Álgebra Linear e Aplicações

- Método de Gram-Schmidt e decomposição QR.

MÉTODO DE GRAM-SCHMIDT

TEOREMA: SEJA $S=\{u_1,\dots,u_n\}\subset\mathbb{R}^m$ UM CONJUNTO LINEARMENTE INDEPENDENTE. LOGO EXISTE UM CONJUNTO ORTONORMAL $\mathcal{B}=\{e_1,\dots,e_n\}$ TAL QUE $[u_1,\dots,u_k]=[e_1,\dots,e_k]$, PARA TODO $k\in\{1,\dots,n\}$.

OBS: AQUI $[z_1,\dots,z_n]=\{\alpha_1 z_1+\dots+\alpha_n z_n:\alpha_1,\dots,\alpha_n\in\mathbb{R}\}$ É O ESPAÇO GERADO POR $\{z_1,\dots,z_n\}$.

DEMO: FAREMOS A PROVA POR INDUÇÃO FINITA.

O TEOREMA VALE PARA $n=1$:

SEJA $S=\{u_1\}$. NESTE CASO, BASTA DEFINIR $e_1=\frac{u_1}{\|u_1\|}$. ASSIM, $\mathcal{B}=\{e_1\}$.

SE O TEOREMA VALE PARA $n\geq 1$, ENTÃO ELE VALE PARA $n+1$:

SEJA $S=\{u_1,\dots,u_n,u_{n+1}\}$. PELA HIPÓTESE DE INDUÇÃO, $\exists\ \{e_1,\dots,e_n\}$ TAL QUE $[u_1,\dots,u_j]=[e_1,\dots,e_j]$, $\forall j\in\{1,\dots,n\}$.

DEFINIÇÃO DE e_{n+1}

DEFINIMOS $\tilde{e}_{n+1}=u_{n+1}-\sum_{j=1}^{n}\langle u_{n+1},e_j\rangle e_j$. NOTE QUE $\tilde{e}_{n+1}\neq 0$, POIS SE $\tilde{e}_{n+1}=0$, ENTÃO $u_{n+1}=\sum_{j=1}^{n}\langle u_{n+1},e_j\rangle e_j\in[e_1,\dots,e_n]=[u_1,\dots,u_n]$. LOGO $u_{n+1}=\alpha_1 u_1+\dots+\alpha_n u_n$.

$\Rightarrow \{u_1,\dots,u_{n+1}\}$ NÃO É L.I. ABSURDO. ASSIM, PODEMOS DEFINIR $e_{n+1}=\frac{\tilde{e}_{n+1}}{\|\tilde{e}_{n+1}\|}$

NOTE QUE O CONJUNTO $\{e_1,\dots,e_{n+1}\}$ É ORTONORMAL.

COMO $\|e_j\|=1$, $\forall j\in\{1,\dots,n+1\}$ E $\langle e_j,e_k\rangle=\delta_{jk}$, PARA $j,k\in\{1,\dots,n\}$, FALTA APENAS MOSTRAR QUE $\langle e_j,e_{n+1}\rangle=0$, $\forall j\in\{1,\dots,n\}$. DE FATO, TEMOS PARA $l\in\{1,\dots,n\}$:

$$\langle e_{n+1},e_l\rangle=\|\tilde{e}_{n+1}\|^{-1}\left[\langle u_{n+1},e_l\rangle-\sum_{j=1}^{n}\langle u_{n+1},e_j\rangle\overbrace{\langle e_j,e_l\rangle}^{\delta_{jl}}\right]=\|\tilde{e}_{n+1}\|^{-1}\left[\langle u_{n+1},e_l\rangle-\langle u_{n+1},e_l\rangle\right]=0.$$

Por fim, vamos mostrar que $[u_1, \dots, u_{n+1}] = [e_1, \dots, e_{n+1}]$.

De fato, $\alpha_1 e_1 + \dots + \alpha_n e_n + \alpha_{n+1} e_{n+1} = \tilde{\alpha}_1 e_1 + \dots + \tilde{\alpha}_n e_n + \frac{\alpha_{n+1}}{\|\tilde{e}_{n+1}\|} u_{n+1}$

$\hookrightarrow e_{n+1} = \frac{1}{\|\tilde{e}_{n+1}\|}\left(u_{n+1} - \sum_{k=1}^{n} (u_{n+1}, e_k) e_k\right).$

$$= \beta_1 u_1 + \dots + \beta_n u_n + \frac{\alpha_{n+1}}{\|\tilde{e}_{n+1}\|} u_{n+1} \in [u_1, \dots, u_n, u_{n+1}].$$

$\hookrightarrow [e_1, \dots, e_n] = [u_1, \dots, u_n]$

Logo $[e_1, \dots, e_{n+1}] \subset [u_1, \dots, u_{n+1}]$.

