# PRO 5971 - Statistical Process Monitoring

## How to measure the performance of a control chart

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# Outline

# RUN LENGTH

The parameters of a control chart: sample size $n$, the control limits (UCL and LCL) and sampling interval (for simplicity, let us consider $h = 1$)

Action: whenever the statistic is plotted out of the control limits, a search for special causes starts.

Performance measure: the number of samples until a signal. In Figure 1, 4 samples for the first signal; 6 samples for second signal; 6 samples for third signal, etc

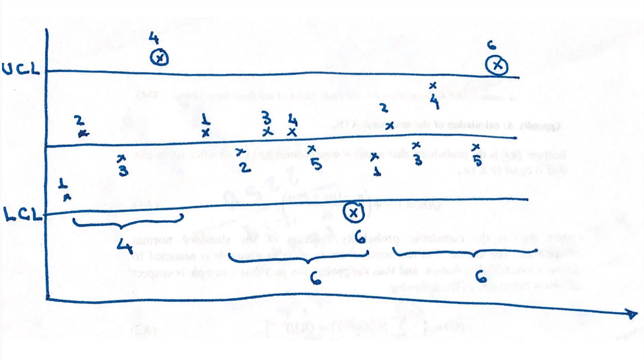


**Figure 1:** # of samples until a signal

# Run length

Y= # of samples until a signal - is a random variable known as RUN LENGTH

If the monitored statistics are **independent** $\rightarrow$ Y follows a Geometric distribution with parameter $p$, the probability to signal

Its probability function is

$$P(Y = y) = p(1 - p)^{y-1}, y = 1, 2, \dots$$

For example: $\begin{cases} P(Y = 1) = p, & \text{the probability to signal at the first sample} \\ P(Y = 2) = p(1 - p), & \text{the probability to signal at the second sample} \\ \dots P(Y = 5) = p(1 - p)^4, & \text{the probability to signal at the fifth sample} \end{cases}$

$E(Y) = \dfrac{1}{p}$ is the average number of samples until a signal.

It is known as AVERAGE RUN LENGTH (ARL) and has been the most used a performance metric in SPC.

Other measures: median run length (MRL); standard deviation run length(SDRL).

# In-control average run length: $ARL_0$

$$\text{When the process is stable (in control or under } H_0) \begin{cases} \alpha - \text{probability to signal} \\ 1-\alpha, \ \text{probability to not signal} \\ ARL_0 = \dfrac{1}{\text{probability to signal}} = \dfrac{1}{\alpha} \end{cases}$$

$$ARL_0 \begin{cases} \text{ is the average number of sample until a signal when the process is in-control} \\ \text{It is desirable to have larger values for } ARL_0 \\ \text{Usually values like 370 or 500 are used as } ARL_0 \text{ in manufacturing process monitoring} \\ \text{IMPORTANT: The value of } ARL_0 \text{ must have a practical context meaning} \end{cases}$$

# Out-of-control average run length: $ARL_1$

When the process is out of control or under $H_1$) $\begin{cases} 1-\beta - \text{probability to signal} \\ \beta, \ \text{probability to not signal} \\ ARL_1 = \dfrac{1}{\text{probability to signal}} = \dfrac{1}{1-\beta} \end{cases}$

$ARL_1 \begin{cases} \text{ is the average number of sample until a signal when the process is out-of-control} \\ \text{It is desirable to have } ARL_1 \approx 1 \end{cases}$

Plan $\overline{X}$ control charts (assuming $\sigma$ known and remains stable) to have $ARL_0 = 200, 370, 500$ with $n = 3, 5, 9$. Obtain $ARL_1$ when $\mu_0$ shifts to $\mu_1 = \mu_0 \pm \delta\sigma$ for $\delta = 0.25, 0.5, 1, 1.5, 2, 2.5$.

Plan $S^2$ control charts (assuming that $\mu$ is stable) to have $ARL_0 = 200, 370, 500$ with $n = 10, 15, 25$. Obtain $ARL_1$ when $\mu_0$ shifts to $\sigma_1^2 = \sigma_0^2\delta$ for $\delta = 0.25, 0.5, 1, 1.5, 2, 2.5$.

Plan $p$ control charts to have $ARL_0 = 200, 370, 500$ with $n = 20, 30, 50$. Obtain $ARL_1$ when $p_0 = 0.05, 0.1$ shifts to $p_1 = p_0\delta$ for $\delta = 0.25, 0.5, 1, 1.5, 2, 2.5$. Use the exact and the approximate distribution of the monitored statistic.

Plan $u$ control charts to have $ARL_0 = 200, 370, 500$ with $n = 10, 20, 30$. Obtain $ARL_1$ when $\lambda_0 = 0.05, 5$ shifts to $\lambda_1 = \lambda_0\delta$ for $\delta = 0.25, 0.5, 1, 1.5, 2, 2.5$. Use the exact and the approximate distribution of the monitored statistic.

In each control chart discuss the behavior of $ARL_1$. Is it possible to find an optimal design for that control chart?

# Dealing with $ARL$ when the monitored statistics are not independent

When the monitored statistics are not independent, the determination of $ARL$ is more complicated.

There are many alternatives for its obtaining.

Three alternatives will be presented, two based on Markov chains and the third by simulation.

# A simple example of Markov chain

Consider a consumer support center. The aim is to discover averagely how many calls a customer needs to make in order to be attended. The process can be summarized as the following steps:

1 - The customer makes a call due to some reason.

2 - The call may be completed or not.

3 - If completed and some attendant is available, the customer is finally attended. Otherwise, the system leaves a message to call latter.

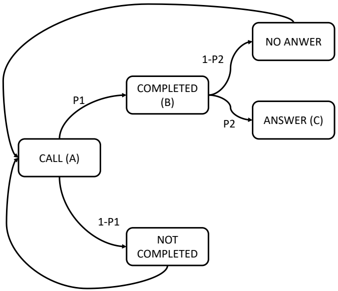


**Figure 2:** Consumer support

# cont - A simple example ....

Markov chain - the current event (state) depends on the previous one. The consumer support process can be described by the states:

- State A - the customer makes a call
- State B - the call is completed
- State C - the attendant answers

The whole process can be represented by a transition matrix of probability

| | A | B | C |
|---|---|---|---|
| A | $p_{AA}$ | $p_{AB}$ | $p_{AC}$ |
| B | $p_{BA}$ | $p_{BB}$ | $p_{BC}$ |
| C | $p_{CA}$ | $p_{CB}$ | $p_{CC}$ |

The index $ij$ in matrix denotes: from state $i$ to state $j$

$p_{ij}$ - denotes probability to reach the current state j conditioned that at previous step was at state i - p(j|i).

For example: $AB$ - from state $A$ to state $B$; $p_{AB}$ - probability to the call be completed conditioned(B) that the customer makes a call (A) - p(B|A).

$p_{BC}$ - probability to the attendant answers (state c) conditioned that the call is completed (state B) - p(C|B)

Assuming $p_{AB} = p_{BC}$=0.5 , the transition matrix of probability is

$$\mathbf{P} = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0.5 & 0.5 & 0 \\ B & 0.5 & 0 & 0.5 \\ C & 0 & 0 & 1 \end{array}$$

The states A and B are transitory and C is absorbent

# Method 1 - ARL by Markov chain - with absorbent state

**P** - the full transitory matrix

**Q** -a square sub matrix of **P** after the removal of rows and columns of absorbent states.

$\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}$

$ARL = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{1}$

In our example, we have

$$\mathbf{N} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

That is $n_{11}$=4, averagely 4 calls are made before be completed; $n_{21}$=2, even completed (state B), averagely more two calls are needed.

$$\mathbf{ARL} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$ARL_0$=6

Details see Brook & Evans (1972)

## Method 2 - ARL by Markov chain - renewal process

The same states can be considered, but the main idea is that whenever the state C is reached, the process is renewed (or feed forward), that is, a new customer will make a call and reaches the state A or B. The new transitory matrix $\mathbf{P}^*$ becomes

$$\mathbf{P}^* = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0.5 & 0.5 & 0 \\ B & 0.5 & 0 & 0.5 \\ C & 0.5 & 0.5 & 0 \end{array}$$

As $s \to \infty$, the rows of $\mathbf{P}^{*\mathbf{s}}$ are equal to a vector $\boldsymbol{\pi} = [\pi_A, \pi_B, \pi_c]$.

Each $\pi_i$ measures the probability to reach the state $i$ for a long term.

The vector $\boldsymbol{\pi} = [\pi_A, \pi_B, \pi_c]$ can be obtained as solution of the system of equations $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}^*$ such that $\pi_A + \pi_B + \pi_c = 1$

In our example, $\boldsymbol{\pi} = [3/6, 2/6, 1/6]$, thus the average run length is

$$\frac{1}{\text{Prob of state C}} = \frac{1}{(1/6)} = 6$$

.

When the process is stable, the non-conformance rate is 0.2. A sample of five items $X_1, X_2, \ldots, X_5$ from a Poisson (0.2) is taken. Thus $Y = \sum_{i=1}^{5} X_i$ follows Poisson (1).

The engineer desires to monitor the average rate. For that, he proposed the criterion: if $Y_i > 4$ or if $2 \leq Y_{i-1} \leq 4$ and $2 \leq Y_i \leq 4$ then the process is stopped to search the special causes.

Define the states of the Markov chain for this process monitoring. Obtain the $ARL_0$.

State A, if $0 \leq Y \leq 1$

State B, if $2 \leq Y \leq 4$

State C, the process is declared out-of-control

Matrix of transitory probabilities

$$\mathbf{P}^* = \begin{array}{c|ccc} & A & B & C \\ \hline A & p_A & p_B & 1 - p_A - p_B \\ B & p_A & 0 & 1 - p_A \\ C & p_A & p_B & 1 - p_A - p_B \end{array}$$

The stationary distribution is

$$\boldsymbol{\pi} = \left[ p_A; \frac{p_B}{1 + p_B}; \frac{1 - p_A - p_A p_B}{1 + p_B} \right]$$

$$ARL_0 = \frac{1 + p_B}{1 - p_A - p_A p_B}$$

## Exercise - ARL by Markov chain - not easy

It is known that Shewhart control chart has good performance to detect large shifts.

So to improve these charts, several contributors have added/included new supplementary rules in the decision criterion.

Let consider an example of supplementary rule for the $\overline{x}$ control chart.

Two sets of limits: control and warning are used: LCL and UCL are respectively the lower ad upper control limits and LWL and UWL, respectively the lower and upper warning limits with $LCL < LWL < UWL < UCL$.

$[LWL, UWL]$ is known as the central region; $[LCL, LWL] \cup [UWL, UCL]$ is the warning region.

# cont - Exercise - ARL by Markov chain - not easy

Decision rules to declare the process as out-of-control:

Rule 1: $\overline{X}_i > UCL$;

Rule 2: $\overline{X}_i < LCL$;

Rule 3: at least two points of the last 3 monitored statistics in the warning region $[LCL, LWL]$; that is, $LCL < \overline{X}_i < LWL$ and $LCL < \overline{X}_{i-1} < LWL$ or $LCL < \overline{X}_i < LWL$, $LCL < \overline{X}_{i-2} < LWL$ and $LWL < \overline{X}_{i-1} < UWL$

Rule 4: at least two points in the last 3 observations in the warning region $[UWL, UCL]$ that is, $UWL < \overline{X_i} < UCL$ and $UWL < \overline{X_{i-1}} < UCL$ or $UWL < \overline{X}_i < UCL$, $UWL < \overline{X}_{i-2} < UCL$ and $LWL < \overline{X}_{i-1} < UWL$

For simplicity let us consider only rules which the monitored statistics fell or above the center line or below the center line, that is, or R1 and R3 or R2 and R4.

The probabilities of a monitored statistic fell in the central region, warning region and above the control limit are respectively: $a_1$, $a_2$ and $1 - a_1 - a_2$.

Define the states of Markov chain and determine an expression for ARL

# Solution - Exercise - ARL by Markov chain - not easy - with absorbent state

States: rules 2 and 4

A - last two observations in the central region

A1 - the last but one observation in the warning region and the last in central region

B - the last but one observation in the central region and the last in warning region

C - the current observation in the action region; the last two observations in the warning region; the last but one in the central region but the third to last and the last observations in the warning region

The transitory matrix

| | A | A1 | B | C |
|---|---|---|---|---|
| A | $a_1$ | 0 | $a_2$ | 1-$a_1$-$a_2$ |
| A1 | $a_1$ | 0 | 0 | 1-$a_1$ |
| B | 0 | $a_1$ | 0 | 1-$a_1$ |
| C | 0 | 0 | 0 | 1 |

# cont - Solution - Exercise - ARL by Markov chain - not easy - with absorbent state

$$(\mathbf{I}-\mathbf{Q})^{-1} = \frac{1}{d}\begin{bmatrix} 1 & a_1 a_2 & a_2 \\ a_1 & 1-a_1 & a_1 a_2 \\ a_1^2 & (1-a_1)a_1 & 1-a_1 \end{bmatrix}$$

$d = 1 - a_1 - a_1^2 a_2$

$$(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{1} = \begin{bmatrix} \dfrac{1+a_1 a_2 + a_2}{1-a_1-a_1^2 a_2} \\[2ex] \dfrac{1-a_1 a_2}{1-a_1-a_1^2 a_2} \\[2ex] \dfrac{1}{1-a_1-a_1^2 a_2} \end{bmatrix}$$

As always we assume that the process start at state A then ARL$=\dfrac{1+a_1 a_2 + a_2}{1-a_1-a_1^2 a_2}$

$ARL_0$ if $a_1$ and $a_2$ are obtained under $\mu_0$;

$ARL_1$ if $a_1$ and $a_2$ are obtained under $\mu_1$;

## cont - Solution - Exercise - ARL by Markov chain - not easy - with absorbent state

The overall $ARL_0$=0.5 $\dfrac{1 + a_1 a_2 + a_2}{1 - a_1 - a_1^2 a_2}$ by symmetry.

$\mu_1 = \mu_0 + \delta\sigma$, thus $ARL_1$ can be expressed as $ARL_1(\delta)$ for the rules 2 and 4

Thus we have $ARL_1(-\delta)$ for the rules 1 and 3,

The overall

$$ARL_1 = \left[\frac{1}{ARL_1(-\delta)} + \frac{1}{ARL_1(\delta)}\right]^{-1}$$

# Solution - Exercise - ARL by Markov chain - not easy - with renewal process

The same states in the previous slides but once state C reached, the process is renewed

The transitory matrix becomes

| | A | A1 | B | C |
|---|---|---|---|---|
| A | $a_1$ | 0 | $a_2$ | 1-$a_1$-$a_2$ |
| A1 | $a_1$ | 0 | 0 | 1-$a_1$ |
| B | 0 | $a_1$ | 0 | 1-$a_1$ |
| C | $a_1$ | 0 | $a_2$ | 1-$a_1$-$a_2$ |

Finding its stationary distribution

$$\boldsymbol{\pi} = \left[\pi_1 = \frac{a_1(a_1+a_2+1}{1+a_2+a_1+a_2} \quad \pi_2 = \frac{(a_1+a_2}{1+a_2+a_1+a_2} \quad \pi_3 = \frac{a_2}{1+a_2+a_1+a_2} \quad \pi_4 = \frac{1-a_1^2a_1-a_1}{1+a_2+a_1+a_2}\right]$$

So $ARL = \dfrac{1}{\pi_4} = \dfrac{1+a_2+a_1+a_2}{1-a_1^2a_1-a_1}$

# ARL by simulation

Some reasons to use the simulation to determine the average run length and also the control (warning) limits:

- the distribution of the monitored statistic is unknown
- the determination of control and warning limits of the control chart are quite complicated or do not have closed forms
- to check the analytical results.

Two runs of simulations are needed:

1 - simulated the monitored statistics when the process is in-control.

Usually for the determination of control and warning limits such that meet the target $ARL_0$ value

2 - simulated the monitored statistics when the process is out-of-control: To calculate $ARL_1$.

# ARL by simulation - a first run

A simplified algorithm for in-control process:

1. Define the decision rules to judge the process as out-of-control
2. Choose the initial control and warning limits, etc,
3. Simulate a large number of the monitored statistics (under in-control parameters)
4. Count the number of the monitored statistics until satisfied the decision rules. This is the first in-control run length $RL_1$, save this count.
5. Repeat the steps 3-4 several times until to get a large number of in-control run lengths (like 10 thousand)
6. Calculate its average as $\widehat{ARL}_0 = \frac{\sum_{i=1}^{10000} RL_i}{10000}$ and verify if it is closer to the target $ARL_0$ value. If it is closer, stop the simulation, you have determined the control limits and use them in the second run of simulation. Otherwise choose other control (warning) limits for step 2 and repeat steps 3-6 .

# ARL by simulation - a second run

A simplified algorithm for out-of-control process:

1. Use the outputs of the first run (in-control simulation) as control and warning limits, etc,
2. Simulate a large number of the monitored statistics (under out-of-control parameters)
3. Count the number of the monitored statistics until satisfied the decision rules. This is the first out-of-control run length $RL_1$; save this count.
4. Repeat steps 2-3 several times until to get a large number of out-of-control run lengths (like 10 thousand)
5. Calculate its average as $\widehat{ARL}_1 = \frac{\sum_{i=1}^{10000} RL_i}{10000}$. This an estimate of $ARL_1$

# Exercises

1 - Choose control and warning limits and other necessary in-control parameters (as in-control mean and variance and sample size) for "the not easy problem". Develop Monte Carlo simulations to check the theoretical results of AR$L_0$ by Markov Chain.

2 - Choose shift sizes for the process mean and check by Monte Carlo simulation if the expression stated for $ARL_1$ is correct.

3 - Now consider that some target value of $ARL_0 = \tau$ is desirable. Search for the control and warning limits by simulation such that minimize $ARL_1$ for a shift of one standard deviation.

# References

Brook, D. & Evans, D. (1972), ‘An approach to the probability distribution of cusum run length’, *Biometrika* pp. 539–549.