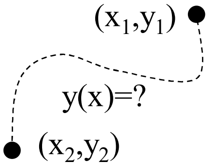
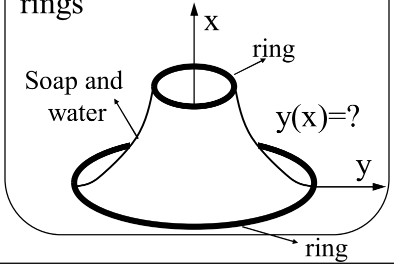
Classical Variational Problems

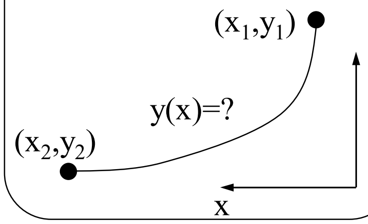
Shortest arch connecting two points



Surface of revolution with the lowest area between two rings



“Brachistochrone” Problem



Dido’s problem – Find the equation that encloses the largest area;

Hamilton’s Principle:
“Lagrange Equation of Motion”

Euler-Lagrange Equation

$$\frac{\partial^2 F}{\partial y'^2}\,y'' + \frac{\partial^2 F}{\partial y \partial y'}\,y' + \frac{\partial^2 F}{\partial x \partial y'} - \frac{\partial F}{\partial y} = 0$$

Special cases

Case 1:
If $F = F(y')$ and $\frac{\partial^2 F}{\partial y'^2} \neq 0$
then: $\frac{\partial^2 F}{\partial y'^2}\,y'' = 0$

$$\boxed{y'' = 0}$$

Case 3:
If $F = F(x,y)$
then: $\boxed{\frac{\partial F}{\partial y} = 0}$

Case 2:
If $F = F(x,y')$
then: $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$\boxed{\frac{\partial F}{\partial y'} = C \text{ (constante)}}$$

Case 4:
If $F = F(y,y')$
then: $\boxed{F - y'\frac{\partial F}{\partial y'} = C \quad \text{(constante)}}$

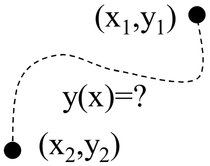
$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} + y'\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] = \frac{\partial F}{\partial x}$$

Classical Variational Problems

Arch with the shortest length connecting two points

$$J(y) = \int_{x_1}^{x_2} \underbrace{\sqrt{1+(y')^2}}_{} dx \qquad y(x_1) = y_1;\ y(x_2) = y_2$$

ds - “arch length”



$$\boxed{\begin{array}{l}\text{Min} \quad J(y)\\ y(x)\end{array}} \Rightarrow \boxed{\delta J(y)=0}$$

Euler-Lagrange Equation:

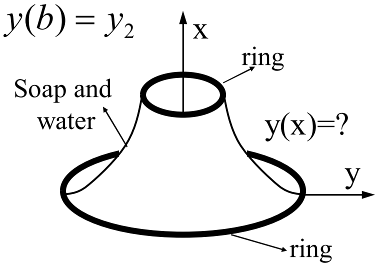
$$y'' = 0 \Rightarrow y = \frac{(y_2 - y_1)(x - x_1)}{(x_2 - x_1)} + y_1$$

Equation of a straight line!!

Classical Variational Problems

Surface of revolution with the shortest area between two rings

$$J(y) = 2\pi\int_{a}^{b} y\sqrt{1+(y')^2}\, dx \qquad y(a) = y_1;\ y(b) = y_2$$



$$\boxed{\begin{array}{l}\text{Min} \quad J(y)\\ y(x)\end{array}} \Rightarrow \boxed{\delta J(y)=0}$$

Euler-Lagrange Equation:

$$\frac{y(y')^2}{\sqrt{1+(y')^2}} - y\sqrt{1+(y')^2} = C_1 \Rightarrow -y = C_1\sqrt{1+(y')^2} \Rightarrow C_1\frac{dy}{\sqrt{y^2 - {C_1}^2}} = dx \Rightarrow$$

$$\Rightarrow \boxed{y = C_1 \cosh\left(\frac{x - C_2}{C_1}\right)};\ \text{where: } \cosh x = \frac{1}{2}(e^{x} + e^{-x})$$

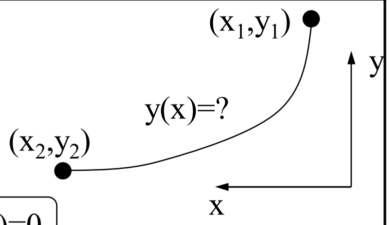
Classical Variational Problems

“Brachistochrone” Problem – What curve between two points subjects a particle, under its own weight, to travel through the path with the shortest time possible?

Time

$$\mathrm{T}=\int_0^T dt \Rightarrow \int_0^L \frac{ds}{v} = \int_{x_1}^{x_2} \frac{\sqrt{1+(\mathrm{y}')^2}}{\mathrm{v}}\,\mathrm{dx}$$

$$y(x_1)=y_1;\ y(x_2)=y_2$$



$$\boxed{\begin{array}{l}\mathrm{Min}\quad \mathrm{T(y)}\\ \mathrm{y(x)}\end{array}} \Rightarrow \boxed{\delta \mathrm{T(y)}=0}$$

From energy conservation:

$$\frac{1}{2}mv^2+mgy=\frac{1}{2}mv_0{}^2+mgy_1 \Rightarrow v=\sqrt{2g(y_1-y)+v_0{}^2} \Rightarrow$$

$$\Rightarrow v=\sqrt{2g(k-y)};\ \mathrm{k}=\mathrm{y}_1+\frac{v_0{}^2}{2g}$$

Classical Variational Problems

$$\text{Therefore:}\mathrm{T}=\frac{1}{\sqrt{2g}}\int_{x_1}^{x_2}\frac{\sqrt{1+(\mathrm{y}')^2}}{\sqrt{\mathrm{k-y}}}\,\mathrm{dx}$$

Euler-Lagrange Equation:

$$\frac{\sqrt{1+(y')^2}}{\sqrt{k-y}}-\frac{(y')^2}{\sqrt{k-y}\sqrt{1+(y')^2}}=C\Rightarrow (k-y)\left[1+(y')^2\right]=\frac{1}{C^2}=C_1\Rightarrow$$

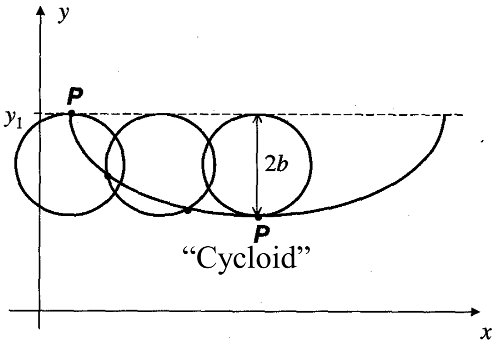
$$\Rightarrow dx=-\frac{\sqrt{k-y}}{C_1-(k-y)}dy;\ k-y=C_1\sin^2\alpha \Rightarrow dx=C_1(1-\cos 2\alpha)d\alpha \Rightarrow$$

$$x(\alpha)=\frac{1}{2}C_1(2\alpha-\sin 2\alpha)+C_2;\quad y(\alpha)=k-C_1\sin^2\alpha$$