Por outro lado, $\alpha_1 u_1 + \dots + \alpha_n u_n + \alpha_{n+1} u_{n+1} = \tilde{\alpha}_1 u_1 + \dots + \tilde{\alpha}_n u_n + \|\tilde{e}_{n+1}\| \alpha_{n+1} e_{n+1}$

$\hookrightarrow u_{n+1} = \|\tilde{e}_{n+1}\| e_{n+1} + \sum_{k=1}^{n} (u_{n+1}, e_k) e_k$ $\hookrightarrow \in [u_1, \dots, u_n]$

$$= \beta_1 e_1 + \dots + \beta_n e_n + \|\tilde{e}_{n+1}\| \alpha_{n+1} e_{n+1} \in [e_1, \dots, e_n, e_{n+1}].$$

$\hookrightarrow [e_1, \dots, e_n] = [u_1, \dots, u_n]$

Logo $[e_1, \dots, e_{n+1}] \supset [u_1, \dots, u_{n+1}]$ $\blacksquare$

O teorema acima nos fornece o seguinte algoritmo.

Algoritmo (Gram-Schmidt)

Entrada: n vetores L.I's $\{u_1, \dots, u_n\}$ de $\mathbb{R}^m$.

Saída: n vetores ortonormais $\{e_1, \dots, e_n\} \subset \mathbb{R}^m$ tais que $[u_1, \dots, u_k] = [e_1, \dots, e_k]$, $\forall\, 1 \leq k \leq n$.

Em particular, $\{e_1, \dots, e_n\}$ é base ortonormal de $[u_1, \dots, u_n]$.

$$\boxed{\begin{array}{l} e_1 = \frac{u_1}{\|u_1\|} \\ \text{Para } j = 2, \dots, n: \\ e_j = u_j - \sum_{k=1}^{j-1} (e_k^T u_j) e_k \\ e_j = \frac{e_j}{\|e_j\|} \end{array}}$$

EXEMPLO 1: $S = \{ \overbrace{(2,1)}^{u_1}, \overbrace{(3,2)}^{u_2} \}$

$$e_1 = \frac{(2,1)}{\|(2,1)\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

$$\tilde{e}_2 = (3,2) - \left\langle (3,2), \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \right\rangle \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = (3,2) - \left(\frac{16}{5}, \frac{8}{5}\right) = \left(-\frac{1}{5}, +\frac{2}{5}\right).$$

$$e_2 = \frac{\left(-\frac{1}{5}, \frac{2}{5}\right)}{\left\|\left(-\frac{1}{5}, \frac{2}{5}\right)\right\|} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\hookrightarrow (3,2) = \left(\frac{16}{5}, \frac{8}{5}\right) + \left(-\frac{1}{5}, \frac{2}{5}\right) = \frac{8}{\sqrt{5}}(2,1) + \frac{1}{\sqrt{5}}\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\mathcal{B} = \{e_1, e_2\} = \left\{ \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \right\}$$

OBS: NOTE QUE $u_1 = \sqrt{5}\, e_1$

$u_2 = \frac{8}{\sqrt{5}} e_1 + \frac{1}{\sqrt{5}} e_2$

OBSERVE QUE:

$[u_1] = [e_1]$

$[u_1, u_2] = [e_1, e_2]$

EXEMPLO 2: $S = \{ \overbrace{(1,0,1)}^{u_1}, \overbrace{(1,0,0)}^{u_2}, \overbrace{(2,1,0)}^{u_3} \}$

$$e_1 = \frac{(1,0,1)}{\|(1,0,1)\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\tilde{e}_2 = (1,0,0) - \left\langle \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (1,0,0) \right\rangle \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = (1,0,0) - \left(\frac{1}{2}, 0, \frac{1}{2}\right) = \left(+\frac{1}{2}, 0, -\frac{1}{2}\right)$$

$$e_2 = \frac{\tilde{e}_2}{\|\tilde{e}_2\|} = \sqrt{2}\,\tilde{e}_2 = \sqrt{2}\left(\frac{1}{2}, 0, -\frac{1}{2}\right) = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right).$$

$$\tilde{e}_3 = (2,1,0) - \left\langle (2,1,0), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \right\rangle \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) - \left\langle (2,1,0), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) \right\rangle \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

$$= (2,1,0) - (1,0,1) - (1,0,-1) = (0,1,0)$$

$$e_3 = \frac{\tilde{e}_3}{\|\tilde{e}_3\|} = (0,1,0).$$

$$\mathcal{B} = \{e_1, e_2, e_3\} = \left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), (0,1,0) \right\}.$$

OBS: NOTE QUE $u_1 = \sqrt{2}\, e_1$ $\qquad [e_1] = [u_1]$

$u_2 = \frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_2$ $\qquad [e_1, e_2] = [u_1, u_2]$

$u_3 = \sqrt{2}\, e_1 + \sqrt{2}\, e_2 + e_3$ $\qquad [e_1, e_2, e_3] = [u_1, u_2, u_3].$

Decomposição QR

Seja A uma matriz $m \times n$. Suponha que as colunas de A sejam vetores linearmente independentes de $\mathbb{R}^m$ (para isto é necessário que $m \geqslant n$).

