Gabarito - prova 2

Mecânica Quântica 1

1- $H(x,y) = \left(\frac{p_x^2}{2m} + \frac{m\omega_x^2 x^2}{2}\right) + \left(\frac{p_y^2}{2m} + \frac{m\omega_y^2 y^2}{2}\right)$

$H(x,y) = H(x) + H(y)$

$$\begin{cases} H(x)\,\psi(x) = E_x\,\psi(x), & E_x = \hbar\omega\left(n_x + \frac{1}{2}\right) \\ H(y)\,\psi(y) = E_y\,\psi(y), & E_y = \hbar\omega\left(n_y + \frac{1}{2}\right) \end{cases}$$

$H(x,y)\,\psi(x,y) = E\,\psi(x,y)$

$\psi(x,y) = \psi(x)\,\psi(y) \rightarrow$ vamos checar que essa é a solução e obter o espectro (E);

$H(x,y)\,\psi(x,y) = [H(x) + H(y)]\,\psi(x)\,\psi(y) = (E_x + E_y)\,\psi(x)\,\psi(y)$

$H(x,y)\,\psi(x,y) = E\,\psi(x,y)$

portanto: $H(x,y) = H(x) + H(y)$

$\psi(x,y) = \psi(x)\,\psi(y)$

$E = E_x + E_y$ //

1- $E = E_x + E_y$

$$E = \hbar\omega_x\left(n_x + \frac{1}{2}\right) + \hbar\omega_y\left(n_y + \frac{1}{2}\right); \quad n_x = 0, 1, 2, \dots$$
$$n_y = 0, 1, 2, \dots$$

A) $\omega_x = \omega_y = \omega$: $E = ?$, degenerescência $= ?$

e $g = ?$

$$E = \hbar\omega\left(n_x + \frac{1}{2}\right) + \hbar\omega\left(n_y + \frac{1}{2}\right)$$

$$E = \hbar\omega(n+1) \quad \text{com } n = n_x + n_y$$
$$n_x = 0, 1, 2, \dots$$
$$n_y = 0, 1, 2, \dots$$

$n = 0 \rightarrow n_x = 0$ e $n_y = 0 \rightarrow g = 1$

$n = 1 \rightarrow \begin{cases} n_x = 1 \text{ e } n_y = 0 \\ n_x = 0 \text{ e } n_y = 1 \end{cases} \rightarrow g = 2$

$n = 2 \rightarrow \begin{cases} n_x = 1 \text{ e } n_y = 1 \\ n_x = 2 \text{ e } n_y = 0 \\ n_x = 0 \text{ e } n_y = 2 \end{cases} \rightarrow g = 3$

generalizando para um estado $|n\rangle$ temos $g = n + 1$

1- B) $\omega_x = 2\omega$ e $\omega_y = \omega$: $\omega_x = 2\omega_y = 2\omega$

$$E = \hbar\omega_x\left(n_x + \frac{1}{2}\right) + \hbar\omega_y\left(n + \frac{1}{2}\right) = 2\hbar\omega\left(n_x + \frac{1}{2}\right) + \hbar\omega\left(n_y + \frac{1}{2}\right)$$

$$E = \hbar\omega\left(n + \frac{3}{2}\right) \text{ com } n = n_y + 2n_x$$

$$n_x = 0, 1, 2, \ldots$$

$$n_y = 0, 1, 2, \ldots$$

$$n = 0 \rightarrow n_x = 0 \text{ e } n_y = 0 \rightarrow g = 1 \text{ e } E_0 = \frac{3\hbar\omega}{2}$$

$$n = 1 \rightarrow n_x = 0 \text{ e } n_y = 1 \rightarrow g = 1 \text{ e } E_1 = \frac{5\hbar\omega}{2}$$

$$\Delta E = \frac{5\hbar\omega}{2} - \frac{3\hbar\omega}{2} = h\nu$$

$$h\nu = \hbar\omega \rightarrow \nu = \frac{h\omega}{2\pi h} \rightarrow \nu = \frac{\omega}{2\pi}$$ //

C) $\omega_x = 2\omega_y = 2\omega$:

$$E = \hbar\omega_y\left(n + \frac{3}{2}\right) \text{ com } n = 2n_x + n_y$$

$$n_x = 0, 1, 2, \ldots$$

$$n_y = 0, 1, 2, \ldots$$

1- C) $n=0 \longrightarrow n_x=0$ e $n_y=0 \longrightarrow g=1$ e $E_0=\frac{3\hbar w}{2}$

$n=1 \longrightarrow n_x=0$ e $n_y=1 \longrightarrow g=1$ e $E_1=\frac{5\hbar w}{2}$

$n=2 \longrightarrow \begin{cases} n_x=1 \text{ e } n_y=0 \\ n_x=0 \text{ e } n_y=2 \end{cases} \longrightarrow g=2$ e $E_2=\frac{7\hbar w}{2}$

O estado mais baixo que apresenta degenerescência é o $|2\rangle$, segundo estado excitado com $g=2$.

