\frac{\partial}{\partial z}\left\{\frac{\partial V(z)}{\partial z} = -(R + j\omega L)\,I(z)\right\}$$

$$= \frac{\partial^2 V(z)}{\partial z^2} = -(R + j\omega L)\frac{\partial I(z)}{\partial z}$$

Note that the **second** telegrapher equation expresses the derivative of $I(z)$ in terms of $V(z)$:

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)\,V(z)$$

Combining these two equations, we get an equation involving $V(z)$ **only**:

$$\frac{\partial^2 V(z)}{\partial z^2} = (R + j\omega L)(G + j\omega C)\,V(z)$$

We can simplify this equation by defining the complex value γ:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

So that:

$$\frac{\partial^2 V(z)}{\partial z^2} = \gamma^2 \, V(z)$$

In a **similar** manner (i.e., begin by taking the derivative of the **second** telegrapher equation), we can derive the differential equation:

$$\frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 \, I(z)$$

We have **decoupled** the telegrapher's equations, such that we now have **two** equations involving **one** function only:

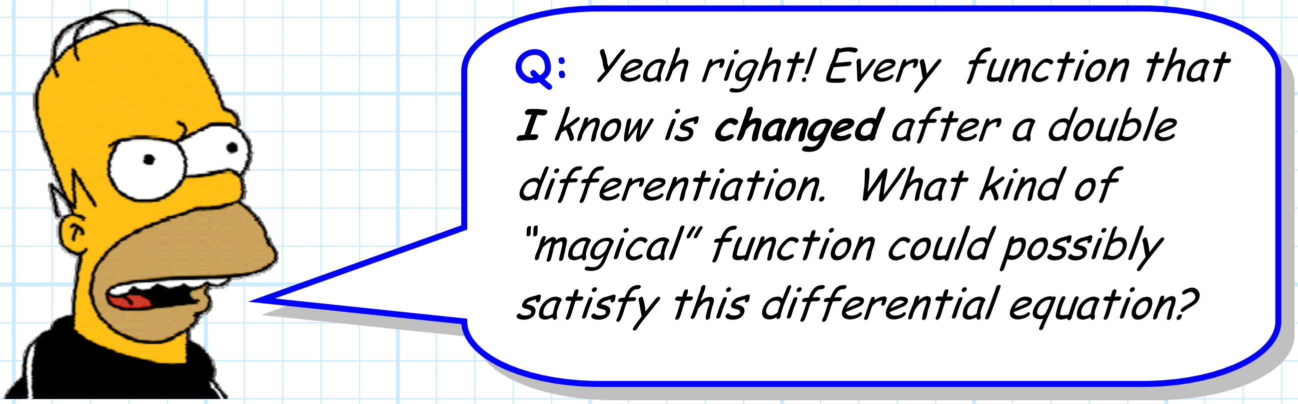
$$\frac{\partial^2 V(z)}{\partial z^2} = \gamma^2 \, V(z)$$

$$\frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 \, I(z)$$

These are known as the **transmission line wave equations**.

Note that value γ is **complex**, and is determined by taking the **square-root** of a **complex** value. Likewise, γ^2 is a **complex** value. Do **you** know how to square a complex number? Can **you** determine the square root of a complex number?

Note only **special** functions satisfy these wave equations; if we take the double derivative of the function, the result is the **original function** (to within a constant γ^2)!



A: Such functions **do** exist !

For example, the functions $V(z) = e^{+\gamma z}$ and $V(z) = e^{-\gamma z}$ each satisfy this transmission line wave equation (**insert** these into the differential equation and see for **yourself**!).

Likewise, since the transmission line wave equation is a **linear** differential equation, a weighted **superposition** of the two solutions is **also a solution** (again, **insert** this solution to and see for **yourself**!):

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

In fact, it turns out that **any and all** possible solutions to the differential equations can be expressed in **this** simple form!

Therefore, the **general** solution to these complex wave equations (and thus the telegrapher equations) are:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

where V_0^+, V_0^-, I_0^+, and I_0^- are **complex constants**.

→ It is **unfathomably** important that **you** understand what this result means!

It means that the functions $V(z)$ and $I(z)$, describing the current and voltage at **all** points z along a transmission line, can **always** be **completely** specified with just **four complex constants** (V_0^+, V_0^-, I_0^+, I_0^-)!!

We can **alternatively** write these solutions as:

$$V(z) = V^+(z) + V^-(z)$$

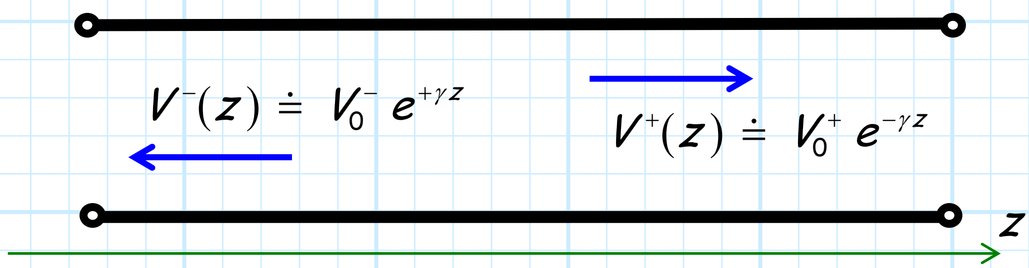
$$I(z) = I^+(z) + I^-(z)$$

where:

$$V^+(z) \doteq V_0^+ e^{-\gamma z} \qquad V^-(z) \doteq V_0^- e^{+\gamma z}$$

$$I^+(z) \doteq I_0^+ e^{-\gamma z} \qquad I^-(z) \doteq I_0^- e^{+\gamma z}$$

The two terms in each solution describe **two waves** propagating in the transmission line, **one** wave ($V^+(z)$ or $I^+(z)$) propagating in one direction ($+z$) and the **other** wave ($V^-(z)$ or $I^-(z)$) propagating in the **opposite** direction ($-z$).