Assim, $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$, $v_j \in \mathbb{R}^m$, $\forall j \in \{1, \dots, n\}$.

Aplicando Gram-Schmidt ao conjunto $S = \{v_1, \dots, v_n\}$, construímos um conjunto ortonormal $\{e_1, \dots, e_n\}$ tal que $[v_1, \dots, v_k] = [e_1, \dots, e_k]$, $k \in \{1, \dots, n\}$.

Portanto, $v_j \in [e_1, \dots, e_j]$. Logo $v_j = \sum_{i=1}^{j} r_{ij} e_i$

Seja $\tilde{R}$ a matriz $n \times n$ definida por $\tilde{R}_{ij} = \begin{cases} r_{ij}, & 1 \leqslant i \leqslant j \\ 0, & i > j \end{cases}$

Logo $v_j = \sum_{i=1}^{n} e_i \tilde{R}_{ij}$. Assim,

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix}}_{:= \tilde{Q}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ 0 & r_{22} & r_{23} & \cdots & r_{2n} \\ \vdots & 0 & r_{33} & \cdots & r_{3n} \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{nn} \end{bmatrix}}_{:= \tilde{R}}$$

Note que $\tilde{Q}^T \tilde{Q} = \begin{bmatrix} - & e_1 & - \\ & \vdots & \\ - & e_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix} = \begin{bmatrix} e_i^T e_j \end{bmatrix} = I$.

PROVAMOS, ASSIM, O SEGUINTE RESULTADO.

TEOREMA 1: SEJA A UMA MATRIZ $m \times n$ CUJAS COLUNAS SÃO VETORES LINEARMENTE INDEPENDENTES EM $\mathbb{R}^m$. LOGO EXISTEM MATRIZES $\tilde{Q}$ $m \times n$ E $\tilde{R}$ $n \times n$ TAIS QUE:

i) $A = \tilde{Q}\tilde{R}$

ii) $\tilde{Q}^T\tilde{Q} = I$, I A IDENTIDADE $n \times n$.

iii) $\tilde{R}$ UMA MATRIZ TRIDIAGONAL SUPERIOR.

COROLÁRIO: SEJA A UMA MATRIZ $m \times n$ CUJAS COLUNAS SÃO VETORES LINEARMENTE INDEPENDENTES EM $\mathbb{R}^m$. LOGO EXISTEM MATRIZES Q $m \times m$ E R $m \times n$ TAIS QUE:

i) $A = QR$

ii) Q É UMA MATRIZ ORTOGONAL

iii) R É UMA MATRIZ TRIANGULAR SOPERIOR.

DEMO: SABEMOS QUE $A = \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ e_1 & \dots & e_n \\ | & & | \end{bmatrix} \begin{bmatrix} R \end{bmatrix}$

SEJAM $e_{n+1}, \dots, e_m \in \mathbb{R}^m$ TAIS QUE $\{e_1, \dots, e_m\}$ SEJA UMA BASE ORTONORMAL DE $\mathbb{R}^m$.

DEFINIMOS A MATRIZ ORTONORMAL $Q = \begin{bmatrix} | & & | \\ e_1 & \dots & e_m \\ | & & | \end{bmatrix}$ E A MATRIZ TRIANGULAR SUPERIOR $R = \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix}$,

EM QUE $\tilde{R}$ É MATRIZ $n \times n$ COMO NO TEOREMA 1 E 0 É A MATRIZ NULA $(m-n) \times n$

ASSIM $QR = \begin{bmatrix} | & & | \\ e_1 & \dots & e_m \\ | & & | \end{bmatrix} \begin{bmatrix} r_{11} & \dots & r_{1m} \\ 0 & r_{21} \dots & r_{2m} \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_{mm} \\ & 0 & \end{bmatrix} = \begin{bmatrix} | & & | \\ \sum_{k=1}^{m} R_{k1} e_k & \dots & \sum_{k=1}^{m} R_{kn} e_k \\ | & & | \end{bmatrix}$

NOTE QUE $\sum_{k=1}^{m} R_{ki} e_k = \sum_{k=1}^{n} \tilde{R}_{ki} e_k = \sum_{k=1}^{i} r_{ki} e_k = v_i$. LOGO $QR = \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix} = A$ ■

Exemplo 1: $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Sabemos que

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \sqrt{5} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{8}{\sqrt{5}} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} + \frac{1}{\sqrt{5}} \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

