$$\therefore\ m\dot{\vec{v}}' = -\nabla U - m\dot{\vec{\omega}}\wedge\vec{r}\,' - 2m\vec{\omega}\wedge\vec{v}\,' - m\,\vec{\omega}\wedge(\vec{\omega}\wedge\vec{r})$$

Oh!

6.3 Exemplos

23/5/23

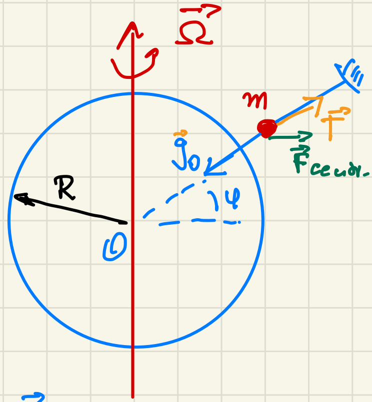
1. Valor local de g

Consideremos um pêndulo parado com respeito à Terra.

$$m\ddot{\vec{r}}\,' = \vec{T} + m\vec{g}_0 - \vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{r}\,')$$

$$= \vec{T} + m[\vec{g}_0 - \underbrace{\vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{r}\,')}_{\vec{F}_{cf}}] = 0$$

parado



$$|\vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{r}\,')| = m\Omega^2 R\cos\varphi = |\vec{F}_{cf}|$$

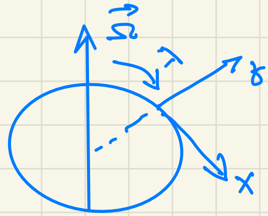
Usando coord. esféricas: $\vec{F}_{cf} = m\Omega^2 R\cos\varphi[\cos\varphi\,\vec{e}_r + \operatorname{sen}\varphi\,\vec{e}_\theta]$

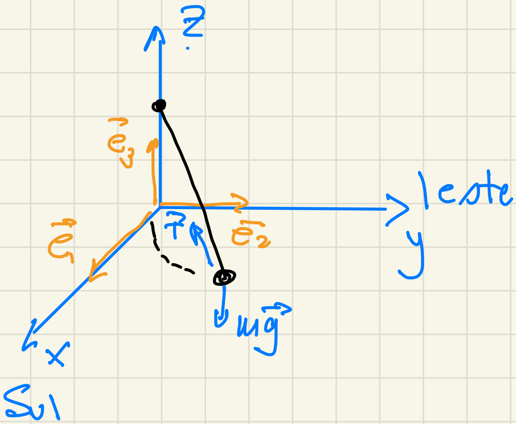
Equilíbrio na direção $\vec{e}_r \Rightarrow T_r = m(g_0 - m\Omega^2 R\cos^2\varphi)$

equilíbrio na direção $\vec{e}_\theta \Rightarrow T_\theta = -m\Omega^2 R\cos\varphi\sin\varphi$

- Como $T_\theta \neq 0 \Rightarrow$ o fio não aponta para O.
- Aparentemente o g local é: $g = g_0 - \Omega^2 R\cos^2\varphi$

2. Pêndulo de Foucault (1851) demonstrou a rotação da Terra!





Eq. de movimento:

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g} - m\frac{d\vec{\Omega}}{dt}\wedge\vec{r} - 2m\vec{\Omega}\wedge\vec{v} - m\vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{r}) \quad ✓$$

Note $\frac{d\Omega}{dt} = 0$ é boa aproximação || força de Coriolis.

$L = 10\,m$

$T \sim 1\,s$

$\Omega = 7{,}3 \times 10^{-5}$ rad/s

e $|\Omega| \ll 1$ tal que $T^2\Omega^2 \simeq 0$

↳ período do pêndulo

Conservando o termo linear em $\vec{\Omega}$:

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g} - 2m\vec{\Omega}\wedge\vec{v} \quad ✓$$

$l \gg x, y, z$

l muito grande

Agora,

$$\vec{T} = T\left[\frac{-x\vec{e}_1 - y\vec{e}_2 - (z-l)\vec{e}_3}{\sqrt{x^2+y^2+(l-z)^2}}\right] \simeq \frac{T}{l}\left[-x\vec{e}_1 - y\vec{e}_2 - (z-l)\vec{e}_3\right]$$

Com $\vec{\Omega} = \Omega\cos\lambda\,\vec{e}_3 - \Omega\sin\lambda\,\vec{e}_1 \quad ✓$

Com isso

$$\vec{\Omega}\wedge\vec{v} = -\Omega\cos\lambda\,\dot{y}\,\vec{e}_1' + \Omega(\cos\lambda\,\dot{x} + \sin\lambda\,\dot{z})\vec{e}_2 - \Omega\sin\lambda\,\dot{y}\,\vec{e}_3 \quad \checkmark$$

Resultando em

$$m\ddot{x} = -\frac{x}{\ell}T + 2m\Omega\cos\lambda\,\dot{y} \quad \checkmark$$

$$m\ddot{y} = -\frac{y}{\ell}T - 2m\Omega(\cos\lambda\,\dot{x} + \sin\lambda\,\dot{z}) \quad \checkmark$$

$$m\ddot{z} = \frac{\ell - z}{\ell}T - mg + 2m\Omega\sin\lambda\,\dot{y} \quad \checkmark$$

$\ell \ggg z$

Como a massa basicamente move-se ao longo do plano x-y,

$$m\ddot{z} = 0 \Rightarrow T = mg - 2m\Omega\sin\lambda\,\dot{y} \;\checkmark \Rightarrow$$

$$\ddot{x} = -\frac{g}{\ell}x + \frac{2\Omega\operatorname{sen}\lambda}{\ell}x\dot{y} + 2\Omega\cos\lambda\,\dot{y} \quad \checkmark$$

$$\ddot{y} = -\frac{g}{\ell}y + \frac{2\Omega\sin\lambda}{\ell}y\dot{y} - 2\Omega\cos\lambda\,\dot{x} \quad \checkmark$$

Este é um sistema de EDO's <u>não linear</u>

Os produtos de Ω, x e $\dot{y}$ são pequenos face aos outros termos

$$\Rightarrow \ddot{x} = -\frac{g}{\ell}x + 2\Omega\cos\lambda\,\dot{y} \;;\; \ddot{y} = -\frac{g}{\ell}y - 2\Omega\cos\lambda\,\dot{x}$$

dois osciladores harmónicos acoplados! ✓

Encontremos a solução: $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{ipt}\right]$

$(g/\ell = \omega_0^2)$

$$\Rightarrow -p^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{bmatrix} -\omega_0^2 & i2\Omega\cos\lambda\, p \\ -i2\Omega\cos\lambda\, p & -\omega_0^2 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$ ✓

Autovalores:

$$\det \begin{bmatrix} p^2-\omega_0^2 & i2\Omega\cos\lambda\, p \\ -i2\Omega\cos\lambda\, p & p^2-\omega_0^2 \end{bmatrix} = 0 \Rightarrow (p^2-\omega_0^2)^2 = 4\Omega^2\cos^2\lambda\, p^2$$ ✓

Definimos $\alpha = \Omega\cos\lambda$.

$$(p^2-\omega_0^2)^2 = 4\alpha^2p^2 \Rightarrow p^4 - (2\omega_0^2+4\alpha^2)p^2 + \omega_0^4 = 0$$ ✓

$$p^2 = \frac{1}{2}\left\{ 2\omega_0^2 + 4\alpha^2 \pm \sqrt{\cancel{4\omega_0^4} + 16\alpha^2\omega_0^2 + 16\alpha^4 - \cancel{4\omega_0^4}} \right\}$$ ✓

Mas $\alpha \ll \omega_0 \Rightarrow p^2 \cong \omega_0^2 \pm 2\alpha\omega_0 = \omega_0^2(1 \pm 2\alpha/\omega_0)$

$$\Rightarrow p = \pm\omega_0(1 \pm \alpha/\omega_0) = \pm(\omega_0 \pm \alpha)$$

Autovetores: $p = \omega_0 + \alpha$.

$$\begin{pmatrix} 2\omega_0\alpha & i2\omega_0\alpha \\ -i2\omega_0\alpha & 2\omega_0\alpha \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0 \qquad x_0 + iy_0 = 0 \Rightarrow y_0 = ix_0$$ ✓

$\begin{pmatrix} 1 \\ i \end{pmatrix}$

$$p = w_0 - \alpha \Rightarrow \begin{pmatrix} -2w_0\alpha & i\,2w_0\alpha \\ -i\,2w_0\alpha & -2w_0\alpha \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0 \qquad \begin{array}{l} -x_0 + i y_0 = 0 \\ y_0 = -i x_0 \end{array}$$

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} \checkmark$$

$$p = -(w_0 + \alpha) \Rightarrow \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \checkmark$$

$$p = -(w_0 - \alpha) \Rightarrow \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \checkmark$$

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = A_1 \begin{pmatrix} 1 \\ +i \end{pmatrix} e^{i(w_0+\alpha)t} + A_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i(w_0-\alpha)t}$$

$$+ A_3 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i(w_0+\alpha t)} + A_4 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-i(w_0-\alpha)t} \qquad \checkmark$$

Condição inicial: $x_0 = 0 \quad \dot{x}_0 = 0 \quad y_0 = L \quad \dot{y}_0 = 0$

$x(0) = 0;\qquad A_1 + A_2 + A_3 + A_4 = 0$

$y(0) = L \qquad i(A_1 - A_2 - A_3 + A_4) = L$

$\dot{x}(0) = 0 \qquad (w_0+\alpha)A_1 + (w_0-\alpha)A_2 - (w_0+\alpha)A_3 - (w_0-\alpha)A_4 = 0$

$\dot{y}(0) = 0 \qquad -(w_0+\alpha)A_1 + (w_0-\alpha)A_2 - (w_0+\alpha)A_3 + (w_0-\alpha)A_4 = 0$

Lembrando que $\alpha \ll \omega_0$ o termo principal da solução é (no notebook foucault)

$$x(t) = L\cos(\omega_0 t)\sin(\alpha t)$$
$$y(t) = L\cos(\omega_0 t)\cos(\alpha t)$$
$$+ \mathcal{O}\left(\frac{\alpha}{\omega_0}\right)$$

Com isso $\vec{r} = L\cos(\omega_0 t)\Big[\underbrace{\operatorname{sen}(\alpha t)\,\vec{e}_1 + \cos(\alpha t)\,\vec{e}_2}_{\hat{n}(t)}\Big]$

$$\vec{r} = L\cos(\omega_0 t)\,\vec{n}(t)$$

No hemisfério norte $\cos\hat{\imath} > 0 \Rightarrow \alpha > 0 \Rightarrow$ rotação no sentido horário! Pense p/ αt pequeno!