PRO5970 - Métodos de Otimização Não Linear

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Optimization models

Main components

Decision variables: x
Objective function: $f(x)$ (measure of the solution “quality”)
Constraints: $g(x) \leq 0$ or $h(x) = 0$

According to Linearity

Optimization problems

- Linear Programming (LP)
- Non Linear Programming (NLP)

According to Linearity

Optimization problems

Linear Programming (LP)

$$
\begin{array}{llll}
\text{max z} = & 3x_1 & -\ x_2 & \\
\text{s.a} & -3x_1 & +3x_2 & \leq 6 \\
 & -8x_1 & +\ 4x_2 & \leq 4 \\
 & x_j \geq 0 & \forall j &
\end{array}
$$

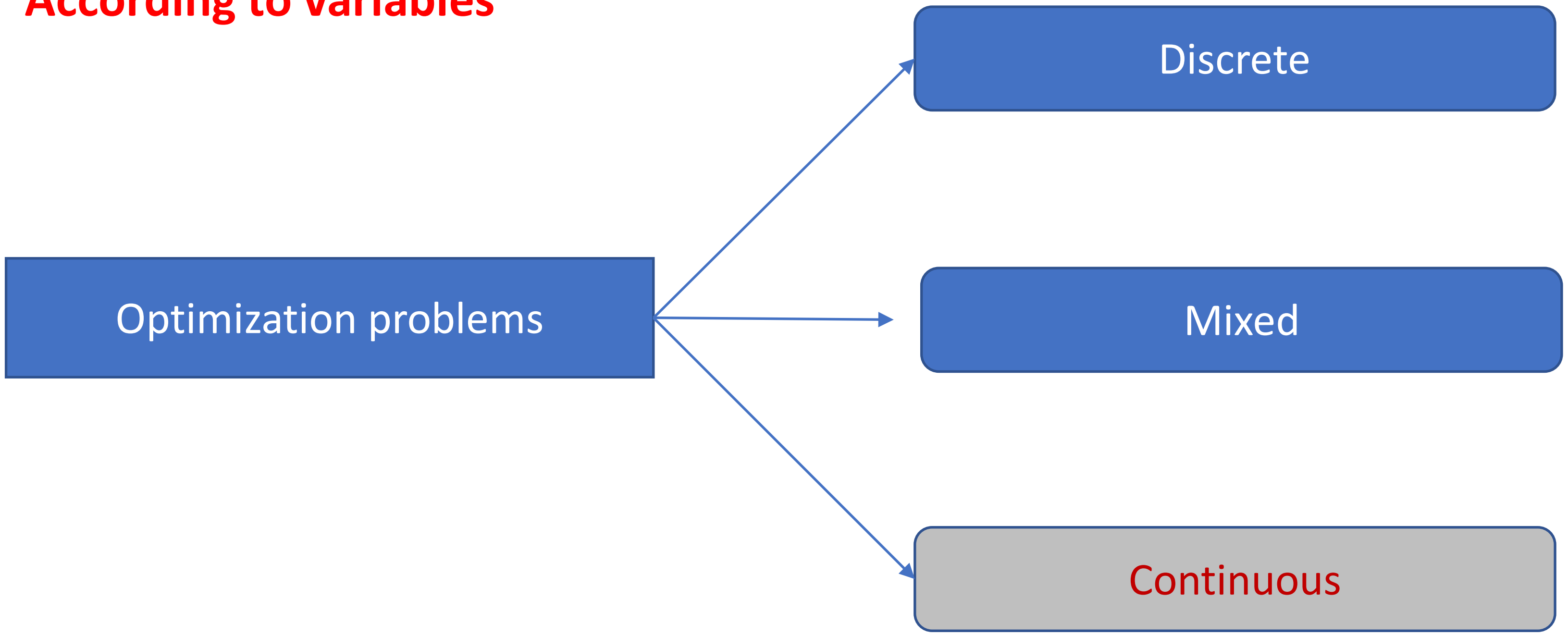
According to Linearity

Optimization problems

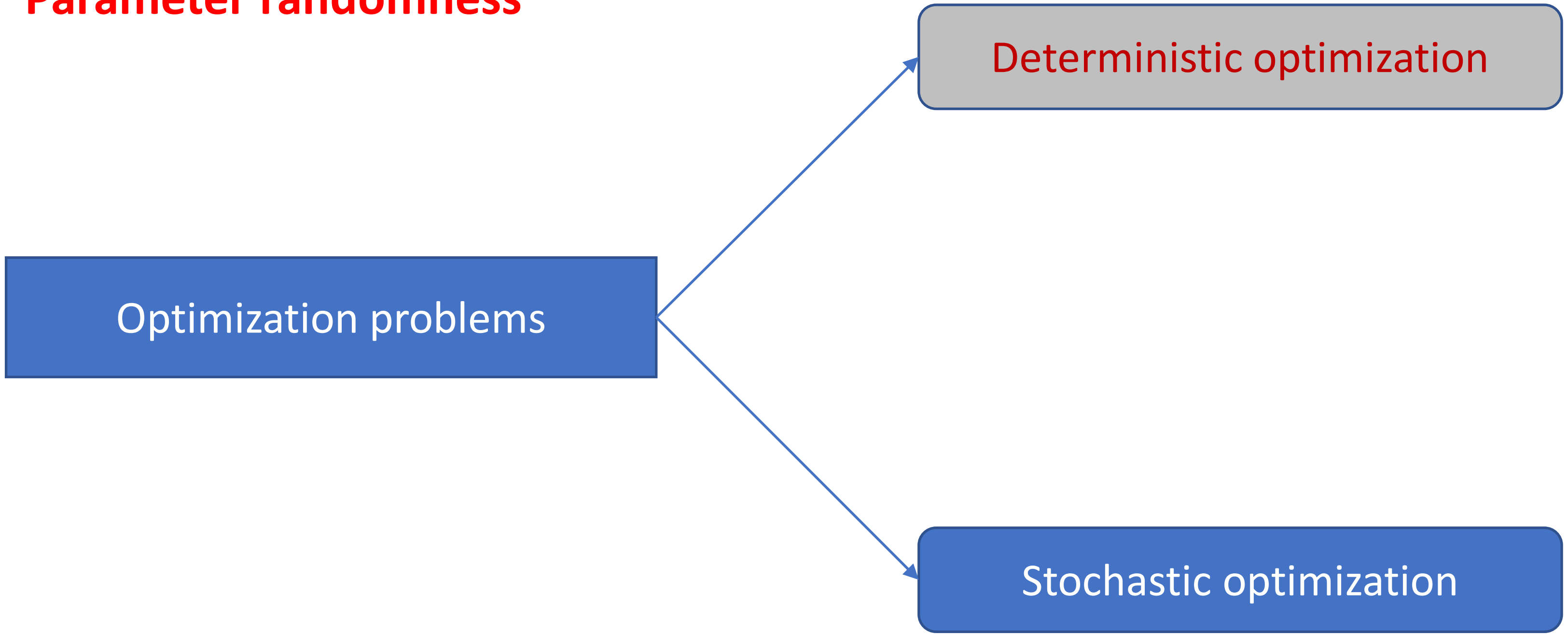
$$\begin{array}{ll} \text{min} & (x_1 - 3)^2 + (x_2 - 2)^2 \\ \text{s.a} & x_1^2 - x_2 \leq 3 \\ & x_2 \leq 1 \\ & x_1 \geq 1 \end{array}$$

Non Linear Programming (NLP)

