Spectral Analysis and Filtering

Cyclical Behavior and Periodicity

As in (1.5), we consider the periodic process

$$x_t = A\cos(2\pi\omega t + \phi) \tag{4.1}$$

for $t = 0, \pm 1, \pm 2, \ldots$, where ω is a frequency index, defined in cycles per unit time with A determining the height or *amplitude* of the function and ϕ, called the *phase*, determining the start point of the cosine function. We can introduce random variation in this time series by allowing the amplitude and phase to vary randomly.

$$x_t = U_1\cos(2\pi\omega t) + U_2\sin(2\pi\omega t), \tag{4.2}$$

where $U_1 = A\cos\phi$ and $U_2 = -A\sin\phi$ are often taken to be normally distributed random variables. In this case, the amplitude is $A = \sqrt{(U_1^2 + U_2^2)}$ and the phase is $\phi = \tan^{-1}(-U_2/U_1)$. From these facts we can show that if, and only if, in (4.1), A and ϕ are independent random variables, where A^2 is chi-squared with 2 degrees of freedom, and ϕ is uniformly distributed on $(-\pi, \pi)$, then U_1 and U_2 are independent, standard normal random variables (see Problem 4.3).

If we assume that U_1 and U_2 are uncorrelated random variables with mean 0 and variance σ^2, then x_t in (4.2) is stationary with mean $\mathrm{E}(x_t) = 0$ and, writing $c_t = \cos(2\pi\omega t)$ and $s_t = \sin(2\pi\omega t)$, autocovariance function

$$\begin{aligned}\gamma_x(h) = \mathrm{cov}(x_{t+h}, x_t) &= \mathrm{cov}(U_1 c_{t+h} + U_2 s_{t+h}, U_1 c_t + U_2 s_t) \\ &= \mathrm{cov}(U_1 c_{t+h}, U_1 c_t) + \mathrm{cov}(U_1 c_{t+h}, U_2 s_t) \\ &\quad + \mathrm{cov}(U_2 s_{t+h}, U_1 c_t) + \mathrm{cov}(U_2 s_{t+h}, U_2 s_t) \\ &= \sigma^2 c_{t+h} c_t + 0 + 0 + \sigma^2 s_{t+h} s_t = \sigma^2 \cos(2\pi\omega h),\end{aligned} \tag{4.3}$$

using Footnote 4.1 and noting that $\mathrm{cov}(U_1, U_2) = 0$. From (4.3), we see that

$$\mathrm{var}(x_t) = \gamma_x(0) = \sigma^2 .$$

Thus, if we observe $U_1 = a$ and $U_2 = b$, an estimate of σ^2 is the sample variance of these two observations, which in this case is simply $S^2 = \frac{a^2+b^2}{2-1} = a^2 + b^2$.

The random process in (4.2) is function of its frequency, ω. For $\omega = 1$, the series makes one cycle per time unit; for $\omega = .50$, the series makes a cycle every two time units; for $\omega = .25$, every four units, and so on. In general, for data that occur at discrete time points, we will need at least two points to determine a cycle, so the highest frequency of interest is .5 cycles per point. This frequency is called the *folding frequency* and defines the highest frequency that can be seen in discrete sampling. Higher frequencies sampled this way will appear at lower frequencies, called *aliases*; an example is the way a camera samples a rotating wheel on a moving automobile in a movie, in which the wheel appears to be rotating at a different rate, and sometimes backwards (the *wagon wheel effect*). For example, most movies are recorded at 24 frames per second (or 24 Hertz). If the camera is filming a wheel that is rotating at 24 Hertz, the wheel will appear to stand still.

[4.1] $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$.

Consider a generalization of (4.2) that allows mixtures of periodic series with multiple frequencies and amplitudes,

$$x_t = \sum_{k=1}^{q} [U_{k1} \cos(2\pi\omega_k t) + U_{k2} \sin(2\pi\omega_k t)], \tag{4.4}$$

where U_{k1}, U_{k2}, for $k = 1, 2, \ldots, q$, are uncorrelated zero-mean random variables with variances σ_k^2, and the ω_k are distinct frequencies. Notice that (4.4) exhibits the process as a sum of uncorrelated components, with variance σ_k^2 for frequency ω_k. As in (4.3), it is easy to show (Problem 4.4) that the autocovariance function of the process is

$$\gamma_x(h) = \sum_{k=1}^{q} \sigma_k^2 \cos(2\pi\omega_k h), \tag{4.5}$$

and we note the autocovariance function is the sum of periodic components with weights proportional to the variances σ_k^2. Hence, x_t is a mean-zero stationary processes with variance

$$\gamma_x(0) = \text{var}(x_t) = \sum_{k=1}^{q} \sigma_k^2, \tag{4.6}$$

exhibiting the overall variance as a sum of variances of each of the component parts.