Q: *So just what* ***are*** *the complex values* $V_0^+, V_0^-, I_0^+, I_0^-$?

A: Consider the wave solutions at **one** specific point on the transmission line—the point $z = 0$. For example, we find that:

$$
\begin{aligned}
V^+(z=0) &= V_0^+ e^{-\gamma(z=0)} \\
&= V_0^+ e^{-(0)} \\
&= V_0^+ (1) \\
&= V_0^+
\end{aligned}
$$

In other words, V_0^+ is simply the **complex** value of the wave function $V^+(z)$ **at the point** ***z*** **= 0** on the transmission line!

Likewise, we find:

$$V_0^- = V^-(z=0)$$

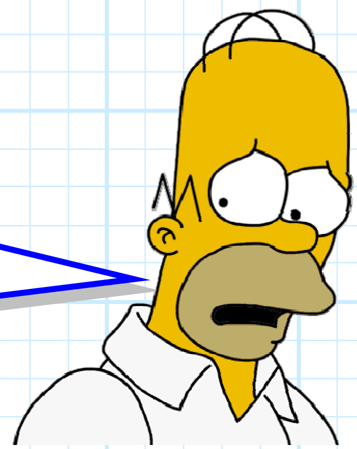
$$I_0^+ = I^+(z=0)$$

$$I_0^- = I^-(z=0)$$

Again, the four complex values V_0^+, I_0^+, V_0^-, I_0^- are **all** that is needed to determine the voltage and current at any and all points on the transmission line.

More specifically, **each** of these four complex constants completely specifies **one** of the four transmission line wave functions $V^+(z)$, $I^+(z)$, $V^-(z)$, $I^-(z)$.

Q: *But what **determines** these wave functions? How do we **find** the values of constants V_0^+, I_0^+, V_0^-, I_0^-?*



A: As you might expect, the voltage and current on a transmission line is determined by the devices **attached** to it on either end (e.g., active sources and/or passive loads)!

The precise values of V_0^+, I_0^+, V_0^-, I_0^- are therefore determined by satisfying the **boundary conditions** applied at **each end** of the transmission line—much more on this **later**!

The Characteristic Impedance of a Transmission Line

So, from the telegrapher's differential equations, we know that the complex current $I(z)$ and voltage $V(z)$ **must** have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

Let's insert the expression for $V(z)$ into the first telegrapher's equation, and **see what happens**!

$$\frac{d\,V(z)}{dz} = -\gamma\, V_0^+ e^{-\gamma z} + \gamma\, V_0^- e^{+\gamma z} = -(R + j\omega L)\, I(z)$$

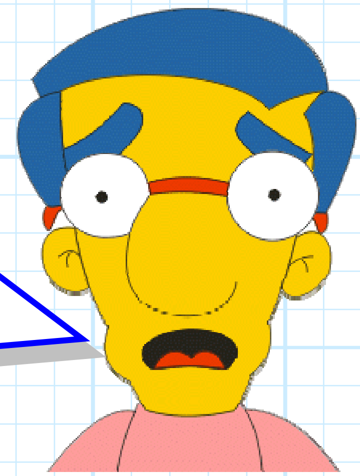
Therefore, rearranging, $I(z)$ must be:

$$I(z) = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

Q: ***But wait*** *! I thought we already knew current $I(z)$. Isn't it:*

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \quad ??$$

How can ***both*** *expressions for $I(z)$ be true??*



A: Easy ! Both expressions for current are **equal** to each other.

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

For the above equation to be true for **all** z, I_0 and V_0 must be related as:

$$I_0^+ e^{-\gamma z} = \left(\frac{\gamma}{R + j\omega L}\right) V_0^+ e^{-\gamma z} \quad \text{and} \quad I_0^- e^{+\gamma z} = \left(\frac{-\gamma}{R + j\omega L}\right) V_0^- e^{+\gamma z}$$

Or—recalling that $V_0^+ e^{-\gamma z} = V^+(z)$ (etc.)—we can express this in terms of the **two propagating waves**:

$$I^+(z) = \left(\frac{+\gamma}{R + j\omega L}\right) V^+(z) \quad \text{and} \quad I^-(z) = \left(\frac{-\gamma}{R + j\omega L}\right) V^-(z)$$

Now, we note that since:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

We find that:

$$\frac{\gamma}{R+j\omega L}=\frac{\sqrt{(R+j\omega L)(G+j\omega C)}}{R+j\omega L}=\sqrt{\frac{G+j\omega C}{R+j\omega L}}$$

Thus, we come to the **startling** conclusion that:

$$\frac{V^{+}(z)}{I^{+}(z)}=\sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad \text{and} \quad \frac{-V^{-}(z)}{I^{-}(z)}=\sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

Q: *What's so startling about **this** conclusion?*

A: Note that although the magnitude and phase of each propagating wave is a **function** of transmission line **position** z (e.g., $V^{+}(z)$ and $I^{+}(z)$), the **ratio** of the voltage and current of **each wave** is independent of position—a **constant** with respect to position z!

Although $V_0^{\pm}$ and $I_0^{\pm}$ are determined by **boundary conditions** (i.e., what's connected to either end of the transmission line), the **ratio** $V_0^{\pm}/I_0^{\pm}$ is determined by the parameters of the transmission line **only** (*R, L, G, C*).