According to variables



Parameter randomness



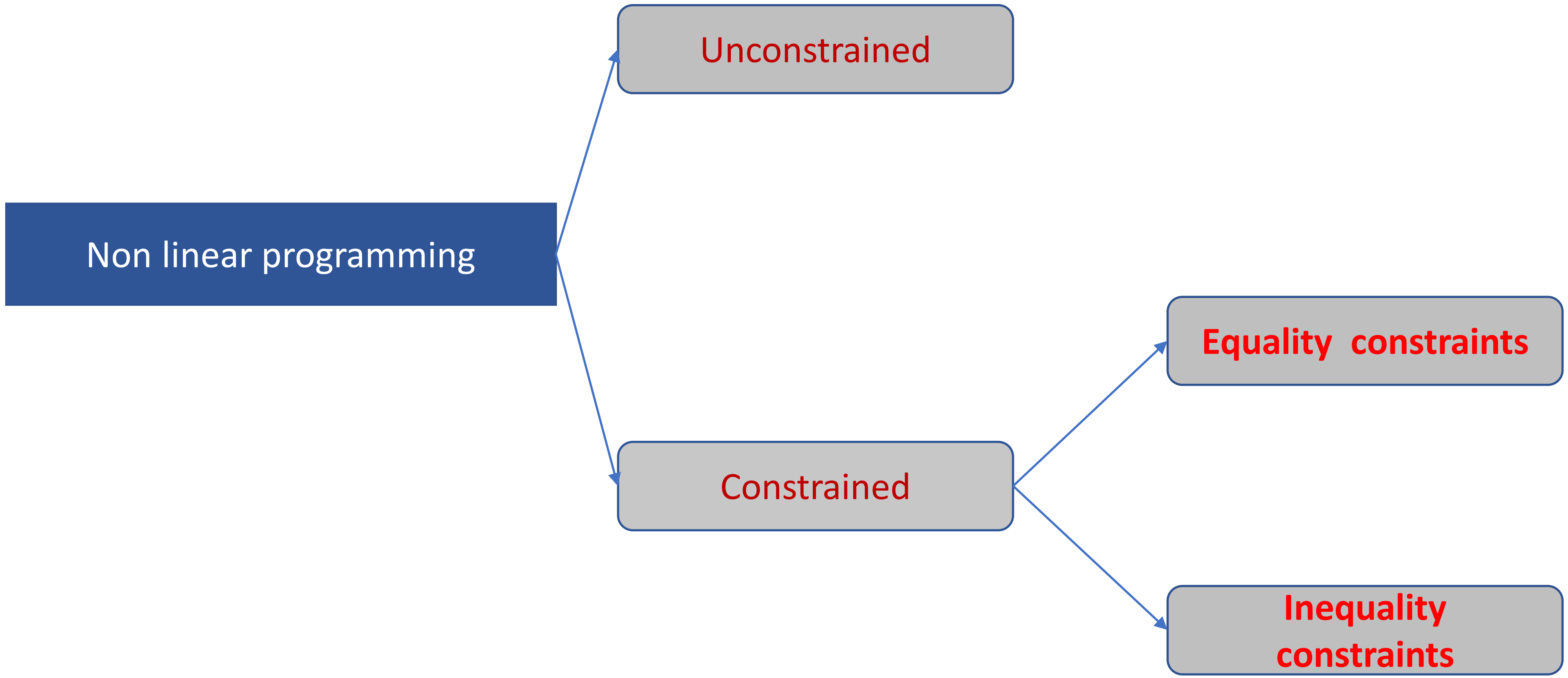
Parameter randomness

Optimization problems

- objective and constraint functions $f_i(x, \omega)$ depend on optimization variable x *and* a random variable ω
- ω models
  - parameter variation and uncertainty
  - random variation in implementation, manufacture, operation
- value of ω is not known, but its distribution is
- goal: choose x so that
  - constraints are satisfied on average, or with high probability
  - objective is small on average, or with high probability