As in the simple case, if we observe $U_{k1} = a_k$ and $U_{k2} = b_k$ for $k = 1, \ldots, q$, then an estimate of the kth variance component, σ_k^2, of $\text{var}(x_t)$, would be the sample variance $S_k^2 = a_k^2 + b_k^2$. In addition, an estimate of the total variance of x_t, namely, $\gamma_x(0)$ would be the sum of the sample variances,

$$\hat{\gamma}_x(0) = \hat{\text{var}}(x_t) = \sum_{k=1}^{q} (a_k^2 + b_k^2)\,. \tag{4.7}$$

Hold on to this idea because we will use it in Example 4.2.

Example 4.1 A Periodic Series

Figure 4.1 shows an example of the mixture (4.4) with $q = 3$ constructed in the following way. First, for $t = 1, \ldots, 100$, we generated three series

$$x_{t1} = 2\cos(2\pi t\, 6/100) + 3\sin(2\pi t\, 6/100)$$
$$x_{t2} = 4\cos(2\pi t\, 10/100) + 5\sin(2\pi t\, 10/100)$$
$$x_{t3} = 6\cos(2\pi t\, 40/100) + 7\sin(2\pi t\, 40/100)$$

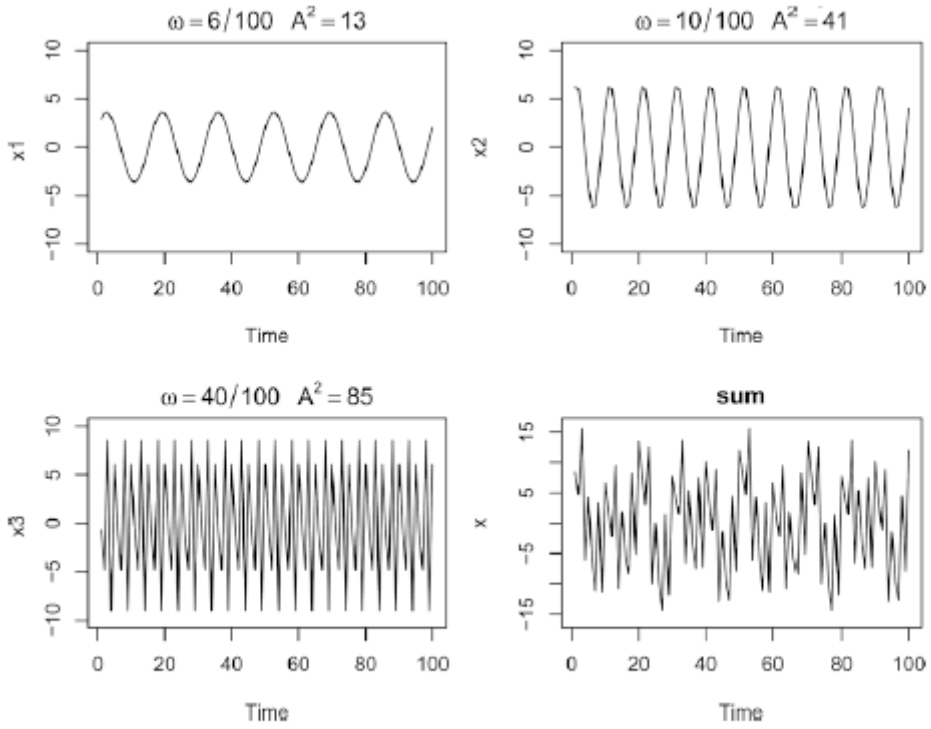


Fig. 4.1. Periodic components and their sum as described in Example 4.1.

These three series are displayed in Figure 4.1 along with the corresponding frequencies and squared amplitudes. For example, the squared amplitude of x_{t1} is $A^2 = 2^2 + 3^2 = 13$. Hence, the maximum and minimum values that x_{t1} will attain are $\pm\sqrt{13} = \pm 3.61$.

