2.5 – The Quarter-Wave Transformer

Reading Assignment: ***pp. 73-76***

By now you've noticed that a quarter-wave length of transmission line ($\ell = \lambda/4$, $2\beta\ell = \pi$) appears often in microwave engineering problems.

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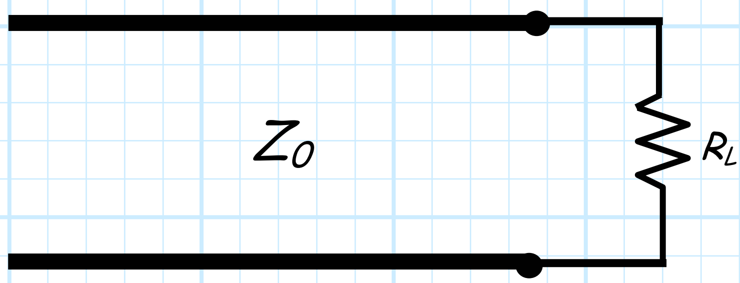
HO: The Quarter-Wave Transformer

Q: Why does the quarter-wave matching network work—after all, the quarter-wave line is mismatched at both ends?

A: HO: Multiple Reflection Viewpoint

The Quarter-Wave Transformer

Say the end of a transmission line with characteristic impedance Z_0 is terminated with a **resistive** (i.e., real) load.

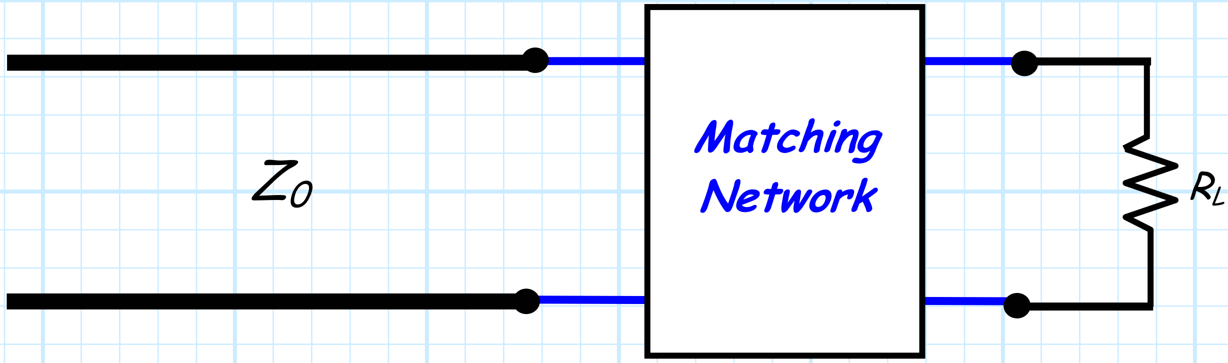


We typically would like **all** power traveling down the line to be **absorbed** by the load R_L.

But if $R_L \neq Z_0$, the line is **unmatched** and some of the incident power will be **reflected**.

Q: *Can* ***all*** *incident power be delivered to a resistive load* ***if*** $R_L \neq Z_0$*??*

A: **Yes**! We can insert a **matching network** between the transmission line and the load.



A matching network is a **lossless**, **2-port** device. Its job is to **transform** the load R_L (or even Z_L) to a value Z_0.