Stochastic optimization

According to the constraints



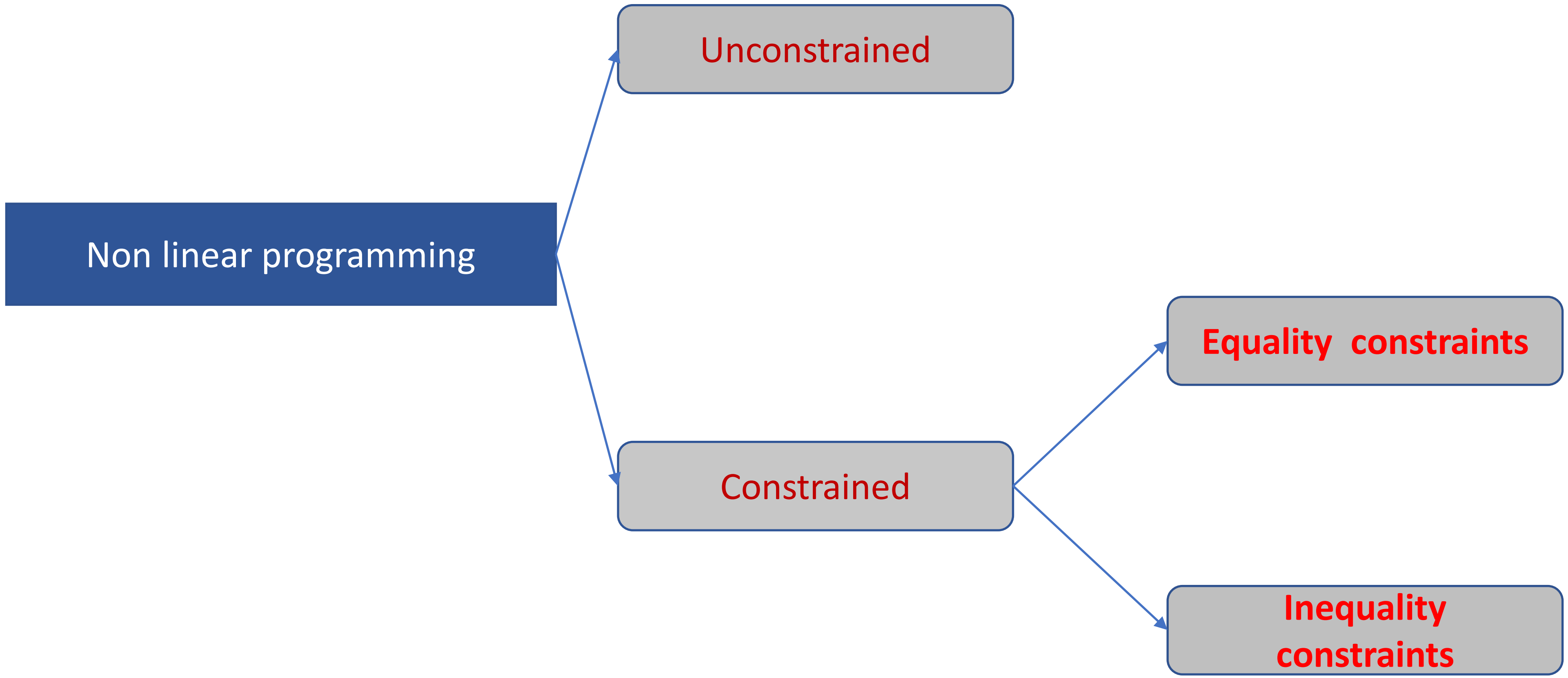
According to the constraints

Non linear programming

Unconstrained

$$f(x) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1}$$

According to the constraints



According to the constraints

$$\begin{aligned} &\text{maximize} && x_1 x_2 \\ &\text{subject to} && x_1^2 + 4x_2^2 = 1 \end{aligned}$$

Non linear programming

Constrained

Equality constraints

Inequality constraints

$$\begin{aligned} &\text{minimize} && f(x_1, x_2) = x_1^2 + x_2^2 + x_1 x_2 - 3x_1 \\ &\text{subject to} && x_1, x_2 \geq 0 \end{aligned}$$

According to the convexity (NLP)

https://www.sfu.ca/~ssurjano/optimization.html

- Non linear programming
  - Non convex
  - Convex
    - Equality constraints
    - Inequality constraints
    - Unconstrained

Our main focus

Non linear optimization problems

Continuous variables

Deterministic optimization

Convex problems

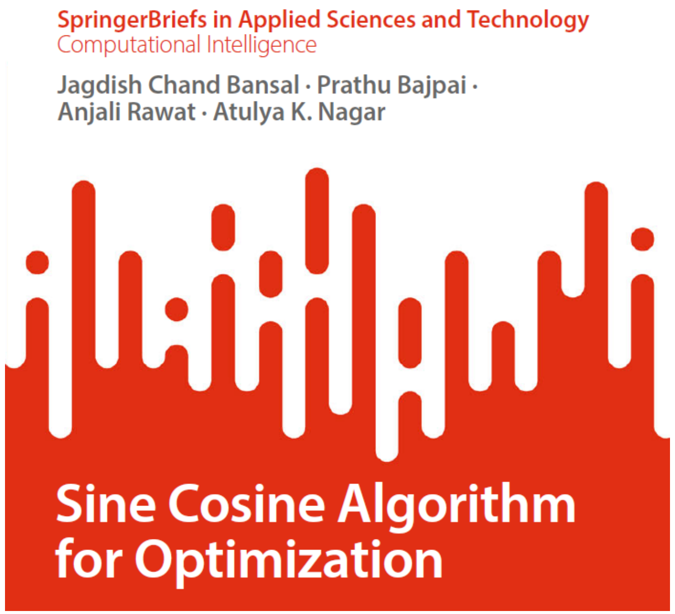
Unconstrained

Inequality constraints

Equality constraints

Table 1.1 Classification of optimization problems

Classification criterion	Optimization problem	Features
Nature of objective function and/or constraints	Linear	Linear objective function and constraints
	Nonlinear	Nonlinear objective unction and/or constraints
	Convex	Convex objective function and feasible set
	Quadratic	Quadratic objective function and linear constraints
	Stochastic	Probabilistically determined problem variables and/or parameters
	Deterministic	Decision variables and/or parameters are known accurately
	Non smooth	Either objective function or the constraints, or both, are not differentiable
Nature of the search space	Discrete	Discrete decision variables
	Continuous	Real decision variables
	Mixed integer	Both real and integer decision variables
Nature of the optimization problem	Dynamic	Objective function varying with time
	Multi-objective	More than one objective function
	Single-objective	Exactly one objective function
Existence of constraints	Constrained	At least one constraint is involved
	Unconstrained	No constraints