Finally, we constructed

$$x_t = x_{t1} + x_{t2} + x_{t3}$$

and this series is also displayed in Figure 4.1. We note that x_t appears to behave as some of the periodic series we saw in Chapters 1 and 2. The systematic sorting out of the essential frequency components in a time series, including their relative contributions, constitutes one of the main objectives of spectral analysis. The R code to reproduce Figure 4.1 is

```
x1 = 2*cos(2*pi*1:100*6/100)  + 3*sin(2*pi*1:100*6/100)
x2 = 4*cos(2*pi*1:100*10/100) + 5*sin(2*pi*1:100*10/100)
x3 = 6*cos(2*pi*1:100*40/100) + 7*sin(2*pi*1:100*40/100)
x  = x1 + x2 + x3
par(mfrow=c(2,2))
plot.ts(x1, ylim=c(-10,10), main=expression(omega==6/100~~~A^2==13))
plot.ts(x2, ylim=c(-10,10), main=expression(omega==10/100~~~A^2==41))
plot.ts(x3, ylim=c(-10,10), main=expression(omega==40/100~~~A^2==85))
plot.ts(x,  ylim=c(-16,16), main="sum")
```

Example 4.2 Estimation and the Periodogram

For any time series sample $x_1, \ldots, x_n$, where n is odd, we may write, *exactly*

$$x_t = a_0 + \sum_{j=1}^{(n-1)/2} \left[a_j \cos(2\pi t\, j/n) + b_j \sin(2\pi t\, j/n) \right], \tag{4.8}$$

for $t = 1, \ldots, n$ and suitably chosen coefficients. If n is even, the representation (4.8) can be modified by summing to $(n/2 - 1)$ and adding an additional component given by $a_{n/2} \cos(2\pi t\, \frac{1}{2}) = a_{n/2}(-1)^t$. The crucial point here is that (4.8) is exact for any sample. Hence (4.4) may be thought of as an approximation to (4.8), the idea being that many of the coefficients in (4.8) may be close to zero.

Using the regression results from Chapter 2, the coefficients a_j and b_j are of the form $\sum_{t=1}^n x_t z_{tj} / \sum_{t=1}^n z_{tj}^2$, where z_{tj} is either $\cos(2\pi t\, j/n)$ or $\sin(2\pi t\, j/n)$. Using Problem 4.1, $\sum_{t=1}^n z_{tj}^2 = n/2$ when $j/n \neq 0, 1/2$, so the regression coefficients in (4.8) can be written as ($a_0 = \bar{x}$),

$$a_j = \frac{2}{n} \sum_{t=1}^{n} x_t \cos(2\pi t j/n) \quad \text{and} \quad b_j = \frac{2}{n} \sum_{t=1}^{n} x_t \sin(2\pi t j/n).$$

We then define the *scaled periodogram* to be

$$P(j/n) = a_j^2 + b_j^2, \tag{4.9}$$

and it is of interest because it indicates which frequency components in (4.8) are large in magnitude and which components are small. *The scaled periodogram is simply the sample variance at each frequency component and consequently is an estimate of* σ_j^2 corresponding to the sinusoid oscillating at a frequency of $\omega_j = j/n$. These particular frequencies are called the *Fourier* or *fundamental frequencies*. Large values of $P(j/n)$ indicate which frequencies $\omega_j = j/n$ are predominant in the series, whereas small values of $P(j/n)$ may be associated with noise. The periodogram was introduced in Schuster (1898) and used in Schuster (1906) for studying the periodicities in the sunspot series (shown in Figure 4.22).

Fortunately, it is not necessary to run a large regression to obtain the values of a_j and b_j because they can be computed quickly if n is a highly composite integer. Although we will discuss it in more detail in Section 4.3, the *discrete Fourier transform (DFT)* is a complex-valued weighted average of the data given by[4.2]

$$\begin{aligned} d(j/n) &= n^{-1/2} \sum_{t=1}^{n} x_t \exp(-2\pi i t j/n) \\ &= n^{-1/2} \left(\sum_{t=1}^{n} x_t \cos(2\pi t j/n) - i \sum_{t=1}^{n} x_t \sin(2\pi t j/n) \right), \end{aligned} \tag{4.10}$$

for $j = 0, 1, \ldots, n-1$, where the frequencies j/n are the Fourier or fundamental frequencies. Because of a large number of redundancies in the calculation, (4.10) may be computed quickly using the *fast Fourier transform (FFT)*. Note that