→ This ratio is an important **characteristic of a transmission line**, called its **Characteristic Impedance** Z_0.

$$Z_0 \doteq \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

We can therefore describe the current and voltage along a transmission line as:

$$V(z) = V_0^+ \, e^{-\gamma z} + V_0^- \, e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

or equivalently:

$$V(z) = Z_0 \, I_0^+ \, e^{-\gamma z} - Z_0 \, I_0^- \, e^{+\gamma z}$$

$$I(z) = I_0^+ \, e^{-\gamma z} + I_0^- \, e^{+\gamma z}$$

Note that instead of characterizing a transmission line with **real** parameters R, G, L, and C, we can (and typically do!) describe a transmission line using **complex** parameters Z_0 and γ.

The Complex Propagation Constant γ

Recall that the activity along a transmission line can be expressed in terms of two functions, functions that we have described as **wave** functions:

$$V^{+}(z) = V_0^{+}\, e^{-\gamma z}$$

$$V^{-}(z) = V_0^{-}\, e^{+\gamma z}$$

where γ is a **complex constant** that describe the properties of a transmission line. Since γ is complex, we can consider both its **real** and **imaginary** components.

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \doteq \alpha + j\beta$$

where $\alpha = \text{Re}\{\gamma\}$ and $\beta = \text{Im}\{\gamma\}$. Therefore, we can write:

$$V^{+}(z) = V_0^{+}\, e^{-\gamma z} = V_0^{+}\, e^{-(\alpha + j\beta)z} = V_0^{+}\, e^{-\alpha z}\, e^{-j\beta z}$$

Q: *What **are** these constants α and β? What do they **physically** represent?*

A: Remember, a complex value can be expressed in terms of its **magnitude** and **phase**. For example:

$$V_0^+ = \left|V_0^+\right| e^{j\phi_0^+}$$

Likewise:

$$V^+(z) = \left|V^+(z)\right| \; e^{j\,\phi^+(z)}$$

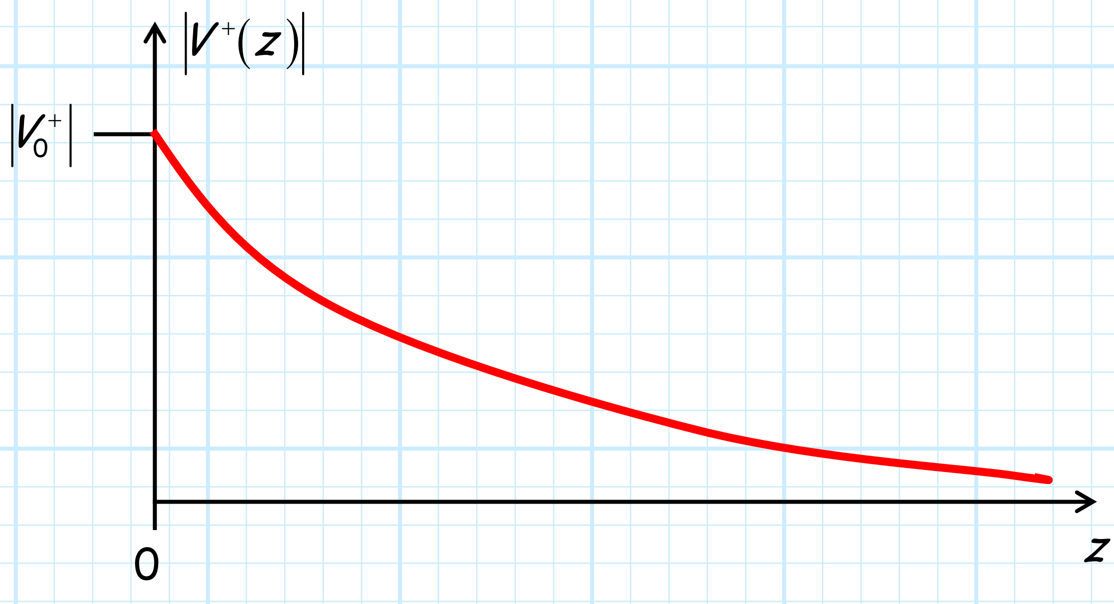
And since:

$$\begin{aligned} V^+(z) &= V_0^+ \; e^{-\alpha z} \; e^{-j\beta z} \\ &= \left|V_0^+\right| e^{j\phi_0^+} \; e^{-\alpha z} \; e^{-j\beta z} \\ &= \left|V_0^+\right| e^{-\alpha z} \; e^{j\left(\phi_0^+ - \beta z\right)} \end{aligned}$$

we find:

$$\left|V^+(z)\right| = \left|V_0^+\right| e^{-\alpha z} \qquad\qquad \phi^+(z) = \phi_0^+ - \beta z$$

It is evident that $e^{-\alpha z}$ **alone** determines the **magnitude** of wave $V^+(z) = V_0^+ \, e^{-\gamma z}$ as a function of position *z*.



Therefore, α expresses the **attenuation** of the signal due to the loss in the transmission line. The larger the value of α, the greater the exponential attenuation.

Q: *So what* ***is*** *the constant β? What does* ***it*** *physically mean?*

A: Recall

$$\phi^{+}(z) = \phi_0^{+} - \beta z$$

represents the relative **phase** of wave $V^{+}(z)$; a **function** of transmission line **position** z. Since phase ϕ is expressed in **radians**, and z is distance (in meters), the value β must have **units** of:

$$\beta = \frac{\phi}{z} \qquad \frac{\text{radians}}{\text{meter}}$$

Thus, if the value β is **small**, we will need to move a **significant distance** Δz down the transmission line in order to observe a change in the relative phase of the oscillation.

Conversely, if the value β is **large**, a significant change in relative phase can be observed if traveling a **short** distance $\Delta z_{2\pi}$ down the transmission line.