Logo $Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ e $R = \begin{bmatrix} \sqrt{5} & \frac{8}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \frac{8}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix}$

Exemplo 2: $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Sabemos que $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Logo $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ e $R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Aplicação MMQ: Se $A = QR$, então $A^T A \lambda = A^T b$ equivale a

$$R^T Q^T Q R \lambda = R^T Q^T b \Leftrightarrow R^T R \lambda = R^T Q^T b \Leftrightarrow \underbrace{R\lambda = Q^T b}_{\text{Fácil de resolver! } R \text{ é triangular superior.}}$$

EXEMPLO: ACHE $\lambda = (\alpha_1, \alpha_2)$ TAL QUE $\alpha_1(1,0,1) + \alpha_2(1,0,0)$ SEJA A MELHOR APROXIMAÇÃO DE $(1,2,3)$, ISTO É, λ MINIMIZA $(1-\alpha_1-\alpha_2)^2 + 2^2 + (3-\alpha_1)^2$.

QUEREMOS "RESOLVER" (NO SENTIDO DE MMQ) O SISTEMA ABAIXO:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}_{\lambda} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{b}.$$

SABEMOS QUE

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}_{R}$$

ASSIM, λ DEVE SATISFAZER $R\lambda = Q^T b$, OU SEJA,

$$\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$$

LOGO $\alpha_2 = -\sqrt{2}\sqrt{2} = -2$

$$\sqrt{2}\alpha_1 + \frac{1}{\sqrt{2}}\alpha_2 = 2\sqrt{2} \Rightarrow \sqrt{2}\alpha_1 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2} \Rightarrow \alpha_1 = 3.$$

PORTANTO, $3 \times (1,0,1) - 2(1,0,0) = (1,0,3)$ É O VETOR PROCURADO.

Observação Final: Se $A = QR$, podemos achar facilmente os elementos de R.

Seja
$$\begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}}_{\text{colunas ortonormais.}} \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \vdots \\ 0 & & r_{nn} \end{bmatrix},$$
então

$$u_1 = r_{11} q_1 \quad \Rightarrow \quad q_1^T u_1 = r_{11} q_1^T q_1 \Rightarrow r_{11} = q_1^T u_1$$

$$u_2 = r_{12} q_1 + r_{22} q_2 \quad \Rightarrow \quad q_1^T u_2 = r_{12} q_1^T q_1 + r_{22} q_1^T q_2 \Rightarrow r_{12} = q_1^T u_2 \;\text{ e }\; q_2^T u_2 = r_{12} q_2^T q_1 + r_{22} q_2^T q_2 \Rightarrow r_{22} = q_2^T u_2.$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$u_k = \sum_{j=1}^{k} r_{jk} q_j \quad \Rightarrow \quad q_i^T u_k = \sum_{j=1}^{k} r_{jk} q_i^T q_j = \sum_{j=1}^{k} r_{jk} \delta_{ij} = r_{ik} \Rightarrow \boxed{r_{ij} = q_i^T u_j}$$
← Esta é a fórmula procurada!

Exemplo 1:
$$\underbrace{\begin{array}{cc} u_1 & u_2 \\ \left[\begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix}\right. & \left.\right] \end{array}}_{A} = \underbrace{\begin{array}{c} q_1 \quad q_2 \\ \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \end{array}}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}}_{R}$$

$r_{11} = q_1^T u_1 = \langle (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}), (2,1) \rangle = \sqrt{5}$, $r_{12} = q_1^T u_2 = \langle (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}), (3,2) \rangle = \frac{8}{\sqrt{5}}$, $r_{22} = q_2^T u_2 = \langle (-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}), (3,2) \rangle = \frac{1}{\sqrt{5}}$.

$$\Rightarrow R = \begin{bmatrix} \sqrt{5} & \frac{8}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix}$$

Exemplo 2:
$$\begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \qquad\qquad \begin{array}{ccc} q_1 & q_2 & q_3 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

$$\left.\begin{aligned}
r_{11} &= q_1^T u_1 = \langle (\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}), (1,0,1) \rangle = \sqrt{2} \\
r_{12} &= q_1^T u_2 = \langle (\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}), (1,0,0) \rangle = \tfrac{1}{\sqrt{2}} \\
r_{13} &= q_1^T u_3 = \langle (\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}), (2,1,0) \rangle = \sqrt{2} \\
r_{22} &= q_2^T u_2 = \langle (\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}), (1,0,0) \rangle = \tfrac{1}{\sqrt{2}} \\
r_{23} &= q_2^T u_3 = \langle (\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}), (2,1,0) \rangle = \sqrt{2} \\
r_{33} &= q_3^T u_3 = \langle (0,1,0), (2,1,0) \rangle = 1
\end{aligned}\right\} \Rightarrow \quad R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$