PRO 5971

Statistical Process Monitoring: Multivariate process monitoring

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Outline

Multivariate process monitoring - Introduction

- Simultaneous Monitoring parameters of two or more related quality characteristics
- The use of separate control chart for each parameter may be misleading
- α^*=type I error for the joint control procedure:
  - p statistically independent quality characteristics and α is the type I error for each $\overline{X}$, then $\alpha^* = 1 - (1 - \alpha)^p$)
  - But if p s are not independent, the above equation does not hold.

About multivariate normal distribution

- Consider p variables, given by $\mathbf{X}' = (X_1 \; X_2 \; \ldots \; X_p)$
- With its respective means $\boldsymbol{\mu}' = (\mu_1 \; \mu_2 \; \ldots \; \mu_p)$
- And their variances and covariances described by a matrix $\boldsymbol{\Sigma}_{\mathbf{p}\times\mathbf{p}}$
- The multivariate normal probability density function is

$$f(\mathbf{X}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}} \exp^{-1/2(\mathbf{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})}$$

About multivariate normal distribution - random sample

- A random sample of size n: $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$
- Sample mean vector

$$\overline{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i = [\, \overline{X}_1 \ \overline{X}_2 \ \ \ldots \ \overline{X}_p]'$$

Monitoring a vector of means

Monitor a vector of means: T^2 Control Chart

- When $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are known
- Monitored statistic

$$\chi_0^2 = n(\overline{\mathbf{X}} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\overline{\mathbf{X}} - \boldsymbol{\mu})$$

- Upper control limit: UCL=$\chi^2_{\alpha,p}$

T^2 Control Chart

- In practice, it is usually necessary to estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$
- Assuming the process is in-control, take m samples of size n
- Obtain

$$\overline{x}_{jk} = \frac{1}{n}\sum_{i=1}^{n} x_{ijk},\ S_{jk}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_{ijk} - \overline{x}_{jk})^2$$

$$j = 1, \ldots, p;\, k = 1, \ldots, m$$

- and the covariance between quality characteristics j and h in the k-th sample

$$S_{jhk} = \frac{1}{n-1}\sum_{i=1}^{n}(x_{ijk} - \overline{x}_{jk})(x_{ihk} - \overline{x}_{hk}),\ k = 1, \ldots, m;\ j \neq h$$

T^2 Control Chart

- Estimates of mean, variance and covariance are respectively given as:

$$\overline{\overline{x}}_j = \frac{1}{m}\sum_{k=1}^{m}\overline{x}_{jk};\ \overline{S}_j^2 = \frac{1}{m}\sum_{k=1}^{m}S_{jk}^2;\ j = 1, \dots, p$$

$$\overline{S}_{jh} = \frac{1}{m}\sum_{k=1}^{m}S_{jhk},\ j \neq h$$

- $\overline{\overline{x}}_j$ is the j-th element of the vector $\overline{\overline{\mathbf{x}}}$, an unbiased estimator of the vector $\boldsymbol{\mu}$
- $\overline{S}_j^2$ is the j-th element of diagonal of the matrix $\mathbf{S}$ and $\overline{S}_{jh}$ is the jh-th element of the same matrix. Matrix $\mathbf{S}$ is an unbiased estimator of $\boldsymbol{\Sigma}$

T^2 Control Chart

- This procedure is called Hotelling T^2 control chart
- The monitored statistics is

$$T^2 = n(\bar{\mathbf{x}} - \bar{\bar{\mathbf{x}}})' \mathbf{S}^{-1} (\bar{\mathbf{x}} - \bar{\bar{\mathbf{x}}})$$

T^2 Control Chart

- Careful selection of the control limit must be taken for T^2 statistic
- It depends on the phases of control chart usage
- Phase 1 - use of charts for establishing control; that is, testing whether the process was in control when the m preliminary subgroups were drawn and the estimates computed - called retrospective analysis

$$UCL = \frac{p(m-1)(n-1)}{mn-m-p+1} F_{\alpha,p,mn-m-p+1}$$

T^2 Control Chart

- Phase 2 - the chart is used for monitoring a future production

$$UCL = \frac{p(m+1)(n-1)}{mn-m-p+1} F_{\alpha,p,mn-m-p+1}$$

T^2 Control chart: interpreting out-of-control signals

- One difficult in any multivariate control chart - practical interpretation of the signals
- Which of p variable is responsible for the signal?
- Standard practices:
- Alt (1985): plot univariate $\overline{X}$ charts on the individual variables with Bonferroni-type control limits (use $z_{\alpha/(2p)}$ in place of $z_{\alpha/2}$)

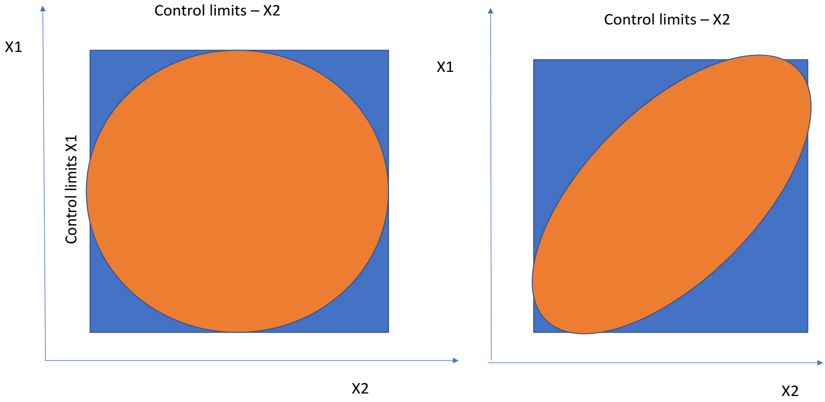

Figure 1: Two no-correlated and correlated variables- ellipsoid contours

T^2 Control chart: interpreting out-of-control signals

- One difficult in any multivariate control chart - practical interpretation of the signals
- Which of p variable is responsible for the signal?
- Standard practices:
  - Runger et al. (1996): Decomposition of T^2 into components that reflects the contribution of each individual variable:

    $$d_i = T^2 - T^2_{(i)}; T^2_{(i)}$$

    is the statistic for all variables except the ith one, i=1,. . . , p

Data for exercise

Table 1: Data for exercises

Sample	Average stiffness	Average bending
1	263.52	469.05
2	265.62	470.37
3	265.87	474.68
4	261.72	470.66
5	263.51	466.9
6	261.42	469.76
7	266.14	470.73
8	268.85	469.1
9	258.11	459.74
10	261.52	465.17
11	262.28	466.25
12	267.86	474.19
13	263.32	473.24
14	263.08	474.02
15	262.19	474.76
16	269.99	465.22
17	268.21	468.46
18	272.85	467.62
19	269.93	466.16
20	278.79	474.16

Exercise

- Stiffness and bending strength are important quality characteristics for lumber manufacturing plant and they should be jointing monitored. Twenty preliminary samples (each one with 10 units) are collected. The data are summarized in Table 1
- Use $\alpha = 0.005$ and determine the control limits for monitoring $\boldsymbol{\mu_0} = \begin{bmatrix} 265 \\ 470 \end{bmatrix}$ with $\boldsymbol{\Sigma_0} = \begin{bmatrix} 100 & 66 \\ 66 & 121 \end{bmatrix}$
- If some samples are judged as out-of-control, use the individual control chart adequately set the individual type error I as $\alpha = 00027$ (by Bonferroni inequality) to identify which dimension is out-of-control
- Repeat the procedure proposed by Runger et al. (1996) for the samples are judged as out-of-control to identify which quality characteristics is responsible for the signal

Data for Exercise

Table 2: Data for exercises

Sample	Average Tensile strength	Average Diameter	Variance S_1^2	Variance S_2^2	Covariance S_{12}
1	115.25	1.04	1.25	0.87	0.8
2	115.91	1.06	1.26	0.85	0.81
3	115.05	1.09	1.3	0.9	0.82
4	116.21	1.05	1.02	0.85	0.81
5	115.9	1.07	1.16	0.73	0.8
6	115.55	1.06	1.01	0.8	0.76
7	114.98	1.05	1.25	0.78	0.75
8	115.25	1.1	1.4	0.83	0.8
9	116.15	1.09	1.19	0.87	0.83
10	115.92	1.05	1.17	0.86	0.95
11	115.92	0.99	1.45	0.79	0.78
12	115.75	1.06	1.24	0.82	0.81
13	114.9	1.05	1.26	0.55	0.72
14	116.01	1.07	1.17	0.76	0.75
15	115.83	1.11	1.23	0.89	0.82
16	115.29	1.04	1.24	0.91	0.83
17	115.47	1.03	1.2	0.95	0.7
18	115.58	1.05	1.18	0.83	0.79
19	115.72	1.06	1.31	0.89	0.76
20	115.4	1.04	1.29	0.85	0.68