Q: *How far must we move along a transmission line in order to observe a change in relative phase of 2π radians?*

A: We can easily determine this distance ($\Delta z_{2\pi}$, say) from the transmission line characteristic β.

$$2\pi = \phi(z + \Delta z_{2\pi}) - \phi(z) = \beta\, \Delta z_{2\pi}$$

or, rearranging:

$$\Delta z_{2\pi} = \frac{2\pi}{\beta} \qquad \Rightarrow \qquad \beta = \frac{2\pi}{\Delta z_{2\pi}}$$

The **distance** $\Delta z_{2\pi}$ over which the relative phase changes by 2π **radians**, is more specifically known as the **wavelength** λ of the propagating wave (i.e., $\lambda \doteq \Delta z_{2\pi}$):

$$\lambda = \frac{2\pi}{\beta} \qquad \Rightarrow \qquad \beta = \frac{2\pi}{\lambda}$$

The value β is thus essentially a **spatial frequency**, in the same way that ω is a **temporal** frequency:

$$\omega = \frac{2\pi}{T}$$

Note T is the **time** required for the phase of the oscillating signal to change by a value of 2π radians, i.e.:

$$\omega T = 2\pi$$

And the **period** of a sinewave, and related to its **frequency** in Hertz (cycles/second) as:

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

Compare these results to:

$$\beta = \frac{2\pi}{\lambda} \qquad\qquad 2\pi = \beta\lambda \qquad\qquad \lambda = \frac{2\pi}{\beta}$$

Q: *So, just how **fast** does this wave propagate down a transmission line?*

We describe wave velocity in terms of its **phase velocity**—in other words, how **fast** does a specific value of absolute phase ϕ seem to **propagate** down the transmission line.

Since velocity is change in distance with respect to **time**, we need to first express our propagating wave in its real form:

$$\begin{aligned} v^+(z,t) &= Re\left\{V^+(z)\,e^{-j\omega t}\right\} \\ &= \left|V_0^+\right| cos\left(\omega t - \beta z + \phi_0^+\right) \end{aligned}$$

Thus, the absolute phase is a function of **both** time and frequency:

$$\phi^+(z,t) = \omega t - \beta z + \phi_0^+$$

Now let's set this phase to some **arbitrary** value of ϕ_c radians.

$$\omega t - \beta z + \phi_0^+ = \phi_c$$

For **every** time t, there is **some** location z on a transmission line that has this phase value ϕ_c. That location is evidently:

$$z = \frac{\omega t + \phi_0^+ - \phi_c}{\beta}$$

Note as **time increases**, so to does the **location** z on the line where $\phi^+(z,t) = \phi_c$.

The **velocity** v_p at which this phase point moves down the line can be determined as:

$$v_p = \frac{dz}{dt} = \frac{d\left(\frac{\omega t + \phi_0^+ - \phi_c}{\beta}\right)}{dt} = \frac{\omega}{\beta}$$

This wave velocity is the **velocity of the propagating wave**!

Note that the value:

$$\frac{v_p}{\lambda} = \frac{\omega}{\beta}\frac{\beta}{2\pi} = \frac{\omega}{2\pi} = f$$

and thus we can conclude that:

$$v_p = f\lambda$$

as well as:

$$\beta = \frac{\omega}{v_p}$$

Q: *But these results were derived for the $V^+(z)$ wave; what about the* ***other*** *wave $V^-(z)$?*

A: The results are essentially the **same**, as each wave depends on the same value β.

The only **subtle difference** comes when we evaluate the phase velocity. For the wave $V^-(z)$, we find:

$$\phi^-(z,t) = \omega t + \beta z + \phi_0^-$$

Note the **plus sign** associated with βz!

We thus find that some arbitrary phase value will be located at location:

$$z = \frac{-\phi_0^- + \phi_c - \omega t}{\beta}$$

Note now that an **increasing time** will result in a **decreasing** value of **position** z. In other words this wave is propagating in the direction of decreasing position z—in the **opposite** direction of the $V^+(z)$ wave!

This is **further** verified by the derivative:

$$v_p = \frac{dz}{dt} = \frac{d\left(\frac{-\phi_0^- + \phi_c - \omega t}{\beta}\right)}{dt} = -\frac{\omega}{\beta}$$

Where the **minus sign** merely means that the wave propagates in the –z direction. Otherwise, the **wavelength** and **velocity** of the two waves are **precisely** the same!

The Lossless Transmission Line

Say a transmission line is **lossless** (i.e., $R = G = 0$); the transmission line equations are then **significantly** simplified!

Characteristic Impedance

$$
\begin{aligned}
Z_0 &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \\
&= \sqrt{\frac{j\omega L}{j\omega C}} \\
&= \sqrt{\frac{L}{C}}
\end{aligned}
$$

Note the characteristic impedance of a **lossless** transmission line is purely **real** (i.e., Im{Z_0} =0)!

Propagation Constant

$$
\begin{aligned}
\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\
&= \sqrt{(j\omega L)(j\omega C)} \\
&= \sqrt{-\omega^2 LC} \\
&= j\,\omega\sqrt{LC}
\end{aligned}
$$

The wave propagation constant is purely **imaginary**!

In other words, for a **lossless** transmission line:

$$\alpha = 0 \quad \text{and} \quad \beta = \omega\sqrt{LC}$$

Note that since $\alpha = 0$, **neither** propagating wave is **attenuated** as they travel down the line—a wave at the **end** of the line is as large as it was at the **beginning**!

And this **makes sense**!

Wave attenuation occurs when **energy is extracted** from the propagating wave and turned into **heat**. This can **only** occur if resistance and/or conductance are present in the line. If $R = G = 0$, then **no attenuation** occurs—that why we call the line **lossless**.

Voltage and Current

The **complex functions** describing the magnitude and phase of the voltage/current at every location z along a transmission line are for a **lossless** line are:

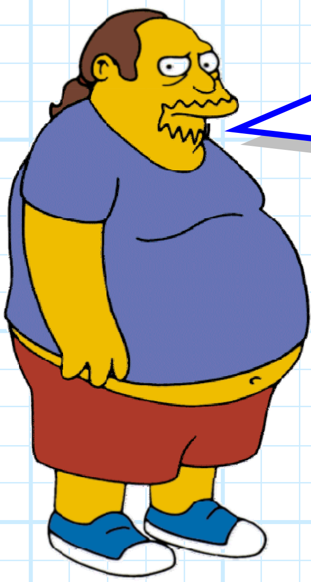
$$V(z) = V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

Wavelength and Phase Velocity

We can now **explicitly** write the wavelength and propagation velocity of the two transmission line waves in terms of transmission line parameters L and C:

$$\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{LC}} \qquad\qquad v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$



Q: *Oh **please**, continue wasting my valuable time. We both know that a **perfectly** lossless transmission line is a physical **impossibility**.*

A: True! However, a **low-loss** line **is** possible—in fact, it is **typical**! If $R \ll \omega L$ and $G \ll \omega C$, we find that the lossless transmission line equations are excellent **approximations**!

Unless **otherwise** indicated, **we will use the lossless equations** to **approximate** the behavior of a **low-loss** transmission line.

The lone **exception** is when determining the attenuation of a **long** transmission line. For that case we will use the approximation:

$$\alpha \approx \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right)$$

where $Z_0 = \sqrt{L/C}$.

A summary of lossless transmission line equations

$$Z_0 = \sqrt{\frac{L}{C}} \qquad \gamma = j\omega\sqrt{LC}$$

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z} \qquad I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

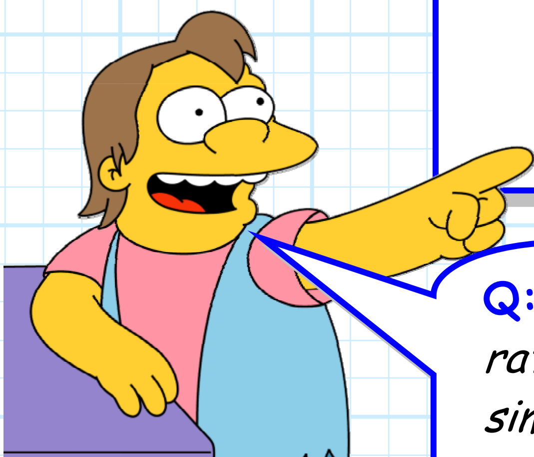
$$V^+(z) = V_0^+ e^{-j\beta z} \qquad V^-(z) = V_0^- e^{+j\beta z}$$

$$\beta = \omega\sqrt{LC} \qquad \lambda = \frac{1}{f\sqrt{LC}} \qquad v_p = \frac{1}{\sqrt{LC}}$$

Line Impedance

Now let's define **line impedance** $Z(z)$, a **complex** function which is simply the ratio of the complex line **voltage** and complex line **current**:

$$Z(z)=\frac{V(z)}{I(z)}$$



Q: *Hey! I know what this is! The ratio of the voltage to current is simply the* ***characteristic impedance*** *Z_0, right ???*

A: NO! The line impedance $Z(z)$ is (generally speaking) **NOT** the transmission line **characteristic** impedance Z_0 !!!

→ It is **unfathomably important** that you understand this!!!!

To see why, recall that:

$$V(z)=V^{+}(z)+V^{-}(z)$$

And that:

$$I(z) = \frac{V^{+}(z) - V^{-}(z)}{Z_0}$$

Therefore:

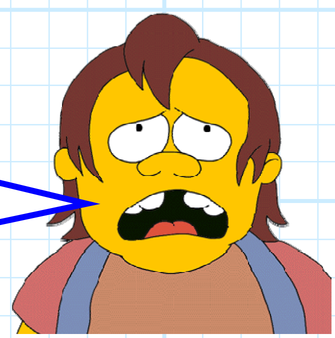
$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \left(\frac{V^{+}(z) + V^{-}(z)}{V^{+}(z) - V^{-}(z)} \right) \neq Z_0$$

Or, more specifically, we can write:

$$Z(z) = Z_0 \left(\frac{V_0^{+} e^{-j\beta z} + V_0^{-} e^{+j\beta z}}{V_0^{+} e^{-j\beta z} - V_0^{-} e^{+j\beta z}} \right)$$

Q: *I'm confused! Isn't:*

$$V^{+}(z) / I^{+}(z) = Z_0 \;???$$



A: Yes! That **is** true! The ratio of the voltage to current for **each** of the two propagating waves is $\pm Z_0$. However, the ratio of the **sum** of the two voltages to the **sum** of the two currents is **not** equal to Z_0 (generally speaking)!

This is actually confirmed by the equation above. Say that $V^{-}(z) = 0$, so that only **one** wave ($V^{+}(z)$) is propagating on the line.

In this case, the ratio of the **total** voltage to the total current is simply the ratio of the voltage and current of the **one** remaining wave—the **characteristic impedance** Z_0!

$$Z(z) = \frac{V(z)}{I(z)} = Z_0\left(\frac{V^+(z)}{V^+(z)}\right) = \frac{V^+(z)}{I^+(z)} = Z_0 \quad \text{(when } V^-(z) = 0\text{)}$$

Q: *So, it appears to me that characteristic impedance Z_0 is a **transmission line parameter**, depending **only** on the transmission line values L and C.*

*Whereas **line impedance** is $Z(z)$ depends the magnitude and phase of the two propagating waves $V^+(z)$ and $V^-(z)$--values that depend **not only** on the transmission line, but also on the two things **attached** to either **end** of the transmission line!*

Right !?

A: **Exactly!** Moreover, note that characteristic impedance Z_0 is simply a **number**, whereas line impedance $Z(z)$ is a **function** of position (z) on the transmission line.

<u>The Reflection Coefficient</u>

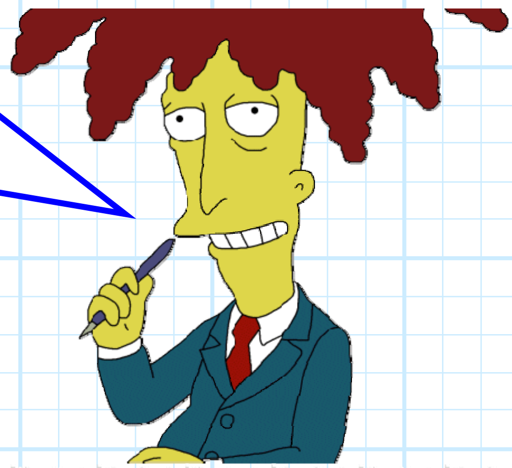
So, we know that the transmission line **voltage** $V(z)$ and the transmission line **current** $I(z)$ can be related by the **line impedance** $Z(z)$:

$$V(z) = Z(z)\, I(z)$$

or equivalently:

$$I(z) = \frac{V(z)}{Z(z)}$$

Q: *Piece of cake! I fully understand the concepts of* ***voltage, current*** *and* ***impedance*** *from my* ***circuits*** *classes. Let's move on to something more important (or, at the very least, more* ***interesting****).*



Expressing the "activity" on a transmission line in terms of **voltage**, **current** and **impedance** is of course **perfectly** valid.

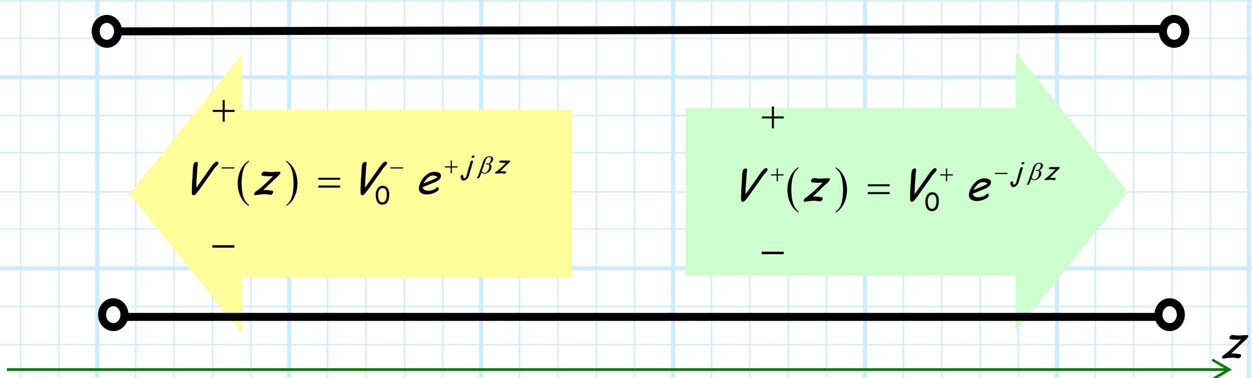
However, let us look **closer** at the expression for each of these quantities:

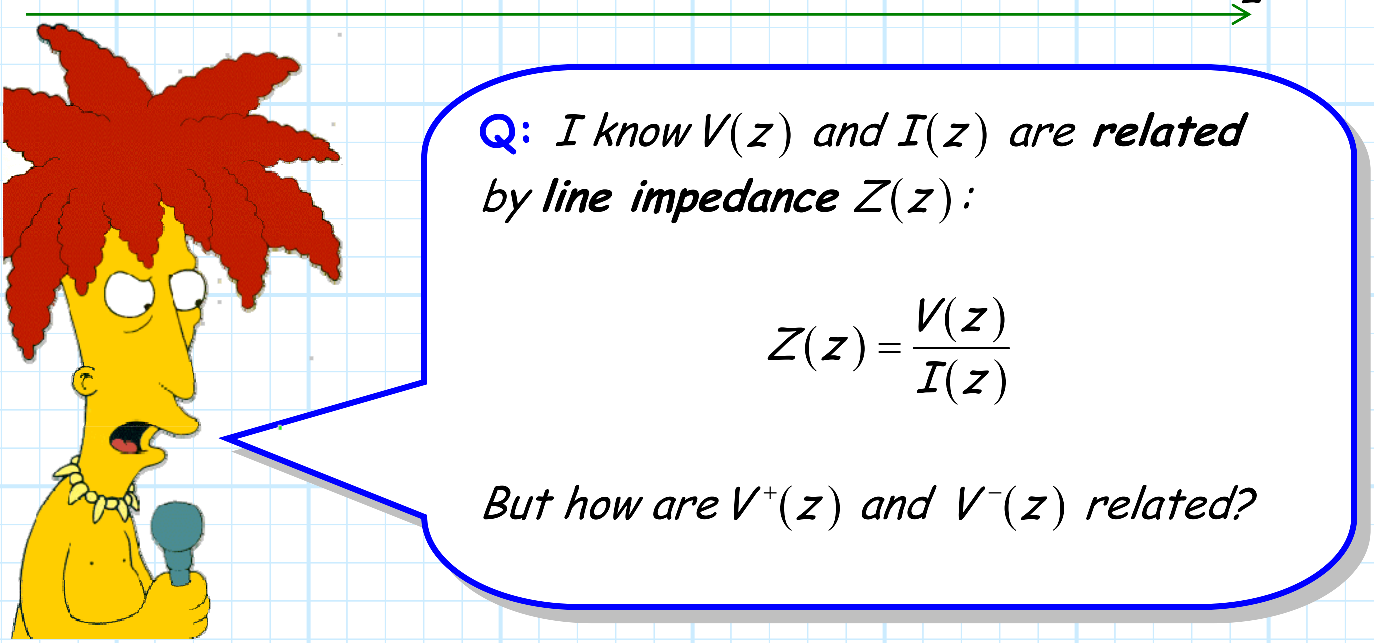
$$V(z) = V^+(z) + V^-(z)$$

$$I(z) = \frac{V^+(z) - V^-(z)}{Z_0}$$

$$Z(z) = Z_0 \left(\frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)} \right)$$

It is evident that we can **alternatively** express all "activity" on the transmission line in terms of the two transmission line **waves** $V^+(z)$ and $V^-(z)$.





A: Similar to line impedance, we can define a new parameter—the **reflection coefficient** $\Gamma(z)$—as the **ratio** of the two quantities:

$$\Gamma(z) \doteq \frac{V^-(z)}{V^+(z)} \quad \Rightarrow \quad V^-(z) = \Gamma(z)\, V^+(z)$$

More specifically, we can express $\Gamma(z)$ as:

$$\Gamma(z) = \frac{V_0^-\, e^{+j\beta z}}{V_0^+\, e^{-j\beta z}} = \frac{V_0^-}{V_0^+}\; e^{+j2\beta z}$$

Note then, the value of the reflection coefficient at $z=0$ is:

$$\Gamma(z=0) = \frac{V^-(z=0)}{V_0^+(z=0)}\; e^{+j2\beta(0)} = \frac{V_0^-}{V_0^+}$$

We define this value as Γ_0, where:

$$\Gamma_0 \doteq \Gamma(z=0) = \frac{V_0^-}{V_0^+}$$

Note then that we can alternatively write $\Gamma(z)$ as:

$$\Gamma(z) = \Gamma_0\; e^{+j2\beta z}$$

So now we have **two different** but equivalent ways to describe transmission line activity!

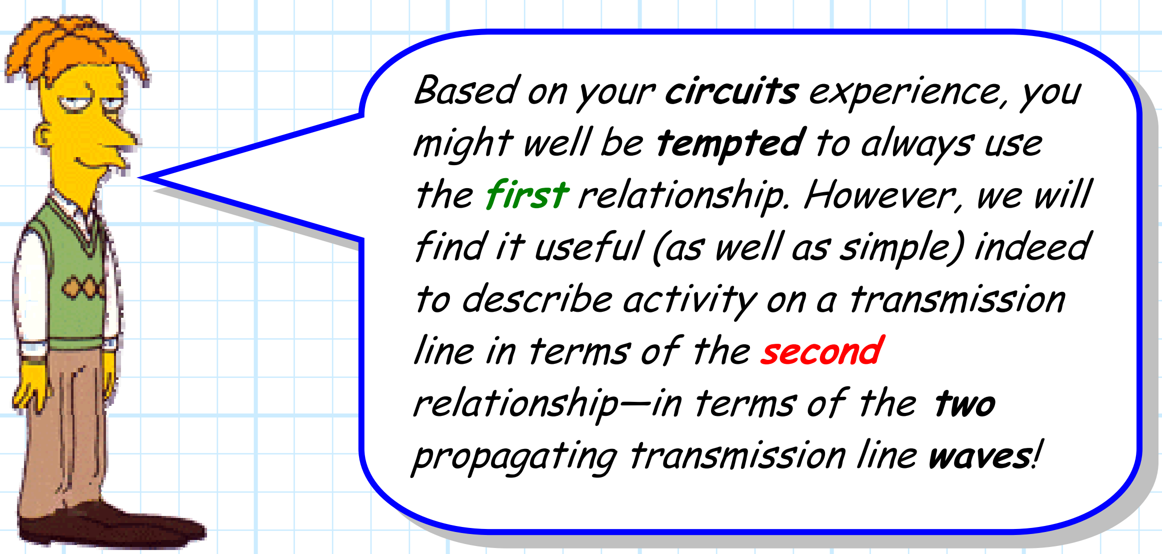
We can use (total) **voltage** and **current**, related by **line impedance**:

$$Z(z) = \frac{V(z)}{I(z)} \quad \therefore \quad V(z) = Z(z)\, I(z)$$

Or, we can use the two propagating **voltage waves**, related by the **reflection coefficient**:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} \quad \therefore \quad V^-(z) = \Gamma(z)\, V^+(z)$$

These are **equivalent** relationships—we can use **either** when describing a transmission line.



V, I, Z or V^+, V^-, Γ ?

Q: *How do I choose* ***which*** *relationship to use when describing/analyzing transmission line activity? What if I make the* ***wrong*** *choice? How will I know* ***if*** *my analysis is correct?*

A: Remember, the two relationships are **equivalent**. There is **no** explicitly wrong or right choice—**both** will provide you with precisely the **same** correct answer!

For example, we know that the total voltage and current can be determined from knowledge wave representation:

$$\begin{aligned} V(z) &= V^+(z) + V^+(z) \\ &= V^+(z)\big(1 + \Gamma(z)\big) \end{aligned}$$

$$\begin{aligned} I(z) &= \frac{V^+(z) - V^+(z)}{Z_0} \\ &= \frac{V^+(z)\big(1 - \Gamma(z)\big)}{Z_0} \end{aligned}$$

Or explicitly using the wave solutions $V^{+}(z)=V_0^{+}e^{-j\beta z}$ and $V^{-}(z)=V_0^{-}e^{+j\beta z}$:

$$
\begin{aligned}
V(z) &= V_0^{+}\, e^{-j\beta z} + V_0^{-}\, e^{+j\beta z} \\
&= V_0^{+}\left(e^{-j\beta z} + \Gamma_0\, e^{+j\beta z}\right)
\end{aligned}
$$

$$
\begin{aligned}
I(z) &= \frac{V_0^{+}\, e^{-j\beta z} - V_0^{-}\, e^{+j\beta z}}{Z_0} \\
&= \frac{V_0^{+}\left(e^{-j\beta z} - \Gamma_0\, e^{+j\beta z}\right)}{Z_0}
\end{aligned}
$$

More importantly, we find that **line impedance** $Z(z)=V(z)/I(z)$ can be expressed as:

$$
\begin{aligned}
Z(z) &= Z_0\,\frac{V^{+}(z)+V^{+}(z)}{V^{+}(z)-V^{+}(z)} \\
&= Z_0\left(\frac{1+\Gamma(z)}{1-\Gamma(z)}\right)
\end{aligned}
$$

Look what happened—the line impedance can be **completely** and unambiguously expressed in terms of **reflection coefficient** $\Gamma(z)$!