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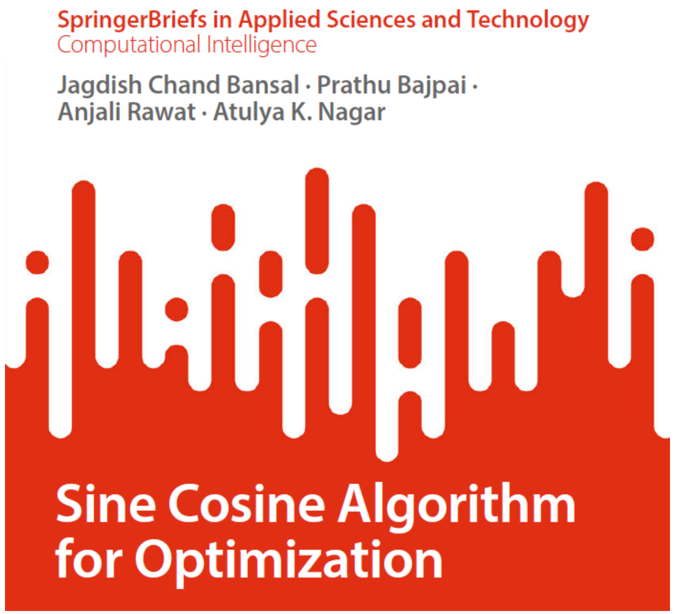

Optimization methods

Traditional

Stochastic

Optimization methods

Traditional

- Start from a randomly chosen initial solution and use specific deterministic rules for changing the solutions’ position in the search space.
- Most of them utilize the gradient information of the objective function.
- The initial solutions always follow the same path for the same starting position and converge to the fixed final position, irrespective of the number of runs.
- Provide a mathematical guarantee that a given optimization problem can be solved with a required level of accuracy within a finite number of steps.
- There exist sufficient literature on traditional optimization methods where different methods are capable of handling various types of optimization problems.
- Sometimes fail to handle optimization problems.
- Usually, these methods rely on the properties like continuity, differentiability, smoothness, and convexity of the objective function and constraints (if any). The absence of any of these properties makes traditional methods incapable of handling such optimization problems.
- Moreover, there are optimization problems for which no information is available about the objective function; these problems are referred as the black-box optimization problem. Traditional optimization methods or deterministic methods also fail to handle such black-box problems.

Optimization methods

Stochastic

- Stochastic or non-deterministic optimization methods contain inherent components of randomness and are iterative in nature.
- These methods utilize stochastic equations which are based on different stochastic processes and utilize different probability distributions. The stochastic nature of these equations governs the path of the solutions in the search space. In different runs of these algorithms, a solution can follow different paths, despite having a fixed initial position.
- Stochastic optimization methods do not always guarantee convergence to a fixed optimal position in the search space.
- In fact, these methods look for near optimal solution in a predefined fixed number of iterations.
- N number of independent runs are simulated to ensure a statistical reliability to these methods, and in general, the number of runs N = 30 or 51 is used to support the claim of near optimal solution.
- The trade-off for sacrificing the optimal solution by stochastic methods is the fast convergence speed, low computational cost, and less time complexity.
- Random number generators or pseudo-random number generators play an important role in the success of the stochastic methods.

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 & x_1 \geq 1
\end{array}
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https://www.desmos.com/calculator?lang=pt-BR

Analyze the following sets

1. $\mathbb{P} = \{x \in \mathbb{R} | -4 \leq x \leq 1\} \cup \{x \in \mathbb{R} | 2 \leq x \leq 4\}$
2. $\mathbb{W} = \left\{ x \in \mathbb{R} | \begin{array}{c} -4 \leq x \leq 1 \\ -1 \leq x \leq 4 \end{array} \right\}$
3. $\mathbb{M} = \{x \in \mathbb{R}^n | Ax \leq b\}$

$$f\left(\lambda x+(1-\lambda)\, y\right) \leq \lambda f\left(x\right)+(1-\lambda) f\left(y\right)$$

1. $f(x) =< c, x > \; x \in \mathbb{R}^n \; c \in \mathbb{R}^n$
2. $f(X) = max\,\{X - X_0, 0\}$
3. $f(x) = \begin{cases} (x+5)^2 & \text{se} \quad x \leq 0 \\ (x-5)^2 & \text{se} \quad x > 0 \end{cases}$

   Note that $f(x) = \min\left\{(x+5)^2, (x-5)^2\right\}$
4. $f(x) = \max\left\{(x+5)^2, (x-5)^2\right\}$