PRO 5971

Statistical Process Monitoring:
Attribute charts to monitor a process mean

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Outline

Preliminaries

The next slides are results of some researches made by a team of contributors.

The aim is: present the improvements on an attribute chart in order to compete with a variable chart in monitoring a process mean;

Some features between attribute x variable Control Charts

Attribute Control Chart

- Simpler, faster, cheaper and operationally easier
- An alternative if the inspection/experiment is destructive
- Standard Criterion used to classify as non-conforming: the specification limits
- Several quality characteristics can jointly be considered

Variable Control chart

- More expensive and time consuming than an attribute Control Chart
- Smaller sample size than attribute control chart to detect a same level of shift
- Use of measurement information of the process performance
- Preferable in a capability process study

Motivation

- Situations/ Scenarios which the use of attribute chart is adequate to monitor the mean/variability of the process:
  - Spend much time and high cost to get precise measurements
  - Destructive experiments- - items discarded after inspection
- Items are inspected/classified by a device
- No measurement of the quality characteristic is realized
- About the sample size used in the attribute control chart:
  - it is known that a Variable chart will always have a better performance than attribute chart if THE SAMPLE SIZES ARE EQUAL
  - The attribute inspection cost is in general cheaper than a measurement of a quality characteristic;
  - So there is possibility to increase the sample size in an attribute inspection in order to have an equal performance of a variable chart

Trade-off

- On one side:
  - Inspection by attribute is easier, faster, cheaper
- But this type of chart may lead to:
  - Increase in the number of parameters in the design of chart
  - Determination of the parameters may be more complicate/complex
  - Use of algorithms to search optimum parameters as for example, the genetic algorithm may be needed

Some go-no go gauges available in the market

Figure 1: Go-nogo gauge

Some researches

- Attribute control chart to monitor process mean and/or variance is not new subject
- Some pioneering contributions: Tippett (1944), Stevens (1948), Mace (1952), Steiner et al. (1994, 1996) and others
- Some recent attribute charts proposed to monitor mean: np, np_x, $\overline{X}_{rec}$, $\overline{X}^{att}$, $\overline{X}^{tn}$, $\overline{X}^{tn}_{(I)}$, $\overline{X}^{tn}_{(K)}$, $np^{\mu}_{x(I)}$

np chart using specification limits to monitor process mean

- Specification limits - set by engineering team/managers
- Device like gauge go-no go calibrated according to specification limits $\mu \pm k\sigma$ (usually $k = 3$) is used to classify the items
- Items which measures of quality characteristic are out of specification limits are classified as non-conforming
- Results after the classification: Y items classified as non-conforming
- Y - follows a binomial distribution (n,p_i, i=0,1)
- with p_0=probability of a non-conforming item when the process is in-control

$$p_0 = 1 - P(\mu_0 - k\sigma < X < \mu_0 + k\sigma | \mu = \mu_0)$$

$$p_0 = 1 - P(-k < Z < +k)$$

Z, standardized normal distribution

- p_1, probability of a non-conforming item when the process is out-of-control

$$p_1 = 1 - P(\mu_0 - k\sigma < X < \mu_0 + k\sigma | \mu_1 = \mu_0 + \delta\sigma)$$

$$p_1 = 1 - P(-k - \delta < Z < k - \delta)$$

np chart using specification limits to monitor process mean

- There is interest to monitor if the non-conforming fraction has increased
- Decision criterion: the process is said to be in-control if

$$Y \leq UCL_{np}$$

- For a fixed α, UCL_{np} can be determined by the binomial distribution or approximated normal distribution

np chart using specification limits to monitor process mean

- Comparing the sample sizes of $\overline{X}$ and np charts

δ	$n_{\overline{X}}$	n_{np}	$\frac{n_{np}}{n_{\overline{X}}}$
0.25	144	32995	229.13
0.5	36	1730	48.06
0.75	16	262	16.39
1	9	60	6.68
1.5	4	6	1.47

Table 1: Comparing the sample size

- Charts planned to have equal ARL_0 and shifts from μ_0 to $\mu_1 = \mu_0 + \delta\sigma$
- Clearly np chart is not efficient to monitor the process mean using the specification limits;
- Large samples (more costly) mainly for shifts $\delta < 1.5$

np_x chart proposed by Wu et al. (2009)

- Other alternatives must be searched/proposed
- One alternative is np_x chart proposed by Wu et al. (2009)
- Operationally the inspection is exactly equal to the previous np chart
- Each item is classified as approved or disapproved using a device calibrated according to OPTIMIZED DISCRIMINANT or WARNING LIMITS: LDL or LWL, UDL or UWL in replacement of the specification limits
- IMPORTANT: Item classified as disapproved does not indicate that it is non-conforming or defective but only the characteristic quality value X $\notin$ [LWL=$\mu_0 - k_w\sigma$; UWL=$\mu_0 + k_w\sigma$]

np_x chart proposed by Wu et al. (2009)

- After the classification: Y, # disapproved items, random variable with binomial distribution (n, p_i, i=0, 1)
- p_0=probability of an item be disapproved when the process is in-control

$$p_0 = 1 - P(\mu_0 - k_w\sigma < X < \mu_0 + k_w\sigma | \mu = \mu_0)$$

$$p_0 = 1 - P(-k_w < Z < +k_w)$$

Z, standardized normal

- p_1, probability of an item be disapproved when the process is out-of-control

$$p_1 = 1 - P(\mu_0 - k_w\sigma < X < \mu_0 + k_w\sigma | \mu_1 = \mu_0 + \delta\sigma)$$

$$p_1 = 1 - P(-k_w - \delta < Z < k_w - \delta)$$

np_x chart proposed by Wu et al. (2009)

- Parameters of the chart: (LWL, UWL)= Warning limits and UCL_{np_x}= control limit of np_x chart
- Objective function

$$LWL^o, UWL^o, UCL^o_{np_x} = argmin\left(ARL_1 = \frac{1}{1-\beta}\right)$$

$$\text{subject to } ARL_0 = \tau = \frac{1}{\alpha}$$

$$\alpha = P(Y > UCL_{np_x}|\mu = \mu_0)$$

$$1-\beta = P(Y > UCL_{np_x}|\mu = \mu_1 = \mu_0 + \delta\sigma)$$

np_x chart proposed by Wu et al. (2009)

- Some plans considering $LWL = \infty$

	$\overline{X}$	np_x		
n	6	6	6	12
k	1.225	1.272	0.867	0.754
UCL	1.225	3	4	7
δ	ARL			
0	740.80	740.58	742.21	740.24
0.25	117.94	156.70	153.54	90.62
0.5	26.37	41.88	41.01	17.55
0.75	8.17	14.08	13.98	5.25
1	3.44	5.91	5.98	2.33
1.25	1.91	3.06	3.15	1.44
1.5	1.33	1.91	1.99	1.13
1.75	1.11	1.40	1.46	1.03
2	1.03	1.17	1.21	1.01
2.25	1.01	1.06	1.09	1.00
2.5	1.00	1.02	1.04	1.00
2.75	1.00	1.01	1.01	1.00

np_x chart proposed by Wu et al. (2009)

- Comparing the performance
  - Sometimes comparing only the ARL_1 values of different plans may not be fair
  - Other criteria as average sampling cost, average sample size need to be included
- Wu et al. (2009) suggest comparing np_x and $\overline{X}$ charts by their average sampling cost

$$\frac{C_{np_x} \times n_{np_x}}{h_{np_x}} = \frac{C_{\overline{X}} \times n_{\overline{X}}}{h_{\overline{X}}}$$

  n_i, C_i and h_i are respectively the sample size, inspection cost and sampling interval of the control charts, $i = np_x; \overline{X}$.

np_x chart proposed by Wu et al. (2009)

- Example : Monitoring the diameters of shafts
  - Parameters of the process: $\mu = 8.018mm; \sigma = 0.012mm$
  - Design : $\overline{X}$ chart
    - Sample size $n_{\overline{X}} = 4$
    - Control limit $UCL_{\overline{X}} = 8.018 = \mu + 3/\sqrt{4}\sigma$
  - Two possibilities of inspection:
    - Use a device
    - Digital Micrometer
  - Average time spent in inspection
    - Device $\overline{t}_{np_x} = 2.125s$;
    - Micrometer $\overline{t}_{\overline{X}} = 9.525s$

- Example: Monitoring the diameter of shafts
- Planning np_x chart

$$\frac{\bar{t}_{\overline{x}}}{\bar{t}_{np_x}} = \frac{9.525}{2.125} = 4.48$$

- Sample size $n_{np_x} = 17 \approx 4.48 \times 4$
- Control and warning limits
  - Warning: UWL=8.0072=$\mu + 0.6\sigma$
  - Control Limit: $UCL_{np_x} = 10$

np_x chart proposed by Wu et al. (2009)

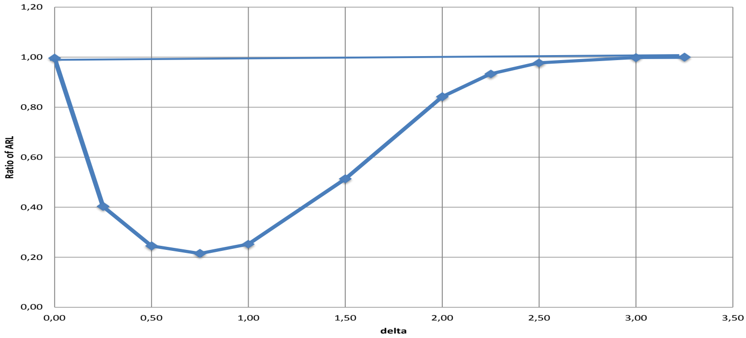

Figure 2: Comparing the plans - Example of shaft diameter monitoring

Weakness and strength of np_x

Strength:

- operational easy, fast, cheap;
- in terms to signal an abnormal situations: only good performance for one-sided shifts

Weakness: not good performance for bilateral shift of mean

$\overline{X}_{rec}$ proposed by Quinino et al. (2015)

- The aim of $\overline{X}_{rec}$: an attribute chart with a better performance than np_x chart
- The quality characteristic is standardized and an attribute inspection procedure is applied
- Each item is classified into 3 classes using a gauge: lower than LWL, above UWL and in the range of [LWL; UWL]
- Results after inspection of n items: N_1 items which values of the quality characteristic are lower than LWL; N_2 above UWL; N_3, in the range of [LWL; UWL]
- Generate - n random values as follow
  - N_1 values from Normal $\sim (-N_1, 1)$
  - N_2 values from Normal $\sim (N_2, 1)$
  - N_3 values from Normal $\sim (0, 1)$
- With the n simulated values calculate the sample mean $\overline{X}_{rec}$
- $\overline{X}_{rec}$ is a mixture of normal distributions; fixed α, LCL and UCL can be determined
- If $LCL < \overline{X}_{rec} < UCL$, is said that the process is in-control

$\overline{X}_{rec}$ chart proposed by Quinino et al. (2015)

- With the process in-control, the values of N_1 and N_2 should be closer resulting a value of $\overline{X}_{rec} \approx 0$
- If the mean shifts, the values of the random variables N_1, N_2, N_3 must change. If $\mu_1 > \mu_0$, it is expected $N_2 > N_1$ and if $\mu_1 < \mu_0$, $N_2 < N_1$
- Objective Function:

$$LWL^o, UWL^o, = argmin[ARL_1(LWL, UWL, LCL, UCL)]$$

$$\text{subject to } ARL_0(LWL, UWL, LCL, UCL) = \tau$$

Table 2: Comparing the performance - equal sample size

	n=6			n=12			n=24		
δ	$\overline{X}$	$\overline{X}_{rec}$	np_x	$\overline{X}$	$\overline{X}_{rec}$	np_x	$\overline{X}$	$\overline{X}_{rec}$	np_x
0.00	370.4	370	370.2	370.4	370	370	23.9	72.59	370.1
0.25	115.9	157.7	251.5	60.69	89.19	239.4	26.36	38.31	72.59
0.50	26.36	45.31	103.6	9.76	17.61	86.63	3.44	6.29	23.9
1.00	3.44	6.665	13.66	1.47	2.39	9.09	1.03	1.89	5.6
1.50	1.33	2.12	3.21	1.01	1.15	2.07	1.00	1.04	1.89
2.00	1.03	1.24	1.42	1.00	1.01	1.14	1.00	1.00	1.05

$\overline{X}$ chart is always better than $\overline{X}_{rec}$ chart;

$\overline{X}_{rec}$ chart is better than np_x

$\overline{X}_{rec}$ chart proposed by Quinino et al. (2015)

- Considering inspection by attributes faster and cheaper, the sample size for $\overline{X}_{rec}$ chart could be increased to have an equal performance of $\overline{X}$ chart
- A search is made to find the min n such that $ARL_{\overline{X}} \sim ARL_{\overline{X}_{rec}}$

Table 3: Matching $\overline{X}$ and $\overline{X}_{rec}$

	$\overline{X}$ and $n = 6$	$\overline{X}_{rec}$ and $n = 9$
0.00	370.4	370.03
0.25	115.87	116.85
0.50	26.36	26.93
1.00	3.44	3.49
1.50	1.33	1.34
2.00	1.03	1.03

$\overline{X}^{att}$ chart proposed by Quinino et al. (2017)

- $\overline{X}^{att}$ is an expansion of $\overline{X}_{rec}$ chart
- Items are classified into 5 classes instead of 3 using a device

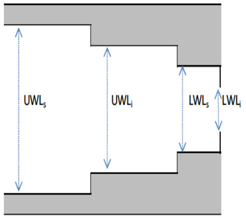

Fig. 1 Go–no-go gauge

Figure 3: Device - 5 classes

- Two sets of discriminant limits are used in the classification satisfying the inequality:

$$LWL_L < LWL_U < \mu_0 < UWL_L < UWL_U$$

- The n items of the sample are classified into:
  - Type 1 - if the value of quality characteristic X $\in (-\infty; LWL_L]$
  - Type 2 - If X $\in (LWL_L; LWL_U]$
  - Type 3 - If X $\in (LWL_U; UWL_L]$
  - Type 4 - If X $\in (UWL_L; UWL_U]$
  - Type 5 - If X $\in (UWL_U; \infty]$
- Result of an inspection :
  - N_1 items of type 1
  - N_2 items of type 2
  - N_3 items of type 3
  - N_4 items of type 4
  - N_5 items of type 5
  - and $n = N_1 + N_2 + N_3 + N_4 + N_5$

$\overline{X}^{att}$ chart proposed by Quinino et al. (2017)

- The statistic $\overline{X}^{att}=$

$$\frac{N_1 Y_1 + N_2 Y_2 + N_3 Y_3 + N_4 Y_4 + N_5 Y_5}{N_1 + N_2 + N_3 + N_4 + N_5}$$

- $Y_1 = (2-a) \times LWL_i$
- $Y_2 = \dfrac{LWL_L + LWL_U}{2}$
- $Y_3 = \dfrac{LWL_U + UWL_L}{2}$
- $Y_4 = \dfrac{UWL_L + UWL_U}{2}$
- $Y_5 = a \times UWL_U$
- $1 < a < 2$

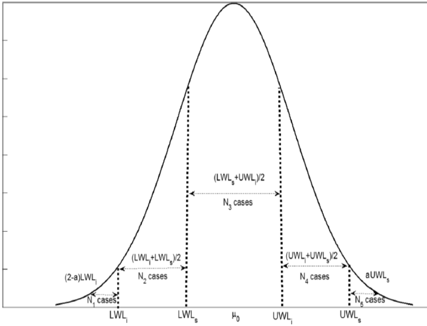

Figure 4: Elements of the statistic $\overline{X}^{att}$

$\overline{X}^{att}$ chart proposed by Quinino et al. (2017) - About $\overline{X}^{att}$:

- Discrete random variable: assumes a finite number of values
- The probability of each value of $\overline{X}^{att}$ is equal to observe the vector $\mathbf{N}=\{N_1, N_2, N_3, N_4, N_5\}$
- The vector **N** follows a multinomial distribution with parameters (n, p_1, p_2, p_3, p_4, p_5) with:
  - $p_1 = P(X \in (-\infty; LWL_L])$
  - $p_2 = P(X \in (LWL_L; LWL_U])$
  - $p_3 = P(X \in (LWL_U; UWL_L])$
  - $p_4 = P(X \in (UWL_L; UWL_U])$
  - $p_5 = P(X \in (UWL_U; \infty])$
- Fixed α, the control limits UCL and LCL can be determined
- Decision criterion: If LCL $< \overline{X}^{att} <$ UCL, the process is declared in-control, otherwise, out-of-control
- The choice of UWL_L, UWL_U, UCL and the constant a:

$$(UWL_i^o, UWL_s^o, UCL^o, a^o) = argmin[ARL_1(UWL_i, UWL_s, UCL, a]$$

Subject to: $ARL_0 = \tau$

Comparing performance among $\overline{X}$, $\overline{X}^{att}$, $\overline{X}_{rec}$

Table 4: Comparing performance varying sample size $n = 4, 5, 6, 7, 8$

	$n=4$			$n=5$			$n=6$			$n=7$			$n=8$		
δ	$\overline{X}$	$\overline{X}^{att}$	$\overline{X}_{rec}$	$\overline{X}$	$\overline{X}^{att}$	$\overline{X}_{rec}$	$\overline{X}$	$\overline{X}^{att}$	$\overline{X}_{rec}$	$\overline{X}$	$\overline{X}^{att}$	$\overline{X}_{rec}$	$\overline{X}$	$\overline{X}^{att}$	$\overline{X}_{rec}$
0.00	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
0.25	155.22	123.97	202.42	133.16	103.36	170.77	115.87	84.67	157.65	101.09	82.50	149.09	90.65	73.92	31.44
0.35	92.32	80.98	135.30	74.76	63.01	105.84	62.01	48.80	94.86	52.40	46.04	88.15	44.95	40.70	70.55
0.50	43.89	44.29	73.70	33.40	31.80	52.89	26.36	23.12	45.31	21.38	20.94	42.23	17.73	18.35	31.44
1.00	6.30	8.45	13.36	4.50	5.43	8.39	3.44	3.75	6.65	2.77	3.22	6.68	2.32	2.68	4.46
1.50	2.00	2.74	4.14	1.57	1.90	2.67	1.33	1.48	2.12	1.20	1.34	2.32	1.12	1.26	1.64
2.00	1.19	1.43	2.01	1.08	1.17	1.47	1.03	1.06	1.24	1.01	1.04	1.38	1.00	1.02	1.12

- Performance in terms of ARL_1
  - $\delta < 0.35$: ARL_1: $\overline{X}^{att} < \overline{X}$
  - $\delta = 0.5$: ARL_1: $\overline{X}^{att} \approx \overline{X}$
  - $\delta > 0.5$: ARL_1: $\overline{X}^{att} > \overline{X}$

- Parameters of the process: μ_0 : $100.00mm$; $\sigma = 10.00mm$
- Parameters of the control chart:
  - Warning limits: $UWL_U = 118.25mm$; $UWL_L = 106.73mm$; LWL_U and LWL_L symmetric;
  - Control limits: $UCL = 124.61mm$; $LCL = 75.39$ with $\alpha = 0.0027$
  - $a = 1.3748$ and Sample size: n=6

Table 5: Results of the last six samples

#	N_1	N_2	N_3	N_4	N_5	$\overline{X}_{att}$	Decision
1	0	0	3	0	3	131.26	out of control
2	0	0	4	2	0	104.16	in control
3	0	1	2	3	0	104.16	in control
4	0	1	4	1	0	100.00	in control
5	1	2	1	1	1	100.20	in control
6	0	0	2	0	4	141.67	out of control
Y_i	51.15	87.51	100	112.9	162.5		

Some weakness of $\overline{X}^{att}$ chart

The gauges with the optimal warning limits might be an obstacle for the practitioners as they have to be searched;

The set warning limit may not be a single one for all shift sizes; it may depend on the shift size.

The gauges may not be available in the market, they have to be ordered appropriately.

The distribution of the $\bar{X}^{att}$ is discrete, which makes it difficult to reach the in-control ARL at the usual values such as 370.

$\overline{X}_{tn}$ proposed by Quinino et al. (2020)

- To overpass these problems, $\overline{X}_{tn}$ is proposed
- Operationally it is exactly equal to the $\overline{X}^{att}$ chart.
- Each item is classified into one of 5 classes **using any available gauge in the market**.
- A random value is generated from a truncated normal distribution in which the lower and the upper truncation points are equal to the limits of the class to which the item was allocated.

Figure 5: Some devices of 5 classes used in industry

- With the n simulated values calculate
$$\bar{X}_{tn} = \frac{\sum_{r=1}^{N_A} x_{Ar} + \sum_{j=1}^{N_B} x_{Bj} + \sum_{k=1}^{N_C} x_{Ck} + \sum_{z=1}^{N_D} x_{Dz} + \sum_{s=1}^{N_E} x_{Es}}{n}$$
- When the process is in control the limits of control for $\bar{X}_{tn}$ is defined as $UCL_{\bar{X}_{tn}} = \mu_0 + Z_{1-\alpha/2}\frac{\sigma_0}{\sqrt{n}}$ and $LCL_{\bar{X}_{tn}} = \mu_0 - Z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}}$.
- The control limits are exactly the same control limits used for the traditional $\overline{X}$ chart.

$\overline{X}_{tn}$ proposed by Quinino et al. (2020) - calculating ARL_1

- The determination of ARL_1 is more complicated as the distribution of $\overline{X}_{tn}|\mu_1 = \mu_0 + \delta\sigma$ does not follow a normal distribution
- Some available alternatives: Monte Carlo simulation; approximation to a normal distribution (by the central limit theorem); numerical analysis of $\overline{X}_{tn}$
- The approximation to normal distribution for $\bar{X}_{tn}$ was verified to be quite good. So it is considered $\bar{X}_{tn} \approx N\left(\mu^*, \dfrac{\sigma^{*2}}{n}\right)$, μ^* is the weighted average of means of each stratum

$$\mu^* = p_A\mu_A^* + p_B\mu_B^* + p_C\mu_C^* + p_D\mu_D^* + p_E\mu_E^*$$

$p_A = P(X < LWL_i)$, $p_B = P(LWL_i < X < LWL_s)$, $p_C = P(LWL_s < X < UWL_i)$, $p_D = P(UWL_i < X < UWL_s)$, $p_E = P(UWL_s < X)$

- The means μ_i^*, $i = A, B, C, D, E$ are obtained using the truncated normal distribution related to each stratum as:

$$\begin{aligned}
\mu_A^* &= \frac{\int_{-\infty}^{LWL_i} xf(x|\mu,\sigma^2)dx}{\int_{-\infty}^{LWL_i} f(x|\mu,\sigma^2)dx};\ \mu_B^* = \frac{\int_{LWL_i}^{LWL_s} xf(x|\mu,\sigma^2)dx}{\int_{LWL_i}^{LWL_s} f(x|\mu,\sigma^2)dx};\ \mu_C^* = \frac{\int_{LWL_s}^{UWL_i} xf(x|\mu,\sigma^2)dx}{\int_{LWL_s}^{UWL_i} f(x|\mu,\sigma^2)dx};\\
\mu_D^* &= \frac{\int_{UWL_i}^{UWL_s} xf(x|\mu,\sigma^2)dx}{\int_{UWL_i}^{UWL_s} f(x|\mu,\sigma^2)dx}; \mu_E^* = \frac{\int_{UWL_s}^{\infty} xf(x|\mu,\sigma^2)dx}{\int_{UWL_s}^{\infty} f(x|\mu,\sigma^2)dx}
\end{aligned}$$

$f(x|\mu,\sigma^2)$ is a probability density function of normal distribution.

$\overline{X}_{tn}$ proposed by Quinino et al. (2020) - calculating ARL_1

- The variance σ^{*2} is also a weighted average of the second centered moment related to μ^* of each stratum

$$\sigma^{*2} = p_A m_A^* + p_B m_B^* + p_C m_C^* + p_D m_D^* + p_E m_E^*$$

with

$$\begin{aligned}
m_A^* &= \frac{\int_{-\infty}^{LWL_i} (x-\mu^*)^2 f(x|\mu,\sigma^2)dx}{\int_{-\infty}^{LWL_i} f(x|\mu,\sigma^2)dx}; m_B^* = \frac{\int_{LWL_i}^{LWL_s} (x-\mu^*)^2 f(x|\mu,\sigma^2)dx}{\int_{LWL_i}^{LWL_s} f(x|\mu,\sigma^2)dx};\\
m_C^* &= \frac{\int_{LWL_s}^{UWL_i} (x-\mu^*)^2 f(x|\mu,\sigma^2)dx}{\int_{LWL_s}^{LWL_i} f(x|\mu,\sigma^2)dx}; m_D^* = \frac{\int_{UWL_i}^{UWL_s} (x-\mu^*)^2 f(x|\mu,\sigma^2)dx}{\int_{UWL_i}^{UWL_s} f(x|\mu,\sigma^2)dx};\\
m_E^* &= \frac{\int_{UWL_s}^{\infty} (x-\mu^*)^2 f(x|\mu,\sigma^2)dx}{\int_{UWL_s}^{\infty} f(x|\mu,\sigma^2)dx}.
\end{aligned}$$

$\overline{X}_{tn}$ proposed by Quinino et al. (2020) - calculating ARL_1

Applying the expressions μ^* and σ^{*2} we are able to obtain the out-of-control average run lengths using the approximation $\bar{X}_{tn} \approx N(\mu^*, \sigma^{*2}/n)$.

Consider $\mu_1 = \mu_0 + \delta\sigma$ which indicates a shift of the process mean signaling an out-of-control condition.

This is equivalent to testing the hypothesis: $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0,\ \mu = \mu_0 + \delta\sigma.$ 21 ARL_1 is obtained as $1/(1-\beta)$ and β is the error of type II expressed as:

$$\begin{aligned}\beta &= P(LCL < \bar{X}_{tn} < UCL | \mu = \mu_1 = \mu_0 + \delta\sigma) \\ &= \Phi\left(\frac{UCL - \mu^*}{\sigma^*/\sqrt{n}}\right) - \Phi\left(\frac{LCL - \mu^*}{\sigma^*/\sqrt{n}}\right)\end{aligned}$$

Φ denotes the cumulative normal standard distribution function.

Simulation versus approximation of $\overline{X}_{tn}$

	$n=5$			$n=6$			$n=7$			$n=8$			$n=9$		
	$\overline{X}$	$\overline{X}_{tn}$		$\overline{X}$	$\overline{X}_{tn}$		$\overline{X}$	$\overline{X}_{tn}$		$\overline{X}$	$\overline{X}_{tn}$		$\overline{X}$	$\overline{X}_{tn}$	
δ		Appr	Simu		Appr	Simu		Appr	Simu		Appr	Simu		Appr	Sim
0	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.37
0.25	133.2	174.2	166.7	115.9	157	150.5	102	142.4	136.6	90.65	130	124.9	81.22	119.3	115.23
0.5	33.4	51.91	50.01	26.36	42.68	41.2	21.38	35.81	34.72	17.73	30.55	29.74	14.97	26.41	25.79
1	4.5	7.54	7.43	3.44	5.84	5.75	2.77	4.7	4.65	2.32	3.91	3.87	2	3.33	3.3
1.5	1.57	2.36	2.28	1.33	1.9	1.86	1.2	1.61	1.59	1.12	1.43	1.42	1.07	1.3	1.3
2	1.08	1.29	1.28	1.03	1.14	1.15	1.01	1.07	1.08	1	1.04	1.04	1	1.02	1.02
2.5	1	1.03	1.05	1	1.01	1.02	1	1	1.01	1	1	1	1	1	1

Sample sizes to match performance of $\overline{X}$ and $\overline{X}_{tn}$

	$\overline{X}$	$\overline{X}_{tn}$	$\overline{X}$	$\overline{X}_{tn}$	$\overline{X}$	$\overline{X}_{tn}$	$\overline{X}$	$\overline{X}_{tn}$
δ	$n=4$	$n=7$	$n=5$	$n=8$	$n=6$	$n=9$	$n=7$	$n=10$
0	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.37
0.25	155.2	136.6	133.2	124.9	115.9	115.2	101.99	102.56
0.5	43.89	34.72	33.4	29.74	26.36	25.79	21.38	21.23
1	6.3	4.65	4.5	3.87	3.44	3.3	2.77	2.81
1.5	2	1.59	1.57	1.42	1.33	1.3	1.2	1.19
2	1.19	1.08	1.08	1.04	1.03	1.02	1.01	1.01
2.5	1.02	1.01	1	1	1	1	1	1

Comparing $\overline{X}_{tn}$ and $\overline{X}_{att}$

Table 6: Comparing $\overline{X}_{tn}$ and $\overline{X}_{att}$

	$n=5$		$n=6$		$n=7$		$n=8$		$n=9$	
δ	$\overline{X}_{tn}$	$\overline{X}_{att}$	$\overline{X}_{tn}$	$\overline{X}_{att}$	$\overline{X}_{tn}$	$\overline{X}_{att}$	$\overline{X}_{tn}$	$\overline{X}_{att}$	$\overline{X}_{tn}$	$\overline{X}_{att}$
0	215.3	215.3	198.5	198.5	429.95	429.95	400.08	400.08	232.84	232.84
0.25	104.3	98.49	87.98	78.88	154.37	141.03	133.24	116.16	78.19	67.9
0.5	33.85	30.84	26.75	22.12	38.49	33	31.27	25.2	19.21	15.34
1	5.77	5.26	4.43	3.62	4.92	4.21	3.98	3.24	2.84	2.34
1.5	1.99	1.88	1.64	1.46	1.63	1.49	1.43	1.3	1.24	1.15
2	1.21	1.18	1.11	1.07	1.09	1.06	1.05	1.03	1.02	1.01
2.5	1.03	1.03	1.01	1.01	1.01	1	1	1	1	1

When do you recommend to use $\overline{X}_{tn}$? and $\overline{X}_{att}$?

Some improvements for $\overline{X}_{tn}$ control chart: $\overline{X}_{tn(I)}$ and $\overline{X}_{tn(K)}$

These improvements are proposed by Yamauchi et al. (2022).

The control chart $\overline{X}_{tn(I)}$ employs the following alternative sampling scheme:

1. Inspect a sample of size n_a and compute the statistic $\bar{X}^a_{tn}$
2. If $\bar{X}^a_{tn} \notin [LCL^a_{\bar{X}^a_{tn}}; UCL^a_{\bar{X}^a_{tn}}]$ then the process is declared out-of-control, adjusted and go to Step 1; otherwise go to Step 3.
3. Inspect a sample of size n_b; compute the statistic $\bar{X}^a_{tn}$; if $\bar{X}^b_{tn} \notin [LCL^b_{\bar{X}^a_{tn}}; UCL^b_{\bar{X}^b_{tn}}]$ then the process is declared out-of-control, adjusted.
4. Go to Step 1.

Another improvement: the control chart $\overline{X}_{tn(K)}$

- The improved control chart $\overline{X}_{tn(K)}$ is based on the supplementary rule proposed by Klein (2000).
- In equally spaced time intervals random samples of size n are taken and obtained the statistic $\bar{X}_{tn}$.
- Let $\bar{X}_{tn}^{i-1}$ and $\bar{X}_{tn}^{i}$ be two successive values of $\bar{X}_{tn}$.
- The decision is taken basing on sequences of two successive values of $\bar{X}_{tn}$: if two successive values are on same side and beyond the control limits then the process is decided to be out-of-control, otherwise the process is in-control.
- Specifically, if $\bar{X}_{tn}^{i-1} < LCL_{\bar{X}_{tn}}$ and $\bar{X}_{tn}^{i} < LCL_{\bar{X}_{tn}}$ or $\bar{X}_{tn}^{i-1} > UCL_{\bar{X}_{tn}}$ and $\bar{X}_{tn}^{i} > UCL_{\bar{X}_{tn}}$ then the process is declared out-of-control.
- $LCL_{\bar{X}_{tn}}$ and $UCL_{\bar{X}_{tn}}$ are the control limits satisfying respectively. $P(\bar{X}_{tn} < LCL_{\bar{X}_{tn}}) = r_L$, $P(\bar{X}_{tn} > UCL_{\bar{X}_{tn}}) = r_U$ and $r = P(LCL_{\bar{X}_{tn}} < \bar{X}_{tn} < UCL_{\bar{X}_{tn}})$. When the process is in-control $r_L = r_U$.

Comparison - some results

Table 7: ARL_1 values for the $\bar{X}$, $\bar{X}_{tn}$, $\bar{X}_{tn(K)}$ and $\bar{X}_{tn(I)}$

Shift δ_1	$\bar{X}$	$\bar{X}_{tn}$	$\bar{X}_{tn(K)}$	$\bar{X}_{tn(I)}$	$\bar{X}$	$\bar{X}_{tn}$	$\bar{X}_{tn(K)}$	$\bar{X}_{tn(I)}$
	Sample size n=5				Sample size n=6			
0	370	382	372	370	370	366	371	370
0.2	178	213	**173**	216	159	194	**157**	199
0.4	57	80	**54**	**77**	46	68	**45**	**63**
0.6	21	32	**21**	**28**	16	26	**17**	**21**
0.8	8.9	14	**9.9**	**12**	6.7	11	**7.9**	**8.8**
1	4.5	7.4	**5.7**	**5.9**	3.4	5.8	**4.7**	**4.5**
1.2	2.7	4.2	**3.8**	**3.5**	2.1	3.3	**3.2**	**2.7**
1.4	1.8	2.7	2.9	**2.4**	1.5	2.2	2.6	**2.0**
1.6	1.4	2	2.4	**1.8**	1.2	1.6	2.3	1.6
1.8	1.2	1.5	2.2	1.6	1.1	1.3	2.1	1.5
2	1.1	1.3	2.1	1.5	1	1.1	2	1.5
2.2	1	1.1	2	1.5	1	1.1	2	1.5
2.4	1	1.1	2	1.5	1	1	2	1.5
2.6	1	1	2	1.5	1	1	2	1.5
2.8	1	1	2	1.5	1	1	2	1.5
3	1	1	2	1.5	1	1	2	1.5

Comparing

Table 8: Minimum sample sizes for $\bar{X}_{tn(I)}$ control chart to have equal performance of the standard $\bar{X}$ chart

δ_1	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0.2	$n_a = 5, n_b = 1$ (3)	$n_a = 9, n_b = 1$ (5)	$n_a = 9, n_b = 1$ (5)	$n_a = 15, n_b = 1$ (8)
0.4	$n_a = 5, n_b = 1$ (3)	$n_a = 9, n_b = 1$ (5)	$n_a = 9, n_b = 1$ (5)	$n_a = 13, n_b = 1$ (7)
0.6	$n_a = 5, n_b = 1$ (3)	$n_a = 7, n_b = 1$ (4)	$n_a = 9, n_b = 1$ (5)	$n_a = 13, n_b = 1$ (7)
0.8	$n_a = 5, n_b = 1$ (3)	$n_a = 7, n_b = 1$ (4)	$n_a = 9, n_b = 1$ (5)	$n_a = 11, n_b = 1$ (6)
1	$n_a = 5, n_b = 1$ (3)	$n_a = 7, n_b = 1$ (4)	$n_a = 9, n_b = 1$ (5)	$n_a = 11, n_b = 1$ (6)
1.2	$n_a = 5, n_b = 1$ (3)	$n_a = 7, n_b = 1$ (4)	$n_a = 9, n_b = 1$ (5)	$n_a = 13, n_b = 1$ (7)
1.4	$n_a = 5, n_b = 1$ (3)	$n_a = 7, n_b = 1$ (4)	$n_a = 9, n_b = 1$ (5)	$n_a = 13, n_b = 1$ (7)
1.6	$n_a = 5, n_b = 1$ (3)	$n_a = 7, n_b = 1$ (4)	$n_a = 9, n_b = 1$ (5)	-
1.8	$n_a = 5, n_b = 1$ (3)	$n_a = 9, n_b = 1$ (5)	$n_a = 9, n_b = 1$ (5)	-
2	$n_a = 5, n_b = 1$ (3)	$n_a = 9, n_b = 1$ (5)	$n_a = 9, n_b = 1$ (5)	-
2.2	$n_a = 5, n_b = 1$ (3)	-	-	-
2.4	$n_a = 7, n_b = 1$ (4)	-	-	-
2.6	-	-	-	-
2.8	-	-	-	-
3	-	-	-	-

Sample size

Table 9: Minimum sample size for $\bar{X}_{tn(K)}$ control chart to have equal performance of the standard $\bar{X}$ chart

δ_1	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.2	1	2	3	4	5
0.4	1	2	3	4	5
0.6	1	2	3	4	5
0.8	1	2	3	4	6
1	1	2	3	5	7
1.2	1	2	4	6	8
1.4	1	3	4	7	-
1.6	1	3	5	-	-
1.8	1	3	-	-	-
2	1	-	-	-	-
2.2	1	-	-	-	-
2.4	2	-	-	-	-
2.6	2	-	-	-	-
2.8	2	-	-	-	-
3	-	-	-	-	-

The improved np_x for monitor a process mean: $np^{\mu}_{X(l)}$

This is subject of one of our seminars

Themes for seminars/discussions

When do you recommend to use $\overline{X}_{tn}$? and $\overline{X}_{att}$?

Find in the literature other procedures to improve the $\overline{X}$. List their advantages and weakness.

Find other contributions in the literature of attribute charts to monitor the process mean than those presented and draw their features, advantages and weakness.

Do you think that there is room for another attribute chart? What are your suggestions?

Which should be your choice among those presented attribute charts to monitor the process mean? Why?

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