

PQI – 3303 – Fenômenos de Transporte III

Aula 04

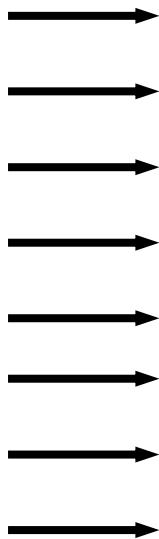
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Modelo - Prandtl

V

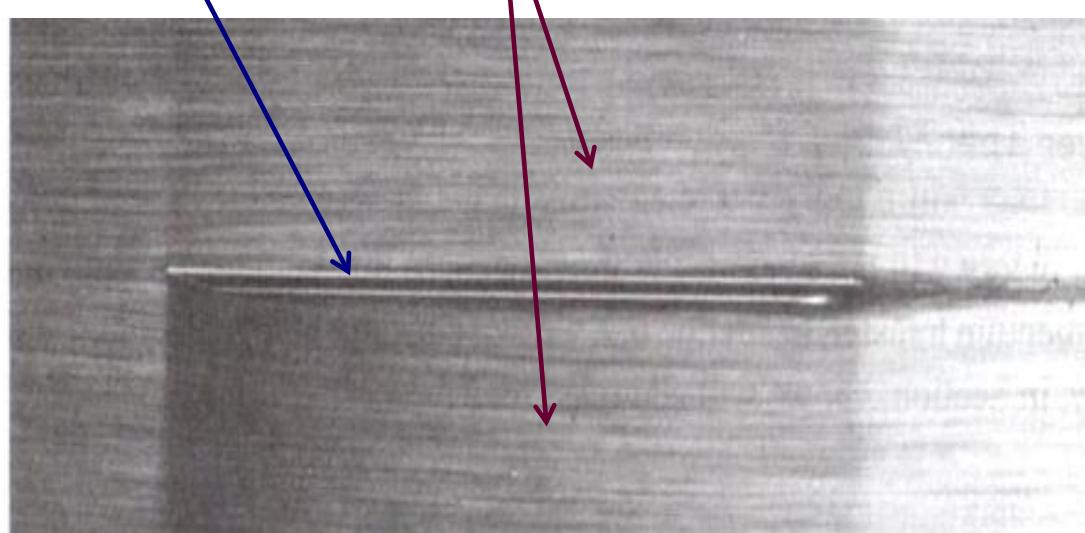
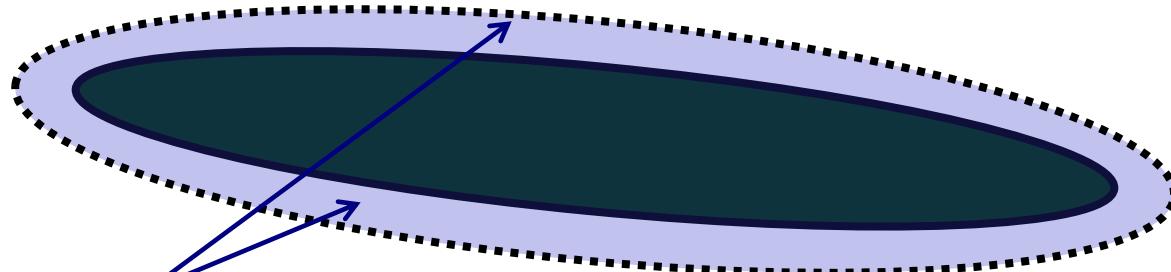
Invísido- Euler



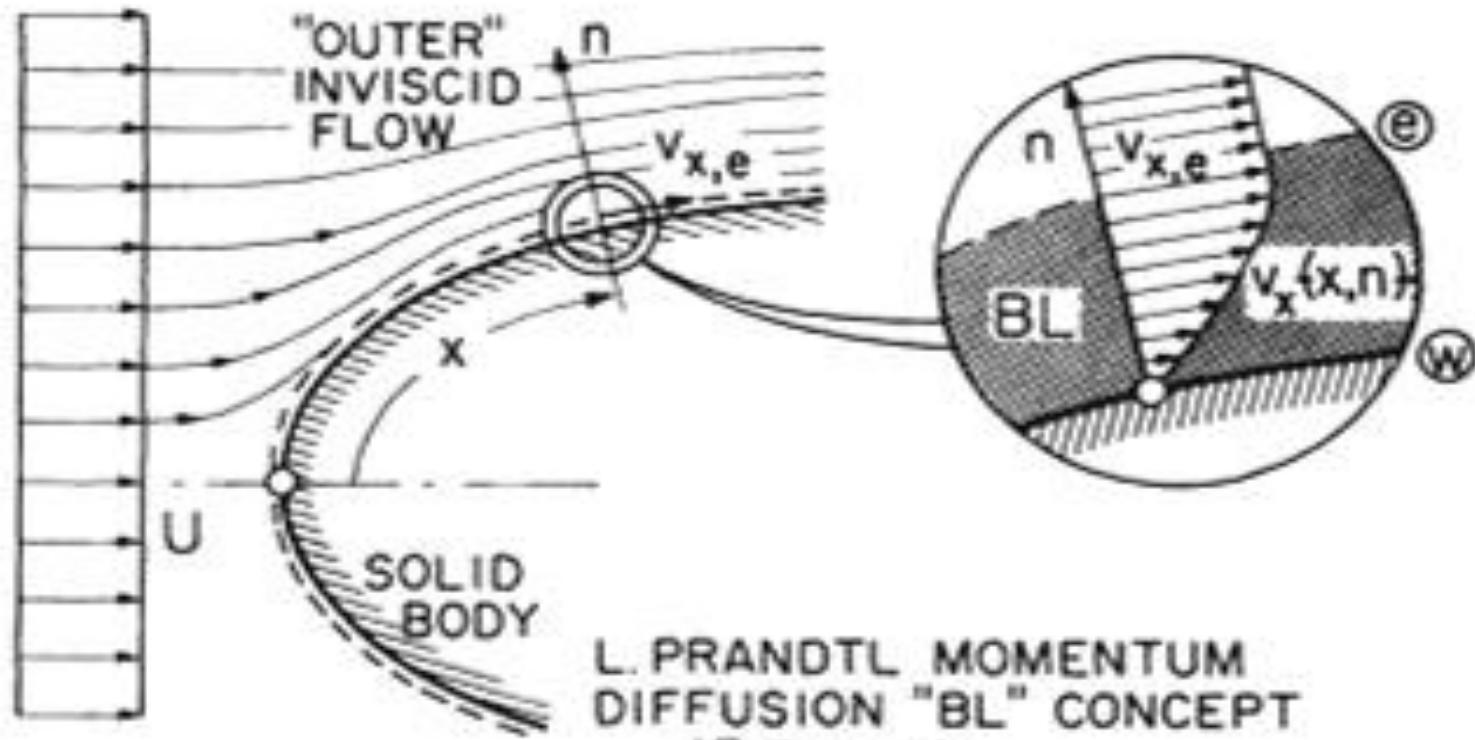
Viscoso -
Navier Stokes

Invísido- Euler

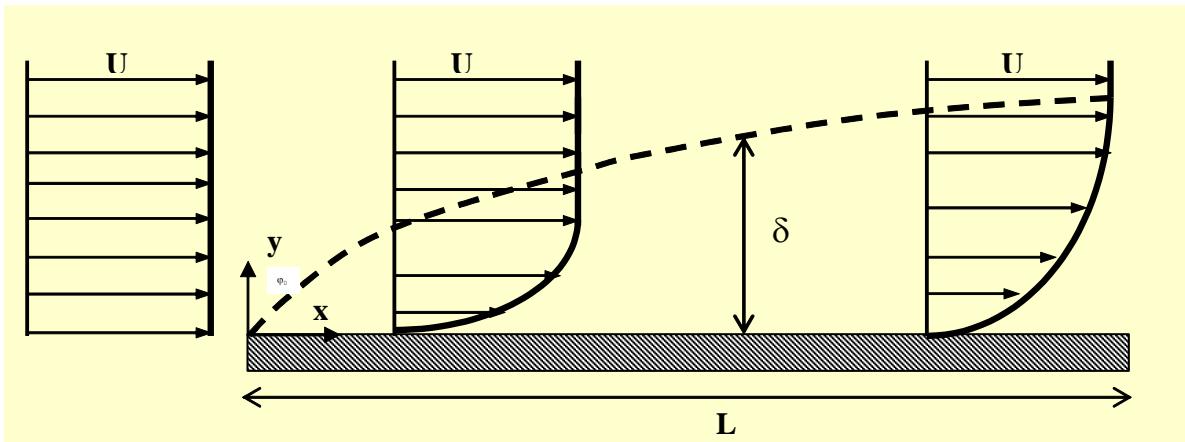
Placa Plana



Modelo - Prandtl



Camada Limite Laminar Hidrodinâmica - Bidimensional



Continuidade

$$\frac{\partial \rho}{\partial t} = - \operatorname{div} \rho \vec{v} \Rightarrow$$