Exercise

- Tensile strength and diameter of a textile fiber are important quality characteristics and they should be jointing monitored. Twenty preliminary samples (each one with 10 units) are collected. The data are summarized in Table 2
- Use $\alpha = 0.001$ and determine the control limits for phase I.
- Which should be the control limits for phase II

T^2 control chart: Questions for seminars

- Describe the procedures: Case 1 - Haridy et al. (2014)- a procedure to build for exact simultaneous confidence intervals
- Case 2 - Jackson (1980): use of control charts based on p principal components
- Case 3: Murphy (1987); Case 4: Chua & Montgomery (1992)- , Case 5-Tracy et al. (1996) Mason et al. (1995, 1996)
- How is the performance of all these methods?
- Find other related contributions in the literature

T^2 control chart for individual observation

- Some industrial settings the subgroup size n=1 like chemical process
- m samples, each of size $n = 1$ are available
- Let $\overline{\mathbf{X}}$ and the matrix $\mathbf{S}$ the sample mean vector and covariance matrix of these observations

$$T^2 = (\mathbf{x} - \overline{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x} - \overline{\mathbf{x}})$$

- Phase 2 control limit:

$$UCL = \frac{p(m+1)(m-1)}{m^2 - mp} F_{\alpha,p,m-p} \text{ or } \chi^2_{\alpha,p} \text{ if m} > 100$$

- Phase 1 control limit

$$UCL = \frac{(m-1)^2}{m} \beta_{\alpha,p/2,(m-p-1)/2}$$

$\beta_{\alpha,p/2,(m-p-1)/2}$ is the upper α percentage of a Beta distribution with parameters p/2,(m-p-1)/2

Monitoring bivariate means by attribute charts

- Some contributions are found in the literature like:
  - np_{xy} and np_w proposed by Ho & Costa (2015) and
  - *Max D* by Melo et al. (2017*b*)
- Like other attribute charts for monitoring a variable quality characteristic, the items are classified using some device.
- In bivariate processes, the classifications are made on the dimensions X and Y.
- What differs among the proposals is the statistics used to monitoring.

np_{xy} and np_w charts proposed by Ho & Costa (2015)

- Some assumptions: the values of the dimensions X and Y are standardized
- Only upper discriminating limit (UDL) is used and equal for the (standardized) dimensions X and Y.
- Items are classified as first, second or third class according to the UDL
  - First class: if (X $<$UDL) and (Y$<$ UDL)
  - Thirs class: if (X $>$ UDL) and (Y$>$ UDL)
  - Otherwise results: the item is classified as

np_{xy} and np_w charts proposed by Ho & Costa (2015)

- Let p_1= P[(X $<$UDL) and (Y$<$ UDL)] - the probability of an item be of the first class
- p_3 =P[(X $>$ UDL) and (Y$>$ UDL)] - the probability of an item be of the third class
- And $p_2 = 1 - p_1 - p_3$= probability of the item be of the second class
- After classification: n_1, n_2, and n_3 items classified as the first, second and third class
- n_1, n_2, n_3 follows a trinomial distribution with parameters: n, p_1, p_2, p_3

np_{xy} and np_w charts proposed by Ho & Costa (2015)

- Control chart np_{xy}: the monitored statistic is $M = n_2 + n_3$
  - The process is declared out of control whenever $M > UCL_{xy}$
  - M follows a binomial distribution with parameters: $n; (1 - p_0)$
- Control chart np_w: the monitored statistic is $W = n_2 + 2n_3$
  - The process is declared out of control whenever $W > UCL_w$

Table 3: np_{xy} and np_w for $n = 6$ and $\rho = 0.8$

k_1	k_2	np_{xy}		np_w			
0.00	0.00	370.31	370.35	370.40	370.40	370.40	370.40
0.00	0.25	149.06	148.41*	170.88	164.82	167.94	166.03
0.00	0.50	53.51*	53.85	81.89	79.78	84.99	84.54
0.00	0.75	19.59*	20.05	40.82	41.68	46.53	48.27
0.00	1.00	8.06*	8.40	20.91	23.15	27.25	30.30
0.25	0.25	83.14	81.87	85.22	77.13	76.32	75.60*
0.25	0.50	38.85	38.52*	43.86	39.18	39.46	39.03
0.25	0.75	16.84*	17.02	23.47	21.52	22.50	22.53
0.25	1.00	7.58*	7.83	13.11	12.70	13.82	14.37
0.50	0.50	23.86	23.61	24.44	21.05	20.78	20.67*
0.50	0.75	12.89	12.92	14.08	12.19*	12.23	12.21
0.50	1.00	6.68*	6.85	8.46	7.59	7.85	7.94
0.75	0.75	8.71	8.76	8.78	7.49	7.42*	7.46
0.75	1.00	5.40	5.51	5.67	4.93*	4.95	5.00
1.00	1.00	4.00	4.10	3.95	3.44	3.42*	3.48
	UCL	3	4	4	6	7	8
	UDL	1.380	0.989	1.602	1.143	0.978	0.736

np_{xy} and np_w charts proposed by Ho & Costa (2015)

Table 4: Comparing np_{xy}, np_w and T^2 control charts

		$\rho = 0.5$						$\rho = 0.0$				
Shifts		T^2	np_xy		np_w		T^2	np_xy		np_w		
k1	k2	n=3	n=3	n=6	n=3	n=6	n=3	n=3	n=6	n=3	n=6
0	0	370.37	370.11	370.2	370.4	370.4	370.37	370.04	370.16	370.4	370.4
0	0.25	161.87	184.93	150.2	194.7	151.31	230.61	179.14	143.99	172.3	131.18
0	0.5	42.26	86.82	55.57	107.2	66.94	90.9	83.46	54.11	84.3	51.42
0	0.75	12.82	40.58	20.92	61.56	32.24	35.18	39.3	21.2	44	22.68
0	1	4.98	19.78	8.72	36.89	16.96	14.99	19.43	9.17	24.8	11.35
0.25	0.25	214.84	111.38	81.44	105.5	67.51	159.77	102.35	70.59	86.6	53.18
0.25	0.5	84.34	61.37	38.38	60.16	32.53	72.26	55.01	32.39	45.5	23.62
0.25	0.75	25.26	32.42	17.11	36.01	17.01	30.4	29.04	14.86	25.3	11.67
0.25	1	8.71	17.21	7.9	22.54	9.68	13.61	15.65	7.24	15	6.47
0.5	0.5	77.1	39.57	23.05	35.43	17.02	41.18	33.88	20.03	25.6	11.8
0.5	0.75	35.08	23.93	12.57	21.96	9.64	20.69	20.14	10.94	15.1	6.5
0.5	1	13.29	14.1	6.68	14.29	5.91	10.46	11.96	6.04	9.5	3.98
0.75	0.75	28.02	16.49	8.35	14.06	5.88	12.41	13.39	6.15	9.5	3.97
0.75	1	14.88	10.87	5.23	9.46	3.88	7.24	8.77	3.9	6.3	2.66
1	1	11.55	7.99	3.83	6.56	2.73	4.82	6.29	2.8	4.3	1.94
UCL		11.829	2	3	3	6	11.829	2	3	3	6
UDL			1.394	0.69	1.394	1.04		1.459	1.531	1.166	0.885

Max D proposed by Melo et al. (2017b)

- Each item is classified as approved or disapproved in respect to each quality characteristic by a gauge
- An item is classified as disapproved in i-th quality characteristic if its value is out of discriminating limits: $w_L; w_U$
- Let $D_i =$ number of disapproved items in i-th quality characteristics in a sample of n units
- The monitor statistic is Max D$=$ max$(D_1, D_2, \ldots, D_p)$
- A signal is triggered whenever Max $D > L$, L, the control limit set to satisfy a performance metric

Max D control chart - bivariate quality characteristics

After the classification:

- n_{11}, # of units disapproved in both quality characteristics;
- n_{10}, # of units disapproved in Y and approved in X;
- n_{01}, # of units approved in Y and disapproved in X and
- n_{00}, # of units approved in both quality characteristics

- with $n = n_{11} + n_{10} + n_{01} + n_{00}$.
- $D_x = n_{11} + n_{01}$, $D_y = n_{11} + n_{10}$, are respectively the number of disapproved units in the quality characteristic X and Y.
- The vector (D_x, D_y) follows a bivariate binomial distribution
- its probability distribution function is expressed as

$$
\begin{aligned}
P(D_y = y, D_x = x|\boldsymbol{\mu}, \boldsymbol{\Sigma}) &= \sum_j \frac{n!}{j!(y-j)!(x-j)!(n-y-x+j)!} \\
&\times P_{11}^{j} P_{10}^{y-j} P_{01}^{x-j} P_{00}^{n-y-x+j}
\end{aligned}
$$

the sum of $j \in [0, 1, \ldots, n]$ must satisfy the restrictions $(y-j) \geq 0$, $(x-j) \geq 0$ and $(n-y-x+j) \geq 0$.