$$|d(j/n)|^2 = \frac{1}{n} \left(\sum_{t=1}^{n} x_t \cos(2\pi t j/n) \right)^2 + \frac{1}{n} \left(\sum_{t=1}^{n} x_t \sin(2\pi t j/n) \right)^2 \tag{4.11}$$

and it is this quantity that is called the *periodogram*. We may calculate the scaled periodogram, (4.9), using the periodogram as

$$P(j/n) = \tfrac{4}{n} |d(j/n)|^2. \tag{4.12}$$

[4.2] Euler's formula: $e^{i\alpha} = \cos(\alpha) + i \sin(\alpha)$. Consequently, $\cos(\alpha) = \frac{e^{i\alpha}+e^{-i\alpha}}{2}$, and $\sin(\alpha) = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}$. Also, $\frac{1}{i} = -i$ because $-i \times i = 1$. If $z = a+ib$ is complex, then $|z|^2 = z\,z^* = (a+ib)(a-ib) = a^2+b^2$; the * denotes conjugation.

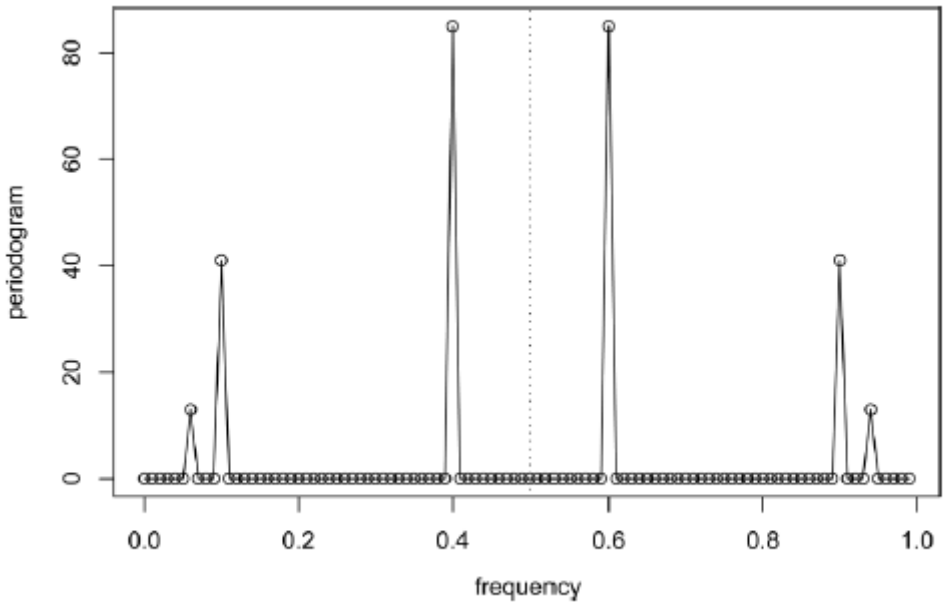


Fig. 4.2. Periodogram of the data generated in Example 4.1.

The scaled periodogram of the data, x_t, simulated in Example 4.1 is shown in Figure 4.2, and it clearly identifies the three components x_{t1}, x_{t2}, and x_{t3} of x_t. Note that

$$P(j/n) = P(1 - j/n), \quad j = 0, 1, \ldots, n - 1,$$

so there is a mirroring effect at the folding frequency of 1/2; consequently, the periodogram is typically not plotted for frequencies higher than the folding frequency. In addition, note that the heights of the scaled periodogram shown in the figure are

$$P(\tfrac{6}{100}) = P(\tfrac{94}{100}) = 13, \quad P(\tfrac{10}{100}) = P(\tfrac{90}{100}) = 41, \quad P(\tfrac{40}{100}) = P(\tfrac{60}{100}) = 85,$$

and $P(j/n) = 0$ otherwise. These are exactly the values of the squared amplitudes of the components generated in Example 4.1.

