Famous trails to Paul Erdős*

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Abstract

The notion of Erdős number has floated around the mathematical research community for more than thirty years, as a way to quantify the common knowledge that mathematical and scientific research has become a very collaborative process in the twentieth century, not an activity engaged in solely by isolated individuals. In this paper we explore some (fairly short) collaboration paths that one can follow from Paul Erdős to researchers inside and outside of mathematics. In particular, we find that all the Fields Medalists up through 1998 have Erdős numbers less than 6, and that over 60 Nobel Prize winners in physics, chemistry, economics, and medicine have Erdős numbers less than 9.

Key words and phrases: *collaboration, Erdős number, Fields Medal, Nobel Prize, research.*

AMS Subject Classification (1991): *primary*: 01A80; *secondary*: 01A60, 01-08, 05C38.

1 An outstanding component of the collaboration graph

The *collaboration graph* C has as vertices all researchers (dead or alive) from all academic disciplines, with an edge joining vertices u and v if u and v have jointly published a research paper or book (with possibly more co-authors). As is the case for any simple (undirected) graph, in C we have a notion of *distance* between two vertices u and v: $d(u, v)$ is the number of edges in the shortest path between u and v, if such a path exits, ∞ otherwise (it is understood that $d(u, u) = 0$).

In this paper we are concerned with the collaboration subgraph centered at Paul Erdős (1913–1996). For a researcher v, the number $d(\text{Paul Erdős}, v)$ is called

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the *Erdős number* of v. That is, PAUL ERDŐS himself has Erdős number 0, and his co-authors have Erdős number 1. People not having Erdős number 0 or 1 who have published with someone with Erdős number 1 have Erdős number 2, and so on. Those who are not linked in this way to PAUL ERDŐS have Erdős number ∞. The collection of all individuals with a finite Erdős number constitutes the *Erdős component* of C.

The Erdős component of C is outstanding for its amazing size and for the manner in which it clusters around ERDŐS. Almost 500 people have Erdős number 1, and over 5000 have Erdős number 2. In the history of scholarly publishing in mathematics, no one has ever matched PAUL ERDŐS's number of collaborators or number of papers (about 1500, almost 70% of which were joint works). With his recent death the man who inspired so much mathematical thinking has—to use his terminology—left, but his legend lives on (see for example two recent biographies [21], [29]). And part of this legend, inside and outside mathematical circles, is perpetuated through the notion of Erdős numbers.

The first explicit mention in the literature of a person's Erdős number appears to be [12], where the reader is assured that PAUL ERDŐS himself was, for a long time, unaware of this entertainment. But the first systematic attempt to study the Erdős component of C was carried out by the second author in [17] and [19] and continues on the Erdős Number Project World Wide Web site [14]. This Web site contains a list of all people with Erdős number 1 (currently 485) and their other co-authors with Erdős number 2 (currently 5337). The files (available also via anonymous ftp, see [15]) are updated annually.

More recently, serious studies of what are now called "small-world networks", of which the collaboration graph is a prime example, have have appeared in *Nature* ([32], [5]) and been reported in *The New York Times* and *Science News*.

It has been surmised that most scientists *must* have a finite Erdős number, but the evidence offered in support has not been really abundant. In [6] the first author contributed new information, and the present paper pursues the matter much further. By skimming through several bibliographic sources we have found that many important people in academic areas—other than mathematics proper—as diverse as physics, chemistry, crystallography, economics, finance, biology, medicine, biophysics, genetics, meteorology, astronomy, geology, aeronautical engineering, electrical engineering, computer science, linguistics, psychology, and philosophy do indeed have finite Erdős numbers. We report on some of these intriguing connections in Section 2. Of course, it cannot be immediately inferred that all people whose areas of expertise reside in the mentioned disciplines, or related ones, must have finite Erdős numbers. But the names first resulting from this kind of browsing are among the most prominent and productive in their respective fields (including more than 60 Nobel Prize winners), and most have had many collaborators over the years. Thus one is led to the conclusion that the majority of researchers in those fields, except for those working in total isolation, probably have finite Erdős numbers.

When referring to all academic or scientific fields, it is really difficult to subject the last statement to a comprehensive testing; thus it should be regarded as a bold—though credible—guess. If we restrict ourselves to authors publishing *mathematical*

research, then the following conjecture

$$(\mathcal{E}) \qquad \left\{ \begin{array}{l} \text{most active mathematical researchers of the twentieth} \\ \text{century have a finite (and rather small) Erdős number} \end{array} \right.$$

seems so plausible that it has been accepted as folklore. Yet the only supportive evidence lies in the following reasoning. ERDŐS had almost 500 co-authors who, in turn, have had over 5000 additional co-authors. The (jointly) published papers by those 5500 people belong to numerous and varied areas of research in the mathematical sciences; therefore $(\mathcal{E})$ *should* be true.

We intend to provide some more conclusive or "hard" evidence in support of $(\mathcal{E})$. To do so we first select a rather high-class sample of the mathematical research community: the winners of the most prestigious awards within this academic discipline, namely, the Fields Medal, the Nevanlinna Prize, the Wolf Prize in Mathematics, and the Steele Prize for Lifetime Achievement. By criss-crossing multiple bibliographic references we have determined that all recipients of these prizes have indeed an Erdős number ≤ 5. Complete tables of upper bounds on these Erdős numbers are presented below in Sections 3 and 4[1]. The respective collaboration paths linking all awardees to PAUL ERDŐS are displayed in full detail for the interested reader in the Web site [14] and in the appendix of this paper. The individuals belonging to these exclusive lists are mathematicians of the highest rank, characterized by being very original, prolific, and influential; most of them have had many disciples, collaborators, and doctoral students. Their impact and influence is not limited to one institution or even to one country or particular epoch (PAUL ERDŐS himself was given the Wolf Prize in 1983–84). Furthermore, these distinctions are conferred with no exclusion of research area (except the Nevanlinna Prize, which is in computer science). The fact that all of these big names are in the Erdős component of C is strong evidence for $(\mathcal{E})$.

Next, we go two steps further: In Section 5 we trace the subject matter of the papers by some researchers known to have a small Erdős number, and we branch out into other academic disciplines. This will give us a more concrete idea of how far the Erdős connection really extends within the mathematical sciences and beyond. Lastly, in Section 6 we pose some open questions. Obviously this work is incomplete (for instance, we have not traced any recent Nobel laureates in physics), and it should not be hard to establish further links with important mathematicians and scientists.

For brevity, we say that a person is *Erdős-n* if his or her Erdős number is n. Thus ERDŐS's co-authors are Erdős-1 and their co-authors (who are not Erdős-1 or Erdős-0) are Erdős-2. The list [15], containing all Erdős-2 individuals and their respective Erdős-1 co-authors, is referred to as the *Erdős-2 list.*

[1]Upper bounds for the Erdős numbers for all Fields Medalists up to 1994 had already been presented by the authors in [6], [14], and [18], but many bounds have been lowered for the present paper.

2 Interesting connections

Without intending to be 100% exhaustive, we have examined several bibliographic databases and historical accounts (*e.g.*, [3], [4], [20], [23], [24], [25], [26], [27], [33], [35], and Internet sites too numerous to list) and discovered that some very conspicuous thinkers and researchers from manifold academic disciplines are in the Erdős component of C.