P_{11}, the probability to be disapproved in both quality characteristics; P_{10}, the probability to be disapproved in Y and approved in X; P_{01}, the probability to be approved in Y and disapproved in X and P_{00}, the probability to be approved in both quality characteristics.

Max D control chart

- If $D_x \leq UCL_{xy}$ and $D_y \leq UCL_{xy}$, the production goes on.
- A signal is triggered if $(D_x > UCL_{xy}) \cup (D_y > UCL_{xy})$.
- Instead of drawing 2 control charts D_x and D_y, we plot $Max\ D = \max\{D_x, D_y\}$
- If $Max\ D > UCL_{xy}$, the signal for adjustment of the process is equivalently triggered.
- The control limit UCL_{xy} is an integer like the np_x Wu et al. (2009)
- Main advantage: the possibility to use a single attribute control chart to monitor two process means even $n = 1$.

The risk of type I, known as α, is expressed as

$$\alpha = 1 - \sum_{i=0}^{UCL_{xy}} \sum_{j=0}^{UCL_{xy}} P(D_x = i, D_y = j | \boldsymbol{\mu_0}, \boldsymbol{\Sigma}).$$

And the risk of type II, β is given by

$$\beta = \sum_{i=0}^{UCL_{xy}} \sum_{j=0}^{UCL_{xy}} P(D_x = i, D_y = j | \boldsymbol{\mu_1}, \boldsymbol{\Sigma}).$$

Max D control chart

Thus for bivariate quality characteristics, the problem can be formulated as

$$w^0_{U_Y}, w^0_{U_X}, w^0_{L_Y}, w^0_{L_X}, UCL^0_{xy} = arg\ min\ ARL_1\ (w_{U_Y}, w_{U_X}, w_{L_Y}, w_{L_X}, UCL_{xy})$$

$$\text{subject to}\ \ ARL_0 = \frac{1}{\alpha}$$

Comparing Max D and T^2

Table 5: Values of ARL_1 of T^2 and *Max D* control charts: $n = 3$.

		ρ											
		0.0			0.3			0.5			0.8		
δ_x	δ_y	d^2	*Max D*	T^2	d^2	*Max D*	T^2	d^2	*Max D*	T^2	d^2	*Max D*	T^2
0.00	0.00	0.00	370.00	370.00	0.00	370.00	370.00	0.00	370.00	370.00	0.00	370.00	370.00
	0.25	0.19	**170.92**	230.39	0.21	**171.16**	221.35	0.25	**171.17**	202.04	0.52	170.30	125.55
	0.50	0.75	**70.11**	90.82	0.82	**70.23**	92.68	1.00	70.09	67.27	2.08	68.44	26.12
	1.00	3.00	**14.09**	14.98	3.30	14.09	12.89	4.00	14.03	9.40	8.33	13.53	2.87
	2.00	12.00	**2.09**	2.76	13.19	2.09	1.58	16.00	2.09	1.32	33.33	2.05	1.01
0.25	0.25	0.38	**111.22**	159.63	0.29	**111.74**	187.26	0.25	**112.70**	202.04	0.21	**115.57**	220.27
	0.50	0.94	**57.43**	72.21	0.78	**57.99**	87.06	0.75	**58.49**	90.92	0.94	**59.76**	72.21
	1.00	3.19	**13.53**	13.60	3.01	**13.60**	14.92	3.25	13.63	13.19	5.52	13.38	5.52
	2.00	12.19	2.09	1.73	12.40	2.09	1.69	14.25	2.09	1.46	27.19	2.05	1.03
0.50	0.50	1.50	**38.92**	41.15	1.15	**39.49**	57.07	1.00	**40.15**	67.24	0.83	**42.17**	81.75
	1.00	3.75	12.23	10.45	3.13	**12.42**	13.99	3.00	**12.59**	14.98	3.75	12.80	10.45
	2.00	12.75	2.07	1.64	12.03	2.08	1.73	13.00	2.08	1.61	22.08	2.05	1.09
1.00	1.00	6.00	7.43	4.82	4.62	7.71	7.42	4.00	**7.99**	9.40	3.33	**8.70**	12.66
	2.00	15.00	1.95	1.40	12.53	1.99	1.67	12.00	2.02	1.76	15.00	2.04	1.40
2.00	2.00	24.00	1.38	1.06	18.46	1.44	1.19	16.00	1.50	1.32	13.33	1.63	1.57
UCL			2	11.827		2	11.827		2	11.827		2	11.827
w_U			1.223			1.222			1.220			1.203	

MSS for *Max D* control chart to outperform T^2 control chart with $n = 3$

Table 6: Minimum sample size (MSS) needed for *MaxD* control chart to outperform T^2 control chart with $n = 3$

		ρ											
		0.0			0.3			0.5			0.8		
		T^2	*Max D*		T^2	*Max D*		T^2	*Max D*		T^2	*Max D*	
δ_x	δ_y	ARL_1	ARL_1	MSS	ARL_1	ARL_1	MSS	ARL_1	ARL_1	MSS	ARL_1	ARL_1	MSS
0.00	0.25	230.39	193.33	2	221.35	193.66	2	202.04	194.02	2	125.55	116.43	7
0.00	0.50	90.82	90.37	2	82.68	70.23	3	67.27	57.38	4	26.12	24.27	9
0.00	1.00	14.98	14.09	3	12.89	9.95	4	9.40	7.35	5	2.87	2.54	11
0.00	2.00	1.76	1.52	4	1.58	1.52	4	1.32	1.31	5	1.01	1.01	11
0.25	0.25	159.63	130.97	2	187.26	131.65	2	202.04	132.82	2	220.27	135.78	2
0.25	0.50	72.21	57.43	3	87.06	74.55	2	90.82	75.29	2	72.21	59.76	3
0.25	1.00	13.60	13.53	3	14.92	13.60	3	13.19	9.70	4	5.52	4.57	7
0.25	2.00	1.73	1.52	4	1.69	1.52	4	1.46	1.31	5	1.03	1.03	9
0.50	0.50	41.15	38.92	3	57.07	52.34	2	67.27	53.28	2	81.75	55.64	2
0.50	1.00	10.45	8.79	4	13.99	12.42	3	14.98	12.59	3	10.45	9.17	4
0.50	2.00	1.64	1.51	4	1.75	1.52	4	1.61	1.52	4	1.09	1.05	8
1.00	1.00	4.82	3.99	5	7.42	5.52	4	9.40	7.99	3	12.66	8.70	3
1.00	2.00	1.40	1.26	5	1.67	1.48	4	1.76	1.49	4	1.40	1.30	5
2.00	2.00	1.06	1.06	5	1.19	1.17	4	1.32	1.21	4	1.57	1.27	4

Comparing *Max D* versus np_{xy}

Table 7: *Max D* versus np_{xy} with $n = 6$.