Change of variables:

$$\underbrace{x(\theta)=a+b(\theta-\sin\theta)\ \ \text{e}\ \ y(\theta)=y_1+\frac{v_0{}^2}{2g}-b(1-\cos\theta)}_{\text{“Cycloid” curve equation}}$$

Classical Variational Problems



$$\frac{v^2}{2}=2g(k-y)\Rightarrow\frac{v^2}{2}=\frac{g}{4b}\left(s_0{}^2-s^2\right)\Rightarrow\quad \mathrm{T}=\int_0^T dt\Rightarrow\int_0^L\frac{ds}{v}=$$

$$=-2\sqrt{\frac{b}{g}}\int_0^L\frac{ds}{\sqrt{s_0{}^2-s^2}}\Rightarrow\boxed{T=\pi\sqrt{\frac{b}{g}}}$$

Therefore, the cycle is independent of the start point: "tautochronism"

"Cycloid pendulum"

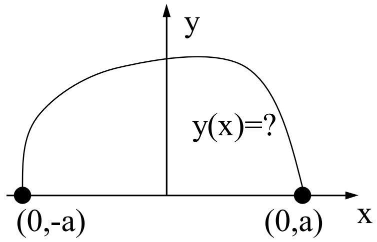
Classical Variational Problems

Isoperimetric problems

Dido's – Find the curve with limited length the involves the largest area;

Area $\mathrm{I(y)}=\int_{-a}^{a} y\mathrm{dx} \qquad y(-a)=0;\ y(a)=0$

Curve length $\mathrm{J(y)}=\int_{-a}^{a}\sqrt{1+(y')^2}\,\mathrm{dx}$



Min I(y)
y(x)
subject to J(y)=πa

➡

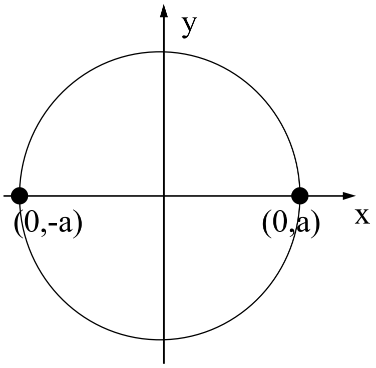
Min I(y)+λJ(y)
y(x)

Classical Variational Problems

Euler-Lagrange Equation:

$$1+\lambda\frac{d}{dx}\left[\frac{y'}{\sqrt{1+(y')^2}}\right]=0\Rightarrow\underbrace{\frac{y''}{\left[1+(y')^2\right]^{3/2}}}_{\text{Curvature = cte.}}=-\frac{1}{\lambda}\Rightarrow x^2+(y-C)^2=a^2$$

Circumference
equation!!



Classical Variational Problems

Hamilton's Principle

Dynamic equilibrium:

$$\boxed{\underset{\mathbf{q}}{\text{Min}}\ \text{I(y)}}\qquad \text{I(y)}=\int_{t_1}^{t_2}L\text{dt}=\int_{t_1}^{t_2}(\text{T - V})\text{dt}\Rightarrow\delta I=\delta\int_{t_1}^{t_2}L\text{dt}=0$$

Generalized coordinates

Euler-Lagrange Equation:

$$\frac{\partial L}{\partial q_j}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)=0,\quad j=1,2,\ldots,m$$

"Euler-Lagrange Equation of Motion"

Optimization Using Variational Calculus

A **Functional** (J) is defined as:

$$J(y(x))=\int_a^b F\big(x,y(x),y'(x)\big)dx \quad \text{where: } y'(x)=\frac{dy(x)}{dx}$$

and: $y(a)=y_a$; $y(b)=y_b$ ➡ kinematic boundary conditions (problem of "fixed ends")

Many physical laws are formulated as extremes of a functional.

Ex.: static equilibrium of a structure ➡ minimal potential energy

<u>Problem</u>: Find an appropriate function that minimizes (or maximizes) the functional ➡ objective of **Variational Calculus**

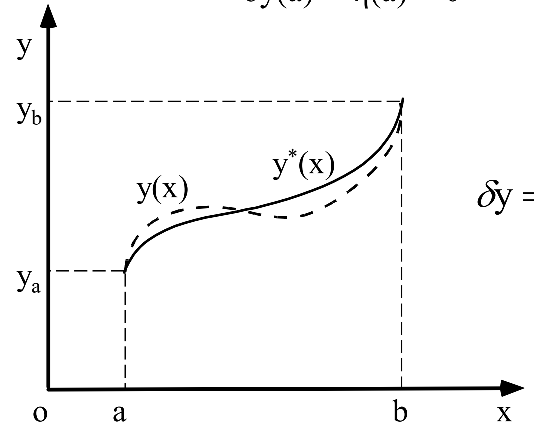
Applications of Variational Calculus: Optimization Theory, FEM formulation, Analytical Mechanics, etc...

Optimization Using Variational Calculus

Solution: Let $y^*(x)$ be the function that minimizes the functional J and satisfies $y^*(a)=y_a$ and $y^*(b)=y_b$. We may write:

$$y(x)=y^*(x)+\varepsilon\eta(x)=y^*(x)+\varepsilon\delta y$$

where ε is small and: $\delta y(a)=\eta(a)=0$ and $\delta y(b)=\eta(b)=0$



$$\delta y=\bar{y}(x)-y^*(x)$$

Optimization Using Variational Calculus

Therefore:

$$J(\varepsilon) = J(y(x,\varepsilon)) = \int_a^b F\Big(x, \underbrace{y^* + \varepsilon\eta}_{y}, \underbrace{y'^* + \varepsilon\eta'}_{y'}\Big)dx$$

The necessary condition for the extreme J, or for the occurrence of minimal J in ε=0 is:

$$\delta J\Big|_{y=y*} = \frac{dJ(\varepsilon)}{d\varepsilon}\Bigg|_{\varepsilon=0} = \int_a^b \left[\frac{\partial F}{\partial y}\frac{dy}{d\varepsilon} + \frac{\partial F}{\partial y'}\frac{dy'}{d\varepsilon}\right]\Bigg|_{\varepsilon=0} dx =$$

$$= \int_a^b \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]\Bigg|_{\varepsilon=0} dx = 0$$

Variational operator δ ≡ Differential operator *d*

Property:

$$\frac{d(\delta y)}{dx} = \delta\left(\frac{dy}{dx}\right) = \delta y'$$

Optimization Using Variational Calculus

Integration by parts:

$$\delta J\Big|_{y=y*} = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\Bigg|_{\varepsilon=0} \delta y dx + \boxed{\frac{\partial F}{\partial y'}\delta y\Bigg|_a^b} = 0$$