$$\frac{\partial \rho}{\partial t} = 0$$

$$\operatorname{div} \vec{v} = 0$$

$$\Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Navier Stokes - bidimensional

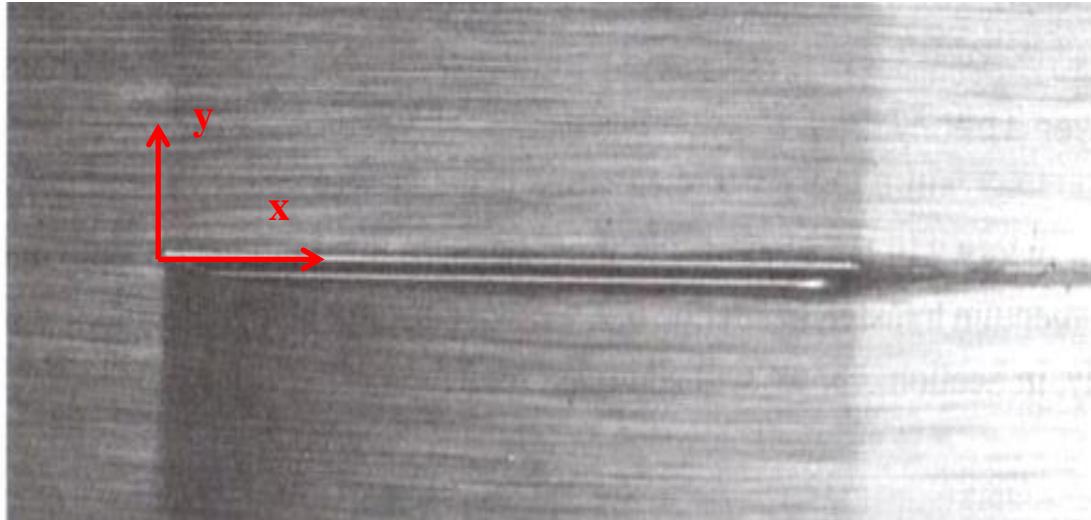
$$\rho \frac{D\vec{v}}{Dt} = \rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla p = \rho \vec{g} - \nabla p + \mu \operatorname{lap} \vec{v}$$

$$\left\{ \begin{array}{l} \rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) \\ \rho \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) = - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) \end{array} \right.$$

Scaling

Escalas de adimensionalização:

- $x \rightarrow L$, $y \rightarrow \delta$
- $v_x \rightarrow U$, $v_y \rightarrow V$
- $\delta \ll L$ (observação experimental)



$$\hat{x} = \frac{x}{L}$$

$$\hat{y} = \frac{y}{\delta}$$

$$\hat{v}_x = \frac{v_x}{U}$$

$$\hat{v}_y = \frac{v_y}{V}$$

SCALING =>

ordem de grandeza das variáveis e das variações ~ 1

“Scaling”

Continuidade:

$$\Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\frac{U}{L} \frac{\partial \hat{v}_x}{\partial \hat{x}} + \frac{V}{\delta} \frac{\partial \hat{v}_y}{\partial \hat{y}} = 0 \quad \Rightarrow \left(\frac{U\delta}{VL} \right) \underbrace{\frac{\partial \hat{v}_x}{\partial \hat{x}}}_{\sim 1} + \underbrace{\frac{\partial \hat{v}_y}{\partial \hat{y}}}_{\sim 1} = 0$$

SCALING => ordem de grandeza das variáveis ~ 1

$$\frac{U\delta}{VL} = 1 \quad \Rightarrow \quad V = \frac{U\delta}{L}$$

$V \ll U$

“Scaling” da Navier- Stokes

Direção x

$$\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

$$\hat{p} = \frac{p}{P}$$

$$\frac{\rho U^2}{L} \left(\hat{v}_x \frac{\partial \hat{v}_x}{\partial \hat{x}} + \hat{v}_y \frac{\partial \hat{v}_x}{\partial \hat{y}} \right) = - \frac{P}{L} \frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\mu U}{\delta^2} \left[\underbrace{\left(\frac{\delta}{L} \right)^2}_{\sim 1} \underbrace{\frac{\partial^2 \hat{v}_x}{\partial \hat{x}^2}}_{\sim 1} + \underbrace{\frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2}}_{\sim 1} \right]$$

“Scaling” da Navier- Stokes

Direção y

$$\rho \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) = - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right)$$

$$\frac{\rho U^2}{L} \frac{\delta}{L} \left(\hat{v}_x \frac{\partial \hat{v}_y}{\partial \hat{x}} + \hat{v}_y \frac{\partial \hat{v}_y}{\partial \hat{y}} \right) = - \frac{P}{\delta} \frac{\partial \hat{p}}{\partial \hat{y}} + \frac{\mu U}{\delta^2} \frac{\delta}{L} \left[\underbrace{\left(\frac{\delta}{L} \right)^2}_{\sim 1} \underbrace{\frac{\partial^2 \hat{v}_y}{\partial \hat{x}^2}}_{\sim 1} + \underbrace{\frac{\partial^2 \hat{v}_y}{\partial \hat{y}^2}}_{\sim 1} \right]$$

$\sim \frac{\delta}{L}$

$\sim \frac{\delta}{L}$

$\sim \frac{\rho U^2}{L} \frac{\delta}{L}$

$$\frac{\rho U^2}{L} \sim \frac{\mu U}{\delta^2} \Rightarrow$$

$$\left(\frac{\delta}{L} \right)^2 \sim \frac{\mu}{\rho U L} \Rightarrow Re \gg 1$$

$\delta \ll L$ (observação experimental)

“Scaling” da Navier- Stokes

Quantidade de movimento na direção x

$$\sim \frac{\rho U^2}{L}$$

$$\frac{\rho U^2}{L} \left(\hat{v}_x \frac{\partial \hat{v}_x}{\partial \hat{x}} + \hat{v}_y \frac{\partial \hat{v}_x}{\partial \hat{y}} \right) = - \frac{P}{L} \frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\mu U}{\delta^2} \left[\frac{\partial^2 \hat{v}_x}{\partial \hat{y}^2} \right]$$

$$Re = \frac{\rho UL}{\mu}$$

Quantidade de movimento na direção y

DESCONSIDERADO

$$\sim \frac{\rho U^2}{L} \frac{\delta}{L}$$

~~$$\frac{\rho U^2}{L} \frac{\delta}{L} \left(\hat{v}_x \frac{\partial \hat{v}_y}{\partial \hat{x}} + \hat{v}_y \frac{\partial \hat{v}_y}{\partial \hat{y}} \right) = - \frac{P}{\delta} \frac{\partial \hat{p}}{\partial \hat{y}} + \frac{\mu U}{\delta^2} \frac{\delta}{L} \left[\frac{\partial^2 \hat{v}_y}{\partial \hat{y}^2} \right]$$~~

$$\frac{\partial \hat{p}}{\partial \hat{y}} \approx 0$$

Equações – Camada Limite Laminar – Placa Plana

$$p = p(x)$$

$$\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = - \frac{dp}{dx} + \mu \left(\frac{\partial^2 v_x}{\partial y^2} \right)$$