The following examples evince the amazing diversity of the scientific collaboration network directly linked to the name of PAUL ERDŐS, providing—in passing—an ample glimpse into the practice of academic collaboration, an aspect of scientific research that has become essential in the twentieth century and has apparently not been systematically addressed in the literature. We shall use double brackets [[ ]] for the 120+ bibliographic references corresponding to the cited collaborative works, which appear in a separate list at the end of this paper (after the normal list of references). The list is also available on the Erdős Number Project World Wide Web site ([8]).

- ALBERT EINSTEIN has Erdős number 2 due to the two joint papers with his Princeton assistant (in the years 1944–48) ERNST G. STRAUS, with whom ERDŐS wrote 20 papers (the first in 1953)[2]. EINSTEIN wrote jointly with about 25 collaborators (see [28]), among them Nobel laureates in physics WOLFGANG PAULI and OTTO STERN.

  At age 20 PAULI had surprised the physics establishment with his brilliant 200-page encyclopedia article on the theory of relativity, a piece of which EINSTEIN wrote a laudatory review. Not surprisingly, their joint paper [[55]] of 1943 (their only joint paper, written during PAULI's stay in Princeton) deals with technical aspects of the general theory of relativity. PAULI received the 1945 Nobel Prize for his 1925 discovery of the so-called *Pauli exclusion principle.* With STERN, EINSTEIN also wrote only one joint article [[57]], when they were both in Prague. STERN was awarded the 1943 Nobel Prize for his discovery of the magnetic moment of the proton.

  EINSTEIN also published with Russian BORIS PODOLSKY and Austrian PAUL EHRENFEST [[52]], one of his closest friends. The well-known EINSTEIN-PODOLSKY-ROSEN paradox, conceived as a thought experiment against the quantum-mechanical conception, originated in their 1935 joint paper [[56]]. Coauthors of PODOLSKY include at least two Nobel laureates: the great British theoretical physicist PAUL DIRAC [[48]] and American chemist LINUS PAULING [[111]]; hence, they are both at most Erdős-4. PAULING received the 1954 chemistry prize for his research on chemical bonding; his classic treatise *The Nature of the Chemical Bond* (1939) is widely considered by chemists the most important work on theoretical chemistry in the twentieth century. As a result

[2] A complete bibliography of ERDŐS's works through about 1996 has been prepared by the second author [16], with annual updates posted on the Erdős Number Project Web site [14].

of his campaign for an international control of nuclear weapons, PAULING was awarded the 1962 Nobel Peace Prize.

A not well known fact is that EINSTEIN wrote jointly with a physician, HANS MÜHSAM, a friend during his Berlin years. In 1923 they published in a medical journal a research paper [[54]] on the experimental determination of the permeability of filters.

There are, moreover, two very curious non-technical joint publications by EINSTEIN. The first is a report about an international bureau of meteorology, written in 1927 with MARIE CURIE and HENDRIK A. LORENTZ and published in the journal *Science* [[39]]. The second is a booklet entitled *Why War?*, which he wrote in 1933 with SIGMUND FREUD [[53]] (see also [28]). It appeared in German, French, and English and was published by the International Institute of Intellectual Cooperation of the League of Nations.

- HENDRIK A. KRAMERS, a Dutch physicist, was one of PAULI's collaborators [[88]] and also wrote with Danish Nobel laureate NIELS BOHR [[22]], one of the pillars of twentieth century scientific thought. Therefore, BOHR is at most Erdős-5. In 1923 BOHR published jointly with DIRK COSTER [[21]], another Dutch physicist, who in the same year co-authored a research paper with GEORGE C. DE HEVESY[3], a Hungarian chemist who went on to receive the Nobel Prize in chemistry in 1943 for his use of isotopes as tracers, a technique which enhanced the understanding of the chemistry of life processes. A distinguished collaborator of BOHR was JOHN A. WHEELER. In 1939 they wrote the seminal work *The mechanism of nuclear fission* [[23]], which made WHEELER the first American involved in the theoretical development of nuclear weapons; in that memoir uranium-235 was singled out for use in a possible atomic bomb.

  Another of KRAMERS' co-authors is LEONARD S. ORNSTEIN [[108]], in turn linked with fellow Dutchman FRITS ZERNIKE [[109]], winner of the 1953 Nobel Prize in physics (for his invention of the phase-contrast microscope). Thus ZERNIKE is at most Erdős-6.

- J. ROBERT OPPENHEIMER is among EHRENFEST's co-authors [[50]], which collaboration makes him at most Erdős-4. OPPENHEIMER is remembered as director of the Los Alamos laboratory during development of the atomic bomb (1943–45) and as director of the Institute for Advanced Study at Princeton (1947–66). ROBERT SERBER, OPPENHEIMER's former student and close collaborator [[106]], is linked to at least two Nobel laureates, American nuclear physicists ERNEST O. LAWRENCE and EDWIN M. MCMILLAN [[28]]. SERBER, LAWRENCE, and MCMILLAN were indispensable members of the Los Alamos scientific team. LAWRENCE was the winner of the 1939 Nobel Prize in physics for his invention of the cyclotron; chemical element 103, lawrencium, is named after him. MCMILLAN shared the 1951 chemistry Nobel Prize for his discovery of element 93, neptunium, the first element heavier than uranium. The above

[3] In that paper [[37]] they reported the discovery of a new chemical element, hafnium.

links show that both LAWRENCE and MCMILLAN have an Erdős number of at most 6.

The astrophysicist HARTLAND SNYDER is a co-author of OPPENHEIMER; in their 1939 joint paper [[107]] the first general-relativistic treatment of black holes was presented.

- MAX BORN, a Nobel laureate in physics (1954), is Erdős-3 through his collaboration with NORBERT WIENER, the creator of cybernetics, whose Erdős number is 2. Their only joint paper [[27]] was written during BORN's visit to MIT in 1925. Among BORN's co-authors we find fellow Germans WERNER HEISENBERG, PASCUAL JORDAN [[24]]—the three of them are the founders of modern quantum mechanics—and MAX VON LAUE [[26]] (the last collaboration might be considered a bit of a stretch, a jointly written technical obituary for physicist MAX ABRAHAM, who died in 1922). For his preeminent role in the foundation of quantum mechanics HEISENBERG was the sole winner of the 1932 Nobel Prize in physics. LAUE had been awarded the 1914 Nobel Prize for his research on the diffraction of X-rays in crystals, which is the origin of solid-state physics (his investigations also demonstrated that X-rays are electromagnetic radiations similar to light).

  Furthermore, HEISENBERG published with the director of his doctoral dissertation, German ARNOLD SOMMERFELD [[81]] (also PAULI's thesis advisor in Munich), who is most remembered for his successful modifications of BOHR's atomic model. One of SOMMERFELD's many co-authors is PETER J. DEBYE (also spelled DEBIJE) [[45]], a Dutch scientist and Nobel laureate in chemistry (1936). DEBYE's Erdős number is at least one lower than implied by this collaboration, however, since he wrote a joint paper with PAULING [[46]].

  We should mention another very famous co-author of BORN, THEODORE VON KÁRMÁN [[25]], the Hungarian-born American research engineer who was one of the founders of the aeronautical and astronautical sciences.

- JOHN VON NEUMANN and ERDŐS never wrote jointly although they were both Hungarian by birth and just 10 years apart in age. Actually, VON NEUMANN did not write with any of ERDŐS's almost 500 co-authors; his Erdős number stands at 3 through his varied collaborations with individuals in the Erdős-2 list (*e.g.*, SALOMON BOCHNER, PAUL HALMOS, HERMAN H. GOLDSTINE). Moreover, VON NEUMANN had very illustrious co-authors, notably DAVID HILBERT, OSWALD VEBLEN, GARRETT BIRKHOFF, PASCUAL JORDAN and Nobel laureate physicists EUGENE WIGNER and SUBRAHMANYAM CHANDRASEKHAR (see [31]).