Assuming the simulated data, `x`, were retained from the previous example, the R code to reproduce Figure 4.2 is

```
P = Mod(2*fft(x)/100)^2;  Fr = 0:99/100
plot(Fr, P, type="o", xlab="frequency", ylab="scaled periodogram")
```

The Spectral Density

Example 4.4 A Periodic Stationary Process

Consider a periodic stationary random process given by (4.2), with a fixed frequency ω_0, say,

$$x_t = U_1 \cos(2\pi\omega_0 t) + U_2 \sin(2\pi\omega_0 t), \tag{4.13}$$

where U_1 and U_2 are uncorrelated zero-mean random variables with equal variance σ^2. The number of time periods needed for the above series to complete one cycle is exactly $1/\omega_0$, and the process makes exactly ω_0 cycles per point for $t = 0, \pm 1, \pm 2, \ldots$. Recalling (4.3) and using Footnote 4.2, we have

$$\gamma(h) = \sigma^2 \cos(2\pi\omega_0 h) = \frac{\sigma^2}{2} e^{-2\pi i\omega_0 h} + \frac{\sigma^2}{2} e^{2\pi i\omega_0 h}$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h} dF(\omega)$$

using Riemann–Stieltjes integration (see Section C.4.1), where $F(\omega)$ is the function defined by

$$F(\omega) = \begin{cases} 0 & \omega < -\omega_0, \\ \sigma^2/2 & -\omega_0 \le \omega < \omega_0, \\ \sigma^2 & \omega \ge \omega_0. \end{cases}$$

The function $F(\omega)$ behaves like a cumulative distribution function for a discrete random variable, except that $F(\infty) = \sigma^2 = \text{var}(x_t)$ instead of one. In fact, $F(\omega)$ is a cumulative distribution function, not of probabilities, but rather of variances, with $F(\infty)$ being the total variance of the process x_t. Hence, we term $F(\omega)$ the *spectral distribution function*. This example is continued in Example 4.9.

Property 4.1 Spectral Representation of an Autocovariance Function

If $\{x_t\}$ is stationary with autocovariance $\gamma(h) = \text{cov}(x_{t+h}, x_t)$, then there exists a unique monotonically increasing function $F(\omega)$, called the spectral distribution function, *with $F(-\infty) = F(-1/2) = 0$, and $F(\infty) = F(1/2) = \gamma(0)$ such that*

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h} \, dF(\omega). \tag{4.14}$$

An important situation we use repeatedly is the case when the autocovariance function is absolutely summable, in which case the spectral distribution function is absolutely continuous with $dF(\omega) = f(\omega)\, d\omega$, and the representation (4.14) becomes the motivation for the property given below.

Property 4.2 The Spectral Density

If the autocovariance function, $\gamma(h)$, of a stationary process satisfies

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty, \tag{4.15}$$

then it has the representation

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h} f(\omega)\, d\omega \quad h = 0, \pm 1, \pm 2, \ldots \tag{4.16}$$

as the inverse transform of the spectral density,

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i\omega h} \quad -1/2 \leq \omega \leq 1/2. \tag{4.17}$$

This spectral density is the analogue of the probability density function; the fact that $\gamma(h)$ is non-negative definite ensures

$$f(\omega) \geq 0$$

for all ω (see Appendix C, Theorem C.3 for details). It follows immediately from (4.12) that

$$f(\omega) = f(-\omega) \quad \text{and} \quad f(\omega) = f(1-\omega),$$

verifying the spectral density is an even function of period one. Because of the evenness, we will typically only plot $f(\omega)$ for $\omega \geq 0$. In addition, putting $h = 0$ in (4.11) yields

$$\gamma(0) = \text{var}(x_t) = \int_{-1/2}^{1/2} f(\omega)\, d\omega,$$

which expresses the total variance as the integrated spectral density over all of the frequencies. We show later on, that a linear filter can isolate the variance in certain frequency intervals or bands.

We note that the autocovariance function, $\gamma(h)$, in (4.16) and the spectral density, $f(\omega)$, in (4.17) are Fourier transform pairs. In particular, this means that if $f(\omega)$ and $g(\omega)$ are two spectral densities for which

$$\gamma_f(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\omega) e^{2\pi i \omega h} \, d\omega = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(\omega) e^{2\pi i \omega h} \, d\omega = \gamma_g(h) \tag{4.18}$$