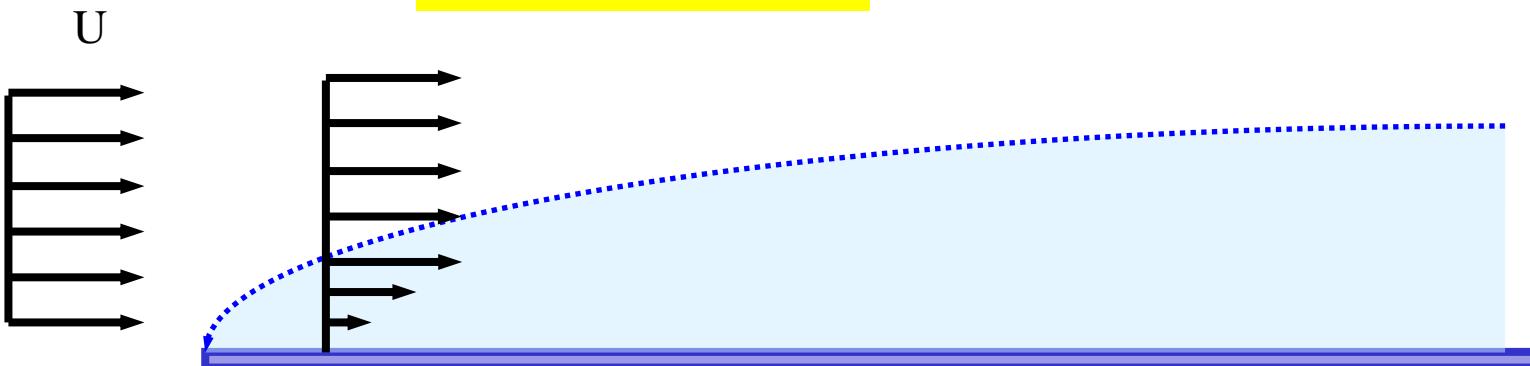
$$\frac{\partial \hat{p}}{\partial \hat{y}} \approx 0$$

Borda da camada limite $v_x \approx U$:

$$\rho \left(U \frac{\partial U}{\partial x} + v_y \frac{\partial U}{\partial y} \right) = - \frac{dp}{dx} + \mu \left(\frac{\partial^2 U}{\partial y^2} \right)$$

EULER - INVÍSCIDO

$$\Rightarrow \rho U \frac{\partial U}{\partial x} = - \frac{dp}{dx}$$



Equações – Camada Limite Laminar – Placa Plana

$$\frac{dp}{dy} \approx 0$$

dp/dx dentro da camada = dp/dx fora da camada

$$\rho U \frac{dU}{dx} = -\frac{dp}{dx}$$

Dentro da camada limite:

VISCOSO

$$p = p(x)$$

$$\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = -\frac{dp}{dx} - \mu \left(\frac{\partial^2 v_x}{\partial y^2} \right)$$

$$\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = \rho U \frac{dU}{dx} + \mu \left(\frac{\partial^2 v_x}{\partial y^2} \right)$$

Placa Plana $U = \text{cte}$: $\rho U \frac{dU}{dx} = 0 = -\frac{dp}{dx}$  **P cte**

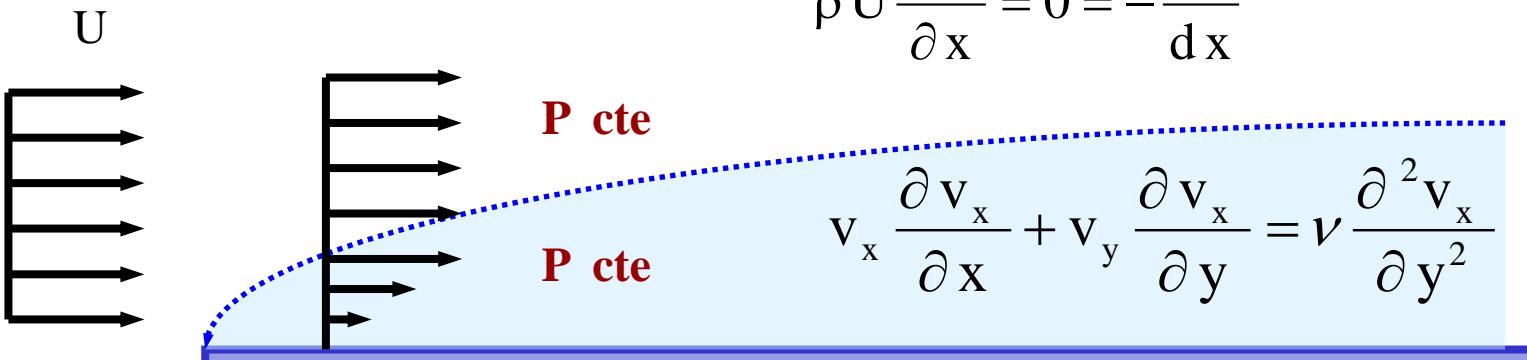
$$\boxed{\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = \mu \left(\frac{\partial^2 v_x}{\partial y^2} \right)}$$

Equações – Camada Limite Laminar – Placa Plana

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

P cte



Camada Limite Laminar – Placa Plana – Equações Aproximadas e Condições de Contorno

Continuidade:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Quantidade de Movimento:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = v \frac{\partial^2 v_x}{\partial y^2}$$

Condições de Contorno:

parede, $y = 0 \rightarrow v_x = 0$
 $v_y = 0$

borda, $y = \infty \rightarrow v_x = U$

Solução de Blasius

$$v_x = \frac{\partial \Psi}{\partial y} \quad ; \quad v_y = -\frac{\partial \Psi}{\partial x}$$

$$\eta = y \sqrt{\frac{U}{v x}}$$

$$f(\eta) = \frac{\Psi}{\sqrt{x v U}}$$

$$f(\eta) \frac{d^2 f(\eta)}{d\eta^2} + 2 \frac{d^3 f(\eta)}{d\eta^3} = 0$$

$\left. \begin{array}{l} \text{parede}, \eta = 0 \rightarrow f = 0 \text{ e } f' = 0 \\ \text{borda}, \eta = \infty \rightarrow f' = 1 \end{array} \right\}$

Solução de Blasius

Solução :

$$v_x = U \frac{df(\eta)}{d\eta}$$

$$v_y = \frac{1}{2} \sqrt{\frac{vU}{x}} \left(\eta \frac{df(\eta)}{d\eta} - f(\eta) \right)$$

$$\frac{v_x}{U} = 0,99 = \frac{df}{d\eta} = 0,99 \Rightarrow \eta = 5 = \delta \sqrt{\frac{U}{vx}}$$