Null, because:
δy(b) = δy(a) = 0

So:

$$\delta J\Big|_{y=y*} = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\Bigg|_{\varepsilon=0} \delta y dx = 0$$

But since δy is arbitrary:

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

Euler-Lagrange
Equation

Optimization Using Variational Calculus

If $\delta y(b) \neq 0$ and $\delta y(a) \neq 0$ then we shall have:

$$\left[\frac{\partial F}{\partial y'}\right]_{x=a} = 0 \quad \text{and} \quad \left[\frac{\partial F}{\partial y'}\right]_{x=b} = 0$$

"Natural boundary conditions"

More generic cases:

- Functional with higher order derivatives:

$$J(y(x)) = \int_a^b F\big(x, y(x), y'(x), y''(x) \ldots, y^{(n)}(x)\big)dx$$

and:

$$y(a) = y_a ; y'(a) = y'_a ; y''(a) = y''_a ; \ldots ; y^{(n-1)}(a) = y_a^{(n-1)}$$

$$y(b) = y_b ; y'(b) = y'_b ; y''(b) = y''_b ; \ldots ; y^{(n-1)}(b) = y_b^{(n-1)}$$

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \ldots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0}$$

Euler-Lagrange Equations

Optimization Using Variational Calculus

- Functional with many unknown functions: $\mathbf{y}(x)=\{y_1(x),y_2(x),\ldots,y_n(x)\}^T$

$$J(\mathbf{y}(x)) = \int_a^b F\big(x, \mathbf{y}(x), \mathbf{y}'(x), \mathbf{y}''(x) \ldots, \mathbf{y}^{(n)}(x)\big)dx$$

and:

$$\mathbf{y}(a) = \mathbf{y}_a ; \mathbf{y}'(a) = \mathbf{y}'_a ; \mathbf{y}''(a) = \mathbf{y}''_a ; \ldots ; \mathbf{y}^{(n-1)}(a) = \mathbf{y}_a^{(n-1)}$$

$$\mathbf{y}(b) = \mathbf{y}_b ; \mathbf{y}'(b) = \mathbf{y}'_b ; \mathbf{y}''(b) = \mathbf{y}''_b ; \ldots ; \mathbf{y}^{(n-1)}(b) = \mathbf{y}_b^{(n-1)}$$

$$\boxed{\nabla_y F - \frac{d}{dx}\big(\nabla_{y'} F\big) + \frac{d^2}{dx^2}\big(\nabla_{y''} F\big) + \ldots + (-1)^n \frac{d^n}{dx^n}\big(\nabla_{y^{(n)}} F\big) = 0}$$

where:

$$\nabla_y F = \left(\frac{\partial F}{\partial y_1}, \frac{\partial F}{\partial y_2}, \ldots, \frac{\partial F}{\partial y_m}\right)^T$$

Euler-Lagrange Equations

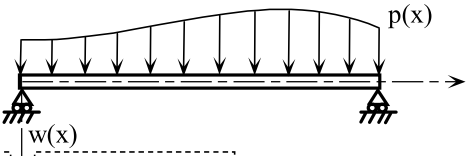
- Functional with an unknown function with multiple variables:

$$J(w(x,y)) = \iint_\Omega F\big(x, y, w(x,y), w_x(x,y), w_y(x,y)\big)dxdy$$

Example

Beam equation:

In equilibrium:



$$\underbrace{\boxed{\Pi}}_{\textbf{Potential Energy}}=\underbrace{\boxed{\frac{1}{2}\int_0^L EI(x)\left(\frac{d^2w}{dx^2}\right)^2 dx}}_{\textbf{Elastic Energy}}-\underbrace{\boxed{\int_0^L p(x)w(x)dx}}_{\textbf{External work}}$$

$$\delta\Pi\Big|_{w=w^*}=0\Rightarrow\delta\Pi=\int_0^L EIw''\delta w''dx-\int_0^L p\delta w dx=\int_0^L\left(EIw''\right)''\delta w dx+$$

$$-\int_0^L p\delta w dx+EIw''\delta w'\Big|_0^L-\left(EIw''\right)'\delta w\Big|_0^L=0$$

Where δw is arbitrary (but admissible): $\boxed{\left(EIw''\right)''=p(x)}$ **Beam equation**

and for $x=0,L$ $\begin{cases}EIw''=0 & \text{or}\quad \delta w'=0\\ (EIw'')'=0 & \text{or}\quad \delta w=0\end{cases}$

Optimization Using Variational Calculus

Constraints ➡ Lagrange Multipliers

- Integral Restriction:

$$\int_a^b g[y(x)]dx=c\Rightarrow\quad L=J+\lambda\left[\int_a^b g[y(x)]dx\text{ - }c\right]$$

- Pointwise constraint:

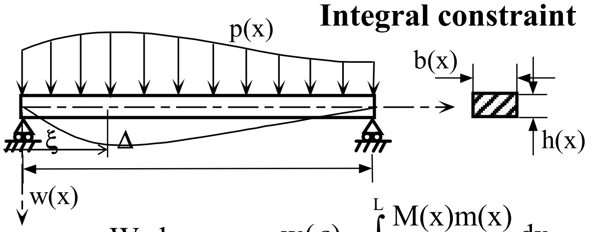
$$h_i\left(x_1,\ldots,x_n,y_1,\ldots,y_p,\frac{\partial y_1}{\partial x_1},\ldots,\frac{\partial y_p}{\partial x_n}\right)=0,\qquad i=1,\ldots,m\ \Rightarrow$$

$$\Rightarrow L=\int_v\left(f+\sum_{i=1}^m\lambda_i(\mathbf{x})h_i\right)dv$$

- Note: $\sum_{i=1}^{\infty}\lambda_i(\mathbf{x})h_i\Delta v_i\Rightarrow\int_v\lambda_i(\mathbf{x})h_i dv$

Exemple

Integral constraint



$$\underset{A(x)}{\text{Min}} \; V = \int_0^L A(x)dx$$
$$\text{s.t.} \quad w(\varepsilon) - \Delta = 0$$

We have: $$w(\varepsilon) = \int_0^L \frac{M(x)m(x)}{EI(x)} dx$$

So: $$L(A,\lambda) = \int_0^L A(x)dx + \lambda\left[\int_0^L \frac{M(x)m(x)}{EI(x)} dx \text{ - } \Delta\right]$$

$$I(x) = \alpha\left[A(x)\right]^n \begin{cases} n = 1 \Rightarrow I = \frac{bh^3}{12} = \frac{h^2}{12}bh = \frac{h^2}{12}A \\ n = 2 \Rightarrow I = \frac{h}{12b}(bh)^2 = \frac{h}{12b}A^2 \\ n = 3 \Rightarrow I = \frac{1}{12b^2}\left(b^3h^3\right) = \frac{1}{12b^2}A^3 \end{cases}$$