for all $h = 0, \pm 1, \pm 2, \ldots$, then

$$f(\omega) = g(\omega). \tag{4.19}$$

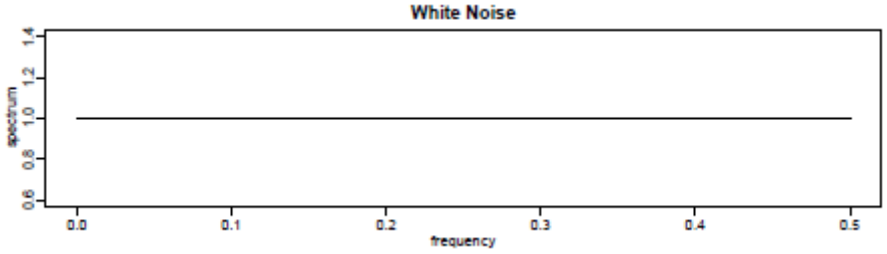
Finally, the absolute summability condition, (4.15), is not satisfied by (4.5), the example that we have used to introduce the idea of a spectral representation. The condition, however, is satisfied for ARMA models.

Example 4.5 White Noise Series

As a simple example, consider the theoretical power spectrum of a sequence of uncorrelated random variables, w_t, with variance σ_w^2. A simulated set of data is displayed in the top of Figure 1.8. Because the autocovariance function was computed in Example 1.16 as $\gamma_w(h) = \sigma_w^2$ for $h = 0$, and zero, otherwise, it follows from (4.17), that

$$f_w(\omega) = \sigma_w^2$$

for $-1/2 \leq \omega \leq 1/2$. Hence the process contains equal power at all frequencies. This property is seen in the realization, which seems to contain all different frequencies in a roughly equal mix. In fact, the name white noise comes from the analogy to white light, which contains all frequencies in the color spectrum at the same level of intensity. The top of Figure 4.4 shows a plot of the white noise spectrum for $\sigma_w^2 = 1$. The R code to reproduce the figure is given at the end of Example 4.7.



Since the linear process is an essential tool, it is worthwhile investigating the spectrum of such a process. In general, a linear filter uses a set of specified coefficients, say a_j, for $j = 0, \pm 1, \pm 2, \ldots$, to transform an input series, x_t, producing an output series, y_t, of the form

$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |a_j| < \infty. \tag{4.20}$$

The form (4.20) is also called a *convolution* in some statistical contexts. The coefficients are collectively called the *impulse response function*, and the Fourier transform

$$A(\omega) = \sum_{j=-\infty}^{\infty} a_j \, \mathrm{e}^{-2\pi i \omega j}, \tag{4.21}$$

is called the *frequency response function*. If, in (4.20), x_t has spectral density $f_x(\omega)$, we have the following result.

Property 4.3 Output Spectrum of a Filtered Stationary Series

For the process in (4.20), if x_t has spectrum $f_x(\omega)$, then the spectrum of the filtered output, y_t, say $f_y(\omega)$, is related to the spectrum of the input x_t by

$$f_y(\omega) = |A(\omega)|^2 \, f_x(\omega), \tag{4.22}$$

where the frequency response function $A(\omega)$ is defined in (4.21).

Proof: The autocovariance function of the filtered output y_t in (4.20) is

$$\begin{aligned}
\gamma_y(h) &= \mathrm{cov}(x_{t+h}, x_t) \\
&= \mathrm{cov}\left(\sum_r a_r x_{t+h-r}, \sum_s a_s x_{t-s} \right) \\
&= \sum_r \sum_s a_r \gamma_x(h - r + s) a_s \\
&\overset{(1)}{=} \sum_r \sum_s a_r \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{2\pi i \omega (h-r+s)} f_x(\omega) d\omega \right] a_s \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_r a_r \mathrm{e}^{-2\pi i \omega r} \right) \left(\sum_s a_s \mathrm{e}^{2\pi i \omega s} \right) \mathrm{e}^{2\pi i \omega h} f_x(\omega) \, d\omega \\
&\overset{(2)}{=} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{2\pi i \omega h} \underbrace{|A(\omega)|^2 f_x(\omega)}_{f_y(\omega)} \, d\omega,
\end{aligned}$$

where we have, (1) replaced $\gamma_x(\cdot)$ by its representation (4.16), and (2) substituted $A(\omega)$ from (4.21). The result holds by exploiting the uniqueness of the Fourier transform.

Property 4.4 The Spectral Density of ARMA

If x_t is ARMA(p, q), $\phi(B)x_t = \theta(B)w_t$, its spectral density is given by

$$f_x(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2} \tag{4.23}$$

where $\phi(z) = 1 - \sum_{k=1}^{p} \phi_k z^k$ and $\theta(z) = 1 + \sum_{k=1}^{q} \theta_k z^k$.