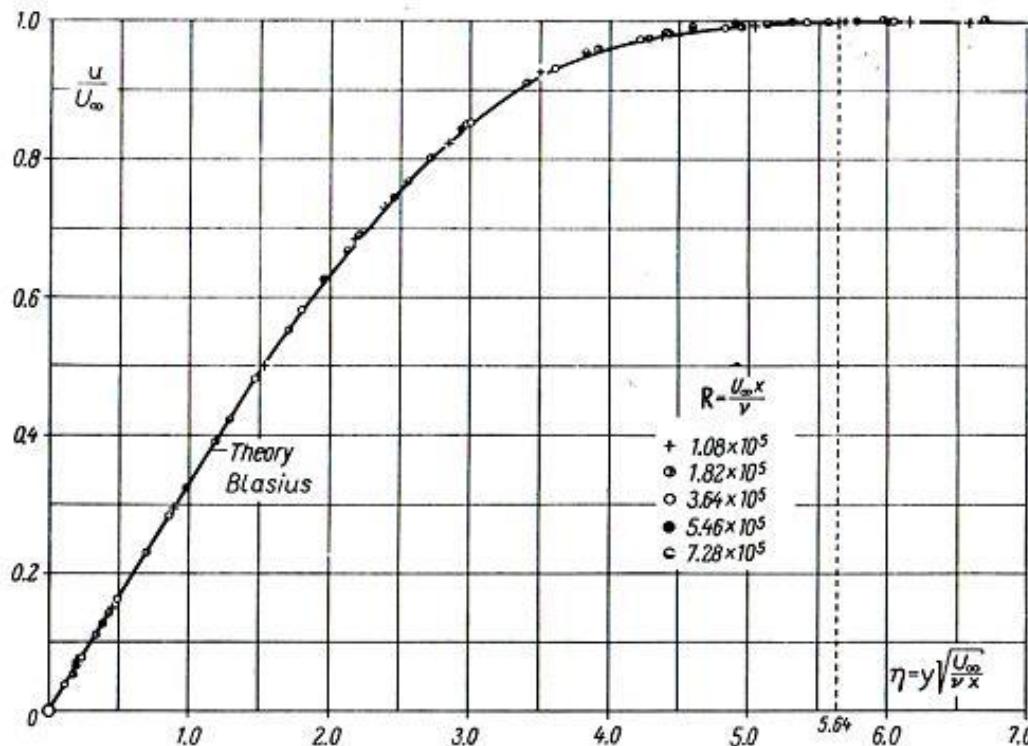
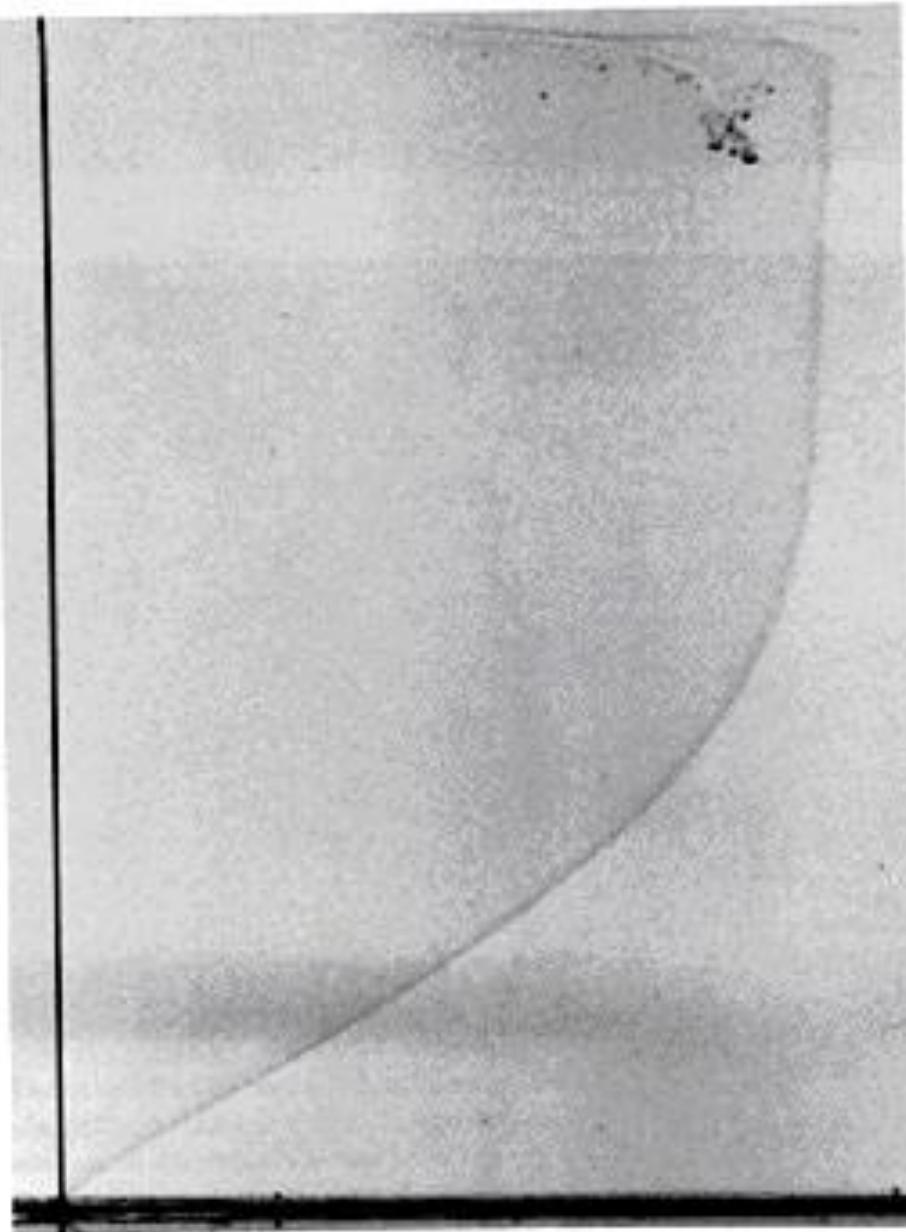


Fig. 7.9. Velocity distribution in the laminar boundary layer on a flat plate at zero incidence, as measured by Nikuradse [20]

$$\delta = 5 \sqrt{\frac{vx}{U}} = \frac{5x}{\sqrt{Re_x}}$$

$$Re_{crítico} = 5.10^5$$

Perfil de Velocidades- Camada Limite



30. Blasius boundary-layer profile on a flat plate. The tangential velocity profile in the laminar boundary layer on a flat plate, discovered by Prandtl and calculated accurately by Blasius, is made visible by tellurium. Water is flowing at 9 cm/s. The Reynolds number is 500 based on distance from the leading edge, and the displacement thickness is about 5 mm. A fine tellurium wire perpendicular to the plate at the left is subjected to an electrical impulse of a few milliseconds duration. A chemical reaction produces a slender colloidal cloud, which drifts with the stream and is photographed a moment later to define the velocity profile. Photograph by F. X. Wortmann.

Solução de Blasius

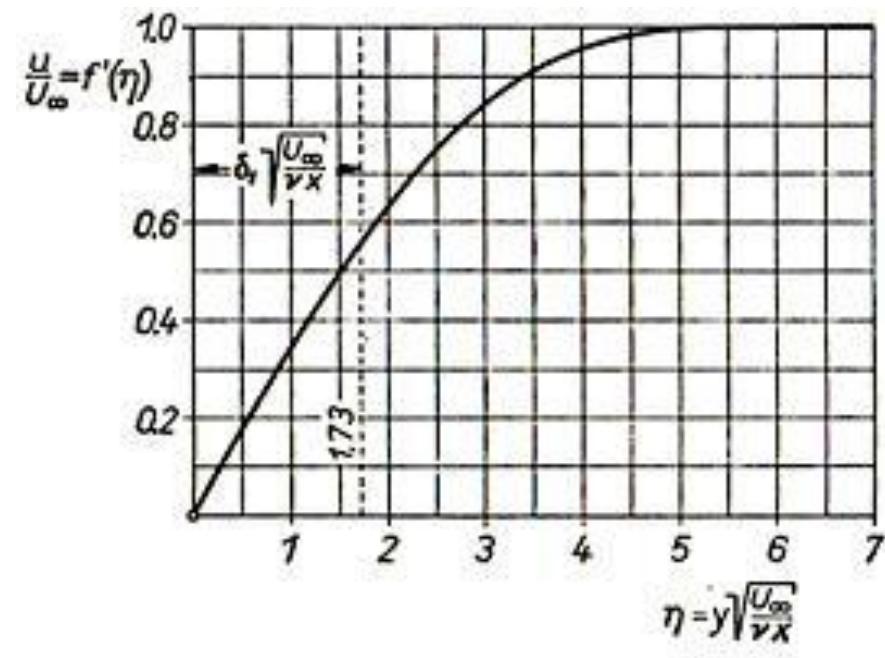


Fig. 7.7. Velocity distribution in the boundary layer along a flat plate, after Blasius [2]