  With HILBERT, VON NEUMANN wrote about the mathematical foundations of quantum mechanics [[82]] shortly after HEISENBERG had proposed his quantum scheme, known as matrix mechanics. HILBERT, VON NEUMANN and HEISENBERG were at Göttingen at that time. WIGNER was also a Hungarian and a friend of VON NEUMANN since childhood; most of their joint papers deal with

quantum mechanics as well. Through WIGNER we find a path to one of the biggest names in quantum physics, Austrian ERWIN SCHRÖDINGER, who shared the 1933 Nobel Prize with DIRAC for their introduction of wave equations in quantum mechanics. This is the route: WIGNER with R. F. O'CONNELL [[83]] with JOHN TREVOR LEWIS [[65]] with JAMES MCCONNELL [[94]] with SCHRÖDINGER [[100]]. Thus, SCHRÖDINGER is at most Erdős-8.

Another of VON NEUMANN's collaborators was the Austrian economist OSKAR MORGENSTERN, with whom he wrote in 1944 the very influential work *Theory of Games and Economic Behaviour* [[126]]. This book, which went into several editions, stimulated a worldwide development of the mathematical aspects of game theory and its applications to fields such as economics, politics, military science, operations research, business, law, sports, and biology (see [30]). The applicability and relevance of game theory are such that at least three Nobel Prizes in economics have been awarded to game-theorists; all of them are in the Erdős component of C (as we shall demonstrate below), even though they are not directly linked with either VON NEUMANN or MORGENSTERN.

MORGENSTERN, in turn, wrote jointly with JOHN G. KEMENY [[86]], the creator (along with THOMAS E. KURTZ) in the mid-1960s of BASIC, a very popular general-purpose programming language.

In the 1950s VON NEUMANN participated, at the Institute for Advanced Study at Princeton, in the first efforts to apply digital computers to the problem of weather prediction. He worked closely with the leader in that field, the American meteorologist JULE G. CHARNEY, from which collaboration the joint article [[34]] was produced.

- GEORGE UHLENBECK, the noted Dutch-American physicist, has Erdős number 2. He is best known for having postulated, along with SAMUEL GOUDSMIT, the concept of electron spin, which led to major changes in atomic theory and quantum mechanics. Their famous joint paper [[125]] was published in 1925 when they were graduate students in physics at the University of Leiden in the Netherlands (both of them were pupils of EHRENFEST).

  Among UHLENBECK's co-authors we encounter at least two Nobel physicists: American WILLIS E. LAMB (1955 prize) [[105]], whose experimental work spurred refinements in the quantum theories of electromagnetic phenomena, and Italian-born ENRICO FERMI (1938 prize) [[61]], one of the chief architects of the nuclear age. FERMI had legions of co-authors and collaborators in Europe and the United States; one of them was his former student in Rome EMILIO SEGRÈ [[59]], who would become a Nobel laureate himself. SEGRÉ and his colleague at the University of California, Berkeley, American OWEN CHAMBERLAIN, discovered the antiproton in 1955 and for that feat were awarded the Nobel Prize in physics in 1959 (they also published jointly [[31]]). The above links show that the Erdős numbers of FERMI, SEGRÈ and CHAMBERLAIN are at most 3, 4, and 5, respectively. (Actually, if we are willing to use technical reports in establishing collaboration links, then we can lower FERMI's Erdős

Nobel Prize in physics	Year	Erdős number
Max von Laue	1914	4
Albert Einstein	1921	2
Niels Bohr	1922	5
Louis de Broglie	1929	5
Werner Heisenberg	1932	4
Paul A. Dirac	1933	4
Erwin Schrödinger	1933	8
Enrico Fermi	1938	3
Ernest O. Lawrence	1939	6
Otto Stern	1943	3
Isidor I. Rabi	1944	4
Wolfgang Pauli	1945	3
Frits Zernike	1953	6
Max Born	1954	3
Willis E. Lamb	1955	3
John Bardeen	1956	5
Walter H. Brattain	1956	6
William B. Shockley	1956	6
Chen Ning Yang	1957	4
Tsung-dao Lee	1957	5

Nobel Prize in physics	Year	Erdős number
Emilio Segrè	1959	4
Owen Chamberlain	1959	5
Robert Hofstadter	1961	5
Eugene Wigner	1963	4
Richard P. Feynman	1965	4
Julian S. Schwinger	1965	4
Hans A. Bethe	1967	4
Luis W. Alvarez	1968	6
Murray Gell-Mann	1969	3
John Bardeen	1972	5
Leon N. Cooper	1972	6
John R. Schrieffer	1972	5
Aage Bohr	1975	5
Ben Mottelson	1975	5
Leo J. Rainwater	1975	7
Steven Weinberg	1979	4
Sheldon Lee Glashow	1979	2
Abdus Salam	1979	3
S. Chandrasekhar	1983	4
Norman F. Ramsey	1989	3

Nobel Prize in economics	Year	Erdős number
Paul A. Samuelson	1970	6
Kenneth J. Arrow	1972	3
Tjalling C. Koopmans	1975	4
Gerard Debreu	1983	3
Franco Modigliani	1985	5
Robert M. Solow	1987	6
Harry M. Markowitz	1990	2
Merton H. Miller	1990	4
John C. Harsanyi	1994	8
John F. Nash	1994	4
Reinhard Selten	1994	7
Robert C. Merton	1997	7

Nobel Prize in chemistry	Year	Erdős number
Peter J. Debye	1936	5
George De Hevesy	1943	7
Otto Diels	1950	7
Kurt Alder	1950	6
Edwin M. McMillan	1951	6
Glenn T. Seaborg	1951	5
Linus Pauling*	1954	4
Walter Gilbert	1980	4
Jerome Karle	1985	4
Herbert A. Hauptman	1985	3

*Also received the 1962 Nobel Peace Prize

Nobel Prize in Physiology/Medicine	Year	Erdős number
Francis H. C. Crick	1962	7
James D. Watson	1962	8

Table 1. Upper bounds on Erdős numbers of some Nobel Prize winners.

	Main research field	Erdős number
Walter Alvarez	Geology	7
Rudolf Carnap	Philosophy	4
Jule G. Charney	Meteorology	4
Noam Chomsky	Linguistics	4
Freeman J. Dyson	Quantum physics	2
George Gamow	Nuclear physics and cosmology	5
Stephen Hawking	Relativity and cosmology	7
Pascual Jordan	Quantum physics	4
Theodore von Kármán	Aeronautical engineering	4
John Maynard Smith	Biology	4
Oskar Morgenstern	Economics	4
J. Robert Oppenheimer	Nuclear physics	4
Roger Penrose	Relativity and cosmology	8
Jean Piaget	Psychology	3
Karl Popper	Philosophy	5
Edwin E. Salpeter	Astrophysics	5
Claude E. Shannon	Electrical engineering	3
Arnold Sommerfeld	Atomic physics	5
Edward Teller	Nuclear physics	4
George Uhlenbeck	Atomic physics	2
John A. Wheeler	Nuclear physics	5

Table 2. Upper bounds on Erdős numbers of some distinguished scholars.

number to 2, since he published a Los Alamos technical report with Stanislaw Ulam.) Another co-author of Segrè is the American nuclear chemist Glenn T. Seaborg [[120]], who received one half of the chemistry Nobel Prize in 1951 for his research on transuranium elements (the other co-winner was McMillan, whom we mentioned in connection with Oppenheimer).