2- A) $\Delta=\sum_{n=0}^{\infty}\langle 1|x^2|n\rangle\langle n|x^2|1\rangle$

Como $\sum_{n=0}^{\infty}|n\rangle\langle n|=\mathbb{1}$, $\Delta=\langle 1|x^2\,\mathbb{1}\,x^2|1\rangle$

$$\Delta=\langle 1|x^4|1\rangle$$

$\hat{x}^4=? \longrightarrow \hat{x}=\sqrt{\frac{\hbar}{2mw}}\,(\hat{a}+\hat{a}^{+})$

$$x^2=\frac{\hbar}{2mw}\left(\hat{a}^2+\hat{a}^{+2}+\hat{a}\hat{a}^{+}+\hat{a}^{+}\hat{a}\right)$$

$$\hat{x}^4=\frac{\hbar^2}{4m^2w^2}\Big(\hat{a}^4+\hat{a}^2\hat{a}^{+2}+\hat{a}^3\hat{a}^{+}+\hat{a}^2\hat{a}^{+}\hat{a}+\hat{a}^{+2}\hat{a}^2+\hat{a}^{+4}+\hat{a}^{+2}\hat{a}\hat{a}^{+}$$
$$+\hat{a}^{+3}\hat{a}+\hat{a}\hat{a}^{+}\hat{a}^{2}+\hat{a}\hat{a}^{+3}+\hat{a}\hat{a}^{+}\hat{a}\hat{a}^{+}+\hat{a}\hat{a}^{+2}\hat{a}+\hat{a}^{+}\hat{a}^3+\hat{a}^{+}\hat{a}\hat{a}^{+2}$$
$$+\hat{a}^{+}\hat{a}^2\hat{a}^{+}+\hat{a}^{+}\hat{a}\hat{a}^{+}\hat{a}\Big)$$

2- A)

$$\langle 1|x^4|1\rangle = \frac{\hbar^2}{4m^2w^2}\Big\{\langle 1|\hat{a}^4|1\rangle + \langle 1|\hat{a}^2\hat{a}^{+2}|1\rangle + \langle 1|\hat{a}^3\hat{a}^+|1\rangle + \langle 1|\hat{a}^2\hat{a}^+\hat{a}|1\rangle$$

$$+\langle 1|\hat{a}^{+2}\hat{a}^2|1\rangle + \langle 1|\hat{a}^{+4}|1\rangle + \langle 1|\hat{a}^{+2}\hat{a}\hat{a}^+|1\rangle + \langle 1|\hat{a}^{+3}\hat{a}|1\rangle$$

$$+\langle 1|\hat{a}\hat{a}^+\hat{a}^2|1\rangle + \langle 1|\hat{a}\hat{a}^{+3}|1\rangle + \langle 1|\hat{a}\hat{a}^+\hat{a}\hat{a}^+|1\rangle + \langle 1|\hat{a}\hat{a}^{+2}\hat{a}|1\rangle + \langle 1|\hat{a}^+\hat{a}^3|1\rangle$$

$$+\langle 1|\hat{a}^+\hat{a}\hat{a}^{+2}|1\rangle + \langle 1|\hat{a}^+\hat{a}^2\hat{a}^+|1\rangle + \langle 1|\hat{a}^+\hat{a}\hat{a}^+\hat{a}|1\rangle\Big\}$$

$$\langle 1|\hat{x}^4|1\rangle = \frac{\hbar^2}{4m^2w^2}\Big\{\langle 1|\hat{a}^2\hat{a}^{+2}|1\rangle + \langle 1|\hat{a}\hat{a}^+\hat{a}\hat{a}^+|1\rangle + \langle 1|\hat{a}\hat{a}^{+2}\hat{a}|1\rangle$$

$$+\langle 1|\hat{a}^+\hat{a}^2\hat{a}^+|1\rangle + \langle 1|\hat{a}^+\hat{a}\hat{a}^+\hat{a}|1\rangle\Big\}$$

$$\langle 1|\hat{x}^4|1\rangle = \frac{\hbar^2}{4m^2w^2}\cdot\{6+4+2+2+1\}$$

$$\Delta = \langle 1|\hat{x}^4|1\rangle = \frac{15\hbar^2}{4m^2w^2}$$

solução alternativa para a questão 2A:

$\langle 1|\hat{x}^2|n\rangle = ?$ e $\langle n|\hat{x}^2|1\rangle = ?$

$$\langle 1|\hat{x}^2|n\rangle = \frac{\hbar}{2mw}\langle 1|\hat{a}^2 + \hat{a}^{+2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a}|n\rangle$$