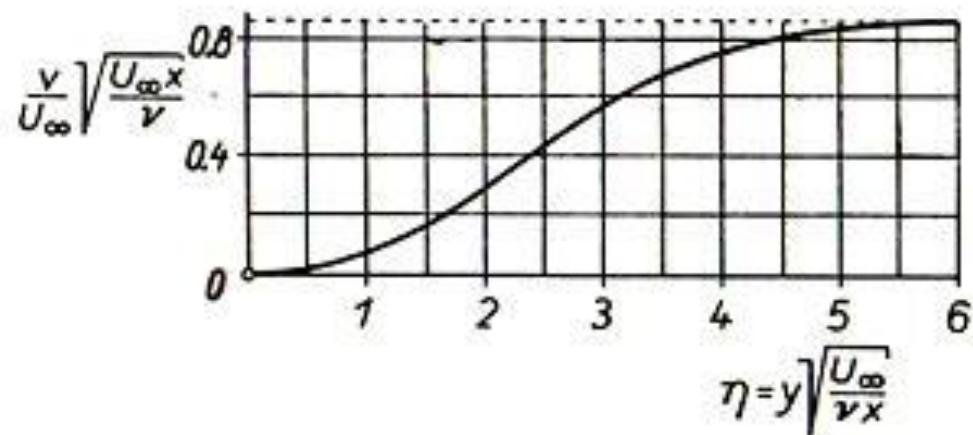


Fig. 7.8. The transverse velocity component in the boundary layer along a flat plate

Solução de Blasius

$\eta = y \sqrt{\frac{U_\infty}{\nu x}}$	f	$f' = \frac{u}{U_\infty}$	f''
0	0	0	0.33206
0.2	0.00664	0.06641	0.33199
0.4	0.02656	0.13277	0.33147
0.6	0.05974	0.19894	0.33008
0.8	0.10611	0.26471	0.32739
1.0	0.16557	0.32979	0.32301
1.2	0.23795	0.39378	0.31659
1.4	0.32298	0.45627	0.30787
1.6	0.42032	0.51676	0.29667
1.8	0.52952	0.57477	0.28293
2.0	0.65003	0.62977	0.26675
2.2	0.78120	0.68132	0.24835
2.4	0.92230	0.72899	0.22809
2.6	1.07252	0.77246	0.20646
2.8	1.23099	0.81152	0.18401
3.0	1.39682	0.84605	0.16136
3.2	1.56911	0.87609	0.13913
3.4	1.74696	0.90177	0.11788
3.6	1.92954	0.92333	0.09809
3.8	2.11605	0.94112	0.08013
4.0	2.30576	0.95552	0.06424
4.2	2.49806	0.96696	0.05052
4.4	2.69238	0.97587	0.03897
4.6	2.88826	0.98269	0.02948
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591

$\eta = y \sqrt{\frac{U_\infty}{\nu x}}$	f	$f' = \frac{u}{U_\infty}$	f''
5.2	3.48189	0.99425	0.01134
5.4	3.68094	0.99616	0.00793
5.6	3.88031	0.99748	0.00543
5.8	4.07990	0.99838	0.00365
6.0	4.27964	0.99898	0.00240
6.2	4.47948	0.99937	0.00155
6.4	4.67938	0.99961	0.00098
6.6	4.87931	0.99977	0.00061
6.8	5.07928	0.99987	0.00037
7.0	5.27926	0.99992	0.00022
7.2	5.47925	0.99996	0.00013
7.4	5.67924	0.99998	0.00007
7.6	5.87924	0.99999	0.00004
7.8	6.07923	1.00000	0.00002
8.0	6.27923	1.00000	0.00001
8.2	6.47923	1.00000	0.00001
8.4	6.67923	1.00000	0.00000
8.6	6.87923	1.00000	0.00000
8.8	7.07923	1.00000	0.00000

Fator de atrito

Coeficiente de arraste (fator de atrito) na placa:

$$C_D = \frac{(\tau_{yx})_{y=0}}{\frac{1}{2} \rho U^2}$$

$$C_D = \frac{\mu}{\frac{1}{2} \rho U^2} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)_{y=0} = \frac{\mu}{\frac{1}{2} \rho U^2} U \sqrt{\frac{U}{x v}} \frac{d^2 f}{d \eta^2}(0)$$

Para $y = 0$ ($\eta=0$) $\Rightarrow f''(0) = 0,332$ resulta:

$$C_D = 0,664 \sqrt{\frac{v}{U x}} \rightarrow C_D = 0,664 \text{ } \text{Re}_x^{-1/2}$$

Camada Limite Laminar Térmica e Mássica

Equação de Conservação

$$\frac{1}{Sr} \frac{\partial \hat{\rho} \hat{\phi}}{\partial \hat{t}} + \operatorname{div} \hat{\rho} \left(\hat{\vec{v}} \hat{\phi} - \frac{1}{Pe} \operatorname{grâd} \hat{\phi} \right) = \frac{\dot{\sigma}_{\forall_{\phi}} L}{\rho_0 \Delta \phi v_0}$$

Regime permanente

sem produção

Incompressível

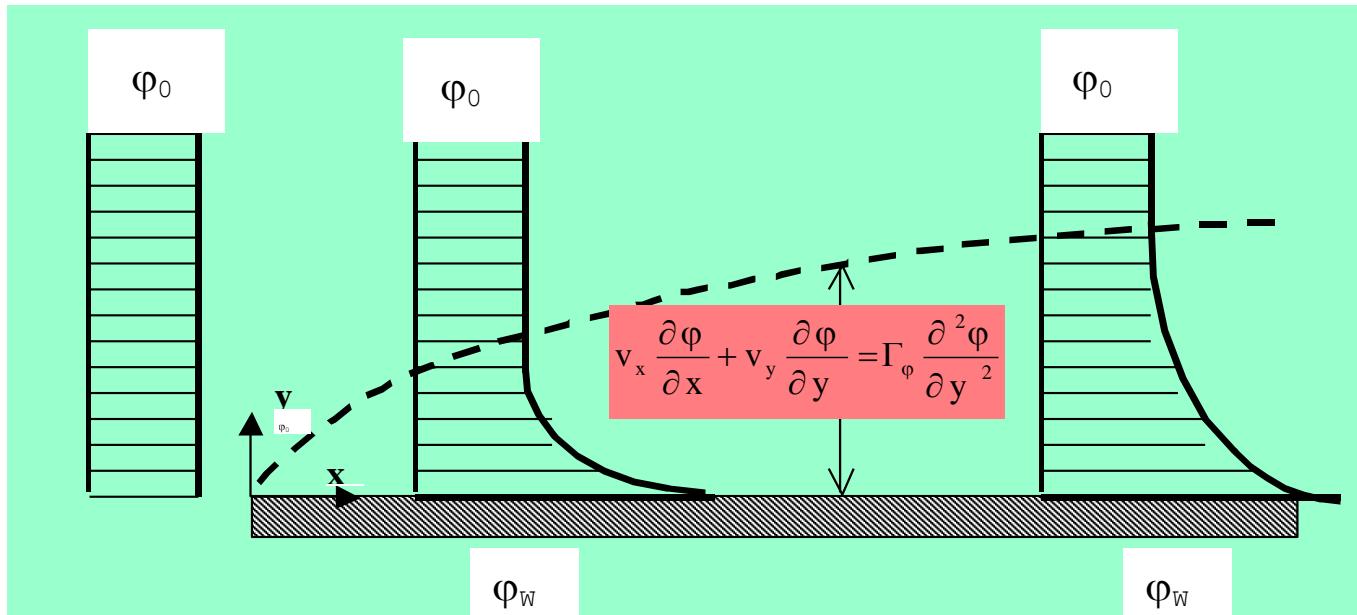
$$\hat{\vec{v}} \cdot \operatorname{grâd} \hat{\phi} - \frac{1}{Pe} \hat{\lambda} \hat{p} \hat{\phi} = 0$$

$$Re \hat{\vec{v}} \cdot \operatorname{grâd} \hat{\phi} - \frac{\Gamma_{\phi}}{v} \hat{\lambda} \hat{p} \hat{\phi} = 0$$