Example

For n=1: $$\delta L = \int_0^L \left[1 - \lambda\frac{M(x)m(x)}{\alpha EA^2(x)}\right]\delta A dx + \delta\lambda\left[\int_0^L \frac{M(x)m(x)}{EI(x)} dx - \Delta\right]$$

Since δA is arbitrary amongst admissible functions:

$$1 - \lambda\frac{M(x)m(x)}{\alpha EA^2(x)} = 0 \Rightarrow A(x) = \lambda^{\frac{1}{2}}\left(\frac{Mm}{\alpha E}\right)^{\frac{1}{2}}$$

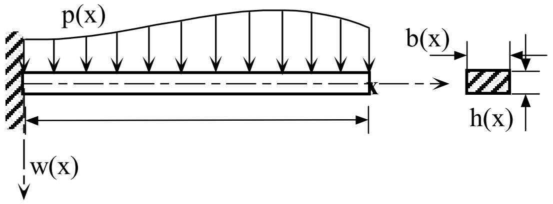
Replacing in: $$w(\varepsilon) - \Delta = 0 \Rightarrow \lambda^{\frac{1}{2}} = \frac{1}{\Delta}\int_0^L \left(\frac{Mm}{\alpha E}\right)^{\frac{1}{2}} dx \Rightarrow$$

$$A^*(x) = \frac{1}{\alpha E\Delta}\left[\int_0^L \left(M(\eta)m(\eta)\right)^{\frac{1}{2}} d\eta\right]\left(M(x)m(x)\right)^{\frac{1}{2}} \Rightarrow$$

$$\Rightarrow V^* = \frac{1}{\alpha E\Delta}\left[\int_0^L \left(M(\eta)m(\eta)\right)^{\frac{1}{2}} d\eta\right]^2$$

Example

Pointwise constraint



$$\min_{A(x),\,w(x)} \int_0^L w(x)dx \quad \text{(compliance)}$$

s.t.

$$V = \int_0^L A(x)dx = V_0$$

Equilibrium Equation

Equilibrium equation:

$$[s(x)w'']'' - q(x) = 0 \begin{cases} \text{at } x = 0: \quad w = 0 \text{ and } w' = 0 \\ \text{at } x = L: \; sw'' = 0 \text{ and } s'w'' + sw''' = 0 \;\; (sw'')'\big|_{x=L} = 0 \end{cases}$$

where: $s(x) = EI(x) = E\alpha A^n(x) \quad n = 1,2, \text{ or } 3$

The Lagrangian is:

$$L(w(x), s(x), \lambda_1, \lambda_2(x)) = \int_0^L w(x)dx + \lambda_1\left[\int_0^L A(x)dx - V_0\right] +$$

$$-\int_0^L \lambda_2(x)\left[sw'''' + 2s'w''' + s''w'' - q\right]dx$$

Example

So:

$$\delta L = \int_0^L \delta w dx + \lambda_1\left[\int_0^L \delta A dx\right] + \delta\lambda_1\left[\int_0^L A dx - V_0\right] +$$

$$-\int_0^L \delta\lambda_2(x)\left[sw'''' + 2s'w''' + s''w'' - q\right]dx +$$

$$-\int_0^L \lambda_2(x)\left[\delta s w'''' + s\delta w'''' + 2\delta s'w''' + 2s'\delta w''' + \delta s''w'' + s''\delta w''\right]dx = 0$$

Integrating by parts and gathering terms that multiply the variational quantities $\delta w, \delta s, \delta\lambda_1$, e $\delta\lambda_2$ theEuler-Lagrange Equations are obtained:

δw: $(\lambda_2'' s)'' - 1 = 0$ (1) (adjoint equation)

δs: $\lambda_1 \dfrac{dA}{ds} - \lambda_2'' w'' = 0$ (2) (optimality condition)

$\delta\lambda_1$: $\displaystyle\int_0^L A(x)dx - V_0 = 0$ (3)

$\delta\lambda_2$: $(sw'')'' - q = 0$ (4) (equilibrium equation)

Example

and the boundary conditions at x=0 and x=L:

$$\delta s = 0 \quad \text{or} \quad \lambda_2 w''' - \lambda_2' w'' = 0$$

$$\delta s' = 0 \quad \text{or} \quad \lambda_2 w'' = 0$$

$$\delta w = 0 \quad \text{or} \quad \lambda_2''' s + \lambda_2'' s' = 0$$

$$\delta w' = 0 \quad \text{or} \quad \lambda_2'' s = 0$$

$$\delta w'' = 0 \quad \text{or} \quad -\lambda_2 s' + \lambda_2' s = 0$$

$$\delta w''' = 0 \quad \text{or} \quad \lambda_2 s = 0$$

Natural boundary conditions

Considering the boundary conditions of the problem:

$$\delta w(L) \neq 0, \quad \delta w'(L) \neq 0, \quad s(0) \neq 0, \quad \delta w(0) = 0, \quad \delta w'(0) = 0, \quad \delta w''(L) = 0,$$

$$\delta w'''(L) = 0, \quad \delta w''(0) \neq 0, \delta w'''(0) \neq 0$$

Thus, the above conditions collapse to:

$$\lambda_2(0) = 0 \quad \text{and} \quad \lambda_2'(0) = 0$$

$$\lambda_2''(L)s(L) = 0 \quad \text{and} \quad \lambda_2'''(L)s(L) + \lambda_2'' s'(L) = (\lambda_2'' s)'\Big|_{x=L} = 0$$

Example

Integrating (1) and (4) twi: $s\lambda_2'' = \frac{1}{2}(x - L)^2$ e $sw'' = p(x)$

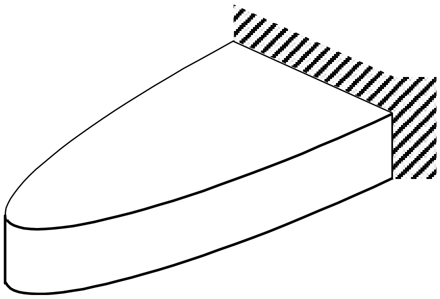
Hence: $s(x)^2 w'' = p(x)s(x) \Rightarrow \lambda_2'' w'' = \frac{(x - L)^2 p(x)}{2s(x)^2}$

Replacing in (2): $s(x)^2 \frac{dA}{ds} = \frac{(x - L)^2 p(x)}{2\lambda_1}$

Solving this equation, we find $A(\lambda_1)$. Replacing into the volume constrain we find $\lambda_1(x)$, and therefore $A(x)$.