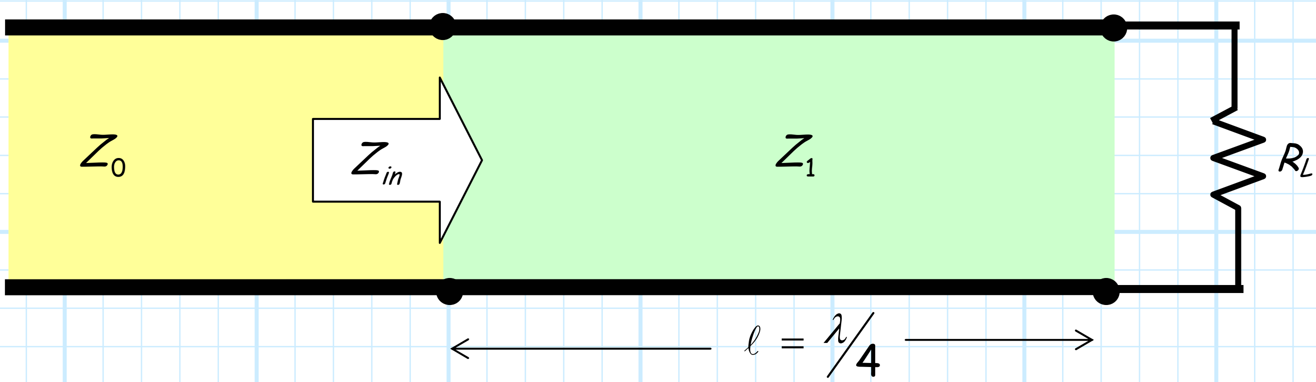
In other words, we want the **input** impedance of the matching network to be $Z_{in}=Z_0$, so that $\Gamma_{in}=0$--**no reflection**!

Since **none** of the incident power is **reflected**, and none is **absorbed** by the lossless matching network, it **all** must be absorbed by the **load** R_L !

Q: *These matching networks sound too good to be true. Exactly how do we **build** them?*

A: There are **many** methods and ways, but perhaps the easiest is the **quarter-wave transformer**.

First, insert a transmission line with characteristic impedance Z_1 and length $\ell=\lambda/4$ (i.e., a quarter-wave line) **between** the load and the Z_0 transmission line.



The $\lambda/4$ line **is** the **matching network**!

Q: *But what about the characteristic impedance Z_1; what **should** its value be??*

A: Remember, the quarter wavelength case is one of the **special** cases that we studied. We know that the **input** impedance of the quarter wavelength line is:

$$Z_{in} = \frac{(Z_1)^2}{Z_L} = \frac{(Z_1)^2}{R_L}$$

Thus, if we wish for Z_{in} to be numerically equal to Z_0, we find:

$$Z_{in} = \frac{(Z_1)^2}{R_L} = Z_0$$

Solving for Z_1, we find its **required** value to be:

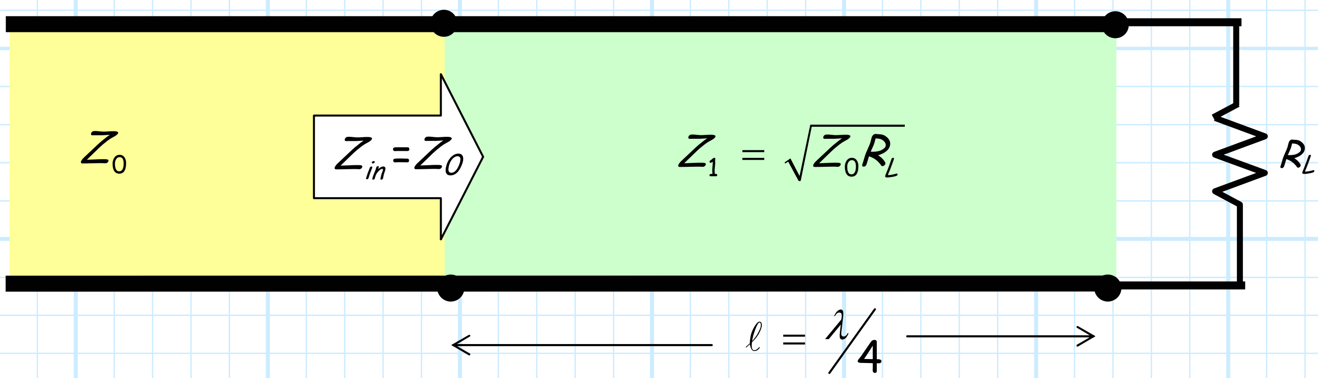
$$(Z_1)^2 / R_L = Z_0$$

$$(Z_1)^2 = Z_0 R_L$$

$$Z_1 = \sqrt{Z_0 R_L}$$

In other words, the characteristic impedance of the quarter wave line is the **geometric average** Z_0 and R_L!