Camada Limite Laminar Térmica e Mássica

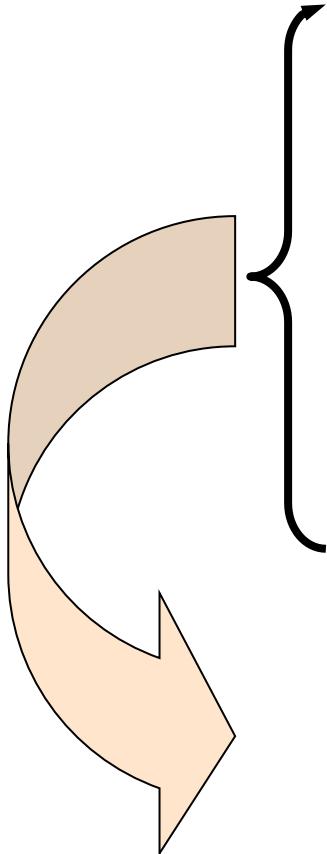
$$\operatorname{Re} \hat{\vec{v}} \cdot \operatorname{grad} \hat{\phi} - \frac{\Gamma_\varphi}{\nu} \operatorname{lap} \hat{\phi} = 0$$

$$\left(v_x \frac{\partial \varphi}{\partial x} + v_y \frac{\partial \varphi}{\partial y} \right) = \Gamma_\varphi \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) \Rightarrow v_x \frac{\partial \varphi}{\partial x} + v_y \frac{\partial \varphi}{\partial y} = \Gamma_\varphi \frac{\partial^2 \varphi}{\partial y^2}$$



Camada Limite Laminar da propriedade φ

Equacionamento generalizado para a propriedade ϕ



$$\left\{ \begin{array}{l} \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \\ v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = v \frac{\partial^2 v_x}{\partial y^2} \\ v_x \frac{\partial \phi}{\partial x} + v_y \frac{\partial \phi}{\partial y} = \Gamma_\phi \frac{\partial^2 \phi}{\partial y^2} \end{array} \right.$$

Condições de Contorno:
 parede, $y = 0 \rightarrow v_x = 0$
 $v_y = 0$
 $\phi = \phi_w$

$y = \infty \rightarrow v_x = U$
 $\phi = \phi_0$

$$\frac{d^2 \hat{\phi}}{d\eta^2} + \frac{v}{\Gamma_\phi} \frac{f(\eta)}{2} \frac{d\hat{\phi}}{d\eta} = 0$$

Condições de Contorno:
 parede, $\eta = 0 \rightarrow \hat{\phi} = 0$
 $\eta = \infty \rightarrow \hat{\phi} = 1$

$$\hat{\phi} = \frac{\int_{\zeta=0}^{\zeta=\eta} \exp \left[-\frac{1}{2} \frac{v}{\Gamma_\phi} \int_{\varepsilon=0}^{\varepsilon=\zeta} f(\varepsilon) d\varepsilon \right] d\zeta}{\int_{\zeta=0}^{\zeta=\infty} \exp \left[-\frac{1}{2} \frac{v}{\Gamma_\phi} \int_{\varepsilon=0}^{\varepsilon=\zeta} f(\varepsilon) d\varepsilon \right] d\zeta}$$

Número de Prandtl e Schmidt

$$\text{Pr}^{1/3} \approx \frac{\delta}{\delta_T}$$

$$\text{Sc}^{1/3} \approx \frac{\delta}{\delta_\omega}$$

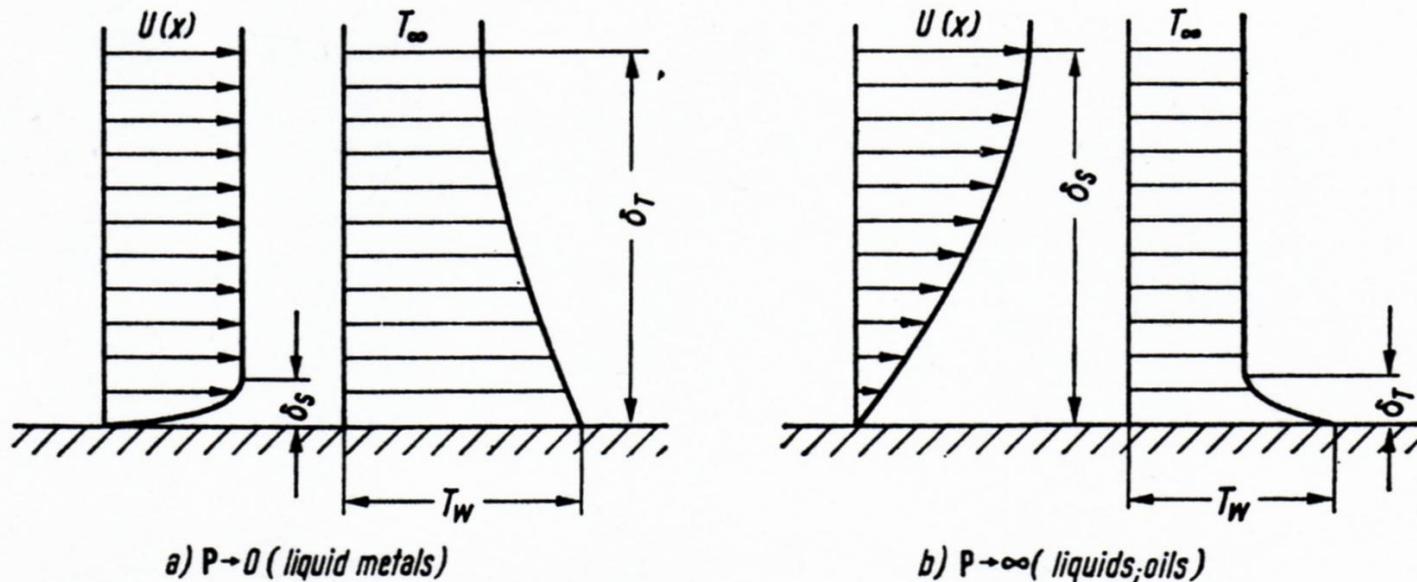
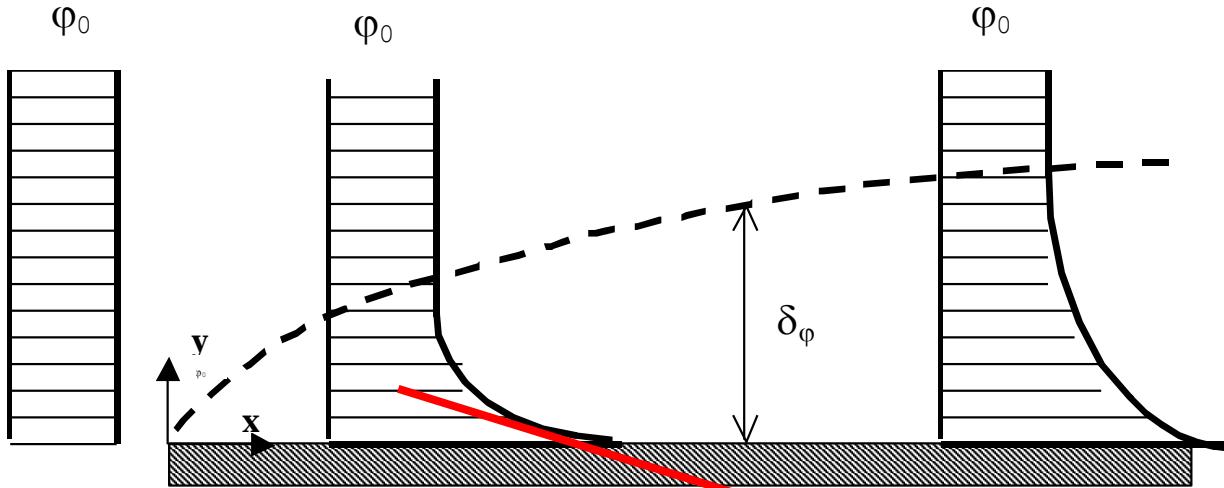