If $q(x)=q_0$ and $n=1$:

$$A^*(x) = \frac{3V_0(x - L)^2}{L^3} \Rightarrow \frac{\int_0^L w(x)dx}{\int_0^L w(x)dx} = \frac{5}{9}$$



Optimization Using Variational Calculus

- Inequality constraints

$$g_i\left(x_1,\dots,x_n,y_1,\dots,y_p,\frac{\partial y_1}{\partial x_1},\dots,\frac{\partial y_p}{\partial x_n}\right)\geq 0,\qquad i=1,\dots,m$$

Transforming into an equality:

$$g_i\left(\mathbf{x},\dots,\frac{\partial y_p}{\partial x_n}\right)-t_i^{\,2}(\mathbf{x})=0,\qquad i=1,\dots,m$$

where t_i is an auxiliary function. Thus: $L=\int_\Omega\left(f+\sum_{i=1}^{m}\lambda_i(g_i-t_i^{\,2})\right)dv$

Stationary condition of L:

$$\delta L\Big|_{y=y^*}=\int_\Omega(\cdots)\delta y\,dv+\sum_{i=1}^{n}\left[\int_\Omega(\cdots)\delta\lambda_i\,dv-2\int_\Omega\lambda_i t_i\delta t_i\,dv\right]=0$$

Contribution from t_i in the variation of the functional: $-2\int_\Omega t_i\lambda_i\delta t_i\,dv$

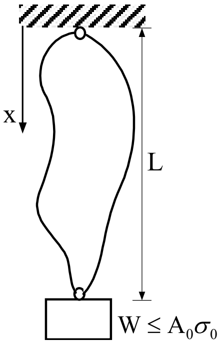
Since δt_i is arbitrary (admissible): $t_i\lambda_i=0\Rightarrow\boxed{\lambda_i g_i=0}$ $(t_i=0\Leftrightarrow g_i=0)$

“Complementarity condition”

Thus: $\begin{cases}\text{se } g_i\neq 0\Rightarrow\lambda_i=0 & \text{(constraint is not critical)}\\ \text{se } g_i=0\Rightarrow\lambda_i\neq 0 & \text{(constraint is critical)}\end{cases}$

Example

$$\begin{aligned}&\underset{A(x)}{\text{Min}}\ \int_0^L A(x)dx=V_0\\ &\text{s.t.}\quad A(x)\sigma_0-P(x)\geq 0\\ &\qquad\ \ A-A_0\geq 0\\ &\qquad\ \ P'+\rho A=0\qquad P(L)=W\end{aligned}$$



$$L(A(x),P(x),\lambda_1,\lambda_2,\lambda_3)=\int_0^L A(x)dx+\int_0^L\lambda_1(A\sigma_0-P)dx+$$

$$+\int_0^L\lambda_2(A-A_0)dx+\int_0^L\lambda_3(P'+\rho A)dx\Rightarrow\delta L=\int_0^L\delta A dx+\int_0^L\lambda_1(\delta A\sigma_0-\delta P)dx+$$

$$+\int_0^L\lambda_2\delta A dx+\int_0^L\lambda_3(\delta P'+\rho\delta A)dx+\int_0^L\delta\lambda_3(P'+\rho A)dx$$

Example

Integrating by parts to convert $\delta P'$ into δP, and the isolating terms in δA and δP, one finds:

$$\delta A: 1+\lambda_1\sigma_0+\lambda_2+\rho\lambda_3=0 \qquad (1)$$

$$\delta P: \lambda_1+\lambda_3'=0 \quad \lambda_3(0)=0 \quad (\text{pois}: \delta P(L)=0) \qquad (2)$$

Together with the complementary conditions $\begin{cases}\lambda_1(A\sigma_0-P)=0\\ \lambda_2(A-A_0)=0\end{cases}$

Solution: Close to x=0: $A > A_0 \Rightarrow \lambda_2 = .0$ Replacing λ_1 of (2) in (1):

$$1-\lambda_3''\sigma_0+\rho\lambda_3=0 \text{ and } \lambda_3(0)=0 \Rightarrow \lambda_3=\frac{e^{\frac{\rho x}{\sigma_0}}-1}{\rho} \Rightarrow \lambda_1=-\frac{e^{\frac{\rho x}{\sigma_0}}}{\sigma_0} \Rightarrow \lambda_1 \neq 0$$

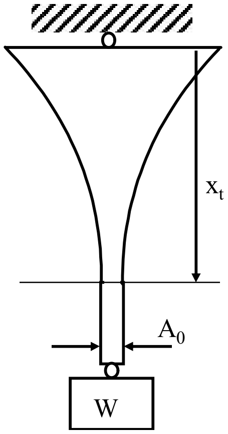
$$\Rightarrow P(x)=A(x)\sigma_0 \Rightarrow A'\sigma_0+\rho A=0 \text{ e } A(x_t)=A_0 \Rightarrow A(x)=A_0 e^{\frac{\rho(x_t-x)}{\sigma_0}}$$

for x<xt

In the inferior extremity $A=A_0$ e from equilibrium equation:

$$P=W+\rho(L-x)A_0 \Rightarrow x_t=L-\frac{A_0\sigma_0-W}{\rho A_0}$$

Example



$$A(x)=A_0 e^{\frac{\rho(x_t-x)}{\sigma_0}} \quad \text{for } x < x_t$$
$$A(x)=A_0 \quad \text{for } x > x_t$$

Other possible formulation:

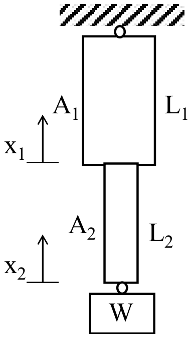
$$\underset{A(x)}{\text{Min}} \ \underset{0\le x\le L}{\text{Max}} \ \sigma(x)$$

Subject to $A-A_0 \ge 0$

$$\int_0^L A(x)dx = V_0$$

$$P'+\rho A=0$$

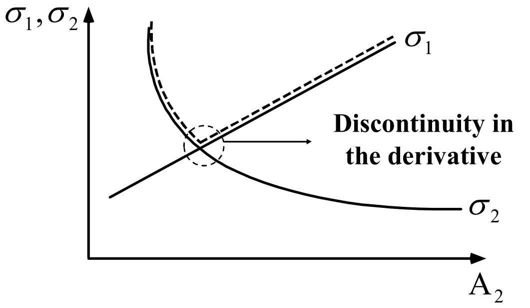
MinMax Problem



$$\min_{A(x)} \max_{0 \le x \le L} \sigma(x)$$

We have:

$$\sigma_1 = \frac{W + A_2 L_2 \rho}{A_1} + x_1 \rho \quad \text{e} \quad \sigma_2 = \frac{W}{A_2} + x_2 \rho$$



Discontinuity in the derivative

MinMax Problem

$$\boxed{\begin{array}{l}\min\limits_{A(x)} \max\limits_{0 \le x \le L} \sigma(x) \\ \text{Subject to } V(x) \le V_0\end{array}} \equiv \boxed{\begin{array}{l}\min\limits_{A(x)} V(x) \\ \text{Subject to } \sigma(x) \le \sigma_0\end{array}}$$

Should σ_0 be properly chosen $\Rightarrow V^* = V_0$

But we often do not have V(x) beforehand. Ex.: reduce stress concentration in an infinite sheet with a hole.