Edward Teller, the Hungarian-born American nuclear physicist who led the development of the world's first thermonuclear weapon, is another co-author of Fermi [[60]]. One of Teller's doctoral students in Chicago was the Chinese-born physicist Chen Ning Yang, who later became assistant to Fermi, publishing joint research work with him [[62]]. Yang and fellow Chinese Tsung-dao Lee received the 1957 Nobel Prize in physics for their work [[91]], [[92]] in discovering violations of the principle of parity conservation, a major discovery in particle physics theory. The just cited collaborations make Yang at most Erdős-4 and Lee at most Erdős-5.

- Freeman J. Dyson, the British-American physicist known by the general public for his writings on extraterrestrial civilizations and his advocacy of space exploration, is a conspicuous member of the Erdős-2 list. He is linked, by way of Richard H. Dalitz [[40]], with the German-American physicist Hans A.

BETHE [[42]], one of the main figures in the development of twentieth century quantum and atomic physics. BETHE was head of the Theoretical Physics Division of the Manhattan Project and was honored with the 1967 Nobel Prize for his explanation of the energy production in the Sun and other stars. The prominent Austrian astrophysicist EDWIN E. SALPETER is included in the large group of BETHE's co-authors and collaborators [[118]].

BETHE is one of the protagonists of a most peculiar joint publication [[2]], one which compels attention for the unique combination of names in its by-line: ALPHER, BETHE, GAMOW. The third author is GEORGE GAMOW, the Ukranian-born nuclear physicist and cosmologist who also made contributions to modern genetic theory; the first author is RALPH ALPHER, one of his students. The paper itself (*The origin of chemical elements*) is actually very important; in it the authors advanced the idea that the chemical elements were synthesized by thermonuclear reactions which took place in a primeval explosion. It was GAMOW who coined the expression "big-bang"; the modern version of this theory, as modified by him and his collaborators, is the leading postulate of cosmology.

From DALITZ we find a path to another Nobel physicist, American ROBERT HOFSTADTER, a co-recipient of the 1961 prize for his investigations of protons and neutrons. This path makes him at most Erdős-5: DALITZ with D. G. RAVENHALL [[41]] with R. HOFSTADTER [[77]].

- SHELDON LEE GLASHOW, an American theoretical physicist and Nobel laureate (1979), has Erdős number 2 for his collaboration [[71]] with the Erdős-1 combinatorialist DANIEL KLEITMAN, his brother-in-law. GLASHOW shares with EINSTEIN the distinction of being, up until now, the only Nobel-winning physicists with Erdős number ≤ 2.

  A co-author of GLASHOW is another Nobel winning physicist (1969): fellow American MURRAY GELL-MANN [[70]], who introduced the concept and the word *quark* for a basic subatomic particle. GELL-MANN also collaborated with the brilliant American physicist and Nobel Prize winner RICHARD FEYNMAN [[63]]. FEYNMAN participated in the Manhattan Project (he was only 25 when he was recruited) and wrote jointly with BETHE [[9]]. The two of them devised the formula for predicting the energy yield of a nuclear explosive.

  FEYNMAN, who had been a pupil of WHEELER at Princeton and published with him [[127]], became a salient figure of postwar physics, receiving the Nobel Prize in 1965 for his very original and far-reaching quantum electrodynamics theory. Sharing the prize with FEYNMAN was JULIAN S. SCHWINGER, who independently formulated a theory of quantum electrodynamics, unaware that FEYNMAN in the United States and SIN-ITIRO TOMONAGA in Japan were working on the same problem. The equivalent theories reconcile quantum mechanics with the special theory of relativity. We can link SCHWINGER to ERDŐS via this path: SCHWINGER with NORMAN F. RAMSEY [[112]] with W. H. FURRY [[68]], the last named being Erdős-2. Thus, SCHWINGER is at most Erdős-4.

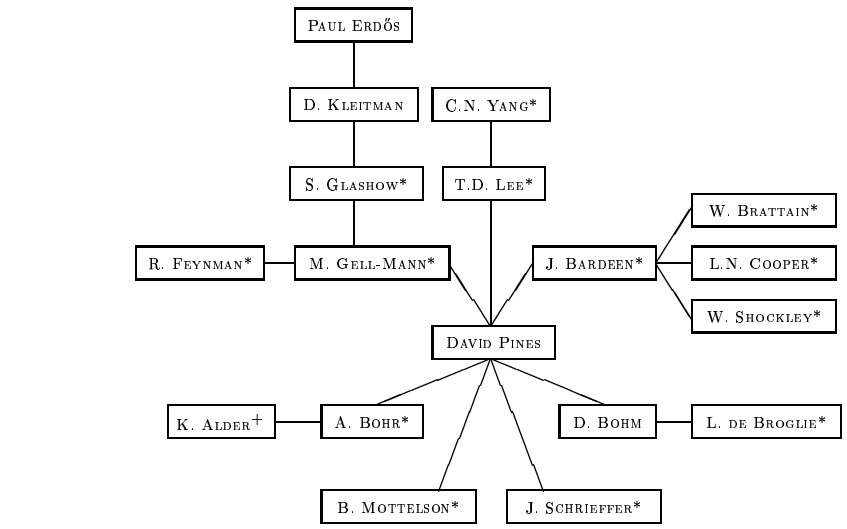

Figure 1. Clustering of Nobel laureates within distance 2 of David Pines.

Curiously enough, Feynman and Schwinger were born in the same year, 1918, in the same city, New York, received the Nobel Prize the same year for the same achievement, and—according to the shortest paths we have been able to find—they also have the same Erdős number, namely 4.

The two co-authors of Schwinger in the above-cited paper [[112]], Americans Isidor I. Rabi and Ramsey, are themselves Nobel laureates. Rabi was given the 1944 Nobel Prize in physics for his 1937 invention of the magnetic resonance method, a technique that made possible applications such as the laser and the nuclear imaging used in diagnostic medicine. Ramsey received one half of the 1989 physics award for his development of a technique called the separated oscillatory fields method, which provides the basis for the cesium atomic clock, the device setting present time standards. Hence, Ramsey is Erdős-3 and Rabi is at most Erdős-4.

- David Pines, an American physicist who publishes on condensed matter theory and theoretical astrophysics, is a key figure in the collaboration graph. As a co-author of Gell-Mann [[58]], he is at most Erdős-4. Pines has co-authored research papers with six Nobel Prize winners and is at distance 2 of eight more. Thus, in the collaboration graph Pines is within distance 2 of 14 different Nobelists (see Figure 1). None of the scientists we mention in the present article (including Erdős himself) clusters so closely around so many Nobel winners.

  Two of the co-authors of Pines are John Bardeen [[14]] and John Robert Schrieffer [[110]], who, along with fellow American Leon N. Cooper, received the 1972 Nobel Prize in physics for their joint theory [[12]], [[13]], known as BCS theory for their surname initials, which was the first successful microscopic theory of superconductivity. When he made his principal contribution to

the BCS theory, SCHRIEFFER was a 26-year-old graduate student at the University of Illinois, where BARDEEN was a professor in the physics and electrical engineering departments.

It should be recalled that BARDEEN had been a co-winner of another Nobel Prize in physics, that of 1956, which he shared with WALTER H. BRATTAIN and WILLIAM B. SHOCKLEY for their research on semiconductors and their joint invention of the transistor. Therefore, both BRATTAIN and SHOCKLEY become at most Erdős-6 because of their joint papers with BARDEEN [[11]], [[15]]. For his significant contributions to science, JOHN BARDEEN was named by *Life Magazine* as one of the 100 most influential people of the twentieth century.

Two additional co-authors of PINES are Nobel winners Danish AAGE BOHR (the son of NIELS BOHR) and Danish-American BEN MOTTELSON [[19]]. They shared the 1975 physics award for the work they made in the early 1950s in determining the asymmetrical shapes of certain atomic nuclei. Their experiments had been inspired by the theories of the American physicist LEO JAMES RAINWATER, who was also a co-recipient of the 1975 Nobel Prize. RAINWATER's Erdős number is finite too, at most 7, via TSUNG-DAO LEE, who, as we saw, is Erdős-5: LEE with C. S. WU [[93]] with RAINWATER [[113]]. Incidentally, PINES has also collaborated with LEE [[90]] (see Figure 1).

A co-author of AAGE BOHR is KURT ALDER [[1]], a German chemist, former student and assistant of OTTO DIELS, along with whom he received the 1950 Nobel Prize in chemistry for their joint method of preparing cyclic organic compounds. That technique, based on the so-called *Diels-Alder reaction,* proved especially useful in the production of synthetic plastics. The first of numerous joint papers they published on this subject appeared in 1928 [[47]]. The Erdős numbers of ALDER and DIELS are, therefore, at most 6 and 7, respectively.

Apart from father NIELS and son AAGE, there is another member of the BOHR family in the Erdős component of C, namely NIELS's younger brother HARALD, who was a prominent mathematician. He was the one who devised the theory of almost periodic functions; the *Bohr-Landau theorem,* concerning the zeta function, is named after him. HARALD's Erdős number is 3 due to his joint paper [[20]] with BORGE JESSEN, whose Erdős number is 2. As a curiosity, we can determine how far from each other are the BOHRs in the collaboration graph C. Using PAUL ERDŐS as a pivotal junction, we can see that the brothers NIELS and HARALD are at distance ≤ 8, whereas the distance between the two Nobel laureates, NIELS and his son AAGE, is ≤ 10; most likely these bounds can be improved.

- ABDUS SALAM and STEVEN WEINBERG shared with the aforementioned GLASHOW the 1979 Nobel Prize in physics for their theoretical work linking the electromagnetic interaction and the so-called weak interaction of elementary particles. SALAM became the first Pakistani to win a Nobel Prize (in any category); he is Erdős-3 due to his joint paper [[117]] with J. C. WARD who

is Erdős-2. WEINBERG is at most Erdős-4 for his many collaborations with SALAM (*e.g.*, [[73]]).

Among SALAM's many co-authors we are able to find another Nobel winner, American molecular biologist WALTER GILBERT[4] [[116]], who shared (with PAUL BERG and FREDERICK SANGER) the 1980 chemistry award for their development of chemical and biological analyses of DNA structure.

- EDWARD WITTEN, the outstanding American theoretical physicist and Fields Medalist in 1990, is at most Erdős-5, as will be shown later. Among his co-authors we find the American physicist LUIS W. ALVAREZ [[4]], who received the Nobel Prize in 1968 for his work on subatomic particles. In 1980, ALVAREZ, his son WALTER (a geologist), and their research team put forward a controversial and widely publicized theory [[3]] explaining—by means of geological considerations—that the extinction of the dinosaurs was the result of an asteroid impact on Earth.

  Another co-author of WITTEN is GARY HOROWITZ [[30]], who in turn has collaborated with STEPHEN HAWKING [[69]], the English theoretical physicist who has become a modern scientific icon and whose 1988 book *A Brief History of Time* was an international best-seller (and later an award-winning movie, as well). HAWKING has also published with fellow British mathematician and physicist SIR ROGER PENROSE [[80]], known for his extensive work on aperiodic tilings and the features of black holes.

- JEAN PIERRE VIGIER, a distinguished French physicist, can be linked to some important scientists; his Erdős number is at most 4 as the following collaborations show: VIGIER with CONSTANTIN PIRON [[64]] with STANLEY P. GUDDER [[76]], GUDDER being an Erdős-2 researcher.

  Among VIGIER's co-authors stands out fellow French physicist Prince LOUIS DE BROGLIE [[29]], who in the mid 1920s developed (in his doctoral dissertation) a revolutionary theory of electron waves, enthusiastically defended by EINSTEIN. Experimental evidence of his theory came a few years afterwards, and DE BROGLIE was awarded the Nobel Prize in 1929. Also a co-author of VIGIER in the work we have just cited is the American physicist and philosopher DAVID BOHM, the last doctoral student of OPPENHEIMER at Berkeley and the originator of the causal interpretation of quantum theory. His 1951 book *Quantum Theory* remains a classic in the field to this day. BOHM is also acknowledged for his joint research with DAVID PINES (see Figure 1); their joint papers [[18]], published under the title *A collective description of electron interactions,* marked the beginning of all current research in plasma-state physics.

  SIR KARL R. POPPER, the eminent Austrian-born British philosopher, is another co-author of VIGIER [[95]]. Popper's most recognized contribution to the

[4] GILBERT's Ph.D. degree is actually in mathematics (from Cambridge University).

philosophy of science is his rejection of the inductive method in the empirical sciences. Ironically, his views on the subject appeared in his 1934 book *The Logic of Scientific Discovery*, published by the Vienna Circle, a European gathering of positivist philosophers who firmly championed inductive empiricism.

- Claude E. Shannon, an American electrical engineer, became famous for his elegant and general mathematical model of "communication", known today as *information theory*. His initial ideas appeared in 1948 in the paper *A mathematical theory of communication* [[121]], which was followed in 1949 by a book, with the same title, co-authored by Warren Weaver [[123]]. The Shannon-Weaver approach developed rapidly, becoming a vital part of cybernetics and finding applications in library science, electrical engineering, optics, thermodynamics, chemical analysis, economics, and even psychology and linguistics.

  Shannon's Erdős number is 3 because of his collaboration [[122]] with Elwyn R. Berlekamp, whose Erdős number is 2.

- Francis H. C. Crick, a British biophysicist, and James D. Watson, an American geneticist and biophysicist, made an epochal discovery when they determined the molecular structure of deoxyribonucleic acid (DNA)—as a double-helix polymer—for which accomplishment they were awarded the 1962 Nobel Prize for Physiology/Medicine. Their findings had been published in two celebrated papers in the British journal *Nature* in April–May 1953.

  In 1957 Crick published a short joint paper [[38]] on information theory, thereby entering the Erdős component of C. Indeed, from that collaboration we are able to demonstrate that Crick's Erdős number is at most 7 due to the following chain of joint works: Crick with J. S. Griffith [[38]] with I. W. Roxburgh [[114]] with P. G. Saffman [[115]] with H. B. Keller [[36]] with K. O. Friedrichs [[66]]. The last named is in the Erdős-2 list.

  Hence, James Watson would be Erdős-8, and from these connections we can conclude that many other active researchers in genetics, biophysics, biochemistry, and related fields have finite Erdős numbers as well.

- Herbert A. Hauptman, a mathematician, and Jerome Karle, a chemist and crystallographer (both American), were awarded the Nobel Prize in chemistry in 1985 for their development of mathematical methods for deducing the molecular structure of biological molecules from the patterns formed when X-rays are diffracted by their crystals. This has been one of the most profitable interdisciplinary enterprises in recent years.

  Hauptman's Erdős number is 3 for his joint publication [[75]] with Fred Gross, who appears in the Erdős-2 list. This makes Karle an Erdős-4 researcher by way of his numerous joint articles with Hauptman (*e.g.*, [[79]]).

- John Maynard Smith, a British biologist, initiated a whole new area of research by his unusual applications of game theory to animal behavior and evolution, with works like *The theory of games and the evolution of animal conflict*

[[98]] and *Evolution and the Theory of Games* [[97]]. It turns out that MAYNARD SMITH has a small Erdős number, at most 4, through JOSEF HOFBAUER [[99]] and HAL L. SMITH [[84]], who is Erdős-2.

- HARRY M. MARKOWITZ, the American finance expert and Nobel laureate in economics, is in the Erdős-2 list (because of his collaboration with ALAN J. HOFFMAN [[85]]) and is the only person having that double distinction to date. He shared the 1990 prize with MERTON H. MILLER and WILLIAM F. SHARPE for their study of financial markets and investment decision making.

  MILLER is at most Erdős-4, through ABRAHAM CHARNES [[32]] and FRED GLOVER [[33]], the latter being Erdős-2. MILLER is also linked [[102]], [[103]] with another Nobel laureate, the Italian economist FRANCO MODIGLIANI, who received the 1985 award for his mathematical analysis of household savings and the dynamics of financial markets.

- HERBERT SCARF, an American economist with Erdős number 2, has published articles on economic analysis with many other renowned economists, such as KENNETH J. ARROW [[6]] and GERARD DEBREU [[44]], both winners of the Nobel Prize in economics (1972 and 1983, respectively). ARROW (American) received the prize for his theories of economic equilibrium, whereas DEBREU (French) was honored for his mathematical treatment of supply and demand theory. A co-author of the latter is the Dutch economist TJALLING C. KOOPMANS [[43]], also a Nobel laureate (1975).

  Additionally, SCARF has published with LLOYD S. SHAPLEY [[124]], one of the major contributors to the development of game theory and a co-author of the American mathematician JOHN F. NASH [[104]], a co-recipient of the 1994 Nobel Prize in economics[5]. NASH shared his prize with the Hungarian-born economist JOHN C. HARSANYI and the German mathematician REINHARD SELTEN, for their beneficial use of game theory in economics (more precisely, for "their pioneering analysis of equilibria in the theory of non-cooperative games"). As can be expected, HARSANYI and SELTEN also have a finite Erdős number. This is the path: KOOPMANS with BECKMAN [[87]] with MARSCHAK [[16]] with SELTEN [[96]] with HARSANYI [[78]]. Therefore, SELTEN is at most Erdős-7 and HARSANYI at most Erdős-8.

  From ARROW we can find a path leading to two more Nobel laureates in economics, the American economists PAUL A. SAMUELSON and ROBERT M. SOLOW. This is the path: ARROW with EDWARD W. BARANKIN [[5]] with ROBERT DORFMAN [[10]] with SAMUELSON and SOLOW [[49]]. Both SAMUELSON and SOLOW made pioneering use of mathematical techniques in economics; SAMUELSON (1970 Nobel Prize) in the theory of general equilibrium and SOLOW (1987 Nobel Prize) in economic growth models.

[5] After this paper was written, NASH also received the 1999 Steele Prize for a Seminal Contribution to Research; see Section 4.

An additional and more recent Nobel connection should be cited: Through his collaboration with SAMUELSON [[101]], the American economist ROBERT C. MERTON, one the recipients of the 1997 Nobel Prize in economics, is at most Erdős-7. MERTON expanded the work of MYRON S. SCHOLES and FISHER BLACK, who had advanced in 1973 a pioneering formula for the valuation of stock options. SCHOLES shared the prize with MERTON, not so BLACK due to his untimely death in 1995[6].

These links show that SCARF is within distance 6 of at least nine Nobel Prize winners in economics, quite a remarkable fact.

- NOAM CHOMSKY, the American linguist and political activist, is one of the most influential figures of twentieth century linguistics. In the period 1956–59 he introduced and elaborated the notion of "phrase structure grammars"—encompassing both natural and artificial languages—which turned into a central concept in linguistic analysis and in the area of computer science known as formal language theory.

  The following path shows that CHOMSKY is at most Erdős-4: CHOMSKY with M. P. SCHUTZENBERGER [[35]] with S. EILENBERG [[51]], the latter being Erdős- 2.

- JEAN PIAGET, the Swiss psychologist who did landmark research in the acquisition of understanding in children, also wrote extensively on zoology, philosophy, and epistemology. Actually, he owns some kind of record, considering that by the age of 15 he had published several papers on sparrows and mollusks, including one at the age of 10 (see [4]). Because of his varied interests we find works such as *Mathematical Epistemology and Psychology* [[17]] written jointly with the Dutch logician EVERT W. BETH. Since BETH's Erdős number is 2, PIAGET's Erdős number is 3.

- RUDOLF CARNAP, the German-born philosopher and the most conspicuous member of the Vienna Circle, is at most Erdős-4, as shown by this path: CARNAP with YEHOSHUA BAR-HILLEL [[7]] with M. PERLES [[8]], the latter being in the Erdős-2 list.

  A student of CARNAP in Prague was WILLARD V. QUINE, an American logician and philosopher, known for undertaking a systematic constructivist analysis of philosophy. He is Erdős-3 due to his collaboration with J. C. C. MCKINSEY [[89]], an Erdős-2 individual. In fact, MCKINSEY was a renowned philosopher himself, who made important contributions from the late 1930s onwards. Another well-known co-author of QUINE is the American philosopher NELSON GOODMAN [[74]], whose work on the philosophy and ontology of art has been very influential.

[6] For the mathematics behind the 1997 Nobel Prize in economics the reader is referred to the very interesting surveys [9] and [10].

Fields Medal	Year	Country of origin	Erdős number
Lars Ahlfors	1936	Finland	5
Jesse Douglas	1936	USA	4
Laurent Schwartz	1950	France	5
Atle Selberg	1950	Norway	2
Kunihiko Kodaira	1954	Japan	2
Jean-Pierre Serre	1954	France	3
Klaus Roth	1958	Germany	2
Rene Thom	1958	France	4
Lars Hormander	1962	Sweden	3
John Milnor	1962	USA	3
Michael Atiyah	1966	Great Britain	4
Paul Cohen	1966	USA	5
Alexander Grothendieck	1966	Germany	5
Stephen Smale	1966	USA	5
Alan Baker	1970	Great Britain	2
Heisuke Hironaka	1970	Japan	4
Serge Novikov	1970	USSR	3
John G. Thompson	1970	USA	3
Enrico Bombieri	1974	Italy	2
David Mumford	1974	Great Britain	2
Pierre Deligne	1978	Belgium	3
Charles Fefferman	1978	USA	2
Gregori Margulis	1978	USSR	5
Daniel Quillen	1978	USA	3
Alain Connes	1982	France	5
William Thurston	1982	USA	4
Shing-Tung Yau	1982	China	2
Simon Donaldson	1986	Great Britain	5
Gerd Faltings	1986	Germany	4
Michael Freedman	1986	USA	4
Valdimir Drinfeld	1990	USSR	5
Vaughan Jones	1990	New Zealand	4
Shigemufi Mori	1990	Japan	3
Edward Witten	1990	USA	3
Pierre-Louis Lions	1994	France	4
Jean Christophe Yoccoz	1994	France	5
Jean Bourgain	1994	Belgium	2
Efim Zelmanov	1994	Russia	4
Richard Borcherds	1998	South Africa/Great Britain	2
William T. Gowers	1998	Great Britain	4
Maxim L. Kontsevich	1998	Russia	4
Curtis McMullen	1998	USA	3

Table 3. Upper bounds on Erdős numbers of the Fields Medalists.

3 Erdős numbers of the Fields Medalists

The *Fields Medal* was established by JOHN CHARLES FIELDS (1863–1932), a Canadian mathematician, to exalt both outstanding achievements and research potential in the mathematical sciences. It has always been granted to mathematicians not older than 40 although the age limit was neither demanded nor suggested by FIELDS himself (see [22]). A minimum of two and a maximum of four medals are awarded on the occasion of the quadrennial International Congress of Mathematicians (the most recent of which was held in Berlin in August, 1998).

The first two medals were conferred at the Oslo (Norway) Congress in 1936 to Finnish mathematician LARS AHLFORS and New Yorker JESSE DOUGLAS, but due to the Second World War no medals were awarded during the next 14 years. The academic distinction resumed in 1950 and to date there have been a total of 42 awardees from 14 different countries.

A careful reading of Table 3 shows that although ERDŐS never wrote jointly with any of the 42 Medalists (a fact perhaps worthy of further contemplation), 10 of them have Erdős number 2 and for none is the number greater than 5. The collaboration paths from which these numbers have been obtained are presented in the Web site [7] and in the appendix to this paper. It is possible that some paths can be lowered still more, but with these data the average Erdős number of the Fields Medalists is 3.52, with standard deviation 1.11 and median 4.

Nevanlinna Prize	Year	Country of origin	Erdős number
ROBERT TARJAN	1982	USA	2
LESLIE VALIANT	1986	Hungary/Great Britain	3
ALEXANDER RAZBOROV	1990	Russia	2
AVI WIGDERSON	1994	Israel	2
PETER SHOR	1998	USA	2

Table 4. Upper bounds on Erdős numbers of the Nevanlinna Prize winners.

4 Erdős numbers of the Steele, Nevanlinna, and Wolf Prize winners

The Fields Medal carries the prestige of a Nobel Prize, but there are many other important international awards for mathematicians. Perhaps the three most renowned, which are acquiring more and more prominence over the years, are the Rolf Nevanlinna Prize, the Wolf Prize in Mathematics, and the Leroy P. Steele Prizes. These prizes were established within a span of twelve years, beginning in 1970.

- Since 1982 the *Rolf Nevanlinna Prize* has been presented, along with the Fields Medal, at the International Congress of Mathematicians every four years ([22]). The funds for the award are granted by the University of Helsinki. This is a very

Wolf Prize in Mathematics	Year	Country of origin	Erdős number
Izrail M. Gelfand	1978	USSR (Russia)	4
Carl L. Siegel	1978	Germany	3
Jean Leray	1979	France	3
André Weil (SP)	1979	France	4
Henri Cartan	1980	France	3
Andrei N. Kolmogorov	1980	Russia	5
Lars Ahlfors (FM)	1981	Finland	5
Oscar Zariski (SP)	1981	USA	3
Hassler Whitney (SP)	1982	USA	2
Mark G. Krein	1982	Ukranian SSR	4
Shiing Shen Chern (SP)	1983–84	China	2
Paul Erdős	1983–84	Hungary	0
Kunihiko Kodaira (FM)	1984–85	Japan	2
Hans Lewy	1984–85	Germany	3
Samuel Eilenberg (SP)	1986	Poland	2
Atle Selberg (FM)	1986	Norway	2
Kiyoshi Ito	1987	Japan	3
Peter D. Lax (SP)	1987	Hungary/USA	3
Friedrich E. Hirzebruch	1988	Germany	3
Lars Hormander (FM)	1988	Sweden	3
Alberto Calderón	1989	Argentina	3
John Milnor (FM)	1989	USA	3
Ennio De Giorgi	1990	Italy	3
Ilya Piatetski-Shapiro	1990	Russia	5
Lennart A. Carleson	1992	Sweden	4
John G. Thompson (FM)	1992	USA	3
Mikhael Gromov	1993	Russia	3
Jacques Tits	1993	Belgium	4
Jurgen K. Moser	1994–95	Germany	3
Robert Langlands	1995–96	Canada	2
Andrew Wiles	1995–96	Great Britain	3
Joseph B. Keller	1997	USA	3
Yakov G. Sinai	1997	Russia	4

(FM): Fields Medalist

(SP): Steele Prize

Table 5. Upper bounds on Erdős numbers of the winners of the Wolf Prize in Mathematics.

Steele Prize (Lifetime Achievement)	Year	Country of origin	Erdős number
Salomon Bochner	1979	Poland	2
Antoni Zygmund	1979	Poland	2
André Weil	1980	France	4
Gerhard P. Hochschild	1980	USA	4
Oscar Zariski	1981	USA	3
Fritz John	1982	Germany	4
Shiing Shen Chern	1983	China	2
Joseph L. Doob	1984	USA	2
Hassler Whitney	1985	USA	2
Saunders Mac Lane	1986	USA	3
Samuel Eilenberg	1987	Poland	2
Deane Montgomery	1988	USA	3
Irving Kaplansky	1989	Canada	1
Raoul Bott	1990	Hungary	3
Armand Borel	1991	Switzerland	4
Peter D. Lax	1992	Hungary/USA	3
Eugene B. Dynkin	1993	Russia/USA	3
Louis Nirenberg	1994	Canada	3
John T. Tate	1995	USA	3
Goro Shimura	1996	Japan	2
Ralph S. Phillips	1997	USA	2
Nathan Jacobson	1998	USA	3

Table 6. Upper bounds on Erdős numbers of the Steele Prize (Lifetime Achievement Award) winners.

restricted distinction in the sense that it is given only to young mathematicians who deal with the mathematical aspects of information science, and only one prize is bestowed per congress.

- The *Wolf Prize* is awarded by the Wolf Foundation, based in Israel ([34]). Each year (since 1978) it gives prizes of $100,000 for outstanding achievements in agriculture, chemistry, medicine, and the arts, as well as mathematics and physics. The Wolf Prize in Mathematics was conferred (in 1984) to Paul Erdős himself, and in addition to his contributions to many fields, the award exalts him "for personally stimulating mathematicians the world over". Erdős is perhaps the only mathematician whose role as an inspiration to other colleagues has been formally recognized for a major award.

- The *Leroy P. Steele Prizes* are awarded by the American Mathematical Society. From 1970 to 1976 one or more prizes were awarded each year for outstanding published mathematical research; in 1977 the Council of the AMS modified the terms under which the prizes are awarded (see [1]). Since then, up to three prizes have been awarded each year in the following categories: (1) Lifetime

Achievement (for the cumulative influence of the total mathematical work of the recipient), (2) Mathematical Exposition (for a book or substantial survey or expository-research paper), and (3) Seminal Contribution to Research (for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field).

We have compiled tables of Erdős numbers for all recipients of the Nevanlinna Prize (Table 4), the Wolf Prize in Mathematics (Table 5), and the Steele Prize for Lifetime Achievement (Table 6) and have found that all these numbers are ≤ 5. Again, one may wonder why KAPLANSKY is the only[7] recipient of any of these prizes who collaborated with PAUL ERDŐS. (As before, the collaboration paths from which these numbers have been obtained are presented in the Web site [7] and this paper's appendix.)

5 How far does the Erdős connection extend?

In this section we first look at the various branches of mathematics to see how they are represented in the Erdős component of C. Next, we consider many other academic disciplines in an effort to determine the scope of the Erdős connection beyond mathematics.