Fig. 12.8. Comparison between the temperature and velocity fields for boundary layers with very small and with very large values of Prandtl number

Perfil de ϕ e Fluxo na Parede



$$\hat{y} = \frac{y}{x}$$

$$\left. \frac{d\hat{\phi}}{d\eta} \right|_{\eta=0} = 0,332 \left(\frac{v}{\Gamma_\phi} \right)^{1/3}$$

$$\hat{\text{grad}} \hat{\phi} \Big|_{\hat{r}=0} = \left. \frac{\partial \hat{\phi}}{\partial \hat{y}} \right|_{\hat{y}=0} = \left. \frac{d\hat{\phi}}{d\eta} \frac{d\eta}{d\hat{y}} \right|_{\hat{y}=0} = x \left. \frac{d\eta}{dy} \frac{d\hat{\phi}}{d\eta} \right|_{\hat{y}=0} = x \sqrt{\frac{U}{vx}} \left. \frac{d\hat{\phi}}{d\eta} \right|_{\eta=0} = \sqrt{\frac{Ux}{v}} \left. \frac{d\hat{\phi}}{d\eta} \right|_{\eta=0}$$

$$\hat{\text{grad}} \hat{\phi} \Big|_{\hat{r}=0} = 0,332 \sqrt{\frac{Ux}{v}} \left(\frac{v}{\Gamma_\phi} \right)^{1/3} = 0,332 \text{Re}_x^{1/2} \left(\frac{v}{\Gamma_\phi} \right)^{1/3}$$

Nusselt e Sherwood

$$\hat{\text{grad}} \hat{\phi} \Big|_{\hat{r}=0} = 0,332 \text{Re}_x^{1/2} \left(\frac{v}{\Gamma_\phi} \right)^{1/3}$$

$\phi = T$

$$\frac{v}{\Gamma} = \frac{v}{\alpha} = \text{Pr}$$
$$\text{Nu} = (\hat{\text{grad}} \hat{T})_{\hat{r}=0}$$

$\phi = w_A$

$$\frac{v}{\Gamma} = \frac{v}{D_{AB}} = \text{Sc}$$
$$\text{Sh} = (\hat{\text{grad}} \hat{w}_A)_{\hat{r}=0}$$

$$\text{Nu} = 0,332 \text{Re}_x^{1/2} \text{Pr}^{1/3}$$

$$\text{Sh} = 0,332 \text{Re}_x^{1/2} \text{Sc}^{1/3}$$

$$Sh_L = 0,664 \text{Re}_L^{1/2} \text{Sc}^{1/3}$$

Camada limite turbulenta

$$\frac{v_x}{U} = \left(\frac{y}{\delta_M} \right)^{1/7}$$

$$\frac{x_A - x_{AW}}{x_{A0} - x_{AW}} = \left(\frac{y}{\delta_M} \right)^{1/7}$$

$$\frac{\delta}{x} = 0,376 \text{Re}_x^{-1/5}$$

$$Sh_L = 0,0292 \text{Re}_x^{4/5} Sc^{1/3}$$

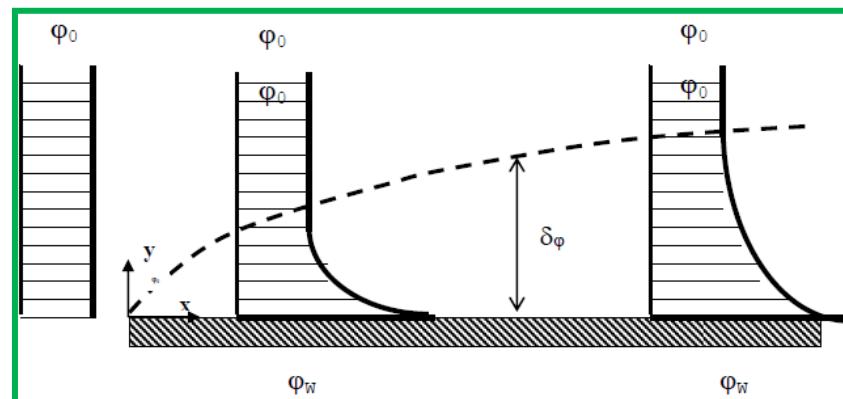
Coeficientes convectivos de TM com injeção/sucção na placa

O fluxo global de A na parede é expresso por:

$$n_{Ay} = J_{Ay} + X_{Ay} n_y = J_{Ay} + \frac{\rho_A}{\rho} \rho v_y = J_{Ay} + \rho_A V_y$$

Para $y = 0$ adota-se o índice s

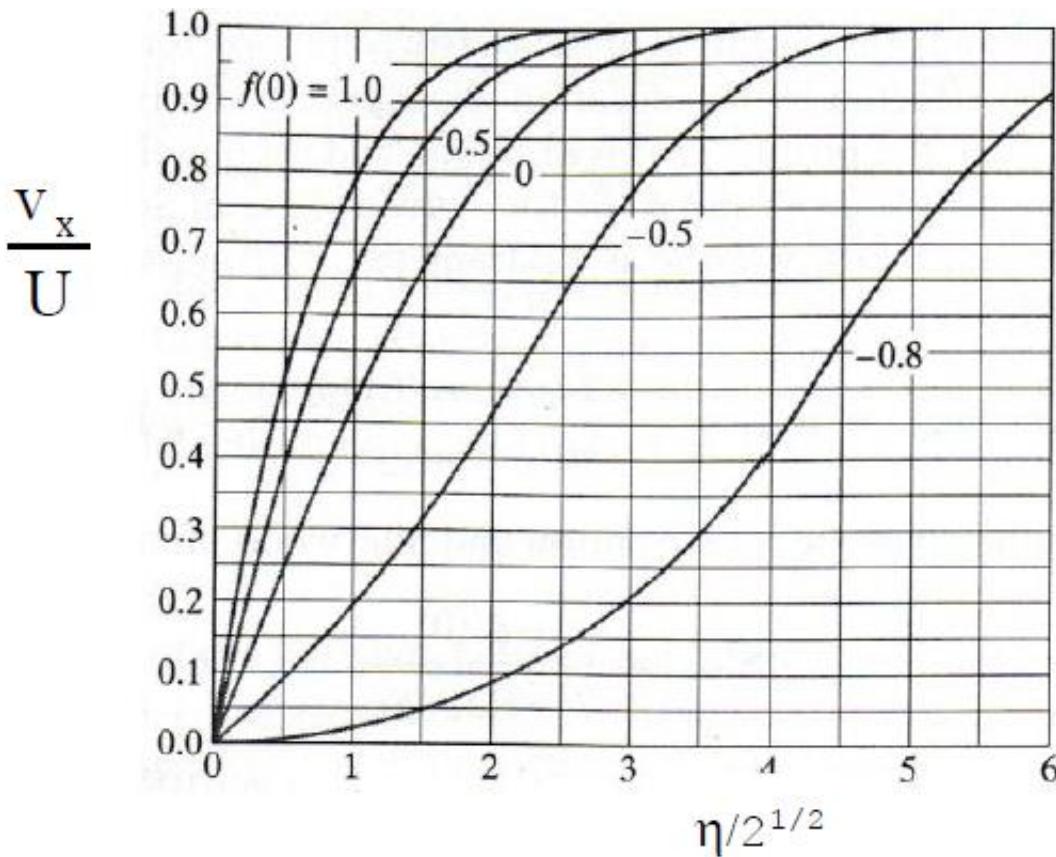
$$n_{Ays} = J_{Ays} + \rho_{As} V_{ys}$$



$$n_{Ays} = k_\rho (\rho_{Aw} - \rho_{A0}) = J_{Ays} + \rho_{As} V_{ys} = k'_\rho (\rho_{Aw} - \rho_{A0}) + \rho_{As} V_{ys}$$

$$k'_\rho = \frac{-\rho D_{AB} (\partial x_A / \partial y)_{y=0}}{\rho (x_{As} - x_{A0})}$$

Coeficientes convectivos de TM com injeção/sucção na placa



◀
$$f(0) = -v_y (2Re_x)^{1/2}/U$$

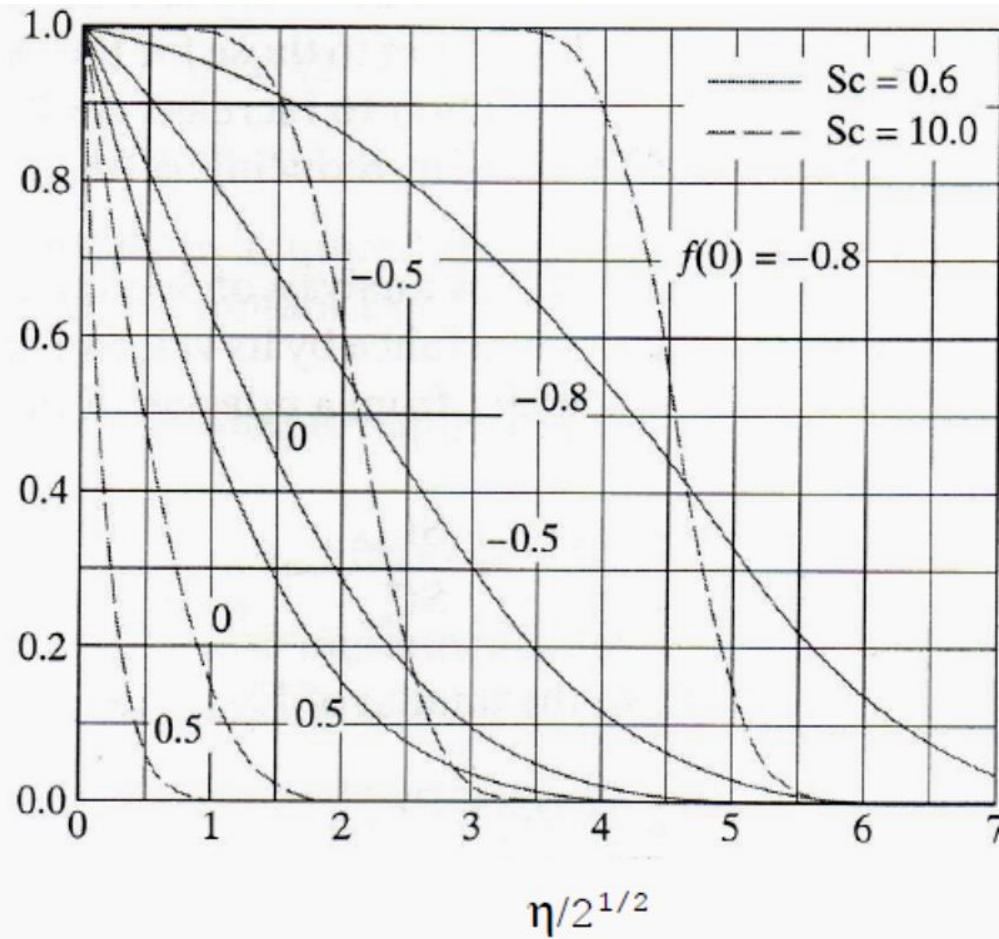
Fonte: adaptado
de A. F. Mills,
Mass Transfer,
Prentice Hall,
2001.

$$\eta = y \sqrt{\frac{U}{v_x}}$$

Perfis de velocidade v_x/U em função de $2^{1/2}\eta$, para diferentes valores do parâmetro $f(0) = -v_y(2Re_x)^{1/2}/U$.

Coeficientes convectivos de TM com injeção/sucção na placa

$$\hat{x}_A = \frac{x_A - x_{AW}}{x_{A0} - x_{AW}}$$

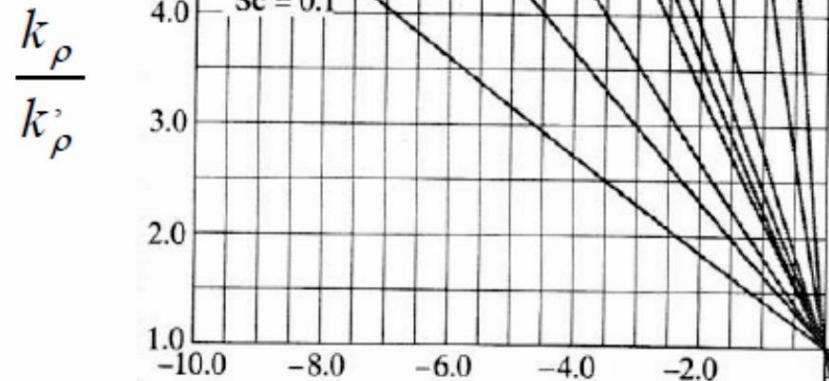


Fonte:
adaptado de
A. F. Mills,
Mass Transfer,
Prentice Hall,
2001.

Perfis de concentração (adimensionalizada) em função de $2^{-1/2}\eta$, para diferentes valores do parâmetro $f(0) = -v_Y(2Re_x)^{1/2}/U$.

Coeficientes convectivos de TM com injeção/sucção na placa

$$n_{Ays} = k_\rho (\rho_{Aw} - \rho_{A0})$$

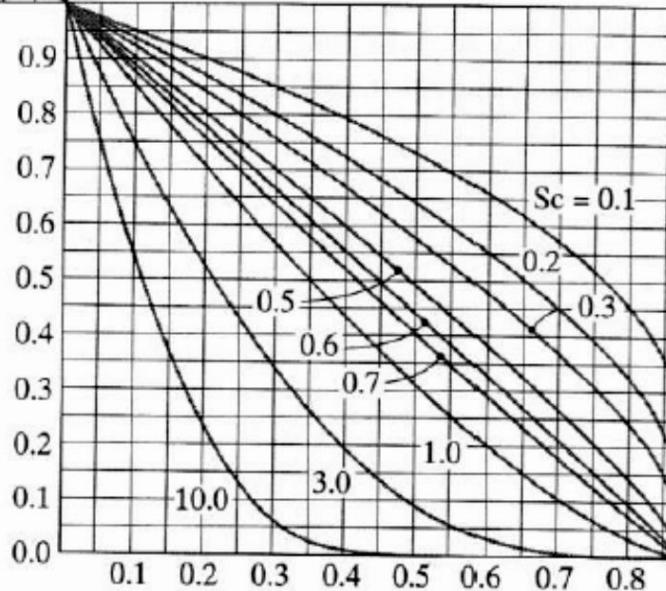


k_ρ e k'_ρ em função do parâmetro $f(0)$.

Fonte: A. F. Mills, *Mass Transfer*, Prentice Hall, 2001.

Sucção $f(0) < 0$

Injeção $f(0) > 0$.



$$\frac{v_y}{U} (2 \text{Re}_x)^{1/2}$$