Thus, alternative formulation (Taylor):

$$\min_{A(x),\, \beta} \beta$$

$$\text{Subject to } \sigma(x) \le \beta, \quad 0 \le x \le L$$

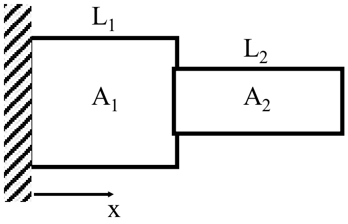
Where β is the unknown limit that we wish as small as possible

$$\boxed{\begin{array}{l}\min\limits_{A(x)} \max\limits_{0 \le x \le L} \sigma(x) \\ \text{Subject to } A - A_0 \ge 0 \\ \qquad \int_0^L A(x)dx = V_0 \\ \qquad P' + \rho A = 0\end{array}} \Rightarrow \boxed{\begin{array}{l}\min\limits_{A(x),\, \beta} \beta \\ \text{Subject to } A(x)\beta - P(x) \ge 0 \\ \qquad A - A_0 \ge 0 \\ \qquad \boxed{\int_0^L A(x)dx = V_0} \\ \qquad P' + \rho A = 0\end{array}}$$

May be removed

Problema MinMax

Ex.:



$$\min_{A_1, A_2, \beta} \ \beta$$

Subject to $\eta' \leq \beta \quad (\sigma = EA\eta')$

$$\sum_{i=1}^{2} A_i L_i = R$$

$$(EA\eta')' + p_0 = 0 \qquad EA\eta'(L_1 + L_2) = 0$$

$$(EA\eta')' + p_0 = 0 \Rightarrow EA\eta' = p_0(L_2 + L_1) - p_0 \Rightarrow \begin{cases} \text{at } x = 0 \Rightarrow \sigma(0) = EA_1\eta'(0) = p_0(L_2 + L_1) \Rightarrow \\ \text{at } x = L_1 \Rightarrow \sigma(L_1) = EA_2\eta'(L_1) = p_0 L_2 \Rightarrow \end{cases}$$

$$\left.\begin{aligned} \Rightarrow A_1 &= \frac{p_0(L_2 + L_1)}{E\beta} \\ \Rightarrow A_2 &= \frac{p_0 L_2}{E\beta} \end{aligned}\right\} \Rightarrow A_1L_1 + A_2L_2 = R \Rightarrow \beta = \frac{p_0\left(L_1{}^2 + L_1L_2 + L_2{}^2\right)}{ER}$$

Other Min Max problem examples:

$$\begin{cases} \min_{A(x)} \max_{0 \leq x \leq L} \ w(x) & \text{(displacement)} \\ \min_{A(x)} \max_{0 \leq x \leq L} |kw(x)| & \text{(elastic foundation)} \end{cases}$$

Optimization Using Variational Calculus

Reformulation of the Objective Function and Constraints

Pointwise constraints ➡ Dificult treatment ➡ Integral constraints

To avoid Min Max problems ➡ Integral formulation

Examples:

- displacement of a point: $y(x_0) \leq y_{max} \Rightarrow \int_0^L \delta(x - x_0) y(x) dx \leq y_{max}$

where: $\delta(x - x_0) = \begin{cases} \infty & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}$ and: $\int_{-\infty}^{+\infty} \delta(x - x_0) dx = 1$

(Dirac Delta)

- $\max_{x \in [0,L]} y(x) \leq y_{max}$ may be replaced by: $\int_0^L p(x) y(x) dx$ (compliance)

or by: $\frac{1}{L} \int_0^L (y^p(x))^{\frac{1}{p}} dx \leq y_{max}$ for $\begin{cases} y(x) \geq 0 & x \in [0,L] \\ p \text{ sufficiently large} \end{cases}$

- Resonance frequency:

$$\begin{cases} (EIy'')'' + \omega^2 A\rho y = 0 & \text{(differential form)} \\ \omega = \min_{y(x) \text{ admissible}} \left\{ \dfrac{\int_0^L EI(y'')^2 dx}{\int_0^L \rho A y^2 dx} \right\} & \text{(integral form)} \end{cases}$$

Optimality Conditions in Classical Problems

Project: Minimization of Compliance (or Maximization of Stiffness)

$$\underset{A(x),\,u(x)}{\text{Min}} \quad \int_\Omega p(x)u(x)dx$$

Subject to

$$\int_\Omega (EAu'v' - pv)dx = 0 \ \ \forall v \in \overline{V}$$

Weak formulation

(or $(EAu')' - p = 0$ and boundary conditions)

$$A(x) - A_{min}$$

$$\int_\Omega Adx - R \le 0$$

$$\Pi = \frac{1}{2}\int_\Omega EAv'^2dx - \int_\Omega pvdx \Rightarrow \delta\Pi = 0 \Rightarrow \int_\Omega (EAv'\delta v' - p\delta v)dx = 0 \Rightarrow$$

$$\Rightarrow \int_\Omega (EAv'\eta' - p\eta)dx = 0$$

In equilibrium: $v = u \Rightarrow \int_\Omega (EAu'\eta' - p\eta)dx = 0$

Optimality Conditions in Classical Problems

Since η is arbitrary amongst admissible functions:

$$\eta = u \Rightarrow \int_\Omega (EAu'^2 - pu)dx = 0 \Rightarrow \Pi\big|_{equil.} = \frac{1}{2}\int_\Omega EAu'^2dx - \int_\Omega pudx = -\frac{1}{2}\int_\Omega pudx =$$

$$= -\frac{1}{2}\int_\Omega EAu'^2dx \Rightarrow \int_\Omega pudx = \int_\Omega EAu'^2dx = -2\Pi\big|_{equil.}$$

Thus, the problem of minimization of compliance is equivalent to:

$$\underset{A(x)}{\text{Min}} \int_\Omega EAu'^2dx$$

s.t.

$$A(x) - A_{min} \ge 0$$

$$\int_\Omega Adx - R \le 0$$

$$\therefore L(A(x), \lambda_A, \Lambda) = \int_\Omega EAu'^2dx + \int_\Omega \lambda_A(A_{min} - A)dx + \Lambda\left(\int_\Omega Adx - R\right) \Rightarrow$$

$$\Rightarrow \delta A{:}\ Eu'^2 = \begin{cases} \Lambda - \lambda_A & \text{in } \Omega_A \ (\lambda_A \ne 0 \Rightarrow A = A_{min}) \\ \Lambda & \text{in } \underline{\Omega} \ \ (\lambda_A = 0 \Rightarrow A > A_{min}) \end{cases} \quad \text{and } \Omega = \Omega_A + \underline{\Omega}$$

Elastic energy is constant

Optimality Conditions in Classical Problems

Project: Maximization of Resonance Frequency (ω)