2-A)

$$\begin{cases} \langle 1|\hat{x}^2|n\rangle = \dfrac{\hbar}{2m\omega}\left\{\langle 1|\hat{a}^2|n\rangle + \langle 1|\hat{a}^{+2}|n\rangle + \langle 1|\hat{a}\hat{a}^+|n\rangle + \langle 1|\hat{a}^+\hat{a}|n\rangle\right\} \\ \langle 1|\hat{x}^2|n\rangle = \dfrac{\hbar}{2m\omega}\left\{\sqrt{n}\sqrt{n-1}\,\delta_{n-2,1} + \cancel{\sqrt{n+1}\sqrt{n+2}\,\delta_{n+2,1}}^{\to 0} + (n+1)\,\delta_{n,1} + n\,\delta_{n,1}\right\} \end{cases}$$

$$\begin{cases} \langle n|\hat{x}^2|1\rangle = \dfrac{\hbar}{2m\omega}\left\{\langle n|\hat{a}^2|1\rangle + \langle n|\hat{a}^{+2}|1\rangle + \langle n|\hat{a}\hat{a}^+|1\rangle + \langle n|\hat{a}^+\hat{a}|1\rangle\right\} \\ \langle n|\hat{x}^2|1\rangle = \dfrac{\hbar}{2m\omega}\left\{\sqrt{2}\sqrt{3}\,\delta_{n,3} + 2\,\delta_{n,1} + \delta_{n,1}\right\} \end{cases}$$

$$\Delta = \sum_{n=0}^{\infty} \langle 1|\hat{x}^2|n\rangle\langle n|\hat{x}^2|1\rangle$$

$$\Delta = \frac{\hbar^2}{4m^2\omega^2}\sum_{n=0}^{\infty}\left\{\sqrt{n(n-1)}\,\delta_{n-2,1} + (2n+1)\,\delta_{n,1}\right\}$$

$$\times\left\{\sqrt{6}\,\delta_{n,3} + 3\,\delta_{n,1}\right\}$$

$$\Delta = \frac{\hbar^2}{4m^2\omega^2}(6+9) \longrightarrow \Delta = \frac{15\hbar^2}{4m^2\omega^2}$$

2-B) $\langle T \rangle_n = \langle n | \hat{T} | n \rangle = \frac{1}{2m} \langle n | \hat{p}^2 | n \rangle$

$$\hat{p} = \frac{\sqrt{2\hbar m \omega}}{2i} (\hat{a} - \hat{a}^+)$$

$$\hat{p}^2 = -\frac{\hbar m \omega}{2} \left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a} - \hat{a}^2 - \hat{a}^{+2} \right)$$

$$\langle n | \hat{p}^2 | n \rangle = \frac{\hbar m \omega}{2} \left\{ \langle n | \hat{a}\hat{a}^+ | n \rangle + \langle n | \hat{a}^+\hat{a} | n \rangle - \cancel{\langle n | \hat{a}^2 | n \rangle} \to 0 - \cancel{\langle n | \hat{a}^{+2} | n \rangle} \to 0 \right\}$$

$$\langle T \rangle_n = \frac{\hbar m \omega}{2(2m)} \left\{ n + 1 + n \right\}$$

$$\boxed{\langle T \rangle_n = \frac{\hbar \omega}{4} \left\{ 2n + 1 \right\}}$$

3- A) $\hat{A} = \hat{A}^+$ e $\hat{B} = \hat{B}^+$: $[A, B]^+ = ?$

hermitiano?

$$[A, B]^+ = (AB)^+ - (BA)^+ = BA - AB$$

3 - A)

$\langle n | [A,B] | n \rangle = \langle n | AB | n \rangle - \langle n | BA | n \rangle$ //

$\langle n | [A,B]^{+} | n \rangle = \langle n | BA | n \rangle - \langle n | AB | n \rangle$ //

$\langle n | [A,B]^{+} | n \rangle = - \langle n | [A,B] | n \rangle$ //

Não é hermitiano, é anti hermitiano.

B) $\hat{H} = \hbar w \left(\hat{a}^{+} \hat{a} + \frac{1}{2} \right)$ e $E = \langle n | H | n \rangle = ?$

$$E = \langle n | \hbar w \left(\hat{a}^{+} \hat{a} + \frac{1}{2} \right) | n \rangle = \hbar w . n + \frac{\hbar w}{2}$$

$$\boxed{E_n = \hbar w \left(n + \frac{1}{2} \right)}$$ //

$$\boxed{E_n = \hbar w \left(n + \frac{1}{2} \right)}$$ //

C) $[\hat{H}, \hat{a}] = ?$ e $[\hat{H}, \hat{a}^{+}] = ?$

$$[\hat{H}, \hat{a}] = \hbar w [\hat{a}^{+}\hat{a}, \hat{a}] + \hbar w \left[\frac{1}{2}, \hat{a} \right]$$

$$[\hat{H}, \hat{a}] = \hbar w \hat{a}^{+} \underbrace{[\hat{a}, \hat{a}]}_{0} + \hbar w \underbrace{[\hat{a}^{+}, \hat{a}]}_{-1} \hat{a}$$

$$\boxed{[\hat{H}, \hat{a}] = - \hat{a} \hbar w}$$ //

3-C) $[H, \hat{a}^+] = \hbar\omega[\hat{a}^+\hat{a}, \hat{a}^+] + \hbar\omega\overbrace{[\tfrac{1}{2}, \hat{a}^+]}^{0}$

$$[H, \hat{a}^+] = \hbar\omega\,\hat{a}^+[\hat{a}, \hat{a}^+] + \hbar\omega[\hat{a}^+, \hat{a}^+]\,\hat{a}$$

$$\boxed{[H, \hat{a}^+] = \hbar\omega\,\hat{a}^+}$$