Example 4.6 Moving Average

As an example of a series that does not have an equal mix of frequencies, we consider a moving average model. Specifically, consider the MA(1) model given by

$$x_t = w_t + .5w_{t-1}.$$

A sample realization is shown in the top of Figure 3.2 and we note that the series has less of the higher or faster frequencies. The spectral density will verify this observation.

The autocovariance function is displayed in Example 3.5, and for this particular example, we have

$$\gamma(0) = (1 + .5^2)\sigma_w^2 = 1.25\sigma_w^2; \quad \gamma(\pm 1) = .5\sigma_w^2; \quad \gamma(\pm h) = 0 \text{ for } h > 1.$$

Substituting this directly into the definition given in (4.17), we have

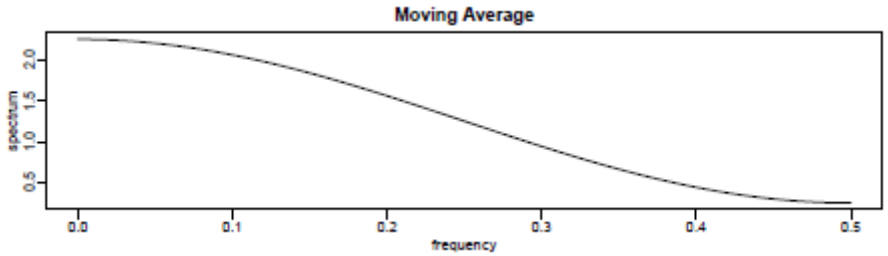
$$\begin{aligned} f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h)\, e^{-2\pi i\omega h} &= \sigma_w^2 \left[1.25 + .5\left(e^{-2\pi i\omega} + e^{2\pi\omega}\right)\right] \\ &= \sigma_w^2 \left[1.25 + \cos(2\pi\omega)\right]. \end{aligned} \tag{4.24}$$

We can also compute the spectral density using Property 4.4, which states that for an MA, $f(\omega) = \sigma_w^2|\theta(e^{-2\pi i\omega})|^2$. Because $\theta(z) = 1 + .5z$, we have

$$\begin{aligned} |\theta(e^{-2\pi i\omega})|^2 &= |1 + .5e^{-2\pi i\omega}|^2 = (1 + .5e^{-2\pi i\omega})(1 + .5e^{2\pi i\omega}) \\ &= 1.25 + .5\left(e^{-2\pi i\omega} + e^{2\pi\omega}\right) \end{aligned}$$

which leads to agreement with (4.24).

Plotting the spectrum for $\sigma_w^2 = 1$, as in the middle of Figure 4.4, shows the lower or slower frequencies have greater power than the higher or faster frequencies.



Example 4.7 A Second-Order Autoregressive Series

We now consider the spectrum of an AR(2) series of the form

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} = w_t,$$

for the special case $\phi_1 = 1$ and $\phi_2 = -.9$. Figure 1.9 shows a sample realization of such a process for $\sigma_w = 1$. We note the data exhibit a strong periodic component that makes a cycle about every six points.