Therefore, a $\lambda/4$ line with characteristic impedance $Z_1 = \sqrt{Z_0 R_L}$ will **match** a transmission line with characteristic impedance Z_0 **to** a resistive load R_L.



Thus, **all power** is delivered to load R_L!

Important Note: We find that $Z_{in} = Z_0$ **only** if the matching if the quarter-wave transmission line is **exactly** one-quarter wavelength in length $\ell = \lambda/4$.

The **problem** with this, of course, is that a physical length ℓ of transmission line is exactly one-quarter wavelength at only **one** frequency f!

Remember, **wavelength** is related to **frequency** as:

$$\lambda = \frac{v_p}{f} = \frac{1}{f\sqrt{LC}}$$

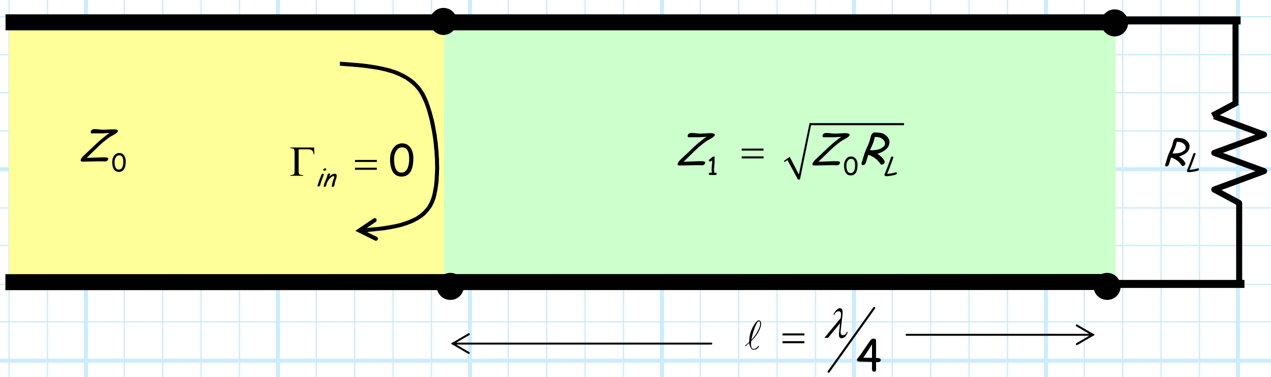
where v_p is the **propagation velocity** of the wave .

For **example**, assuming that $v_p = c$ (c = the speed of light in a vacuum), one wavelength at 1 GHz is 30 cm ($\lambda = 0.3\ m$), while one wavelength at 3 GHz is 10 cm ($\lambda = 0.1\ m$). As a result, a transmission line length $\ell = 7.5\ cm$ is a quarter wavelength for a signal at 1GHz **only**.

Thus, a quarter-wave transformer provides a **perfect** match ($\Gamma_{in} = 0$) at **one** and **only one** signal frequency!