Both *Mathematical Reviews* (MR) [26] and *Zentralblatt für Mathematik* (Zbl) [35] assign a number to each published work representing its primary subject area. For example, combinatorics is 05 and number theory is 11 (to mention the areas in which about 80% of PAUL ERDŐS's works appear). A total of 61 broad categories are currently in use [2]. It turns out—not surprisingly—that all 61 subject classifications are represented in the Erdős component of C. In fact, we can say much more: ERDŐS himself published in at least 27 of these categories, his co-authors published in at least 32 more, and there are people with Erdős number 2 who have published in the remaining two (K-theory and geophysics).

We saw in Section 2 that some outstanding scientists from myriad fields have finite Erdős numbers. We now extend the reach even further. Sophisticated mathematical models and tools have become standard in many fields, not just the obvious ones like the natural sciences, computer science, and engineering, but many social sciences and other areas, as well. The editors at *Mathematical Reviews* regularly scan papers in such periodicals as the *Journal of Mathematical Psychology* and publish summaries or reviews of those papers with mathematical content. The subject classifications for these works include 62 (statistics, which explicitly includes applications to actuarial sciences and financial mathematics, biology, psychology, economics, and social sciences, and engineering topics such as reliability, life testing, and quality control), 90 (which includes economics, operations research, and management science), and 92 (which includes mathematical biology, physiological, cellular and medical topics, genetics and population dynamics, chemistry, social and behavioral sciences, mathematical sociology, anthropology, psychology, and other social and behavioral sciences).

[7] After this paper was written, the 1999 Wolf Prize in Mathematics was awarded to ERDŐS coauthor LÁSZLÓ LOVÁSZ.

It is not hard to find researchers with fairly small Erdős numbers publishing in such journals. For example, SCOTT A. BOORMAN, who has Erdős number at most 7, has papers in both the *Journal of Mathematical Psychology* and the *Journal of Mathematical Sociology*. One also finds his work listed in *Sociological Abstracts*. Certainly hundreds, if not thousands, of statisticians have small Erdős numbers, and they often become co-authors on papers growing out of their consulting work. As another example, PETER C. FISHBURN, whose Erdős number is 1 and who works in a variety of mathematical disciplines, has published in *Management Science* and *Theory and Decision, An International Journal for Methods and Models in the Social and Decision Sciences*.

Probably the best source of links into other disciplines is FRANK HARARY, a co-author of PAUL ERDŐS who himself has over 270 co-authors. HARARY reports[8] that he has published with anthropologists, architects, biologists, chemists, economists, engineers, geographers, journalists (including the grand-nephew of the writer JAMES JOYCE), philosophers, physicians, physicists (including GEORGE UHLENBECK), political scientists, psychologists, scientific writers (including MARTIN GARDNER), and sociologists, among others. He has papers in such MR-scanned journals as *Behavioral Science*, the *Journal of Mathematical Psychology*, the *Journal of Mathematical Sociology*, the *Journal of Theoretical Biology*, *Mind*, *Psychometrika*, *Social Networks*, and *Sociometry*, as well as in periodicals from other fields, including *American Antiquity*, *Current Anthropology*, the *Journal of Chemical Documentation*, the *Journal of Social Psychology*, *Language*, and *Oceania*. Clearly much work remains to be done in exploring collaborations in other disciplines.

6 Final remarks and open questions

It could be thought *a priori* that in order for a mathematician to make his or her entrance into the Erdős component of C it is necessary to have many co-authors. But one of the important conclusions we can draw from the compilation of data for this article is that what really matters is not *how many* people you publish with but *whom* you publish with. That is, *quality* is more significant than *quantity*. We believe this is the case for the following two reasons. First, the most prolific mathematicians of this century seem to have a finite Erdős number (we have presented strong evidence to support that claim), and secondly, the Erdős component has grown so enormous that it already embraces almost everyone.

If that reasoning is not altogether convincing, consider a more dramatic example than the ones presented thus far, the case of the great Austrian logician KURT GÖDEL. In regard to the number of joint papers, GÖDEL is at the other end of the spectrum from ERDŐS: He wrote only one (see [11]), and that is a one-page note (in German) he wrote with KARL MENGER and ABRAHAM WALD [[72]] concerning MENGER's approach to differential and projective geometry. It turns out that WALD's Erdős number is 2. Hence, despite his precarious number of joint papers, GÖDEL still makes his way into the Erdős component of C with a rather small Erdős number.

[8] private communication

We close with some open questions which, even in the era of supercomputing and worldwide information networks, are extremely difficult to answer. The first two were already put forward in [19] but no hint to their possible solution has yet surfaced.

- In the collaboration graph C, what is the second largest component (measured by the number of its vertices)? If we restrict ourselves to looking only at mathematicians, then the second largest component is probably not very large, but it is conceivable that there are large components in other disciplines.

- What are the radius and the diameter of the Erdős component of C (in graph-theoretical terms)? Again the question would be interesting both as applied to all researchers and when restricted to mathematicians.

- The *Nobel-Erdős* number is, at a given moment, the number of Nobel Prize laureates having a finite Erdős number. This number changes as new prizes are awarded and more people enter into the Erdős component of C. We have established that the *Nobel-Erdős* number is ≥ 63 but its exact value (as of, say, the end of 1998) is unknown. Surely our bound is not nearly the best possible.

- The *Erdős span* measures how far back in time the connection with PAUL ERDŐS extends. More precisely, we can define the Erdős span as the smallest number representing the year of birth of a person with a finite Erdős number. All we can say for now is that this number is no greater than 1849, which is the year of birth of GEORG FROBENIUS (1849–1917), the German algebraist who made major contributions to group theory. He developed the theory of finite groups of linear substitutions mostly in collaboration with ISSAI SCHUR (1875–1941) [[67]]. It turns out that SCHUR's Erdős number is 2 because of his 1925 joint paper [[119]] with GABOR SZEGÖ, a co-author of Erdős.

We do not know whether the Erdős span can be traced further back into the early 1800s. What we can be sure of is that the Erdős connection will extend forever into the future.

7 Appendix: Collaboration paths

The collaboration paths for all Fields Medalists (whose names appear in boldface), linking them to PAUL ERDŐS, are displayed below. A number in parentheses represents the year in which a joint research paper or book was published. As for the winners (whose names are marked with an $*$) of the Nevanlinna Prize, the Wolf Prize in Mathematics, or the Steele Prize, we present only the subgraphs for those researchers having Erdős numbers ≥ 3, and the years of collaboration are omitted. For the awardees with Erdős numbers ≤ 2, whose graphs are not displayed here, the reader can trace their paths to ERDŐS by referring to the Erdős-2 list [15].

Erdős*
(1973) E. Netanyahu
(1959) C. Loewner
(1974) L. Nirenberg*
(1955) L. Bers
(1961) **L. Ahlfors***

Erdős*
(1939) A. Wintner
(1941) N. Wiener
(1926) P. Franklin
(1933) **J. Douglas**

Erdős*
(1960) S. Chowla
(1967) **A. Selberg***

Erdős*
(1943) S. Kakutani
(1950) **K. Kodaira***

Erdős*
(1936) H. Davenport
(1955) **K. Roth**

Erdős*
(1945) I. Niven
(1944) S. Eilenberg*
(1966) H. Cartan* (1967) **A. Grothendieck**
(1939) J. Dieudonné
(1949) **L. Schwartz**

Erdős*
(1965) W. Tutte
(1972) H. Whitney*
(1959) A. Dold
(1958) **R. Thom**

Erdős*
(1964) L. Rubel
(1966) J.-L. Lions
(1956) **L. Hormander***

Erdős*
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