More explicitly:

$$
Z(z) = Z_0\,\frac{V_0^{+}\, e^{-j\beta z} + V_0^{-}\, e^{+j\beta z}}{V_0^{+}\, e^{-j\beta z} - V_0^{-}\, e^{+j\beta z}} = Z_0\,\frac{1+\Gamma_0\, e^{+j2\beta z}}{1-\Gamma_0\, e^{+j2\beta z}}
$$

With a little algebra, we find likewise that the wave functions can be determined from $V(z), I(z)$ and $Z(z)$:

$$V^+(z) = \frac{V(z) + I(z)Z_0}{2} = \frac{V(z)}{Z(z)}\left(\frac{Z(z) + Z_0}{2}\right)$$

$$V^-(z) = \frac{V(z) - I(z)Z_0}{2} = \frac{V(z)}{Z(z)}\left(\frac{Z(z) - Z_0}{2}\right)$$

From this result we easily find that the reflection coefficient $\Gamma(z)$ can **likewise** be written directly in terms of **line impedance**:

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$

Thus, the values $\Gamma(z)$ and $Z(z)$ are **equivalent** parameters—if we know **one**, then we can directly determine the **other**!

Q: *So, if they are equivalent, why wouldn't I* ***always*** *use the current, voltage, line impedance representation? After all, I am more* ***familiar*** *and more confident those quantities. The* ***wave*** *representation sort of* ***scares*** *me!*

A: Perhaps I can **convince** you of the value of the **wave** representation.

Remember, the time-harmonic solution to the telegraphers equation simply boils down to **two complex constants**—V_0^+ and V_0^-. Once these complex values have been determined, we can describe **completely** the activity **all** points along our transmission line.

For the **wave representation** we find:

$$V^+(z) = V_0^+ \, e^{-j\beta z}$$

$$V^-(z) = V_0^+ \, e^{+j\beta z}$$

$$\Gamma(z) = \frac{V_0^-}{V_0^+} \, e^{+j\,2\beta\, z}$$

Note that the **magnitudes** of the complex functions are in fact **constants** (with respect to position *z*):

$$\left|V^{+}(z)\right| = \left|V_0^{+}\right|$$

$$\left|V^{-}(z)\right| = \left|V_0^{+}\right|$$

$$\left|\Gamma(z)\right| = \left|\frac{V_0^{-}}{V_0^{+}}\right|$$

While the **relative phase** of these complex functions are expressed as a **simple** linear relationship with respect to z:

$$\boldsymbol{arg}\left\{V^{+}(z)\right\} = -\beta z$$

$$\boldsymbol{arg}\left\{V^{-}(z)\right\} = +\beta z$$

$$\boldsymbol{arg}\left\{\Gamma(z)\right\} = +2\beta z$$

Now, **contrast** this with the complex current, voltage, impedance functions:

$$V(z) = V_0^+ \; e^{-j\beta z} + V_0^- \; e^{+j\beta z}$$

$$I(z) = \frac{V_0^+ \; e^{-j\beta z} - V_0^- \; e^{+j\beta z}}{Z_0}$$

$$Z(z) = Z_0 \frac{V_0^+ \; e^{-j\beta z} + V_0^- \; e^{+j\beta z}}{V_0^+ \; e^{-j\beta z} - V_0^- \; e^{+j\beta z}}$$

With magnitude:

$$|V(z)| = \left| V_0^+ \; e^{-j\beta z} + V_0^- \; e^{+j\beta z} \right| = ??$$

$$|I(z)| = \frac{\left| V_0^+ \; e^{-j\beta z} - V_0^- \; e^{+j\beta z} \right|}{Z_0} = ??$$

$$|Z(z)| = Z_0 \frac{\left| V_0^+ \; e^{-j\beta z} + V_0^- \; e^{+j\beta z} \right|}{\left| V_0^+ \; e^{-j\beta z} - V_0^- \; e^{+j\beta z} \right|} = ??$$

and phase:

$$\textbf{arg}\{V(z)\} = \textbf{arg}\left\{V_0^+ \; e^{-j\beta z} + V_0^- \; e^{+j\beta z}\right\} = ??$$

$$\textbf{arg}\{I(z)\} = \textbf{arg}\left\{V_0^+ \; e^{-j\beta z} - V_0^- \; e^{+j\beta z}\right\} = ??$$

$$\begin{aligned}\textbf{arg}\{Z(z)\} &= \textbf{arg}\left\{V_0^+ \; e^{-j\beta z} + V_0^- \; e^{+j\beta z}\right\} \\ &\quad - \textbf{arg}\left\{V_0^+ \; e^{-j\beta z} - V_0^- \; e^{+j\beta z}\right\} \\ &= ??\end{aligned}$$

Q: *It appears to me that when attempting to describe the activity along a transmission line—as a function of **position** z—it is much **easier** and more **straightforward** to use the **wave** representation.*

*Is my insightful conclusion **correct** (nyuck, nyuck, nyuck)?*

A: Yes it is! However, this does **not** mean that we **never** determine $V(z)$, $I(z)$, or $Z(z)$; these quantities are still **fundamental** and very important—particularly at each **end** of the transmission line!