Multiple Reflection Viewpoint

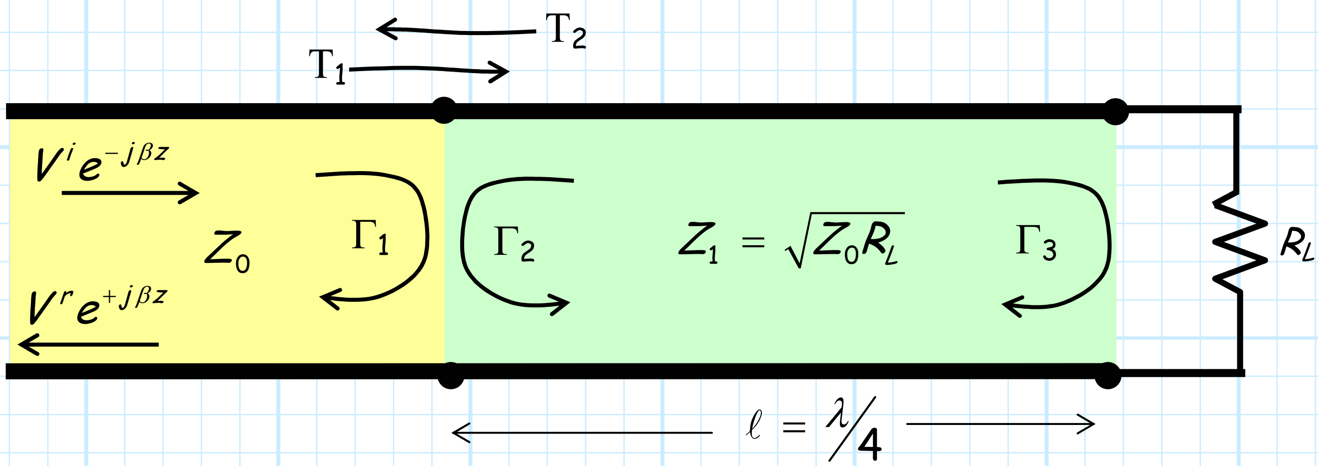
The **quarter-wave** transformer brings up an interesting question in μ-wave engineering.



Q: ***Why*** *is there* ***no*** *reflection at* $z = -\ell$ *? It appears that the line is* ***mismatched*** *at both* $z = 0$ *and* $z = -\ell$*.*

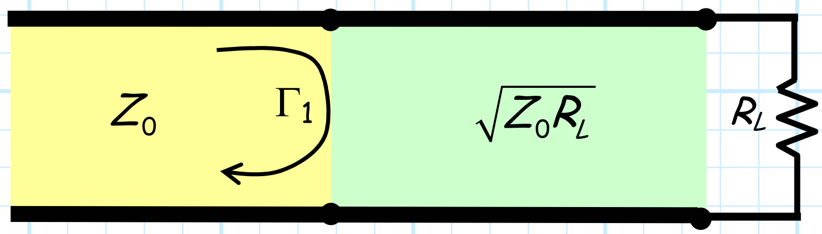
A: In fact there **are** reflections at these mismatched interfaces—an **infinite** number of them!

First, lets **define** a few terms:



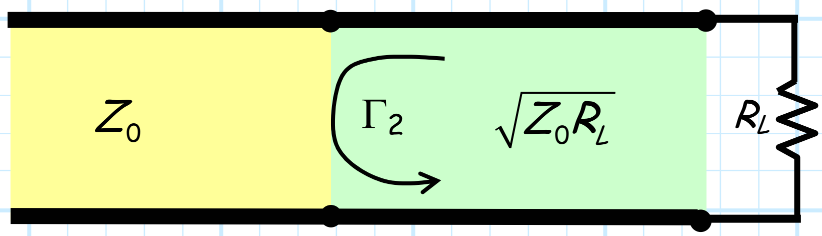
Γ_1 = **partial** reflection coefficient of a wave incident on the $z = -\ell$ interface from the Z_0 line:

$$\Gamma_1 = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$



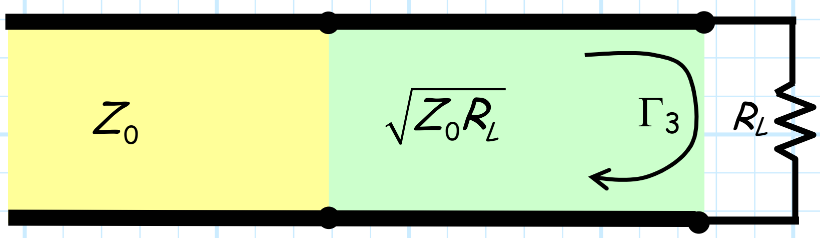
Γ_2 = **partial** reflection coefficient of a wave incident on the $z = -\ell$ interface from the Z_1 line:

$$\Gamma_2 = \frac{Z_0 - Z_1}{Z_0 + Z_1} = -\Gamma_1$$



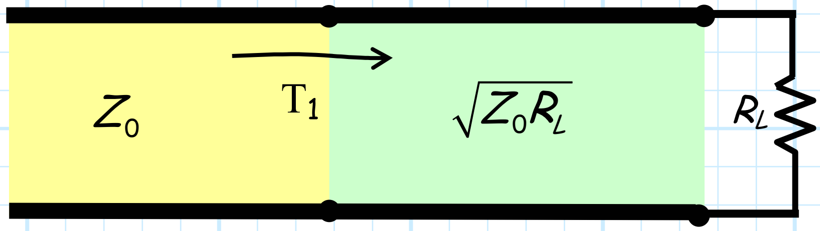
Γ_3 = **partial** reflection coefficient of a wave incident on the $z = -0$ interface from the Z_1 line:

$$\Gamma_3 = \frac{R_L - Z_1}{R_L + Z_1}$$



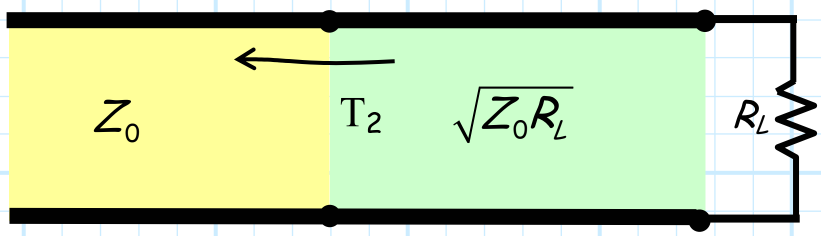
T_1 = **partial** transmission coefficient of a wave incident on the $z = -\ell$ interface from the Z_0 line:

$$T_1 = \frac{2Z_1}{Z_1 + Z_0}$$

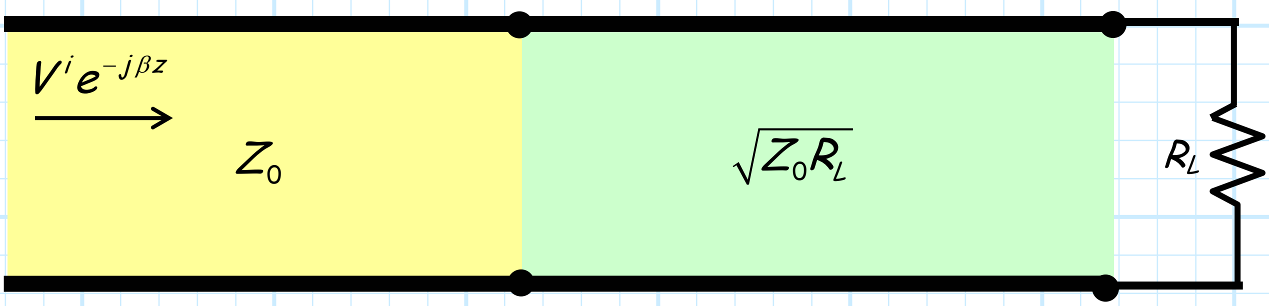


T_2 = **partial** transmission coefficient of a wave incident on the $z = -\ell$ interface from the Z_1 line:

$$T_2 = \frac{2Z_0}{Z_0 + Z_1}$$

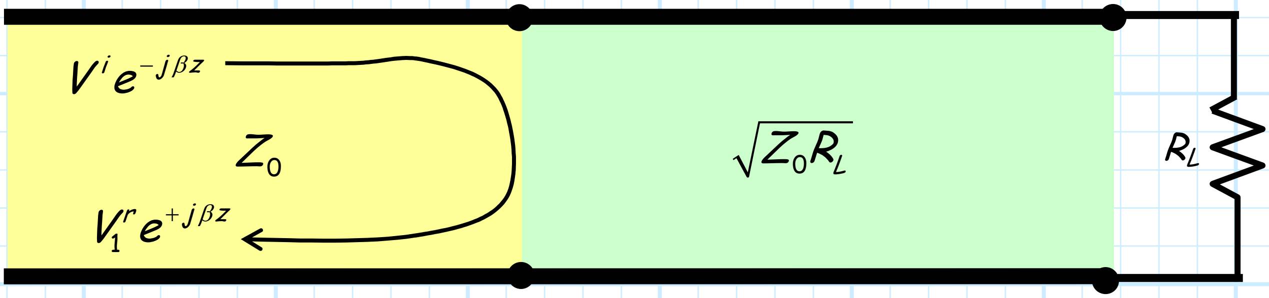


Now let's try to intemperate what **physically** happens when the **incident** voltage wave:



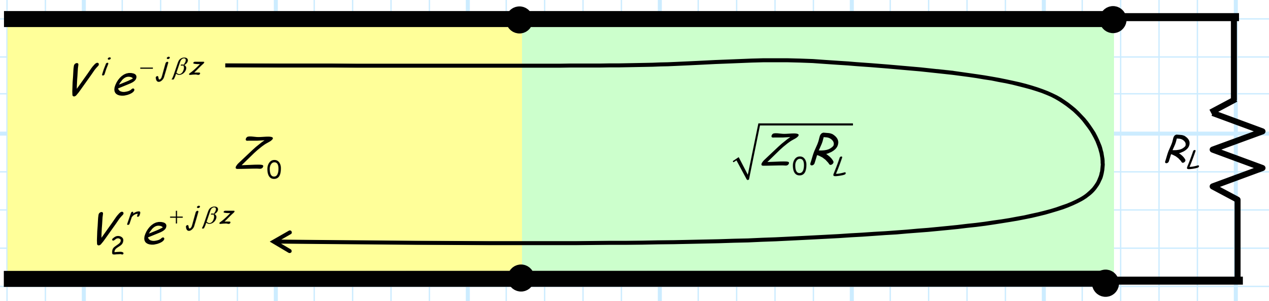
reaches the interface at $z = -\ell$.

1. At $z=-\ell$, the characteristic impedance of the transmission line changes from Z_0 to Z_1. This mismatch creates a **reflected** wave:



where $V_1^r = \Gamma_1 V^i$.

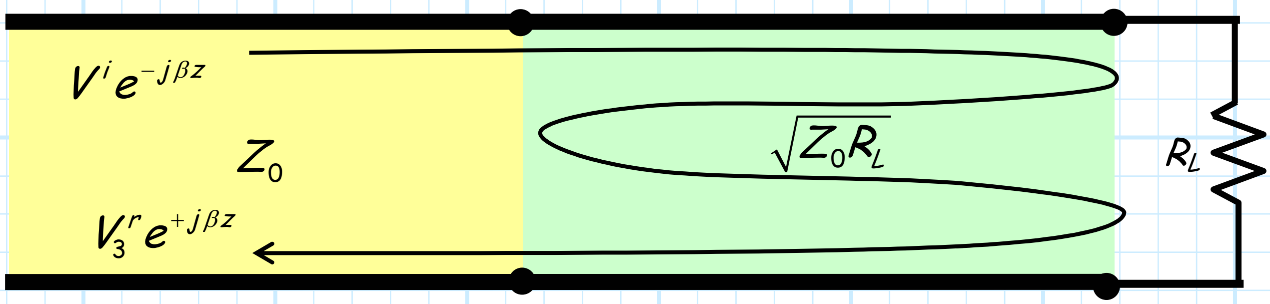
2. However, a **portion** of the incident wave is transmitted (T_1) across the interface at $z=-\ell$, this wave travels a distance of $\beta\ell = 90^\circ$ to the load at $z=0$, where a portion of it is reflected (Γ_3). This wave travels back $\beta\ell = 90^\circ$ to the interface at $z=-\ell$, where a portion is again transmitted (T_2) across into the Z_0 transmission line—**another** reflected wave (V_2^r)!



where we have found that traveling $2\beta\ell = 180^\circ$ has produced a **minus** sign in our result:

$$\begin{aligned} V_2^r &= \mathrm{T}_2\, e^{-j90^\circ}\, \Gamma_3\, e^{-j90^\circ}\, \mathrm{T}_1\, V^i \\ &= -\mathrm{T}_1\, \mathrm{T}_2\, \Gamma_3\, V^i \end{aligned}$$

3. However, a **portion** of this **second** wave is also **reflected** (Γ_2) back into the Z_1 transmission line at $z = -\ell$, where it again travels to $\beta\ell = 90^\circ$ the load, is partially reflected (Γ_3), travels $\beta\ell = 90^\circ$ back to $z = -\ell$, and is partially transmitted into Z_0 (T_2)—our **third** reflected wave!



where:

$$
\begin{aligned}
V_3^r &= \mathrm{T}_2\, e^{-j90^\circ}\, \Gamma_3\, e^{-j90^\circ}\, \Gamma_2\, e^{-j90^\circ}\, \Gamma_3\, e^{-j90^\circ}\, \mathrm{T}_1 V^i \\
&= \mathrm{T}_1\, \mathrm{T}_2 \left(\Gamma_3\right)^2 \Gamma_2 V^i
\end{aligned}
$$

n. We can see that this "bouncing" back and forth can go on **forever**, with each trip launching a **new** reflected wave into the Z_0 transmission line.

Note however, that the **power** associated with each successive reflected wave is **smaller** than the previous, and so eventually, the power associated with the reflected waves will **diminish** to insignificance!

Q: *But, why then is* $\Gamma = 0$ *?*

A: Each reflected wave V_n^r is a **coherent** wave. That is, they all oscillate at same frequency ω; the reflected waves differ only in terms of their **magnitude** and **phase**.

Therefore, to determine the **total** reflected wave, we must perform a **coherent summation** of each reflected wave—a operation easily performed since we have expressed our waves with **complex** notation:

$$V^r e^{+j\beta z} = \sum_{n=1}^{\infty} V_n^r e^{+j\beta z}$$

It can be shown that this infinite series **converges**, with the result:

$$V^r = \left(\frac{\Gamma_1 + \Gamma_1 \Gamma_2 \Gamma_3 - \mathrm{T}_1 \mathrm{T}_2 \Gamma_3}{1 + \Gamma_2 \Gamma_3} \right) V^i$$

Thus, the **total** reflection coefficient is:

$$\Gamma = \frac{V^r}{V^i} = \frac{\Gamma_1 + \Gamma_1 \Gamma_2 \Gamma_3 - \mathrm{T}_1 \mathrm{T}_2 \Gamma_3}{1 + \Gamma_2 \Gamma_3}$$

Using our definitions, it can likewise be shown that the **numerator** of the above expression is:

$$\Gamma_1 + \Gamma_1 \Gamma_2 \Gamma_3 - \mathrm{T}_1 \mathrm{T}_2 \Gamma_3 = \frac{2\left(Z_1^2 - Z_0 R_L\right)}{\left(Z_1 + Z_0\right)\left(R_L + Z_1\right)}$$

It is evident that the numerator (and therefore Γ) will be **zero** if:

$$Z_1^2 - Z_0 R_L \qquad \Rightarrow \qquad Z_1 = \sqrt{Z_0 R_L}$$

Just as we expected!

Physically, this results insures that all the reflected waves add coherently together to produce a **zero value**!

A simple example of this phenomenon is the addition of **two** waves with **equal** magnitude and **opposite** phase (i.e., their phase difference is 180°).

$$cos(\omega t) + cos(\omega t + 180^\circ) = cos(\omega t) - cos(\omega t) = 0$$

Note **all** of our transmission line analysis has been **steady-state** analysis. We assume our signals are **sinusoidal**, of the form $\exp(j\omega t)$. Note this signal exists for **all time** t—the signal is assumed to have been "on" **forever**, and assumed to continue on forever.

In other words, in steady-state analysis, **all** the multiple reflections have long since occurred, and thus have reached a steady state—the reflected wave is **zero**!