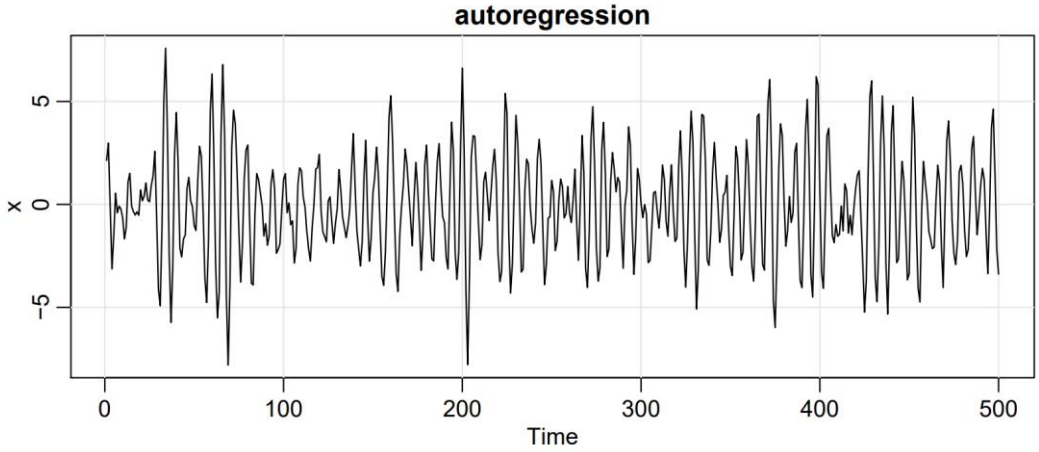


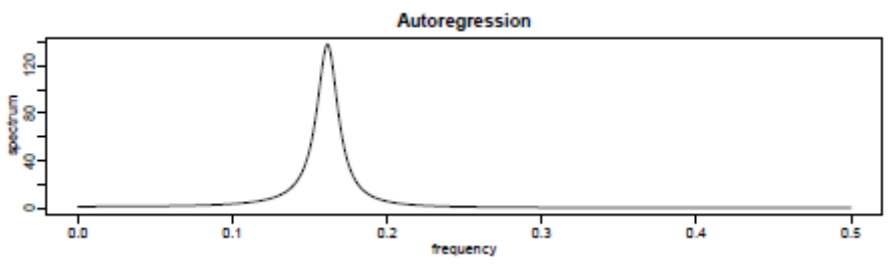
Fig. 1.9. *Autoregressive series generated from model (1.2).*

To use Property 4.4, note that $\theta(z) = 1$, $\phi(z) = 1 - z + .9z^2$ and

$$\begin{aligned} |\phi(e^{-2\pi i\omega})|^2 &= (1 - \mathrm{e}^{-2\pi i\omega} + .9\mathrm{e}^{-4\pi i\omega})(1 - \mathrm{e}^{2\pi i\omega} + .9\mathrm{e}^{4\pi i\omega}) \\ &= 2.81 - 1.9(\mathrm{e}^{2\pi i\omega} + \mathrm{e}^{-2\pi i\omega}) + .9(\mathrm{e}^{4\pi i\omega} + \mathrm{e}^{-4\pi i\omega}) \\ &= 2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega). \end{aligned}$$

Using this result in (4.23), we have that the spectral density of x_t is

$$f_x(\omega) = \frac{\sigma_w^2}{2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)}.$$



The spectral density can also be obtained from first principles, without having to use Property 4.3. Because $w_t = x_t - x_{t-1} + .9x_{t-2}$ in this example, we have

$$\begin{aligned}\gamma_w(h) &= \mathrm{cov}(w_{t+h}, w_t)\\ &= \mathrm{cov}(x_{t+h} - x_{t+h-1} + .9x_{t+h-2},\, x_t - x_{t-1} + .9x_{t-2})\\ &= 2.81\gamma_x(h) - 1.9[\gamma_x(h+1) + \gamma_x(h-1)] + .9[\gamma_x(h+2) + \gamma_x(h-2)]\end{aligned}$$

Now, substituting the spectral representation (4.11) for $\gamma_x(h)$ in the above equation yields

$$\begin{aligned}\gamma_w(h) &= \int_{-1/2}^{1/2} [2.81 - 1.9(e^{2\pi i\omega} + e^{-2\pi i\omega}) + .9(e^{4\pi i\omega} + e^{-4\pi i\omega})]e^{2\pi i\omega h} f_x(\omega)d\omega\\ &= \int_{-1/2}^{1/2} [2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)]e^{2\pi i\omega h} f_x(\omega)d\omega.\end{aligned}$$

If the spectrum of the white noise process, w_t, is $g_w(\omega)$, the uniqueness of the Fourier transform allows us to identify

$$g_w(\omega) = [2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)]\, f_x(\omega).$$

But, as we have already seen, $g_w(\omega) = \sigma_w^2$, from which we deduce that

$$f_x(\omega) = \frac{\sigma_w^2}{2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)}$$

is the spectrum of the autoregressive series.

Property 4.5 Spectral Representation of a Stationary Process

If x_t is a mean-zero stationary process, with spectral distribution $F(\omega)$ as given in Property 4.1, then there exists a complex-valued stochastic process $Z(\omega)$, on the interval $\omega \in [-1/2, 1/2]$, having stationary uncorrelated non-overlapping increments, such that x_t can be written as the stochastic integral (see Section C.4.2)

$$x_t = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega t}\, dZ(\omega),$$

where, for $-1/2 \le \omega_1 \le \omega_2 \le 1/2$,

$$\mathrm{var}\,\{Z(\omega_2) - Z(\omega_1)\} = F(\omega_2) - F(\omega_1).$$