		ρ															
		0.0				0.3				0.5				0.8			
δ_x	δ_y	*Max D*		np_{xy}		*Max D*		np_{xy}		*Max D*		np_{xy}		*Max D*		np_{xy}	
0.00	0.00	370.00	370.00	370.16	370.38	370.00	370.00	370.21	370.29	370.00	370.00	370.21	370.05	370.00	370.00	370.31	370.35
0.00	0.25	128.87	**126.95**	143.99	141.31	129.51	**127.80**	147.69	146.16	129.22	**127.17**	150.16	148.34	**128.60**	128.61	149.06	148.41
0.00	0.50	39.52	**38.81**	54.11	53.46	39.71	**39.06**	56.08	56.15	39.60	**38.80**	55.57	56.89	38.73	**37.98**	53.51	53.85
0.00	0.75	13.79	**13.71**	21.20	21.39	13.83	**13.77**	21.97	22.50	13.78	**13.68**	20.92	22.54	13.42	**13.34**	19.59	20.05
0.00	1.00	5.85	**5.93**	9.17	9.50	**5.87**	5.95	9.45	9.92	**5.85**	5.91	8.72	9.81	**5.71**	5.78	8.06	8.40
0.25	0.25	78.20	76.71	70.59	**67.31**	79.05	77.85	75.03	**72.82**	79.47	78.02	81.44	**76.34**	81.87	**80.30**	83.14	81.87
0.25	0.50	33.07	32.44	32.39	**30.88**	33.47	**32.93**	35.07	34.20	33.76	**33.05**	38.38	36.21	34.38	**33.70**	38.85	38.52
0.25	0.75	12.95	**12.86**	14.86	14.45	13.08	**13.01**	16.13	16.07	13.14	**13.02**	17.11	16.89	13.09	**13.00**	16.84	17.02
0.25	1.00	**5.71**	5.78	7.24	7.24	**5.76**	5.83	7.79	7.97	**5.76**	5.82	7.90	8.24	**5.68**	5.75	7.58	7.83
0.50	0.50	21.10	20.70	20.03	**19.43**	21.57	21.21	20.03	**19.43**	21.99	21.53	23.05	**21.06**	23.19	**22.74**	23.86	23.61
0.50	0.75	10.69	**10.59**	10.94	10.75	10.95	10.87	10.94	**10.75**	11.16	**11.04**	12.57	11.69	11.63	**11.53**	12.89	12.92
0.50	1.00	**5.27**	5.32	6.04	6.08	**5.38**	5.44	6.04	6.08	**5.45**	5.49	6.68	6.52	**5.52**	5.59	6.68	6.85
0.75	0.75	7.27	7.23	6.15	**5.89**	7.55	7.51	7.04	**6.94**	7.78	7.71	8.35	**7.64**	8.37	**8.32**	8.71	8.76
0.75	1.00	4.37	4.40	3.90	**3.81**	4.54	4.57	**4.46**	4.47	**4.67**	4.69	5.23	4.90	**4.95**	4.99	5.40	5.51
1.00	1.00	3.22	3.26	2.80	**2.76**	3.38	3.42	**3.22**	3.24	**3.52**	3.54	3.83	3.57	**3.82**	3.86	4.00	4.10
UCL		3	4	3	4	3	4	3	4	3	4	3	4	3	4	3	4
w_U		1.271	0.866	1.531	1.182	1.272	0.867	1.502	1.138	1.270	0.864	1.469	1.094	1.261	0.856	1.380	0.989

Monitoring bivariate means by attribute+variable Max D-T^2 control chart

- Max D-T^2 chart proposed by Melo et al. (2017*a*)
- The sample of n units is split into 2 sub-samples: n_1 and $n_2 = n - n_1$
- Evaluate n_1 attributively by a gauge and get the statistic Max D
- If Max D $>$ C, then measure n_2 units and calculate T^2. If $T^2 > L$, then the process is stopped for adjustment

Max D-T^2 control chart

In this case, the error of type I of the combined control chart, α is expressed as

$$\alpha = P(Max\ D > C|n_1, \boldsymbol{\mu_0}, \boldsymbol{\Sigma_0}) \times P(T^2 > L|n_2, \boldsymbol{\mu_0}, \boldsymbol{\Sigma_0}) = \alpha_D \times \alpha_T \tag{1}$$

Note that the error of type I is a product of two components, the first related to *Max D*, α_D, and the second part is related to T^2 control chart, α_T.

The type error II, β for the combined control chart is expressed as

$$\beta = 1 - P(D_1 > C \cup \ldots \cup D_q > C \mid n_1, \boldsymbol{\mu_1}, \boldsymbol{\Sigma_0}) \times P(T^2 > L \mid n_2, \boldsymbol{\mu_1}, \boldsymbol{\Sigma_0})$$

Max D-T^2 control chart

Thus the problem can be formulated as

$$C^0, L^0, n_1^0, n_2^0, w_{L_i}^0; w_{U_i}^0, i = 1, \ldots, q = arg\ min\ ARL_1\ (C, L, n_1, n_2, w_{L_i}; w_{U_i}, i = 1, \ldots, q)$$

$$\text{subject to}\ \ ARL_0 = \frac{1}{\alpha}$$

Max D-T^2 control chart

Table 8: Some designs of $Max\ D - T^2$ control chart

							Max D			T^2	
ρ	δ_1	δ_2	n_1	n_2	ASS	ARL_1	C	w	α_D	L	α_{T^2}
0.0	0	0.5	2	4	2.721	57.098	1	0.503	0.180	8.399	0.015
			3	6	4.622	34.972	1	0.705	0.270	9.210	0.010
			6	6	6.404	27.066	3	0.694	0.068	6.438	0.040
			7	5	7.118	27.801	4	0.690	0.024	4.326	0.115
	0.5	0.5	2	4	2.721	21.052	1	0.503	0.180	8.399	0.015
			3	6	4.622	11.617	1	0.705	0.270	9.210	0.010
			6	6	6.649	9.024	3	0.583	0.108	7.378	0.025
			7	5	7.300	9.947	4	0.513	0.060	6.202	0.045
0.5	0	0.5	2	4	3.081	43.741	1	0.237	0.270	9.210	0.010
			3	6	6.243	25.310	1	0.226	0.541	10.597	0.005
			6	6	6.811	20.829	3	0.506	0.135	7.824	0.020
			7	5	7.270	22.868	4	0.524	0.054	5.991	0.050
	0.5	0.5	2	4	2.360	33.236	1	0.758	0.090	7.013	0.030
			3	6	3.811	19.726	1	0.956	0.135	7.824	0.020
			6	6	6.295	14.102	3	0.753	0.049	5.801	0.055
			7	5	7.113	14.418	4	0.691	0.023	4.241	0.120
0.8	0	0.5	2	4	4.162	16.837	1	0.698	0.541	10.597	0.005
			3	6	6.243	8.811	1	0.140	0.541	10.597	0.005
			6	6	9.243	8.073	3	0.316	0.541	10.597	0.005
			7	5	8.351	10.047	3	0.418	0.270	9.210	0.010
	0.5	0.5	2	4	2.270	40.099	1	0.817	0.068	6.438	0.040
			3	6	3.649	24.665	1	0.993	0.108	7.378	0.025
			6	6	6.203	17.075	3	0.813	0.034	5.051	0.080
			7	5	7.073	16.864	4	0.752	0.015	3.375	0.185

Max D-T^2 versus T^2 control charts

Table 9: Values of ARL_1 of $Max\ D - T^2$: $\rho = 0.0$, $\delta_1 = 0.0$ and $\delta_2 = 0.5$.

n_2	Sub-sample size n_1: Attribute chart						
	1	2	3	4	5	6	7
1	132.207 # (1.003)	90.416 (2.003)	70.173 (3.003)	57.402 (4.003)	46.138 (5.003)	38.879 (6.003)	32.972 (7.003)
2	107.162 (1.216)	86.619 (2.031)	70.013 (3.007)	57.342 (4.005)	46.090 (5.005)	38.838 (6.005)	32.938 (7.005)
3	80.928 ★ (1.811)	71.260 (2.270)	62.591 (3.090)	52.704 (4.054)	44.708 (5.024)	38.536 (6.014)	32.834 (7.011)
4	62.426 (3.162)	57.098 • (2.721)	51.488 (3.541)	45.175 (4.216)	39.924 (5.103)	35.262 (6.075)	30.841 (7.044)
5	49.061 (3.703)	46.103 (3.351)	42.221 ♠ (3.901)	38.164 (4.541)	34.645 (5.270)	31.028 (6.193)	27.801 (7.118)
6	39.642 (4.243)	37.913 (5.243)	34.972 (4.622)	32.291 ▲ (4.811)	29.861 ■ (5.541)	27.066 (6.405)	24.693 (7.249)
7	32.741 (4.784)	31.313 (5.784)	29.438 (4.892)	27.501 (5.261)	25.645 (6.261)	23.609 ♣ (6.757)	21.838 (7.473)
T^2 chart n_{T^2}	**202.043** # (1)	**129.684** ★ (2)	**90.824** • (3)	**67.268** ♠ (4)	**51.833** ▲ (5)	**41.150** ■ (6)	**33.445** ♣ (7)

Monitoring a covariance-variance matrix

Monitoring a covariance-variance matrix

Some contributions found in the literature: the first results in Alt (1985) are based on asymptotic distribution of some statistics

Monitoring matrix of covariance-variance: W, NT and $\mathbf{v}_t$ control charts

- The statistic W is calculated

$$W = -pn + pn\ln(n) - n\ln\left(\frac{|\mathbf{A}|}{|\boldsymbol{\Sigma}|}\right) + tr(\boldsymbol{\Sigma}^{-1}\mathbf{A})$$

- $\mathbf{A} = (n-1)\mathbf{S}$, $\mathbf{S}$, the observed matrix of covariance-variance
- The statistic NT

$$NT = \frac{n-1}{2} tr\left(\mathbf{S}\boldsymbol{\Sigma}_0^{-1} - \mathbf{I}\right)^2$$

- W and NT follow asymptotically a Chi-square distribution with $0.5p(p+1)$ degrees of freedom
- The statistic v_t for bivariate process is

$$v_t = P(F_{(2n-4),2(N-k-1)} < \left(\frac{N-k-1}{n-2}\right)\left(\frac{|n\mathbf{S}_t|}{|N\overline{\mathbf{S}}|}\right)^{0.5}$$

k= # of training samples;n=sample size of each training sample;
N=n$\times$k;$\overline{\mathbf{S}} = \frac{\sum_{i=1}^{k}\mathbf{S}_i}{k}$; $\mathbf{S}_i$ is the i-th sample covariance matrix

Monitoring matrix of covariance-variance

- Approach based on the first two moments of $|\mathbf{S}|$
- Central line and control limits build as:

$$CL = E(|\mathbf{S}|) = b_1|\boldsymbol{\Sigma}|, \text{with } b_1 = \frac{1}{(n-1)^p}\prod_{i=1}^{p}(n-i)$$

$$\text{Control limits:} E(|\mathbf{S}|) \pm 3\sqrt{Var(|\mathbf{S}|})$$

$$Var(|\mathbf{S}|) = b_2|\boldsymbol{\Sigma}|^2,$$

$$b_2 = \frac{1}{(n-1)^{2p}}\prod_{i=1}^{p}(n-i)\left[\prod_{j=1}^{p}(n-j-2) - \prod_{j=1}^{p}(n-j)\right]$$

Monitoring matrix of covariance-variance

- Another approach based on asymptotic distribution of $|\mathbf{S}|$
- For $p = 2$,

$$2(n-1)\left(\frac{|\mathbf{S}|}{|\mathbf{\Sigma}|}\right)^{0.5}$$

  follows a Chi-square distribution with (2n-4) degrees of freedom
- Let $\mathbf{S}$, a covariance matrix with n degrees of freedom. Then

$$\sqrt{n}\left(\frac{|\mathbf{S}|}{|\mathbf{\Sigma}|}-1\right)$$

  is asymptotically normally distributed with mean 0 and variance $2p$

Exercises

- Use the data from Table 2 to build control chart to monitor the covariance-variance matrix

Other approaches: VMax

- VMax chart- proposed by Costa & Machado (2009):
- Let $S_i^2 = \sum_{j=1}^{n} \frac{z_{ij}^2}{n}$, $z_{ij} = \frac{X_{ij} - \mu_i}{\sigma_i}$
- VMax=max$(S_1^2, S_2^2, \ldots, S_p^2)$, a signal is triggered whenever $VMax > L$, L, the control limit satisfying some performance metric

Other approaches: VMax

Table 3 The *ARL* for the VMAX chart and for the |**S**| chart (p=2, ρ=0.5)

		n					
		4			5		
		|**S**|	VMAX		|**S**|	VMAX	
			Case I	Case II		Case I	Case II
γ^2	UCL	6.134	4.094	4.094	5.375	3.668	3.668
1.0		200.0	200.0	200.0	200.0	200.0	200.0
1.1		146.8	136.6	143.0	141.4	132.5	139.7
1.2		112.5	92.4	107.0	104.6	86.8	102.4
1.3		89.1	63.9	82.9	80.5	58.3	78.0
1.4		73.3	45.7	66.1	64.1	40.7	61.4
1.5		60.4	33.9	54.1	51.9	29.6	49.6
2.0		30.2	11.6	25.4	24.1	9.62	22.3
3.0		13.6	4.09	10.7	10.2	3.38	9.09
5.0		6.37	1.95	4.77	4.58	1.67	3.98

Figure 2: Comparison VMax versus |**S**|, $p = 2$

Other approaches: VMax

Table 6 The *ARL* for the VMAX chart and for the $|\mathbf{S}|$ chart (p=3, ρ_{12}=ρ_{13}=ρ_{23}=0.5)

		n							
		4				5			
		$\|\mathbf{S}\|$	VMAX			$\|\mathbf{S}\|$	VMAX		
			Case I	Case II	Case III		Case I	Case II	Case III
γ^2	*UCL*	4.050	4.313	4.313	4.313	4.620	3.851	3.851	3.851
1.0		200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.1		160.4	149.9	155.9	157.8	155.2	146.3	153.0	155.2
1.2		135.3	107.7	123.0	127.7	125.3	101.9	118.6	123.8
1.3		116.9	76.8	98.7	105.8	103.5	70.5	93.5	101.3
1.4		102.9	55.6	80.4	89.5	87.3	49.7	75.1	84.7
1.5		89.4	41.3	66.6	77.1	74.5	36.1	61.4	72.3
2.0		54.6	13.7	31.7	42.6	41.6	11.3	27.8	38.6
3.0		29.8	4.55	12.9	20.9	20.7	3.72	10.9	18.2
5.0		15.8	2.06	5.49	9.86	9.93	1.75	4.55	8.35

Figure 3: Comparison VMax versus $|\mathbf{S}|$, $p = 3$

Other approaches: VMax

Table 7 The *ARL* for the VMAX chart (p=4, n=5)

ρ value		Value							
$\rho_{12}=\rho_{13}$		0.5	0.7	0.5	0.7	0.5	0.7	0.5	0.7
$\rho_{14}=\rho_{23}$		0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$\rho_{24}=\rho_{34}$		0.5	0.2	0.5	0.2	0.5	0.2	0.5	0.2
		Case I		Case II		Case III		Case IV	
γ^2	*UCL*	3.980	3.970	3.980	3.970	3.980	3.970	3.980	3.970
1.0		200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.1		152.7	158.6	160.0	162.9	162.5	164.9	164.5	166.0
1.2		112.9	116.9	128.8	130.1	134.7	138.8	135.8	139.1
1.3		79.4	82.1	105.2	108.1	114.6	114.8	118.0	115.0
1.4		56.9	57.9	85.4	87.9	97.3	99.9	99.6	102.7
1.5		41.4	41.9	70.9	73.7	82.9	84.7	88.7	89.9
2.0		12.6	12.8	32.5	33.7	45.3	48.3	52.9	54.3
3.0		3.95	3.92	12.4	13.1	20.9	22.8	27.9	28.4
5.0		1.80	1.80	5.05	5.37	9.49	10.1	13.5	14.2

Figure 4: Comparison VMax versus $|\mathbf{S}|$, $p = 4$

Other approaches: VMax

Table 8 The *ARL* for the VMAX chart and for the chart (p=4, n=5, ρ_{12}=ρ_{13}=ρ_{14}=ρ_{23}=ρ_{24}=ρ_{34}=0.5)

		$\|\mathbf{S}\|$	VMAX			
			Case I	Case II	Case III	Case IV
γ^2	*UCL*	2.000	3.980	3.980	3.980	3.980
1.0		200.0	200.0	200.0	200.0	200.0
1.1		166.8	152.7	160.0	162.5	164.5
1.2		145.6	112.9	128.8	134.7	135.8
1.3		127.9	79.4	105.2	114.6	118.0
1.4		108.5	56.9	85.4	97.3	99.6
1.5		96.9	41.4	70.9	82.9	88.7
2.0		61.1	12.6	32.5	45.3	52.9
3.0		35.7	3.95	12.4	20.9	27.9
5.0		19.2	1.80	5.05	9.49	13.5

Figure 5: Comparison VMax versus $|\mathbf{S}|$, $p = 4$ - equicorrelation case

Other approaches: RMax proposed by Costa & Machado (2011)

- For a sample of n units, let R_i=max $(X_{1i}, X_{2i}, \ldots, X_{ni})$ - min $(X_{1i}, X_{2i}, \ldots, X_{ni})$
- RMax=max$(R_1, R_2, \ldots, R_p)$
- A signal is triggered whenever $RMax > L$, L, the control limit

Other approaches: RMax

Table 6. The ARL for the RMAX chart and for the |**S**| chart ($n = 5$, $\rho_{12} = \rho_{13} = \rho_{23} = 0.5$).

$p = 2$				$p = 3$				
CL		$\|\mathbf{S}\|$	RMAX	CL			$\|\mathbf{S}\|$	RMAX
a_1	a_2	5.375	5.145	a_1	a_2	a_3	4.620	5.294
1.0	1.0	200.0	200.0	1.0	1.0	1.0	200.0	200.0
$\sqrt{1.2}$	1.0	104.6	96.9	$\sqrt{1.2}$	1.0	1.0	125.3	96.9
$\sqrt{1.4}$	1.0	64.1	49.6	$\sqrt{1.4}$	1.0	1.0	87.3	49.6
$\sqrt{2}$	1.0	24.1	13.0	$\sqrt{2}$	1.0	1.0	41.6	13.0
$\sqrt{3}$	1.0	10.2	4.53	$\sqrt{3}$	1.0	1.0	20.7	4.53
$\sqrt{5}$	1.0	4.58	2.08	$\sqrt{5}$	1.0	1.0	9.93	2.08
1.0	1.0	200.0	200.0	1.0	1.0	1.0	200.0	200.0
$\sqrt[4]{1.2}$	$\sqrt[4]{1.2}$	104.6	110.6	$\sqrt[4]{1.2}$	$\sqrt[4]{1.2}$	1.0	125.3	127.4
$\sqrt[4]{1.4}$	$\sqrt[4]{1.4}$	64.1	70.0	$\sqrt[4]{1.4}$	$\sqrt[4]{1.4}$	1.0	87.3	85.4
$\sqrt[4]{2}$	$\sqrt[4]{2}$	24.1	27.8	$\sqrt[4]{2}$	$\sqrt[4]{2}$	1.0	41.6	35.0
$\sqrt[4]{3}$	$\sqrt[4]{3}$	10.2	12.0	$\sqrt[4]{3}$	$\sqrt[4]{3}$	1.0	20.7	14.6
$\sqrt[4]{5}$	$\sqrt[4]{5}$	4.58	5.28	$\sqrt[4]{5}$	$\sqrt[4]{5}$	1.0	9.93	6.17
				1.0	1.0	1.0	200.0	200.0
				$\sqrt[6]{1.2}$	$\sqrt[6]{1.2}$	$\sqrt[6]{1.2}$	125.3	132.0
				$\sqrt[6]{1.4}$	$\sqrt[6]{1.4}$	$\sqrt[6]{1.4}$	87.3	94.4
				$\sqrt[6]{2}$	$\sqrt[6]{2}$	$\sqrt[6]{2}$	41.6	46.4
				$\sqrt[6]{3}$	$\sqrt[6]{3}$	$\sqrt[6]{3}$	20.7	23.0
				$\sqrt[6]{5}$	$\sqrt[6]{5}$	$\sqrt[6]{5}$	9.93	11.0

Figure 6: Comparison RMax versus $|\mathbf{S}|$, $p = 2,\ 3$

Other approaches: RMax

Table 4. The ARL for the RMAX and VMAX charts ($p = 2, \rho = 0.5$).

			n					
			4			5		
			VMAX	RMAX	%	VMAX	RMAX	P_v (%)
a_1	a_2	CL	4.094	4.960	–	3.668	5.145	–
1.0	1.0		200.0	200.0	0	200.0	200.0	–
1.25	1.0		28.5	35.8	25.6	24.7	31.5	27.5
1.25	1.25		16.2	20.5	26.5	13.9	17.9	28.8
1.5	1.0		8.11	10.9	34.4	6.70	9.09	35.7
1.25	1.5		6.94	9.24	33.1	5.78	7.72	33.6
1.5	1.5		4.69	6.24	33.1	3.91	5.20	33.0

Figure 7: Comparison RMax versus Vmax, $p = 2$

Other approaches: RMax

Table 5. The ARL for the RMAX chart and for the VMAX chart ($p = 3$, $\rho_{12} = \rho_{13} = \rho_{23} = 0.5$).

				n					
				4			5		
				VMAX	RMAX	%	VMAX	RMAX	P_v (%)
a_1	a_2	a_3	CL	4.313	5.113	–	3.851	5.294	–
1.0	1.0	1.0		200.0	200.0	0	200.0	200.0	–
1.25	1.0	1.0		34.8	43.9	26.1	30.1	38.8	28.9
1.25	1.25	1.0		19.9	25.6	28.6	17.1	22.3	30.4
1.25	1.25	1.25		14.4	18.4	27.8	12.3	16.0	30.1
1.5	1.5	1.5		4.05	5.38	32.8	3.39	4.45	31.3

Figure 8: Comparison RMax versus VMax, $p = 3$

Other approaches: VMix proposed by Quinino et al. (2012)- for p=2

- Consider W_1 and W_2 two normal correlated random variables
- Let

$$X_1 = Z_1 \text{ and } X_2 = \frac{Z_2 - \rho X_1}{\sqrt{1-\rho^2}},$$

with $Z_1 = \dfrac{W_1 - \mu_1}{\sigma_1}, Z_2 = \dfrac{W_2 - \mu_2}{\sigma_2}$

- $VMix = \dfrac{\sum_{i=1}^{n} X_{1i}^2 + X_{2i}^2}{2n}$, $2n \times VMix$ follows a chi-square distribution with $2n$ degrees of freedom

Other approaches: VMix

(kx; ky)	VMIX	VMAX	|S|	NT	W	vt
(1.1025; 1)	128.583	130.677	140.291	134.201	193.586	144.886
(1.1025; 1.10255)	88.278	97.108	100.448	95.579	188.343	100.321
(1.21; 1)	84.12	82.983	101.976	87.796	181.837	101.926
(1.21; 1.21)	44.625	52.489	56.024	49.595	161.372	55.729
(1.5625; 1)	27.986	24.653	46.395	31.989	103.051	46.275
(1.5625; 1.5625)	10.391	13.359	15.403	12.485	62.937	15.459
(2.25; 1)	7.98	6.700	18.415	9.493	27.419	18.696
(2.25; 2.25)	2.919	3.669	4.529	3.530	11.979	4.556
(4; 1)	2.399	2.134	6.299	2.849	5.067	6.311
(4; 4)	1.266	1.396	1.692	1.421	2.352	1.691

Monitoring Covariance Matrix

A question arises: some statistics used to monitor a vector of mean are also able to monitor covariance matrix?

Some proposals:

- The statistics np_{xy}, np_w and Max D are used to build attribute control charts to monitor covariance by Machado et al. (2018)
- DV Max is a mixed variable-attribute control chart proposed by Machado et al. (2022)

Monitoring covariance matrix: np_{xy}, np_w, Max D, VMax and $|S|$

Table 10: Comparing the performance

			Correlation ρ																			
n	Shifts		0.8					0.5					0.3					0.0				
	a_{11}	a_{22}	MAX D	np_{xy}	np_w	$VMAX$	$\|S\|$	$MAX\ D$	np_{xy}	np_w	$VMAX$	$\|S\|$	$max\ D$	np_{xy}	np_w	$VMAX$	$\|S\|$	$MAX\ D$	np_{xy}	np_w	$VMAX$	$\|S\|$
5	1.00	1.00	369.99	369.99	370.00	370.00	370.00	370.00	369.98	370.00	370	370.00	369.98	369.99	370.00	370.00	370.00	369.99	369.99	370.00	370.00	370.00
	1.00	1.25	52.14	151.96	87.07	34.86	57.82	52.84	150.70	79.98	35.9	124.86	52.81	149.48	66.13	35.93	146.16	52.78	148.69	59.53	35.90	156.75
	1.00	1.50	13.47	69.87	32.25	8.20	23.05	13.72	70.44	26.85	8.45	60.75	13.73	69.82	18.30	8.46	76.05	13.72	69.35	16.25	8.46	84.31
	1.00	1.75	5.93	37.5	15.44	3.70	13.14	6.02	38.38	12.33	3.78	36.17	6.03	38.09	7.75	3.79	46.64	6.03	37.83	7.02	3.79	52.57
	1.00	2.00	3.53	22.92	8.77	2.33	8.95	3.57	23.69	7.05	2.37	24.40	3.57	23.55	4.38	2.37	31.84	3.57	23.39	4.05	2.37	36.17
	1.25	1.25	30.73	89.91	31.29	21.46	19.56	29.22	84.21	28.06	19.8	52.57	28.82	82.14	25.68	19.33	66.46	28.63	80.98	23.52	19.10	74.07
	1.25	1.50	12.05	51.43	14.71	7.71	10.66	11.61	47.88	12.83	7.37	29.34	11.45	46.45	10.92	7.21	38.12	11.37	45.62	9.89	7.12	43.17
	1.25	1.75	5.75	31.12	8.49	3.66	7.19	5.66	29.32	7.25	3.62	19.20	5.60	28.44	5.74	3.57	25.14	5.57	27.91	5.22	3.54	28.64
	1.25	2.00	3.49	20.33	5.64	2.32	5.44	3.47	19.45	4.77	2.32	13.89	3.44	18.91	3.63	2.30	18.16	3.43	18.56	3.35	2.29	20.71
	1.50	1.50	8.16	35.2	8.34	5.42	6.75	7.53	31.68	7.16	4.84	17.88	7.34	30.41	6.26	4.64	23.41	7.24	29.69	5.69	4.54	26.68
	1.50	1.75	4.96	24.06	5.44	3.31	4.98	4.63	21.56	4.65	3.03	12.45	4.51	20.62	3.97	2.92	16.25	4.45	20.09	3.64	2.86	18.52
	1.50	2.00	3.28	17.01	3.94	2.25	3.99	3.12	15.38	3.37	2.14	9.43	3.06	14.71	2.82	2.08	12.21	3.03	14.32	2.61	2.04	13.89
	1.75	1.75	3.8	18.15	3.89	2.63	3.87	3.47	15.94	3.31	2.34	9.06	3.36	15.14	2.87	2.24	11.71	3.31	14.69	2.65	2.19	13.32
	1.75	2.00	2.86	13.8	3.01	2.04	3.22	2.63	12.09	2.58	1.85	7.10	2.55	11.46	2.23	1.78	9.07	2.51	11.1	2.08	1.74	10.27
	2.00	2.00	2.4	11.13	2.45	1.77	2.76	2.19	9.63	2.10	1.59	5.71	2.12	9.09	1.85	1.53	7.20	2.08	8.79	1.74	1.50	8.10
9	1.00	1.00	369.98	370	370.00	370.00	370.00	369.98	370.00	370.00	370	370.00	370.00	369.99	370.00	370.00	370.00	369.99	369.98	370.00	370.00	370.00
	1.00	1.25	36.41	129.24	65.30	21.53	33.39	36.94	128.78	55.18	22.2	88.41	36.93	127.92	47.33	22.24	108.39	36.91	127.36	43.23	22.23	118.72
	1.00	1.50	8.02	51.59	18.51	4.59	11.10	8.15	52.51	13.97	4.71	35.50	8.15	52.17	10.72	4.71	47.00	8.15	51.9	9.80	4.71	53.54
	1.00	1.75	3.42	25.07	7.30	2.17	5.93	3.46	25.93	5.60	2.21	18.86	3.46	25.81	4.28	2.21	25.64	3.46	25.67	4.00	2.21	29.69
	1.00	2.00	2.09	129.24	3.82	1.50	3.97	2.11	128.78	3.11	1.51	11.86	2.11	127.92	2.46	1.51	16.20	2.11	127.36	2.35	1.51	18.86
	1.25	1.25	21.16	72.11	21.21	13.37	9.21	20.07	67.53	17.84	12.2	29.69	19.79	65.98	17.86	11.87	39.70	19.65	65.15	16.57	11.70	45.47
	1.25	1.50	7.34	37.7	8.92	4.41	4.75	7.09	35.20	7.23	4.22	14.71	6.99	34.24	6.85	4.13	20.08	6.94	33.7	6.32	4.07	23.34
	1.25	1.75	3.35	21.02	4.72	2.16	3.20	3.31	19.97	3.81	2.14	9.02	3.28	19.44	3.43	2.12	12.27	3.27	19.13	3.20	2.10	14.29
	1.25	2.00	2.08	37.7	2.98	1.50	2.46	2.07	35.20	2.46	1.50	6.30	2.06	34.24	2.18	1.49	8.46	2.06	33.7	2.07	1.48	9.82
	1.50	1.50	4.96	24.78	4.96	3.18	3.01	4.57	22.33	4.02	2.83	8.32	4.45	21.49	3.94	2.71	11.30	4.38	21.03	3.65	2.65	13.15
	1.50	1.75	2.97	16.09	3.17	2.01	2.27	2.78	14.47	2.60	1.86	5.60	2.71	13.89	2.49	1.80	7.48	2.67	13.56	2.34	1.76	8.66
	1.50	2.00	2	16.09	2.28	1.47	1.89	1.92	14.47	1.91	1.41	4.19	1.88	13.89	1.81	1.38	5.48	1.86	13.56	1.72	1.37	6.30
	1.75	1.75	2.32	11.83	2.32	1.67	1.84	2.13	10.43	1.93	1.51	4.02	2.06	9.94	1.87	1.46	5.25	2.03	9.68	1.77	1.43	6.02
	1.75	2.00	1.79	8.74	1.83	1.38	1.60	1.67	7.70	1.56	1.28	3.16	1.62	7.34	1.51	1.25	4.03	1.6	7.13	1.45	1.23	4.57
	2.00	2.00	1.55	6.95	1.55	1.25	1.44	1.44	6.06	1.35	1.17	2.57	1.40	5.75	1.31	1.14	3.20	1.38	5.58	1.27	1.13	3.59

Sample size

Table 11: Sample size for the *MAX D* and np_w charts to reach an equal performance of the $|S|$ and *VMAX* charts with $n_v = 5$ varying ρ

		MAX D								np_w							
Shifts		$\|S\|$				*VMAX*				$\|S\|$				*VMAX*			
a_{11}	a_{22}	0.8	0.5	0.3	0	0.8	0.5	0.3	0	0.8	0.5	0.3	0	0.8	0.5	0.3	0
1	1.25	5	3	3	3	10	10	10	10	12	3	3	3	24	17	14	13
1	1.5	3	3	3	3	9	9	9	9	9	3	3	3	21	15	13	12
1	1.75	3	3	3	3	9	9	9	9	6	3	3	3	16	13	12	11
1	2	3	3	3	3	8	8	8	8	5	3	3	3	14	12	11	10
1.25	1.25	10	3	3	3	9	10	10	10	10	3	3	3	9	8	8	7
1.25	1.5	6	3	3	3	9	9	9	9	8	3	3	3	11	9	8	8
1.25	1.75	5	3	3	3	9	9	9	9	6	3	3	3	12	10	9	8
1.25	2	4	3	3	3	8	8	8	8	6	3	3	3	12	10	9	9
1.5	1.5	7	3	3	3	9	9	9	9	7	3	3	3	9	8	7	7
1.5	1.75	5	3	3	3	8	9	9	9	6	3	3	3	9	8	8	7
1.5	2	5	3	3	3	8	8	8	8	5	3	3	3	10	8	8	7
1.75	1.75	5	3	3	3	8	8	9	9	6	3	3	3	8	8	7	7
1.75	2	5	3	3	3	8	8	8	8	5	3	3	3	8	8	7	7
2	2	5	3	3	3	8	8	8	8	5	3	3	3	8	7	7	7

DVMax an attribute-variable control chart to monitor a covariance matrix

It is proposed by Machado et al. (2022)

- Monitoring always starts with an attribute chart employing the Max D control chart
- Depending on the outcome, a variable control chart named VMAX chart is run at a second stage to check for process stability
- Two versions were developed: the first version, denoted as the DVMAX_1 chart, two independent samples are used at the two stages of the same inspection; the second version, denoted as the DVMAX_2 chart, the same sample is used at both the first and second stage of the same inspection.

Detailing DVMAX$_1$

1. At each inspection i, a sample Ω_i of items of size n is collected.
2. The sample Ω_i is partitioned in two subsamples: $\Omega_{1,i}$, having size n_1, to be used to run the attribute Max D chart and $\Omega_{2,i}$, having size n_2, to be used to run the variable Vmax chart, if necessary. Of course, $n = n_1 + n_2$.
3. The n_1 items of $\Omega_{1,i}$ undergo two go/no-go gauge tests with respect to the discriminating limits (l_X, u_X) and (l_Y, u_Y): then, the counts of the disapproved items $D_{X,i}$ and $D_{Y,i}$ within $\Omega_{1,i}$ are obtained. The statistic plotted on the Max D control chart is $D_i = \max(D_{X,i}, D_{Y,i})$. If D_i falls below the control limit UCL_D the process is declared to be in-control and the inspection is stopped and goes to step 1.

- 4. Otherwise, the n_2 items of $\Omega_{2,i} = (X_{i,2}, Y_{i,2})$ are measured. Given the $X_{i,2} = (x_{i1}, x_{i2}, \ldots, x_{in_2})^T$ and $Y_{i,2)} = (y_{i1}, y_{i2}, \ldots, y_{in_2})^T$ samples of observations of the two quality characteristics, the statistic plotted on the Vmax control chart is:

$$V_i = \max\left(S^2_{X,i}, S^2_{Y,i}\right)$$

  where $S^2_{X,i} = \sum_{j=1}^{n_2} \frac{z^2_{X,ij}}{n_2}$, $S^2_{Y,i} = \sum_{j=1}^{n_2} \frac{z^2_{Y,ij}}{n_2}$ and $z_{X,ij} = \frac{x_{ij} - \mu_{X0}}{\sigma_{X0}}$, $z_{Y,ij} = \frac{y_{ij} - \mu_{Y0}}{\sigma_{Y0}}$.
- 5. The process is declared to be out of control (in control) when V_i is plotted above (below) the control limit UCL_V.

About $\mathbf{DVMax}_2$

- The DVMAX_2 control chart is similar to the DVMAX_1 chart, but at the first stage the sample Ω_i of n units is entirely used to run the attribute Max D control chart.
- If this chart triggers a signal, then the same n units from Ω_i are measured.

Comparing

Table 12: The ARLs of the Max D, Vmax, $DVmax_1$ and $DVmax_2$ charts ($\rho = 0$)

a_x	a_y	Max D	Vmax	$DVmax_1$	$DVmax_1$	$DVmax_1$	$DVmax_2$	$DVmax_2$	$DVmax_2$
1	1	370.3	370.4	370.4	370.47	370.5	370.4	370.5	370.4
1	1.25	52.78	35.9	28.60	25.13	17.28	35.92	36.93	50.76
1	1.5	13.72	8.46	6.42	5.58	4.18	8.77	8.92	10.88
1	1.75	6.03	3.79	2.94	2.64	2.33	3.96	4.00	4.56
1	2	3.57	2.37	1.93	1.80	1.77	2.49	2.53	2.81
1.25	1.25	28.63	19.1	12.21	10.5	7.14	24.60	26.18	38.80
1.25	1.5	11.37	7.12	4.61	4.00	3.07	8.76	9.34	11.68
1.25	1.75	5.57	3.54	2.42	2.23	2.01	4.07	4.22	4.92
1.25	2	3.43	2.29	1.75	1.63	1.62	2.53	2.62	2.94
1.5	1.5	7.24	4.54	2.86	2.51	2.14	6.02	6.12	11.76
1.5	1.75	4.45	2.86	1.94	1.77	1.69	3.58	3.70	8.32
1.5	2	3.03	2.04	1.58	1.43	1.46	2.40	2.50	2.88
1.75	1.75	3.31	2.19	1.55	1.45	1.47	2.69	2.81	3.46
1.75	2	2.51	1.74	1.33	1.28	1.34	2.06	2.12	2.50
2	2	2.08	1.5	1.21	1.18	1.25	1.75	1.80	2.06
	n_1	5	-	3	3	3	5	5	5
	n_2	-	5	8	10	20	5	5	5
	$\%P_{D_0}$	-	100%	25.0%	20.0%	10.0%	25.0%	20.0%	10.0%

Principal Component chart

- T^2 control chart is effective if p (the number of quality characteristics) is not very large
- As p increases, the performance metric as ARL_1 to detect a specified shift also increases
- It looks like the shift "diluted" in the p-dimensional space of variables
- Most common alternative - monitor by principal component charts

Principal Component chart

- Original variables: $\mathbf{X}$=(X_1, $\ldots X_p$) find new variables $\mathbf{Y}$=(Y_1, $\ldots$, Y_p) as

$$\mathbf{Y} = \mathbf{XC}$$

  c_{ij}, constants to be determined such $\mathbf{Y}$ are no correlated variables

- $\mathbf{C_{p\times p}}$ is determined such that

$$\mathbf{C}'\boldsymbol{\Sigma}\mathbf{C} = \boldsymbol{\lambda}$$

- $\boldsymbol{\lambda}$ - a diagonal matrix, the main diagonal elements $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of the matrix $\boldsymbol{\Sigma}$

Principal component chart

- Properties: $\boldsymbol{\Sigma}$ and $\boldsymbol{\lambda}$:

$$tr(\boldsymbol{\Sigma}) = \sum_{i=1}^{p} \sigma_i^2 = \sum_{i=1}^{p} \lambda_i$$

  σ_i^2 - the variance of the X_i

- $\lambda_1 \ \geq \ \lambda_2 \ \ldots, \geq \ \lambda_p \geq 0$
- λ_i is the variance of the new variable Y_i
- $\mathbf{C} = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p)$, $c_i{=}(c_{1i}, c_{2i}, \ldots, c_{pi})$ is the eigenvector related to the eigenvalue λ_i

Principal Component chart

- For the j-th observation $\mathbf{x}_j = (x_{1j}, \ldots, x_{pj})$
- Principal component scores can be obtained as

$$y_{1j} = c_{11}x_{1j} + \ldots + c_{1p}x_{pj}$$

$$y_{2j} = c_{21}x_{1j} + \ldots + c_{2p}x_{pj}$$

$$\ldots$$

$$y_{pj} = c_{p1}x_{1j} + \ldots + c_{pp}x_{pj}$$

- In general the first r components are retained for analysis such that

$$\frac{\sum_{i=1}^{r} \lambda_i}{\sum_{i=1}^{p} \lambda_i} > k$$

Principal component chart - General framework

- In general the first two principal components are retained
- A 95% (or another level) confidence contour is drawn, and score values of z_{1i} and z_{2i} are plotted.

Principal component chart

Chemical Process Data

	Original Data					
Observation	x_1	x_2	x_3	x_4	z_1	z_2
1	10	20.7	13.6	15.5	0.291681	–0.6034
2	10.5	19.9	18.1	14.8	0.294281	0.491533
3	9.7	20	16.1	16.5	0.197337	0.640937
4	9.8	20.2	19.1	17.1	0.839022	1.469579
5	11.7	21.5	19.8	18.3	3.204876	0.879172
6	11	20.9	10.3	13.8	0.203271	–2.29514
7	8.7	18.8	16.9	16.8	–0.99211	1.670464
8	9.5	19.3	15.3	12.2	–1.70241	–0.36089
9	10.1	19.4	16.2	15.8	–0.14246	0.560808
10	9.5	19.6	13.6	14.5	–0.99498	–0.31493
11	10.5	20.3	17	16.5	0.944697	0.504711
12	9.2	19	11.5	16.3	–1.2195	–0.09129
13	11.3	21.6	14	18.7	2.608666	–0.42176
14	10	19.8	14	15.9	–0.12378	–0.08767
15	8.5	19.2	17.4	15.8	–1.10423	1.472593
16	9.7	20.1	10	16.6	–0.27825	–0.94763
17	8.3	18.4	12.5	14.2	–2.65608	0.135288
18	11.9	21.8	14.1	16.2	2.36528	–1.30494
19	10.3	20.5	15.6	15.1	0.411311	–0.21893
20	8.9	19	8.5	14.7	–2.14662	–1.17849

Figure 9: Data set

Principal Component chart

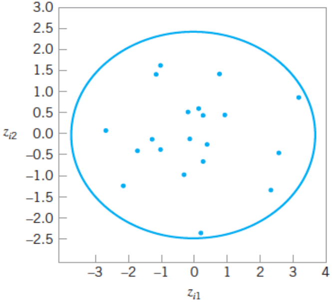

FIGURE 11.16 Scatter plot of the first 20 principal component scores z_{i1} and z_{i2} from Table 11.6, with 95% confidence ellipse.

Figure 10: Plots of the first 20 scores

Principal component chart

			New Data			
Observation	x_1	x_2	x_3	x_4	z_1	z_2
21	9.9	20	15.4	15.9	0.074196	0.239359
22	8.7	19	9.9	16.8	–1.51756	–0.21121
23	11.5	21.8	19.3	12.1	1.408476	–0.87591
24	15.9	24.6	14.7	15.3	6.298001	–3.67398
25	12.6	23.9	17.1	14.2	3.802025	–1.99584
26	14.9	25	16.3	16.6	6.490673	–2.73143
27	9.9	23.7	11.9	18.1	2.738829	–1.37617
28	12.8	26.3	13.5	13.7	4.958747	–3.94851
29	13.1	26.1	10.9	16.8	5.678092	–3.85838
30	9.8	25.8	14.8	15	3.369657	–2.10878

Figure 11: Data set - 2nd part

Principal Component chart

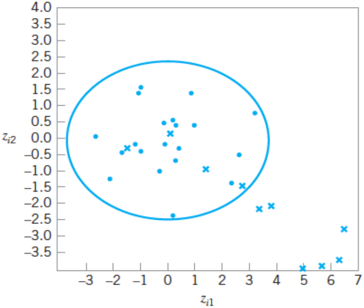

FIGURE 11.17 Principal components trajectory chart, showing the last 10 observations from Table 11.6.

Figure 12: Plots of the last 10 scores

Principal component chart

- If more than 2 components are retained - analysis pairwise scatter plots
- For $r > 4$, may have some difficulties of interpretation of the meaning of the principal components

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