Rayleigh Coefficient (QR):

$$QR = \frac{E_{elástica}}{E_{cinética}} = \frac{\int_0^L EI(x)w''^2dx}{\int_0^L \rho A(x)w^2dx} = \frac{\int_0^L E\alpha\left[A(x)\right]^n w''^2dx}{\int_0^L \rho A(x)w^2dx}$$

Resonance frequency (ω): $\omega^2 = \underset{w(x)}{\mathrm{Min}}\left(QR\right)$

So: $\delta(QR) = 0 \Rightarrow [EIw'']'' - \omega^2\rho Aw = 0$ and boundary conditions

So the problem:

$$\underset{A(x)}{\mathrm{Max}}\ \omega^2 = \underset{w(x)}{\mathrm{Min}}\left(QR\right)$$

Subject to $\boxed{\left[EIw''\right]'' - \omega^2\rho Aw = 0 \ \text{(e condições de contorno)}} \rightarrow$ **redundant**

$$\int_0^L Adx = V_0$$

$$A(x) - A_{min} \geq 0$$

Optimality Conditions in Classical Problems

$$\therefore L(w(x), A(x), \lambda) = \frac{\int_0^L E\alpha\left[A\right]^n w''^2dx}{\int_0^L \rho Aw^2dx} - \lambda\left(\int_0^L Adx - V_0\right) \Rightarrow \delta L = 0 \Rightarrow$$

$$\Rightarrow \frac{2\int_0^L E\alpha\left[A\right]^n w''\delta w''dx}{\int_0^L \rho Aw^2dx} - \frac{2\int_0^L E\alpha\left[A\right]^n w''^2dx}{\left[\int_0^L \rho Aw^2dx\right]^2}\left[\int_0^L \rho Aw\delta wdx\right] + \frac{\int_0^L nE\alpha\left[A\right]^{n-1} w''^2\delta Adx}{\int_0^L \rho Aw^2dx} +$$

$$- \frac{\int_0^L E\alpha\left[A\right]^n w''^2dx}{\left[\int_0^L \rho Aw^2dx\right]^2}\left[\int_0^L \rho w^2\delta Adx\right] + \lambda\left[\int_0^L \delta Adx\right] = 0$$

Optimality Conditions in Classical Problems

Integrating by parts and collecting terms in δA e δw:

Equation of motion: $[E\alpha A^{n} w'']'' - \omega^2 \rho A w = 0$

Boundary conditions: $\delta w = 0 \text{ ou } [E\alpha A^{n} w'']' = 0$; $\delta w' = 0 \text{ ou } [E\alpha A^{n} w''] = 0$

Optimality condition: $nE\alpha A^{n-1} w''^2 - \omega^2 \rho w^2 = \text{cte.}$

Verifying the sufficient condition for n=1. Let:

A and $\bar{A}$, solutions that satisfy the volume constraint, and $\omega, \bar{\omega}, w, e\ \bar{w}$ the corresponding resonance frequencies and respective vibration modes.

Thus:

$$\omega^2 = \frac{\int_0^L E\alpha A w''^2 dx}{\int_0^L \rho A w^2 dx} \text{ and } \bar{\omega}^2 = \frac{\int_0^L E\alpha \bar{A} \bar{w}''^2 dx}{\int_0^L \rho \bar{A} \bar{w}^2 dx} \Rightarrow \bar{\bar{\omega}}^2 = \frac{\int_0^L E\alpha \bar{A} w''^2 dx}{\int_0^L \rho \bar{A} w^2 dx} \geq \bar{\omega}^2 \Rightarrow$$

$$\left(\omega^2 = \underset{w(x)}{\text{Min}} \left(QR\right)\right)$$

Optimality Conditions in Classical Problems

$$\Rightarrow \bar{\bar{\omega}}^2 \int_0^L \rho \bar{A} w^2 dx = \int_0^L E\alpha \bar{A} w''^2 dx \text{ and } \omega^2 \int_0^L \rho A w^2 dx = \int_0^L E\alpha A w''^2 dx \Rightarrow$$

Subtracting:: $\bar{\bar{\omega}}^2 \int_0^L \rho \bar{A} w^2 dx - \omega^2 \int_0^L \rho A w^2 dx = \int_0^L E\alpha (\bar{A} - A) w''^2 dx$

But: $E\alpha w''^2 - \omega^2 \rho w^2 = c \Rightarrow E\alpha w''^2 = c + \omega^2 \rho w^2 \Rightarrow$

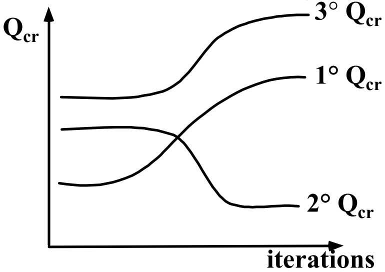
$$\int_0^L E\alpha (\bar{A} - A) w''^2 dx = \int_0^L (\bar{A} - A)(c + \omega^2 \rho w^2) dx = \int_0^L (\bar{A} - A) \omega^2 \rho w^2 dx;$$

$$\left(\int_0^L (\bar{A} - A) dx = 0\right) \Rightarrow \bar{\bar{\omega}}^2 \int_0^L \rho \bar{A} w^2 dx - \omega^2 \int_0^L \rho A w^2 dx = \int_0^L (\bar{A} - A) \omega^2 \rho w^2 dx \Rightarrow$$

$$\Rightarrow \bar{\bar{\omega}}^2 - \omega^2 = 0 \Rightarrow \boxed{\omega^2 \geq \bar{\omega}^2}$$

Optimality Conditions in Classical Problems

Mode inversion during the optimization. Ex.:



A solution: Bimodal Formulation

$$\underset{A(x)}{\text{Max}}\quad \theta = \int_0^L E\alpha[A(x)]^n w''^2 dx$$

$$\text{Subject to} \quad \int_0^L E\alpha[A(x)]^n w_1''^2 dx = \int_0^L E\alpha[A(x)]^n w_2''^2 dx$$

$$\int_0^L w_1'^2 dx = 1; \quad \int_0^L w_2'^2 dx = 1; \int_0^L A dx = V_0; A \geq A_{min}$$

Optimality Conditions in Classical Problems

Equivalent problems:

$$\underset{A(x)}{\text{Min}} \int_0^L A dx$$

Subject to $[EIw'']'' - \omega^2 \rho A w = 0$ (and boundary conditions)

or $\omega^2 = \underset{w(x)}{\text{Min}}$ (QR)

$A(x) - A_{min}$

or

$$\underset{A(x)}{\text{Max}} \quad \beta$$

Subject to $\omega^2 \geq \beta$

$[EIw'']'' - \omega^2 \rho A w = 0$ (and boundary conditions)

$$\int_0^L A dx = V_0$$

$A(x) - A_{min}$

Normalized problem:

$$\underset{A(x)}{\text{Max}} \quad \int_0^L \alpha^p [w'']^2 d\varepsilon$$

Subject to $\int_0^L \alpha^p w^2 d\varepsilon = 1$ (kinetic energy)

$\int_0^L \alpha d\varepsilon = 1$ (volume)

$$\alpha = \frac{AL}{V_0}; \varepsilon = \frac{x}{L}; \lambda = \frac{\omega^2 \rho L^2}{ECV_0}$$

Optimality Conditions in Classical Problems

Project: Maximization of Buckling Load (Q)

Rayleigh’s Coefficient (QR):

$$QR = \frac{\int_0^L EI(x)w''^2 dx}{\int_0^L w'^2 dx} = \frac{\int_0^L E\alpha[A(x)]^n w''^2 dx}{\int_0^L w'^2 dx}$$

Buckling load (Q_{cr}): $Q_{cr} = \underset{w(x)}{\mathrm{Min}}(QR)$

Thus: $\delta(QR) = 0 \Rightarrow [EIw'']'' - Q_{cr}w'' = 0$ and boundary conditions

Consider the problem:

$\underset{A(x)}{\mathrm{Max}} \quad Q_{cr} = \underset{w(x)}{\mathrm{Min}}(QR)$

Subject to $\boxed{[EIw'']'' - Q_{cr}w'' = 0 \text{ (and boundary conditions)}}$ → **redundant**

$\int_0^L Adx = V_0$

$A(x) - A_{min}$

Optimality Conditions in Classical Problems

$$\therefore L(w(x), A(x), \lambda) = \frac{\int_0^L E\alpha[A]^n w''^2 dx}{\int_0^L w'^2 dx} - \lambda\left(\int_0^L Adx - V_0\right) \Rightarrow \delta L = 0 \Rightarrow$$

$$\Rightarrow \frac{2\int_0^L E\alpha[A]^n w''\delta w'' dx}{\int_0^L w'^2 dx} - \frac{2\int_0^L E\alpha[A]^n w''^2 dx}{\left[\int_0^L w'^2 dx\right]^2}\left[\int_0^L w'\delta w' dx\right] + \frac{\int_0^L nE\alpha[A]^{n-1} w''^2 \delta A dx}{\int_0^L w'^2 dx} +$$

$$-\lambda\left[\int_0^L \delta A dx\right] = 0$$

Integrating by parts and gathering terms in δA e δw:

Optimality Conditions in Classical Problems

Eq. of Motion: $[E\alpha A^n w'']'' - Q_{cr}w'' = 0$

Boundary conditions: $\delta w = 0 \;\text{ or }\; [E\alpha A^n w'']' + Q_{cr}w' = 0$
$\delta w' = 0 \;\text{ or }\; [E\alpha A^n w''] = 0$

Optimality condition: $nE\alpha A^{n-1}w''^2 - \lambda\int_0^L w'^2 dx = \text{cte.}$

Equivalent problems:

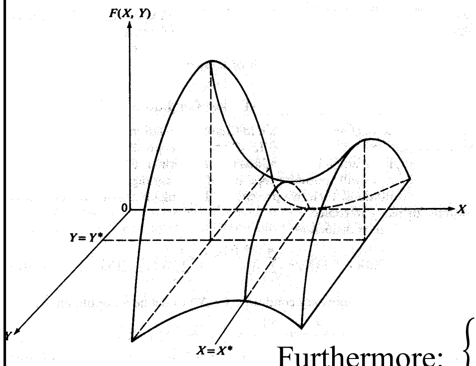
$$\min_{A(x)} \int_0^L A dx$$

Subject to $[EIw'']'' - Q_{cr}w'' = 0$
(and boundary conditions)
$\boxed{\text{or }\; Q_{cr} = \min_{w(x)} \text{ (QR)}}$
$A(x) - A_{min}$

$$\max_{A(x)} \beta$$

Subject to $Q_{cr} \geq \beta$
$[EIw'']'' - Q_{cr}w'' = 0$
(and boundary conditions)
$\int_0^L A dx = V_0$
$A(x) - A_{min}$

Saddle Point



A function F(**x**,**y**) contains a saddle point if:

$F(\mathbf{x}^*, \mathbf{y}) \leq F(\mathbf{x}^*, \mathbf{y}^*) \leq F(\mathbf{x}, \mathbf{y}^*)$

$F(\mathbf{x}^*, \mathbf{y}^*)\begin{cases}\text{minimal in } \mathbf{x}\\ \text{maximal in } \mathbf{y}\end{cases}$

In the case of the Lagrangian: $L(\mathbf{x}^*, \lambda^*)$

Furthermore: $\begin{cases}\nabla_x L(\mathbf{x}^*, \lambda^*) = 0\\ \nabla_\lambda L(\mathbf{x}^*, \lambda^*) = 0\end{cases}$ and $\begin{cases}\nabla_\lambda L(\mathbf{x}^*, \lambda) \leq 0\\ \nabla_x L(\mathbf{x}, \lambda^*) \geq 0\end{cases}$

Solution of the saddle point ➡ Min Max problem:

$$L(\lambda) = \min_{\mathbf{x}} L(\mathbf{x}, \lambda) \Rightarrow \max_{\lambda} L(\lambda) = \max_{\lambda}\min_{\mathbf{x}} L(\mathbf{x}, \lambda)$$

or:

$$L(\mathbf{x}) = \max_{\lambda} L(\mathbf{x}, \lambda) \Rightarrow \min_{\mathbf{x}} L(\mathbf{x}) = \min_{\mathbf{x}}\max_{\lambda} L(\mathbf{x}, \lambda)$$

$$\boxed{\max_{\lambda}\min_{\mathbf{x}} L(\mathbf{x}, \lambda) \equiv \min_{\mathbf{x}}\max_{\lambda} L(\mathbf{x}, \lambda)}$$

Example

Ex.:

$$\begin{array}{ll} \underset{x}{\text{Min}} & \dfrac{1}{x} \\ \text{s.t.} & x-1 \le 0 \\ & x \ge 0 \end{array}$$

The Lagrangian of the problem is given by:

$$L(x,\lambda) = \frac{1}{x} + \lambda(x-1)$$

Calculating the mininimum in x: $\nabla_x L(x,\lambda) = 0 \Rightarrow -\frac{1}{x^2} + \lambda = 0 \Rightarrow$

$$\Rightarrow x = \pm\frac{1}{\sqrt{\lambda}}\,(\text{mas } x \ge 0) \Rightarrow x = \frac{1}{\sqrt{\lambda}} \Rightarrow L(\lambda) = \sqrt{\lambda} + \lambda\left(\frac{1}{\sqrt{\lambda}} - 1\right) = 2\sqrt{\lambda} - \lambda$$

Calculating the maximum in λ: $\nabla_\lambda L(\lambda) = \frac{1}{\sqrt{\lambda}} - 1 \Rightarrow \lambda^* = 1 \Rightarrow x^* = 1$

Saddle point: $\boxed{\therefore \left(x^*, y^*\right) = (1,1)}$

The gradients must be continuous in **x** and λ, otherwise another approach to the Min Max problem must be employed.

Example

Plot of the Lagrangian function: