



Treliças
Quadros espaciais
Rótulas em estruturas planas

COPYRIGHT Valério S. Almeida

AULA DO DIA 18 ABRIL/2023 PROF. ANTONIO LEROSA

TRELIÇAS

Treliça: uma das principais estruturas na engenharia

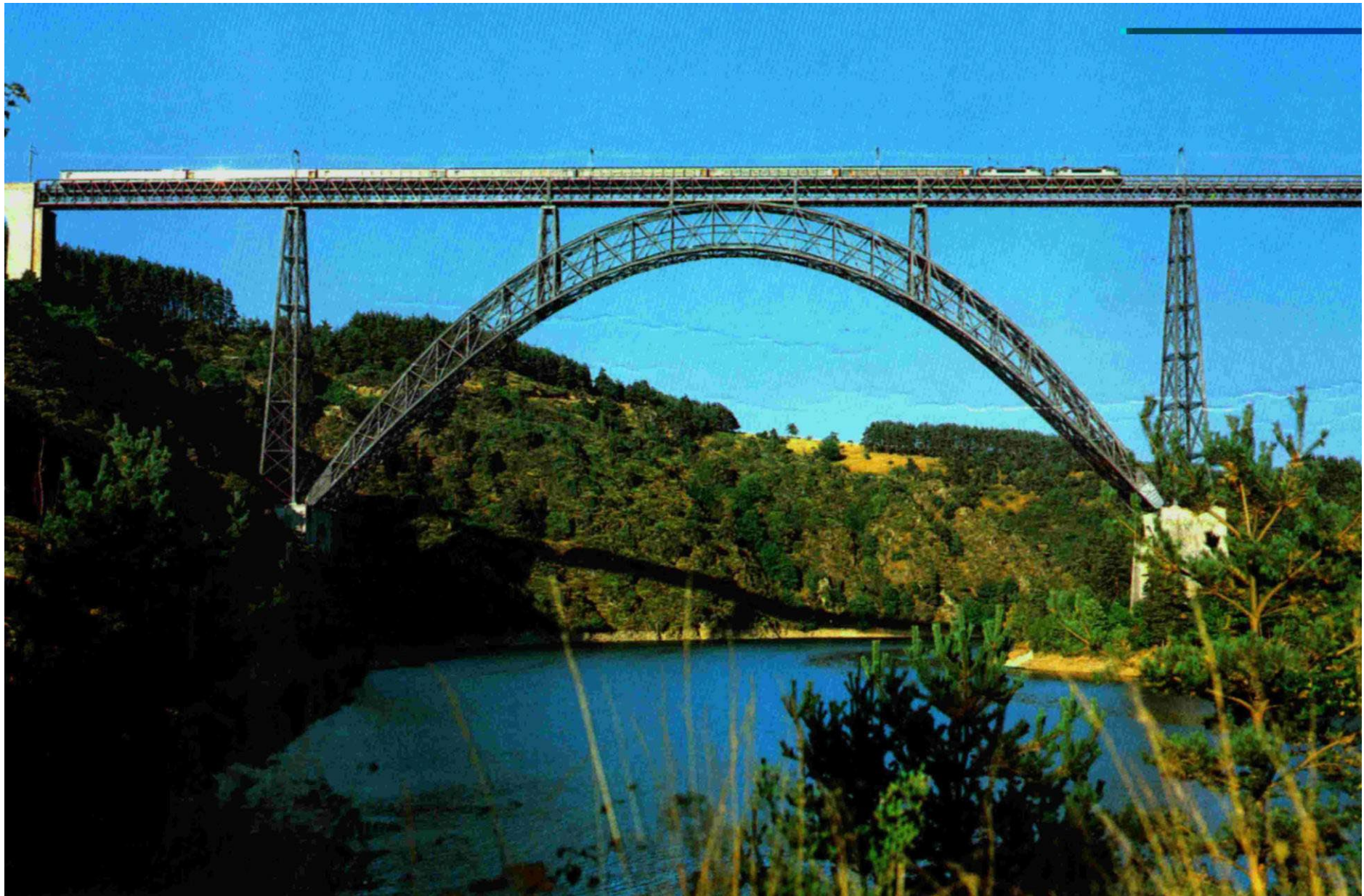
Solução prática econômica



Pontes

Coberturas

Torres



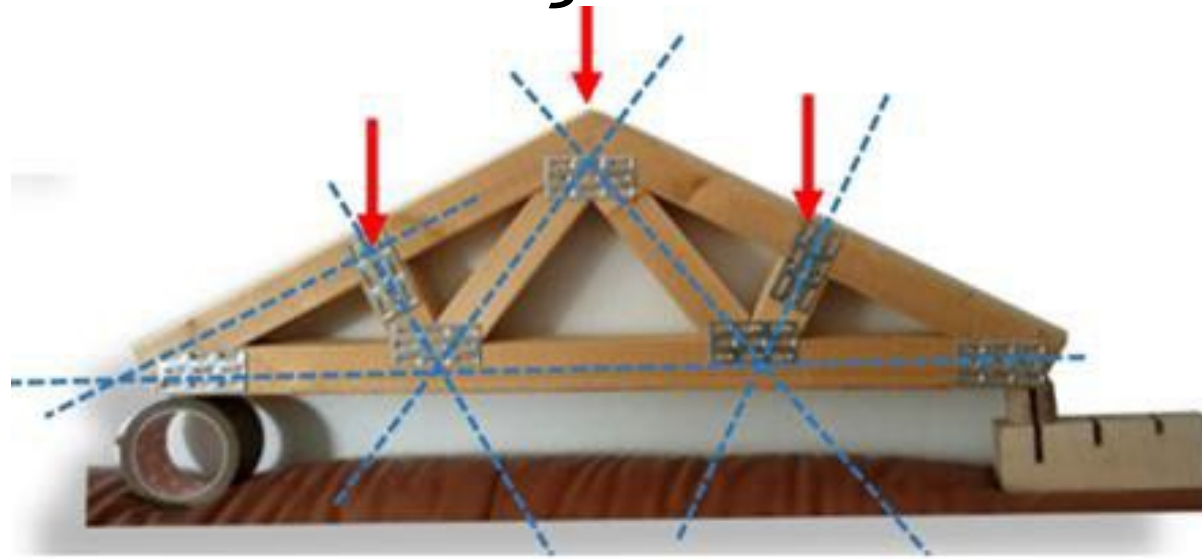




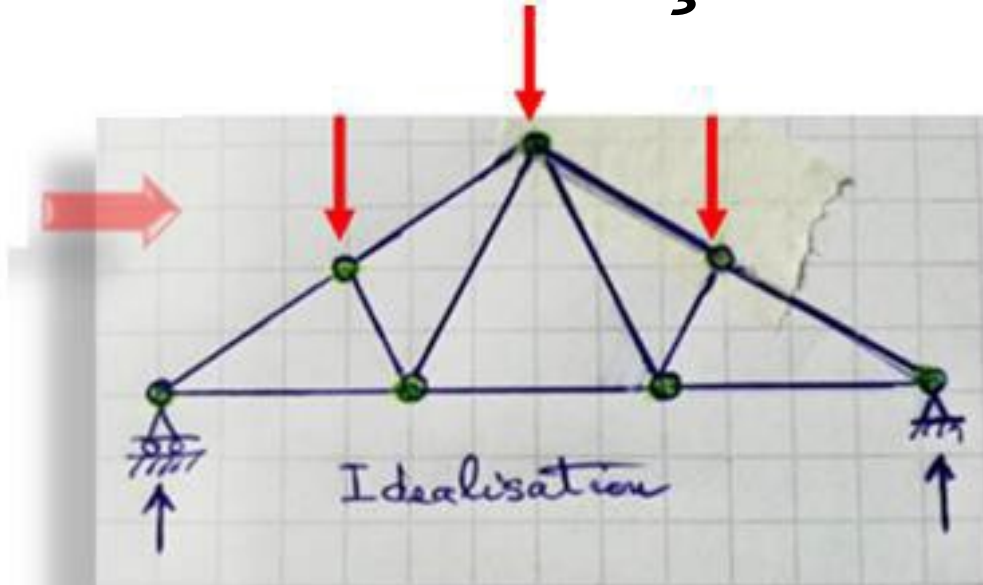
Jätkäsaari, 10.9.2017, DBA



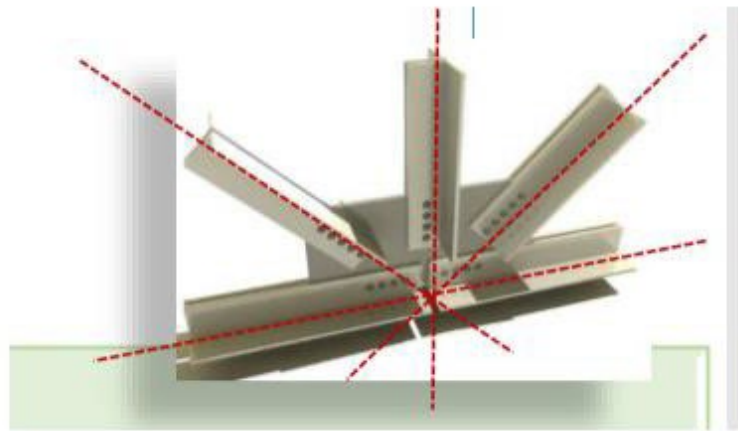
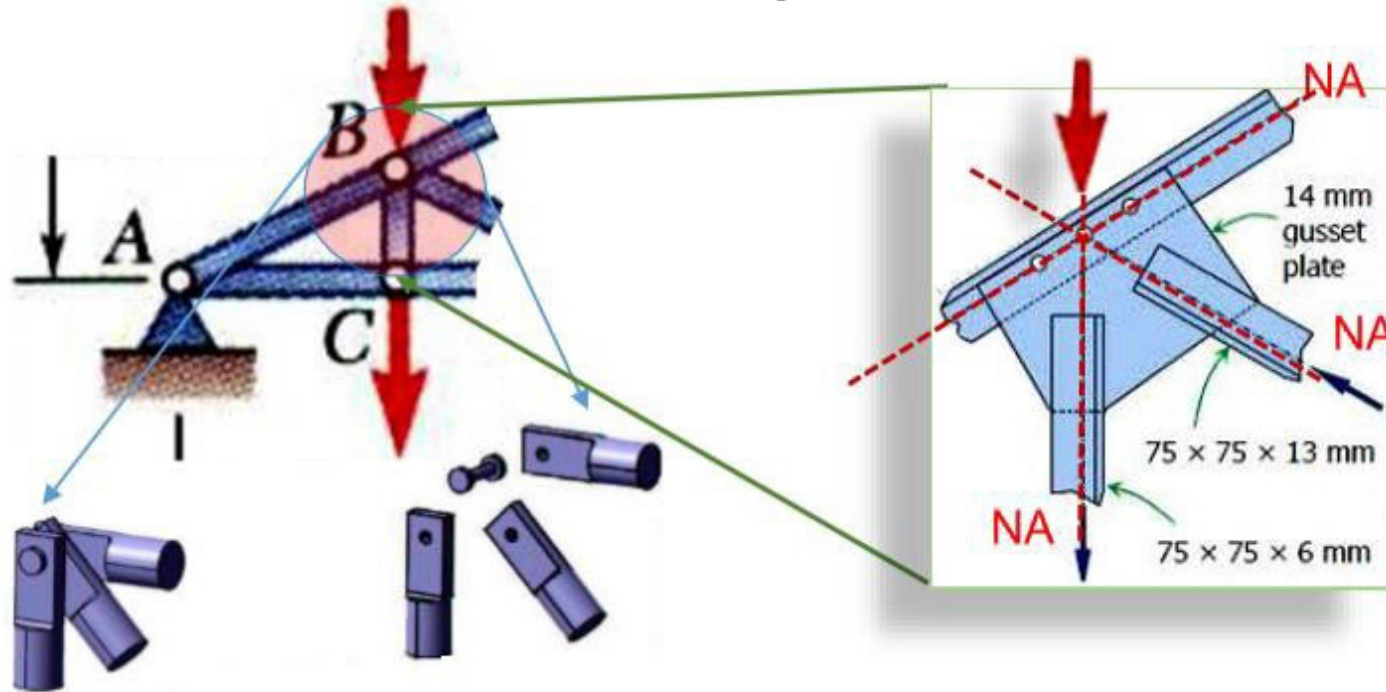
TRELIÇAS IDEAL



Idealização



LIGAÇÃO



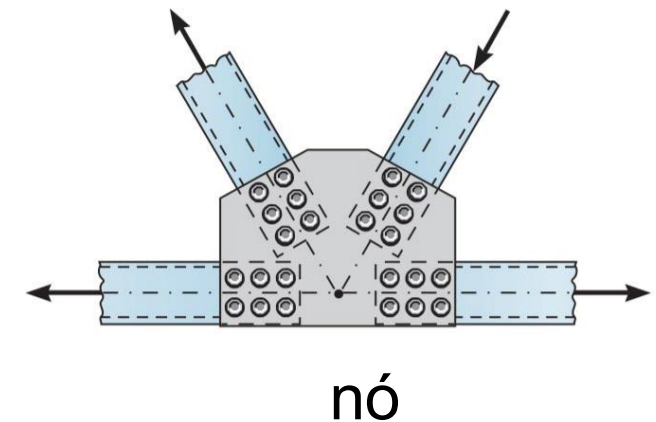
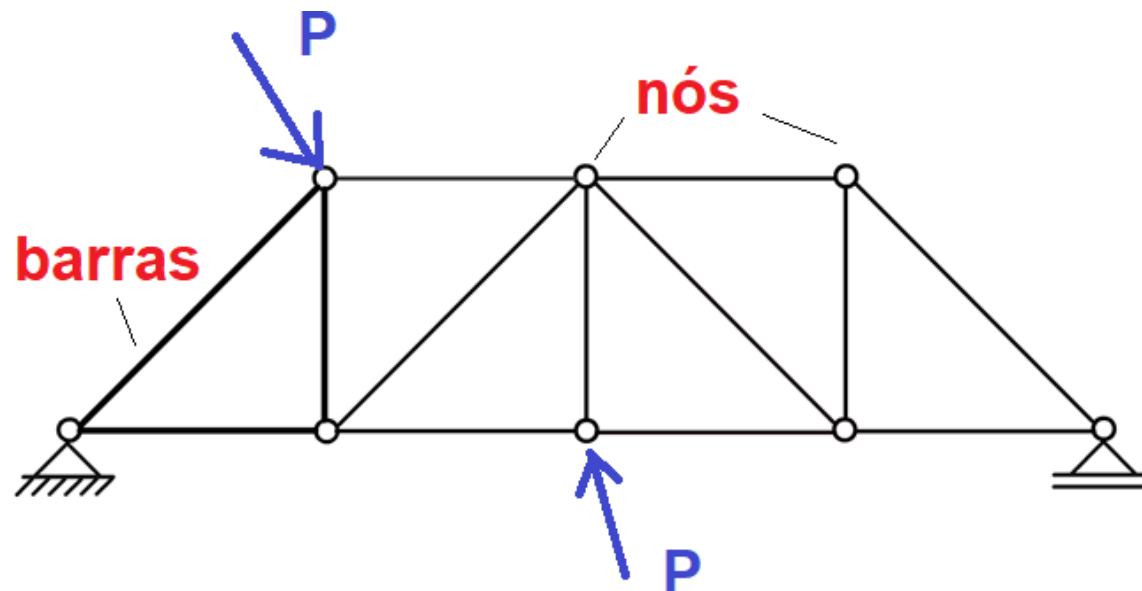
TRELIÇAS

Formadas: barras conectadas por nós

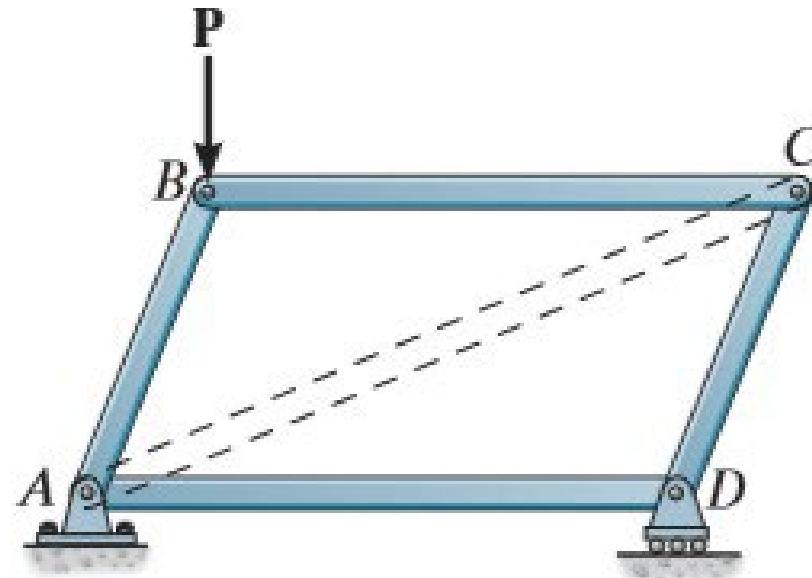
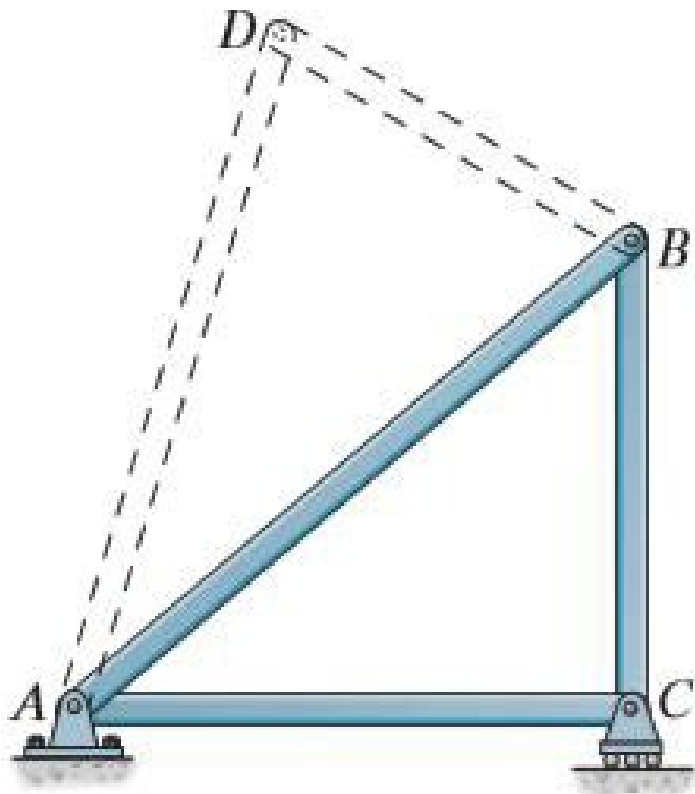
Estruturas com forças somente em nós

Barras esbeltas e retas

Lei de formação básica: triângulos

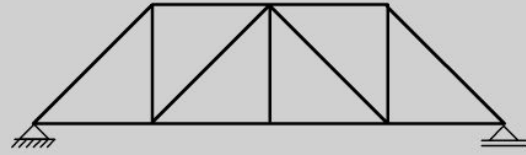


TRELIÇAS - FORMAÇÃO

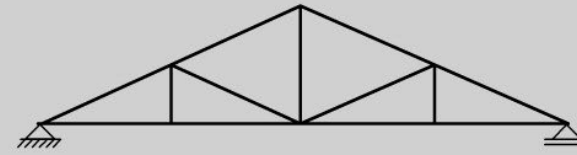


TRELIÇAS - FORMAÇÃO

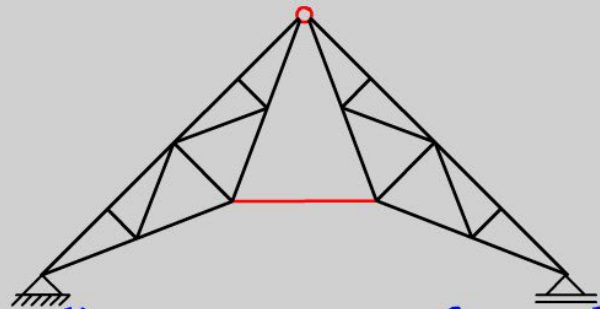
Métodos de formação de treliças



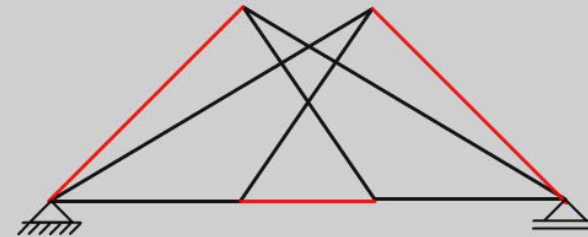
treliça simples utilizada
em pontes



treliça simples utilizada
em coberturas



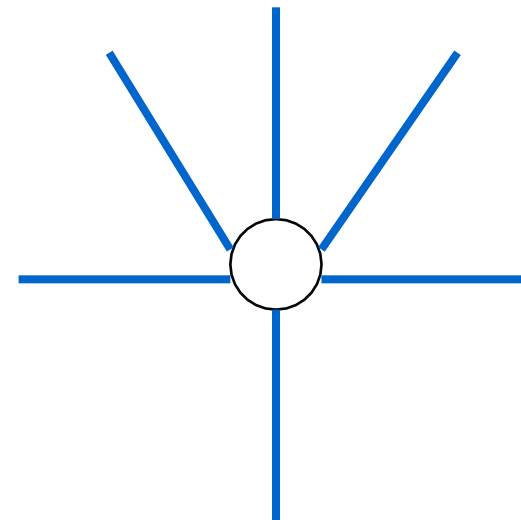
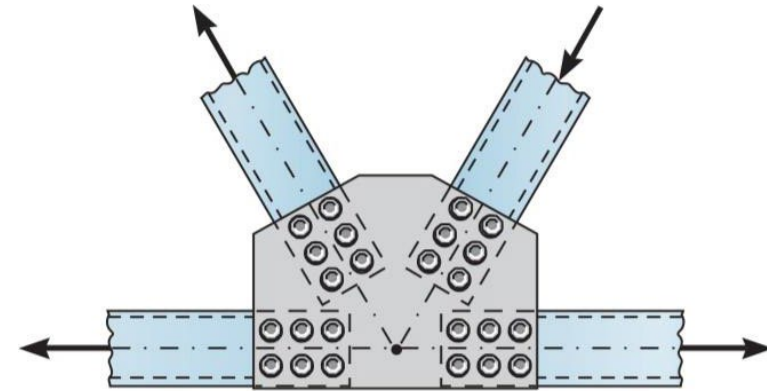
treliças compostas formadas a
partir da união de duas treliças
simples por meio de um nó
comum e uma barra



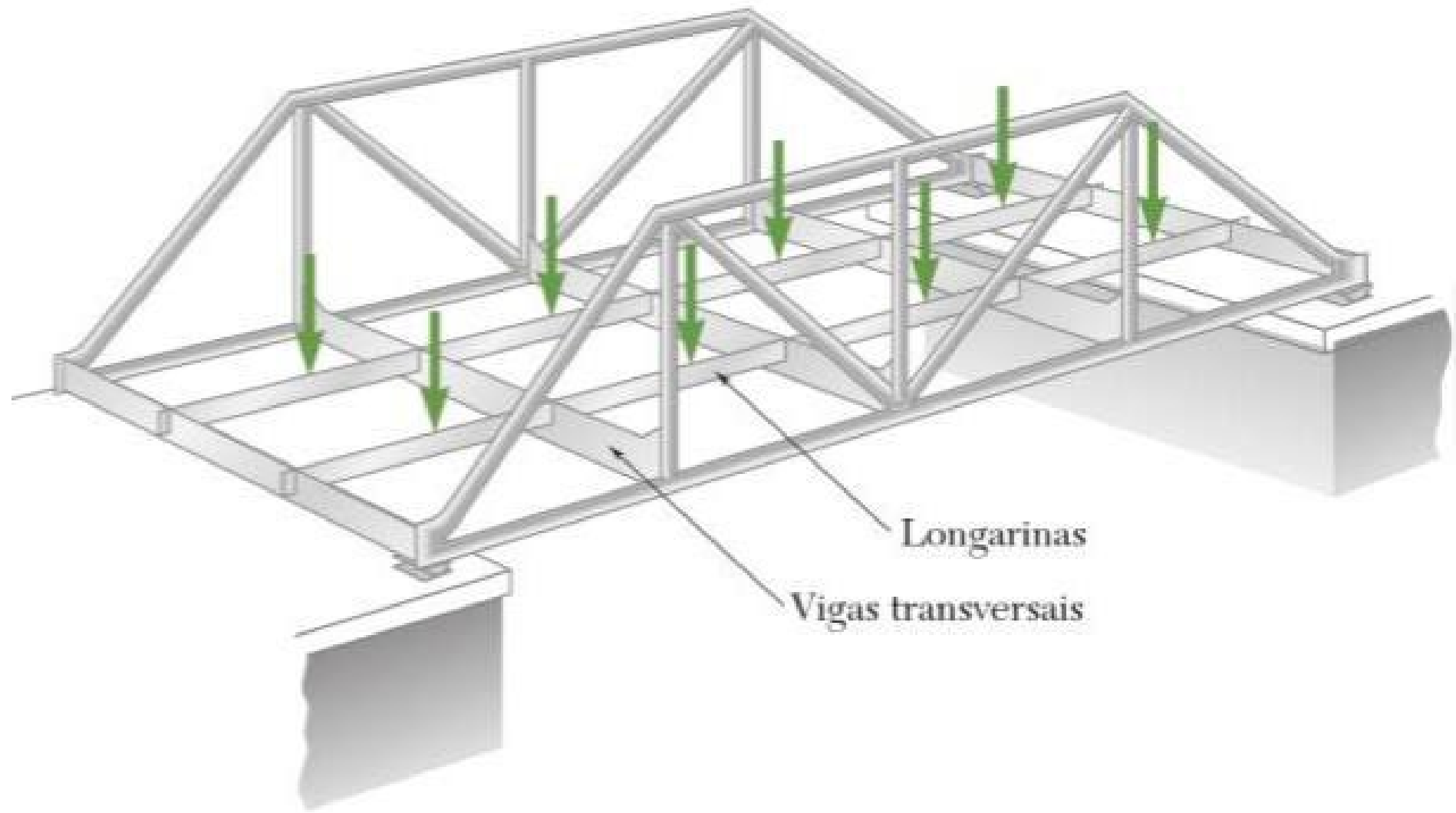
treliças compostas formadas a
partir da união de duas treliças
simples por meio de três barras

TRELIÇAS - LIGAÇÕES

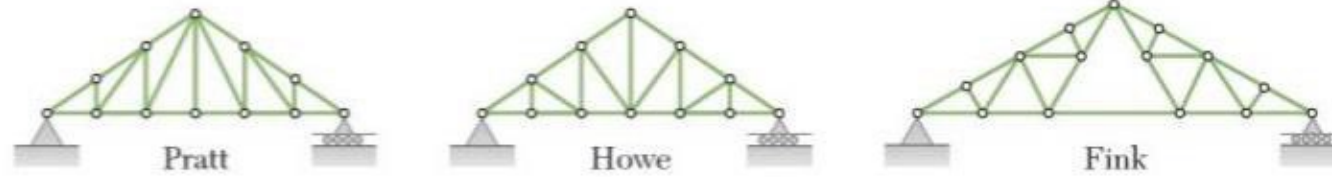
Ligações: nós



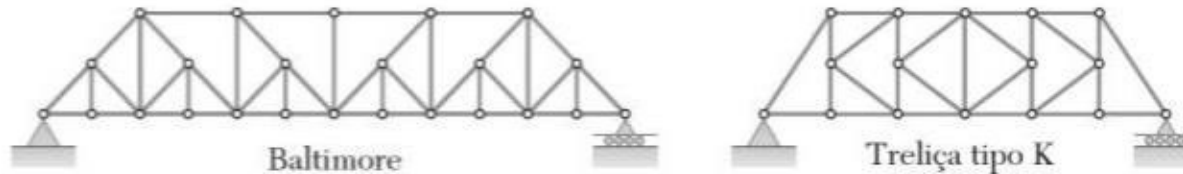
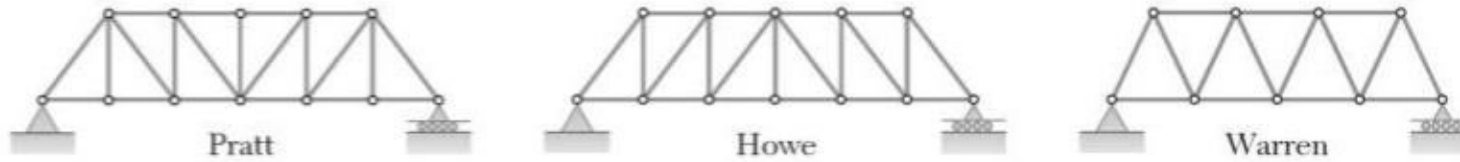
TRELIÇAS – SIMPLIFICA PARA 2D



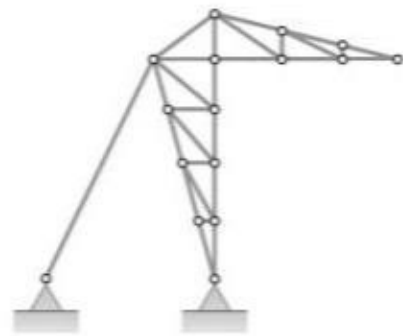
TRELIÇAS: TIPOS



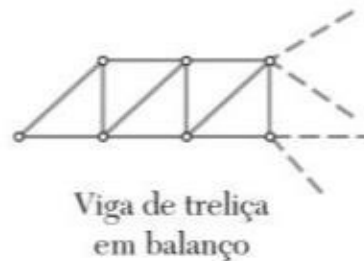
Treliças típicas para telhados



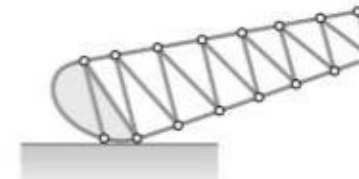
Treliças típicas para pontes



Tipo estádio



Outros tipos de treliças



Basculante

TRELIÇAS - HIPÓTESES

Hipóteses de cálculo: Treliça ideal

- 1) nós são ligações entre as barras e não se considera atrito
- 2) estão sujeitas apenas as forças concentradas aplicadas somente nos nós;
- 3) os eixos das barras coincidem com a reta que une os centros das articulações, não gerando resultante de momento na articulação;
- 4) cada barra apresenta apenas esforços normais.

Os esforços de momento fletor e força cortante são nulos

TRELIÇAS - ESFORÇOS

Barras com esforços apenas NORMAIS



TRELIÇAS

Determinação da Treliça:

nr. barras = b

nr. reações = r

nr. nós = n

$b + r = 2.n$  isostático

$b + r > 2.n$  hiperestático

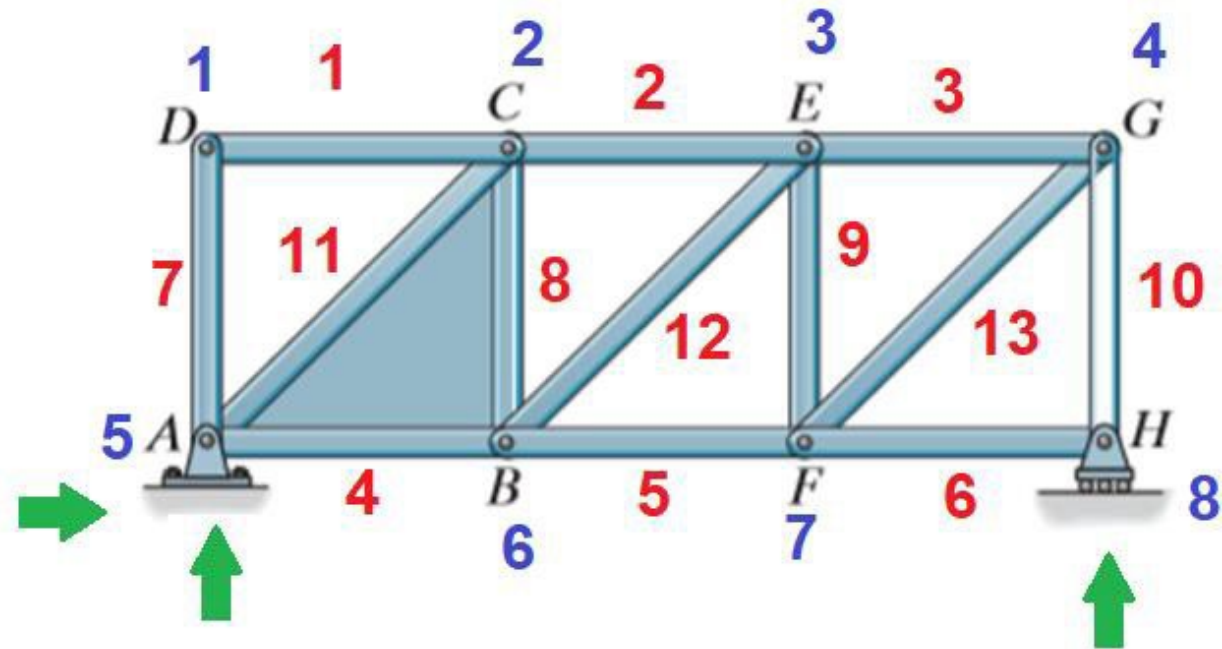
$b + r < 2.n$  hipostático

TRELIÇAS

Determinação da Treliça:

$$b + r = 2.n \quad \longrightarrow \quad \text{isostático}$$

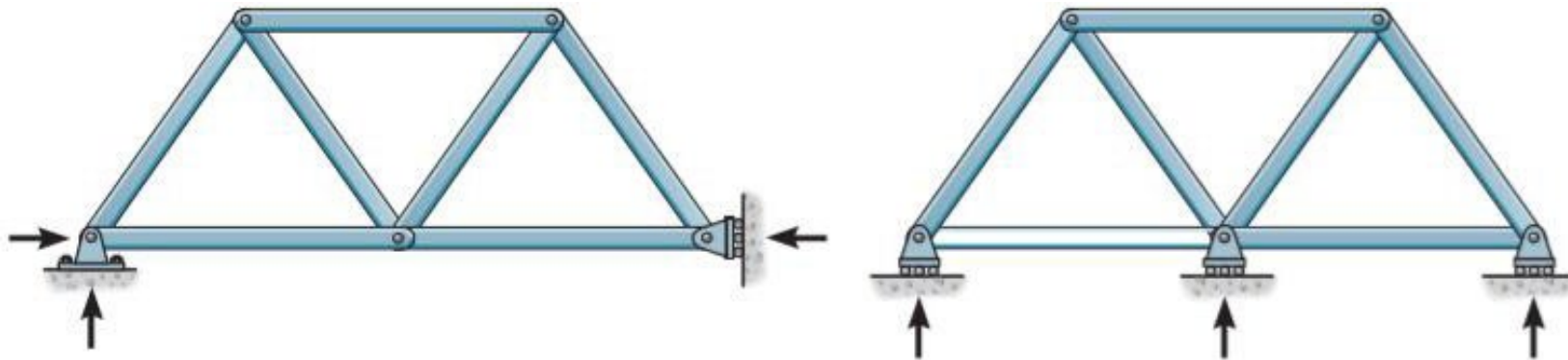
$$b = 13$$
$$r = 3$$
$$n = 8$$



TRELIÇAS

Determinação da Treliça:

$b + r < 2.n$  hipostático



TRELIÇAS

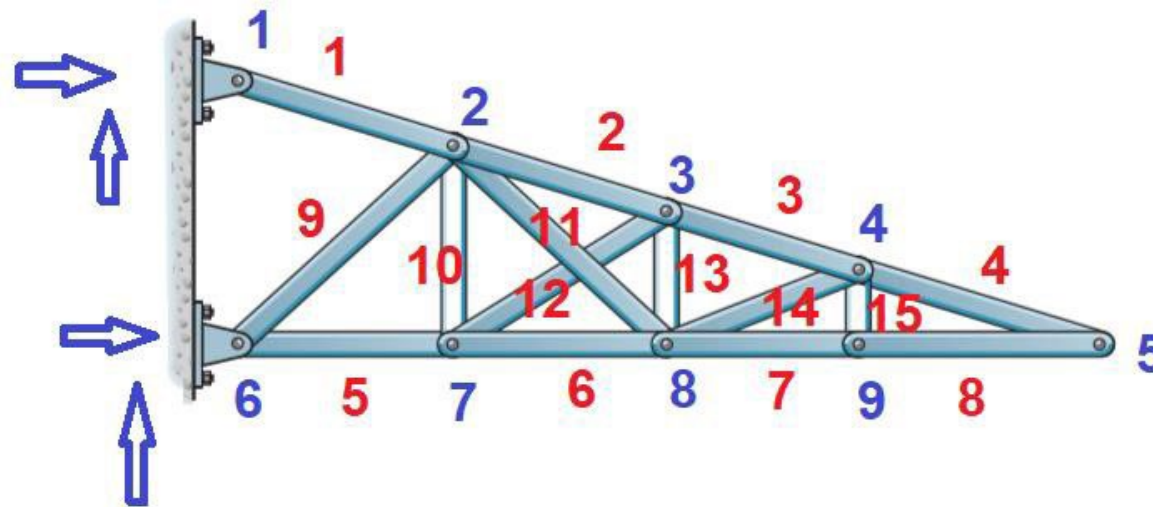
Determinação da Treliça:

$b + r > 2.n$ **➔** hiperestático

$$b = 15$$

$$r = 4$$

$$n = 9$$



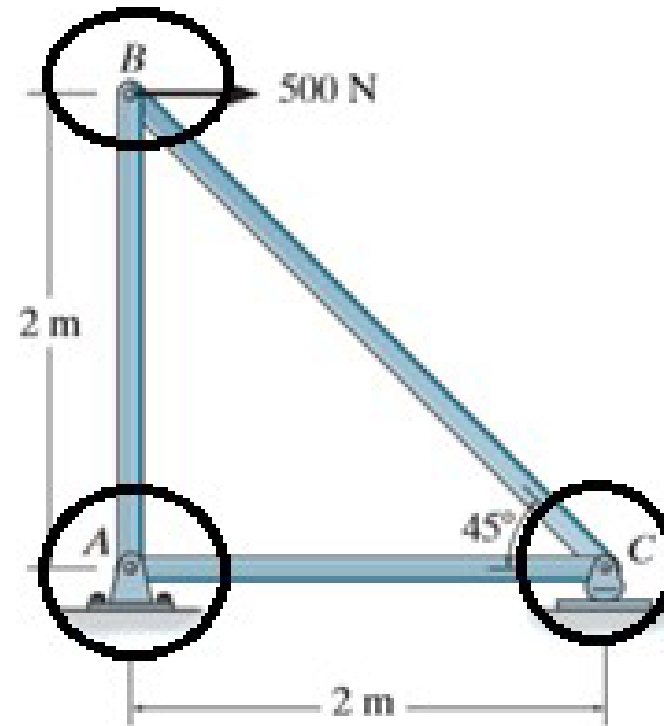
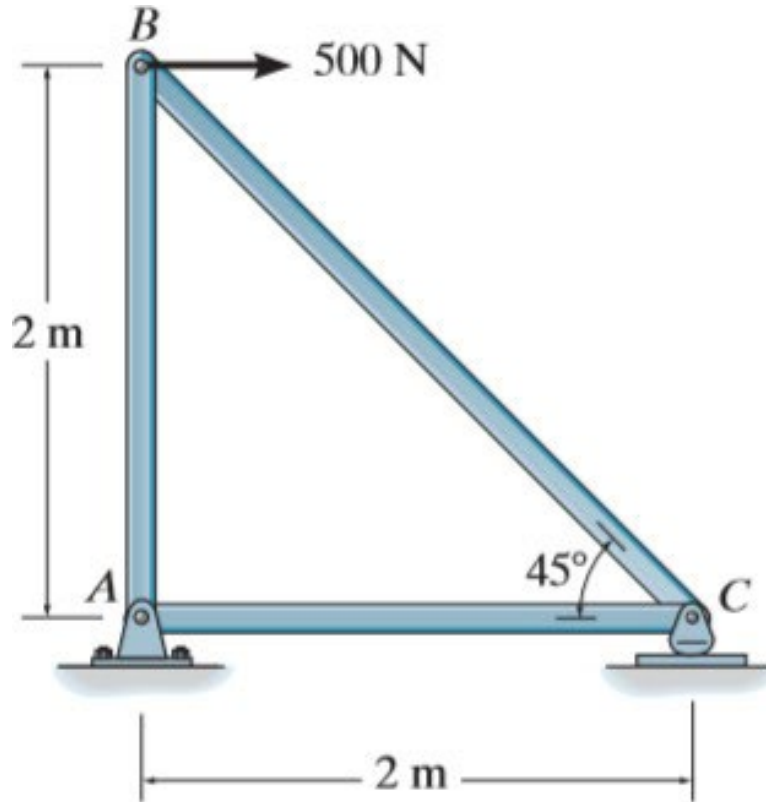
$$15 + 4 > 2.9$$

TRELIÇAS

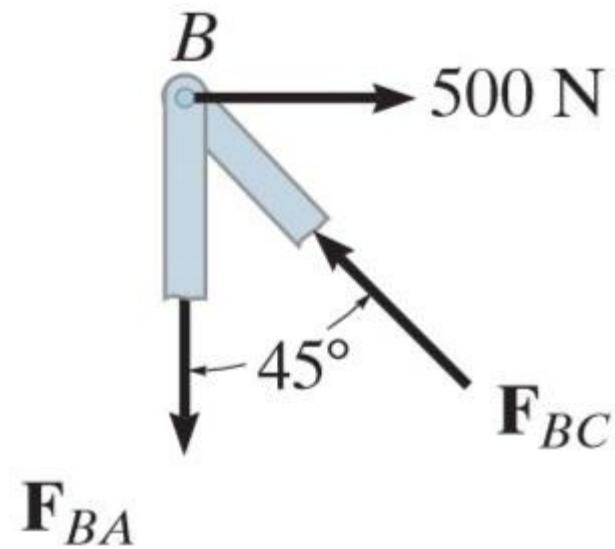
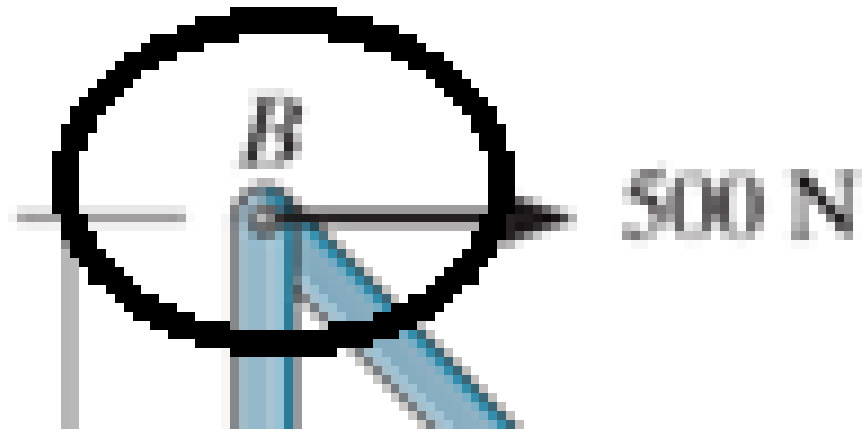
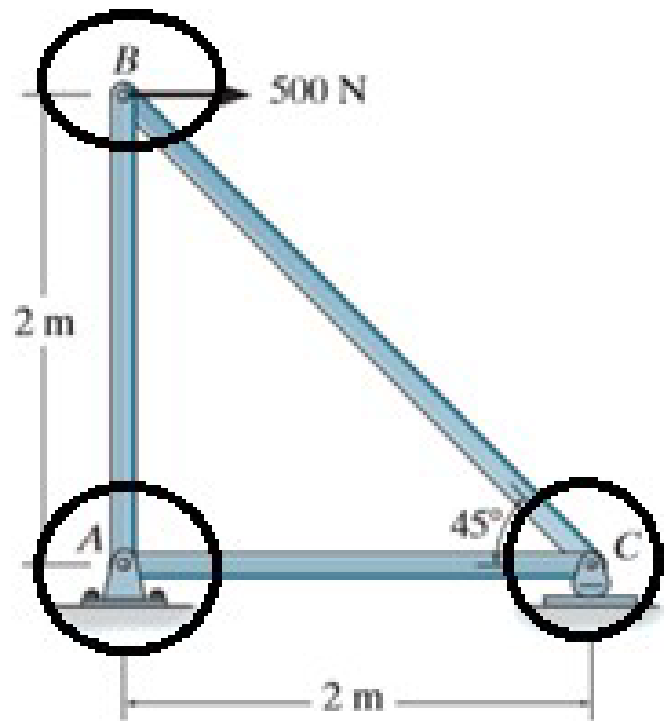
Métodos de cálculo manual

- **Método do equilíbrio dos nós**
- **Método de Ritter ou das seções**

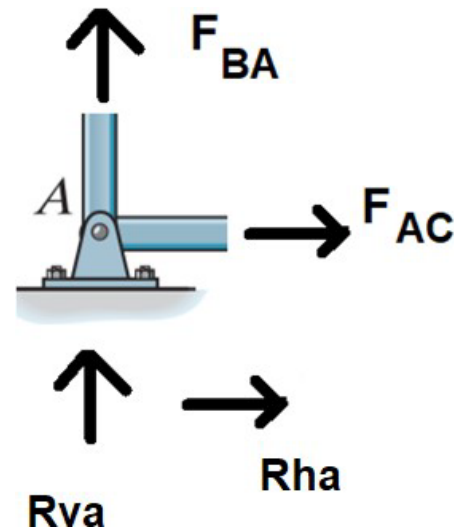
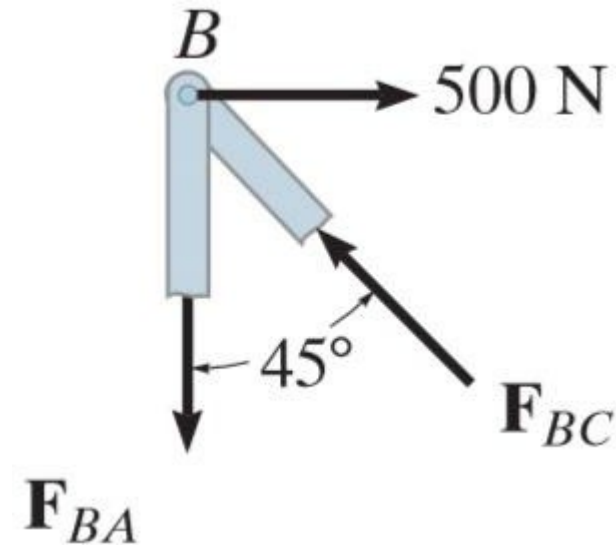
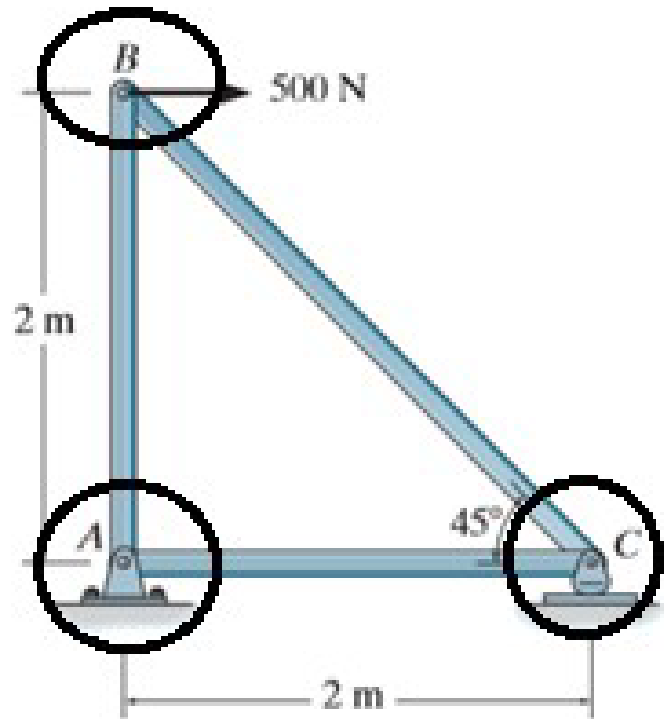
Treliças: método do equilíbrio dos nós



Treliças: método do equilíbrio dos nós



Treliças: método do equilíbrio dos nós



Treliças: método do equilíbrio dos nós

Em função das hipóteses admitidas, cada nó se constitui em um ponto material submetido a um sistema de forças de equilíbrio, onde se podem usar as seguintes equações para cada nó i :

$$\sum F_x^i = 0 \text{ e } \sum F_y^i = 0.$$

Assim, o procedimento de cálculo das forças normais de cada barra da treliça fica:

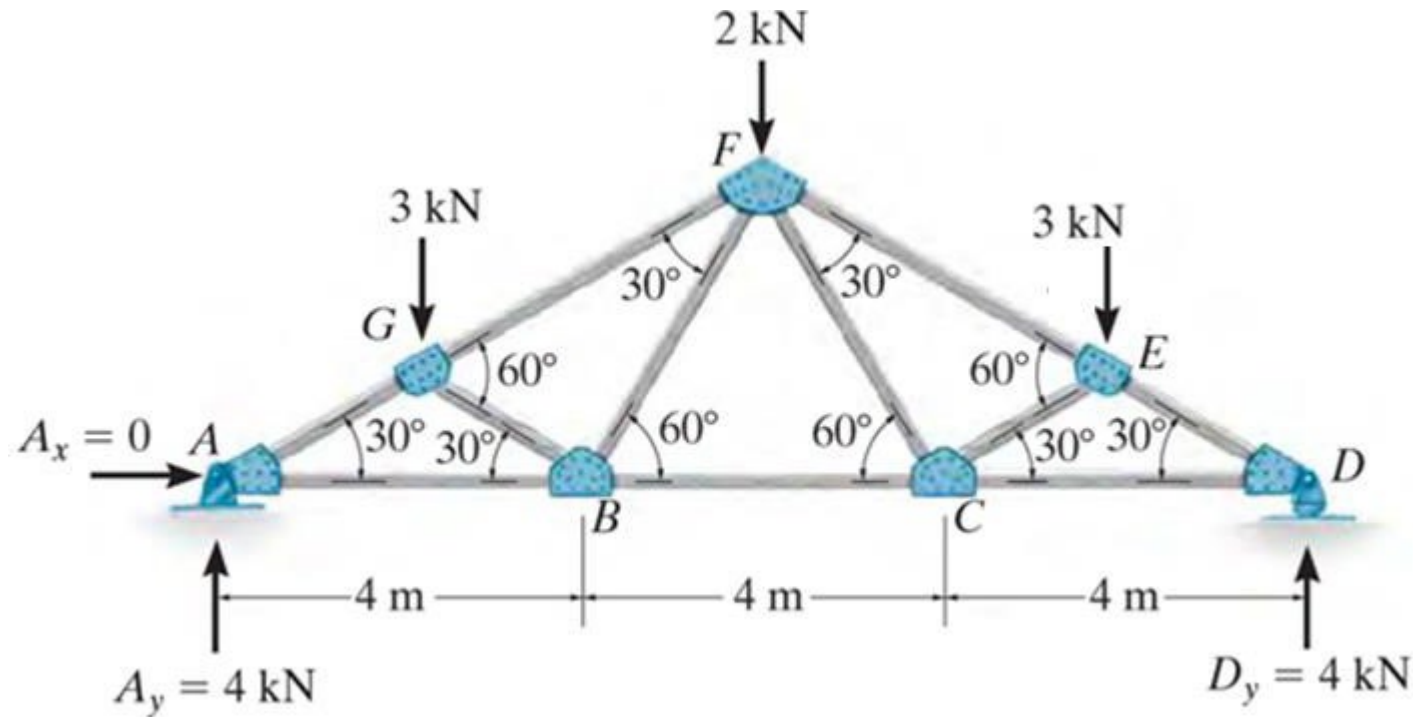
- 1) Determinar as reações ($\sum F_x = 0$, $\sum F_y = 0$ e $\sum M_s = 0$);
- 2) Aplicar as duas equações de equilíbrio em cada nó.

Alguns casos, o nó está com mais de 2 barras conectadas

Resolver sistema linear

Treliças: método do equilíbrio dos nós

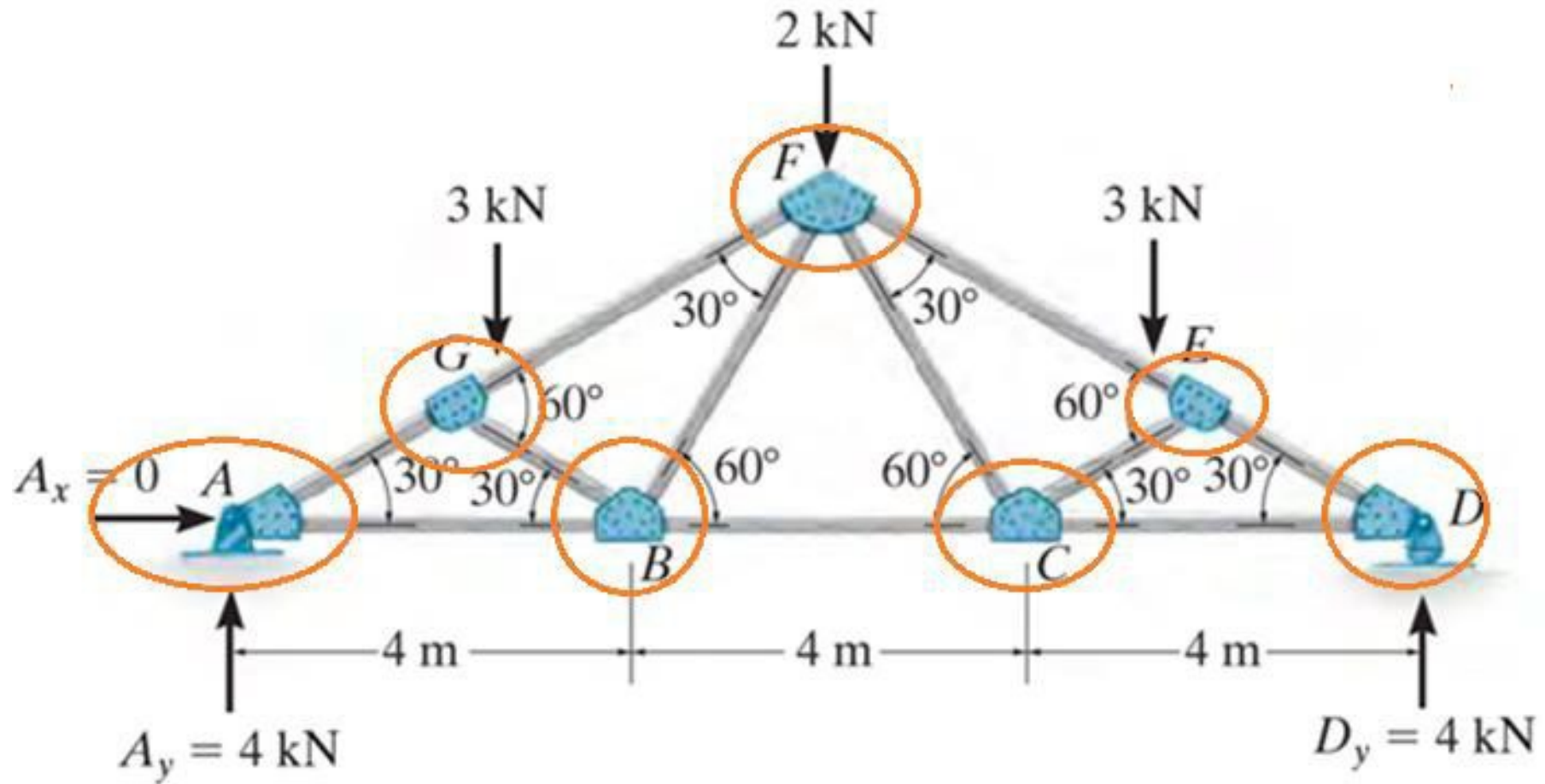
Alguns casos, o nó está com mais de 2 barras conectadas



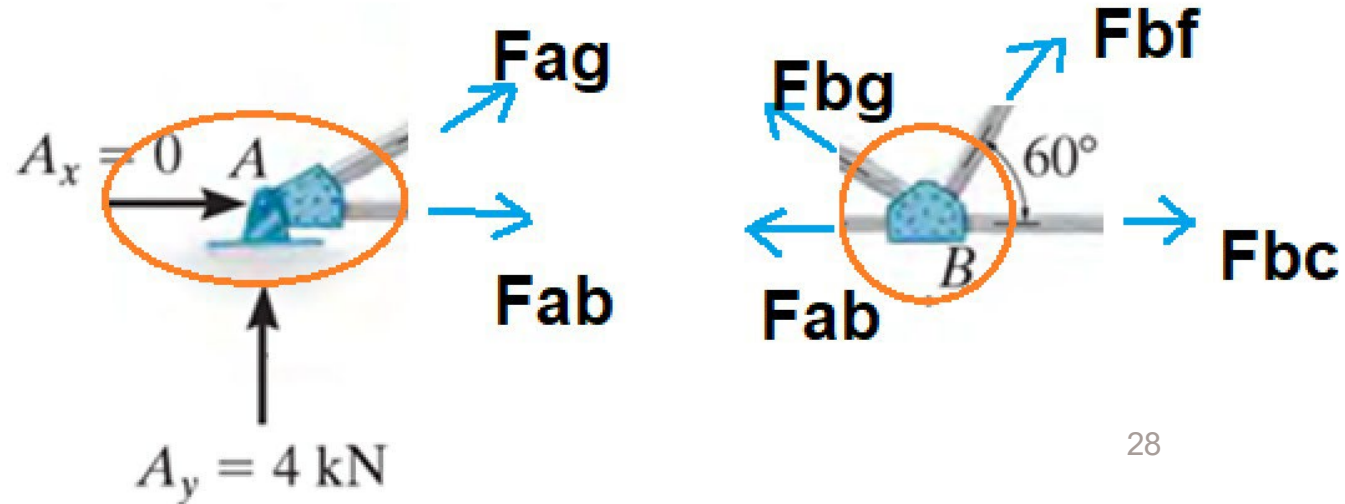
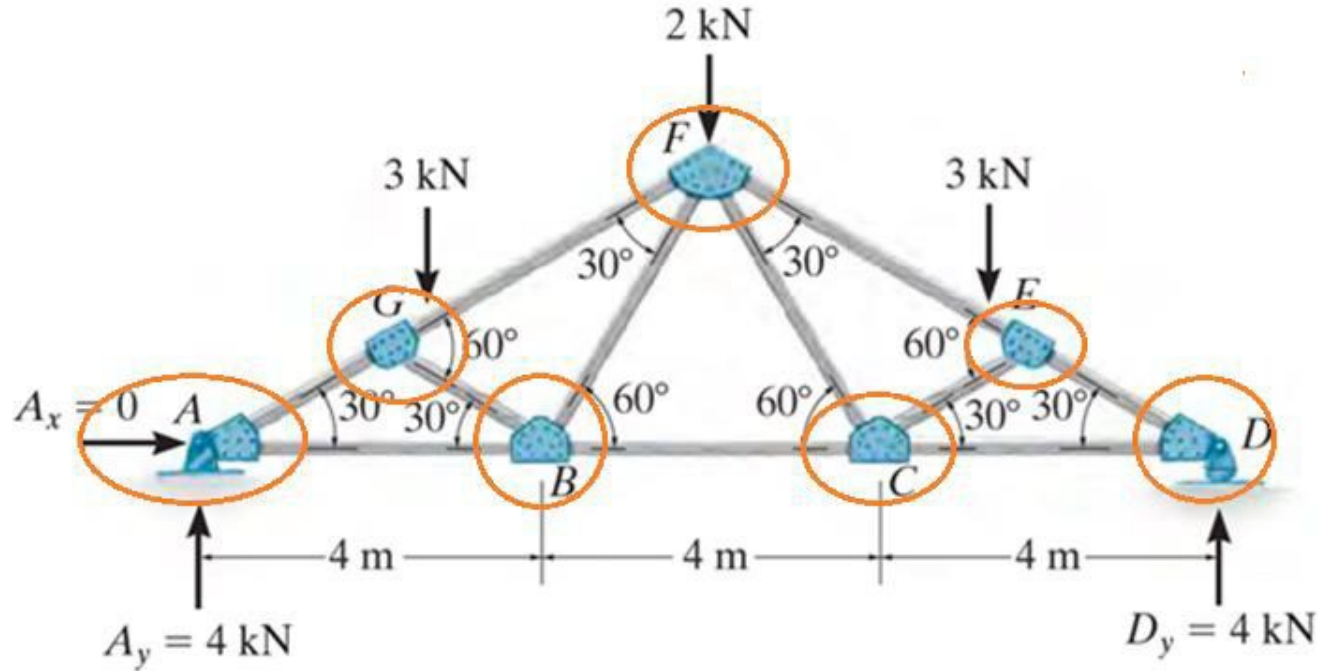
Obtenha reações

Comece por nós que se tem 2 incógnitas: $\sum F_x^i = 0$ e $\sum F_y^i = 0$.

Treliças: método do equilíbrio dos nós



Treliças: método do equilíbrio dos nós

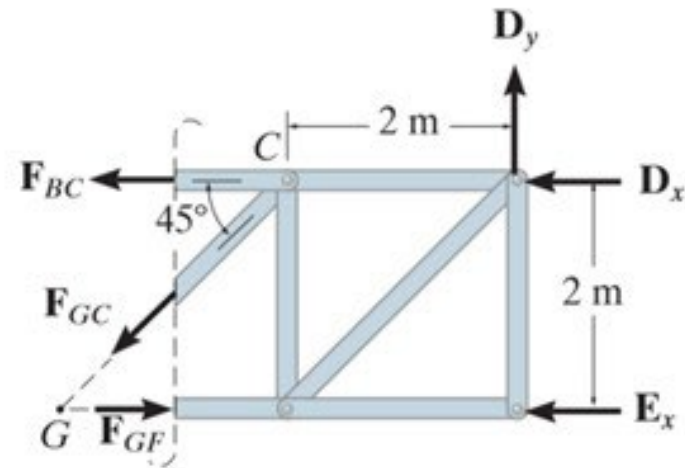
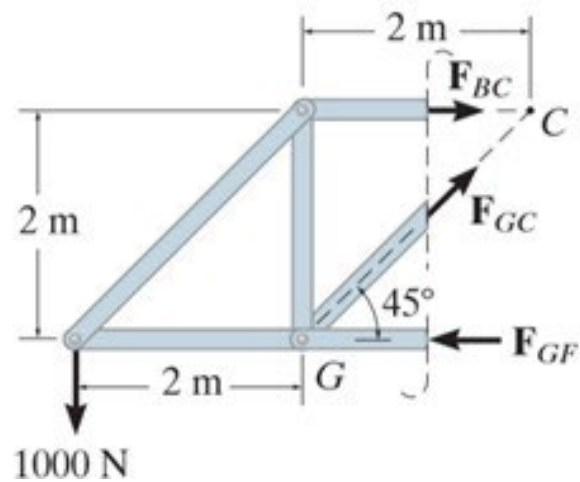
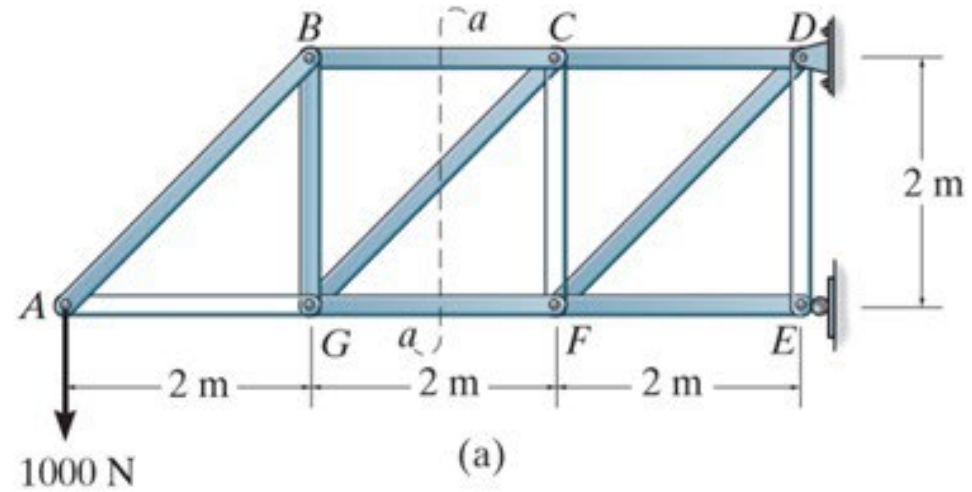


Treliças: Método de Ritter ou das seções

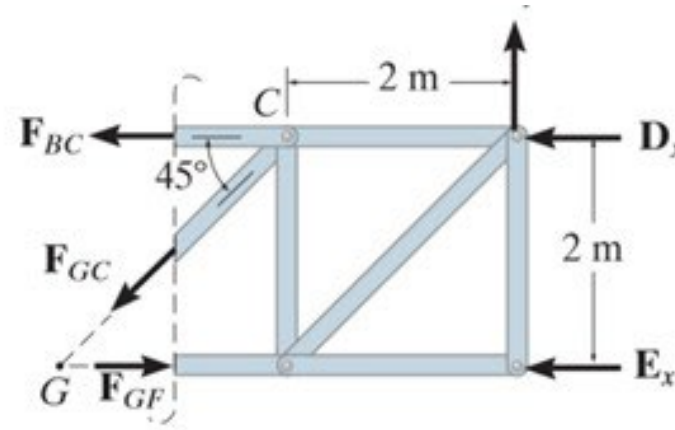
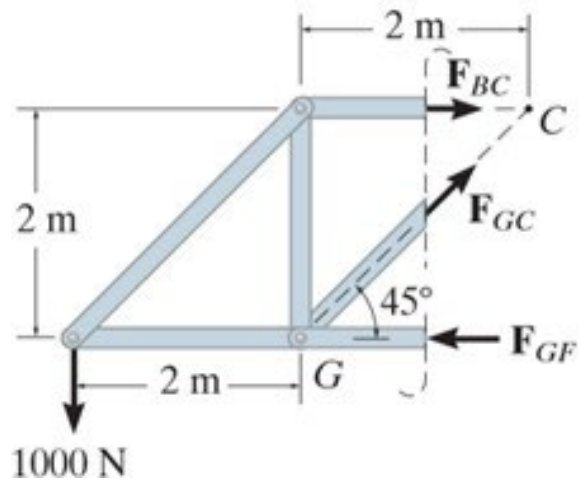
O método é mais empregado quando se queira determinar os esforços normais apenas em determinadas barras. Baseia-se pela aplicação de um corte numa seção imaginária que separe a treliça em duas partes. De modo que as três equações da estática possa ser aplicada no diagrama de corpo livre de cada uma das partes, levando a determinação dos esforços nas seções cortadas, quando possível.

- 1) Determinar as reações ($\sum F_x = 0$, $\sum F_y = 0$ e $\sum M_s = 0$);
- 2) Fazer um seletivo corte, passando pela(s) barra(s) desejada(s), separando a treliça em duas partes, aplicar as 3 equações da estática de equilíbrio na parte desejada, determinando os esforços na(s) barra(s) requerida(s).

Treliças: Método de corte de Ritter ou das seções



Treliças: Método do corte de Ritter ou das seções



$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum M_s = 0$$

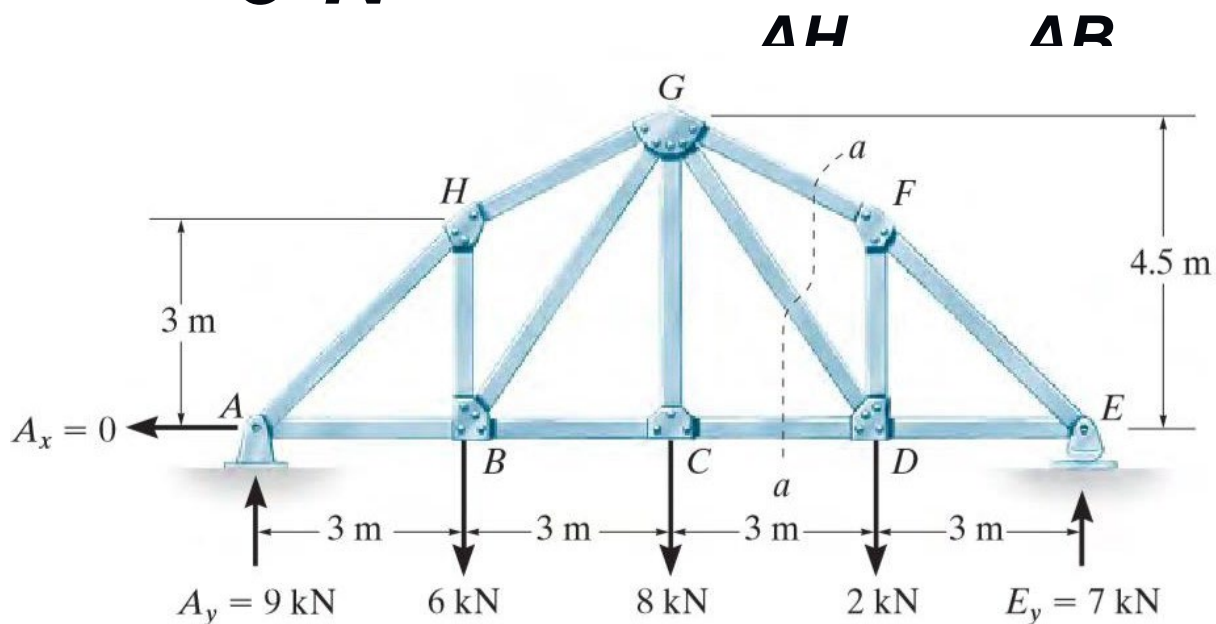
Usar a equação de equilíbrio de interesse

Exemplo 1.0

Diagramas

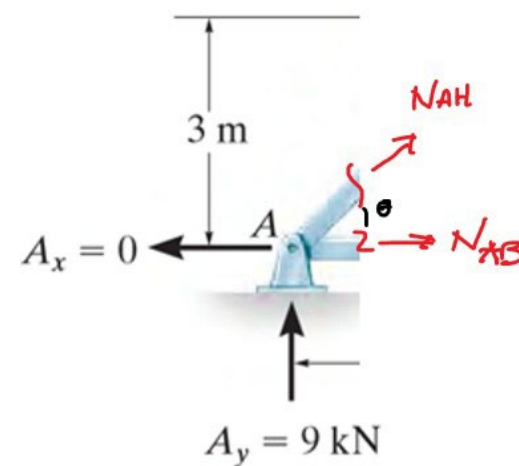
Força N

e N



Obtido reações..

Equilíbrio do nó A



$$\sum F_x = 0: N_{AB} + N_{AH} \cos \theta = 0 \quad (1)$$

$$\sum F_y = 0: 9 + N_{AH} \sin \theta = 0 \quad (2)$$

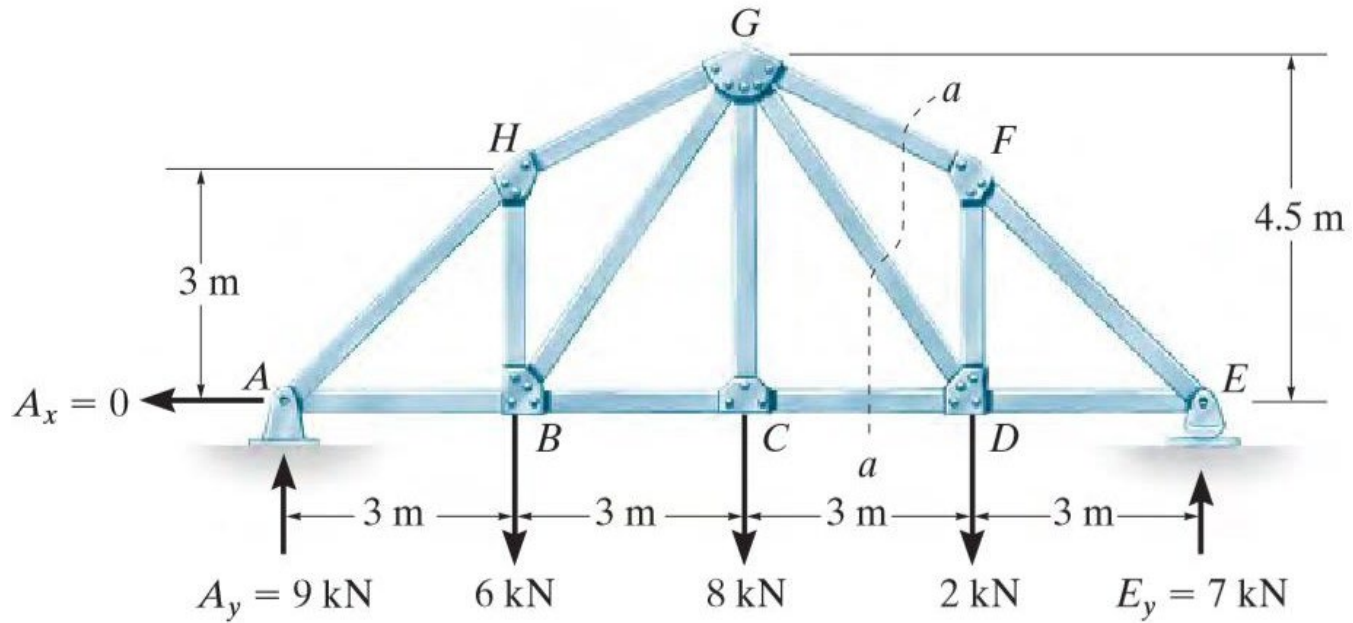
$$N_{AH} = \frac{-9}{\sin \theta} \quad \text{em } (1)$$

$$\text{tg } \theta = \frac{3}{3} \Rightarrow \theta = 45^\circ$$

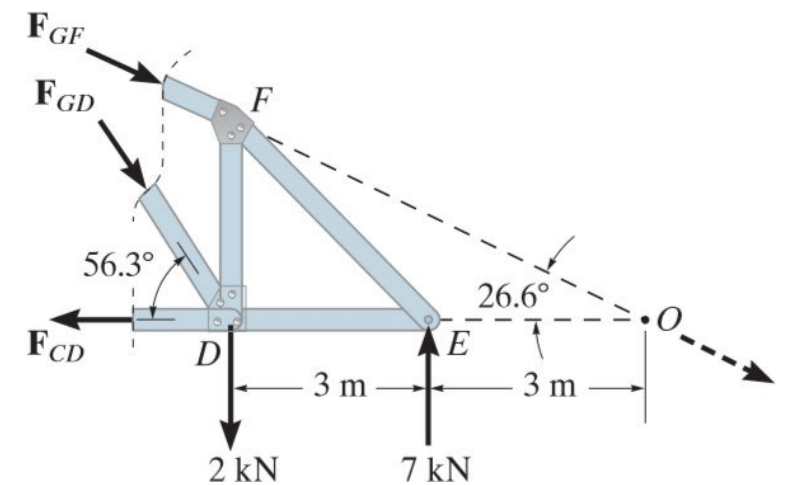
$$N_{AH} = \frac{-9}{\sin 45^\circ} = -12,73 \text{ kN (C)}$$

$$N_{AB} = 9 \text{ kN (T)}$$

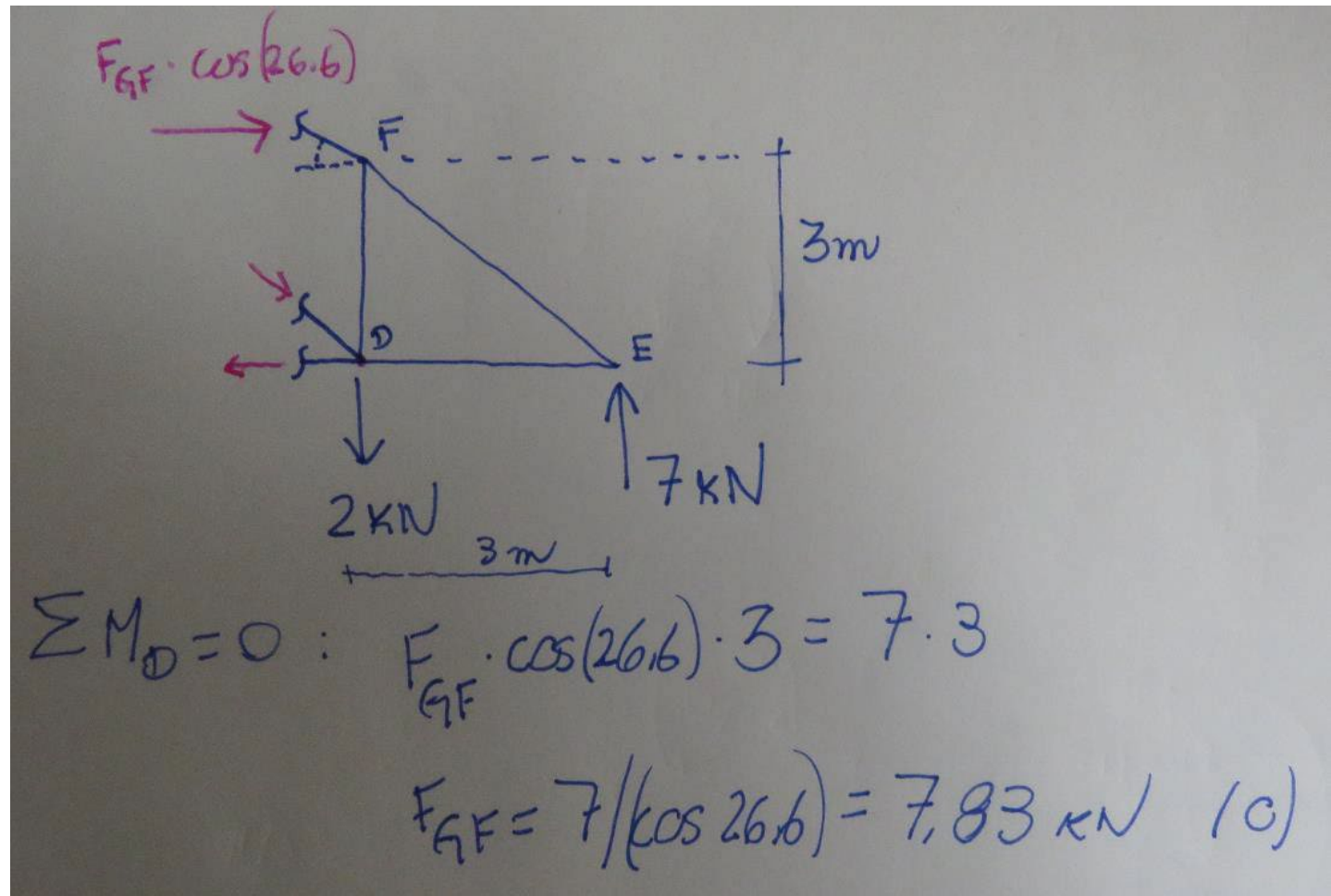
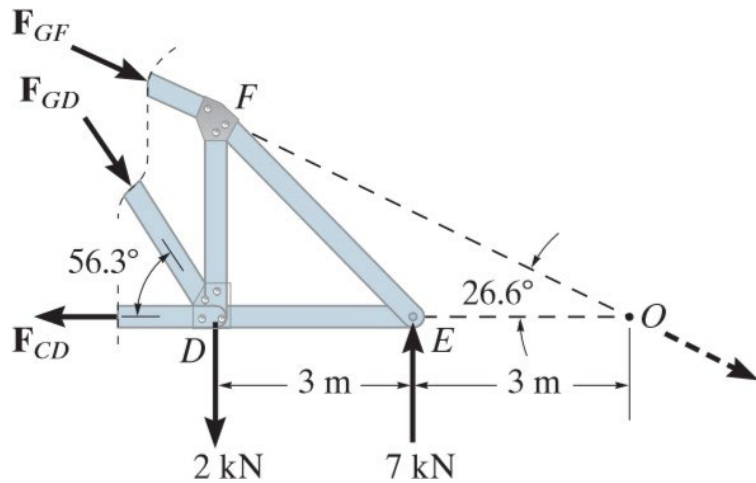
Exemplo 1: Obter N_{FG}



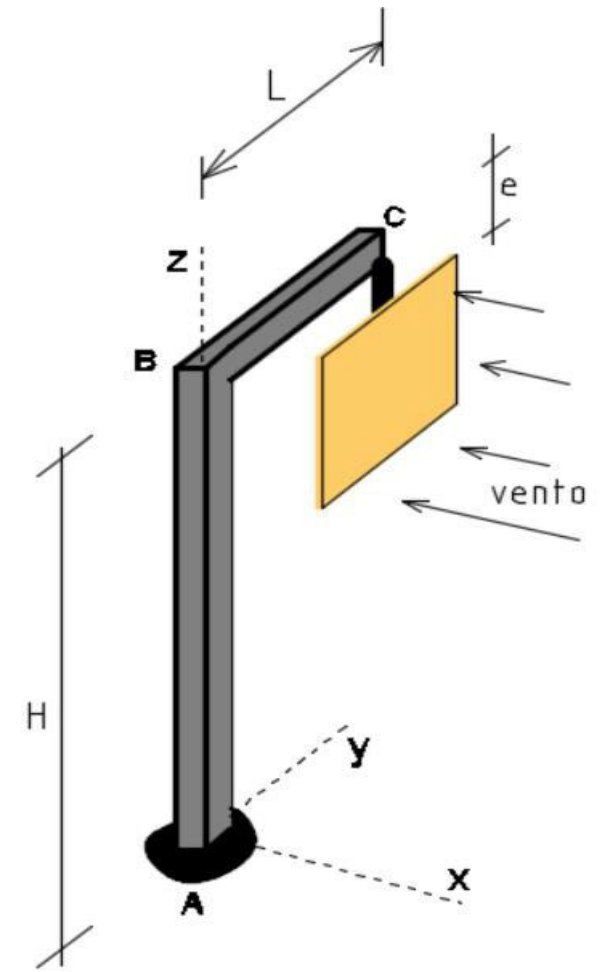
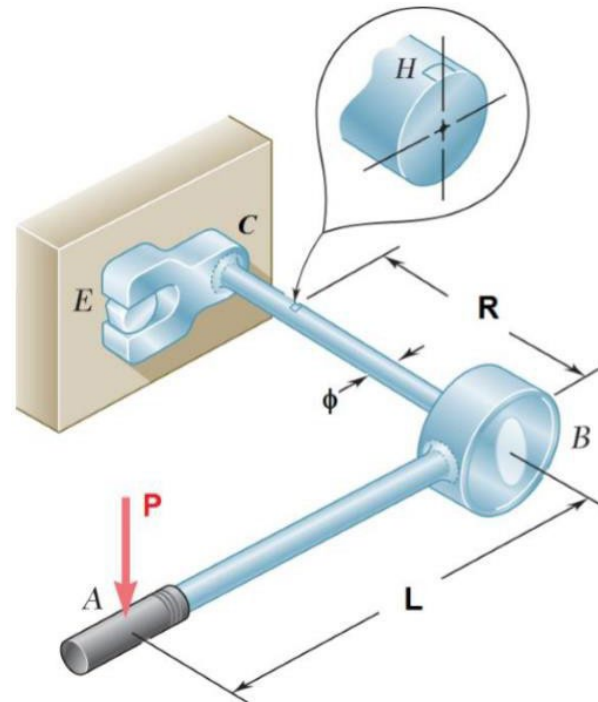
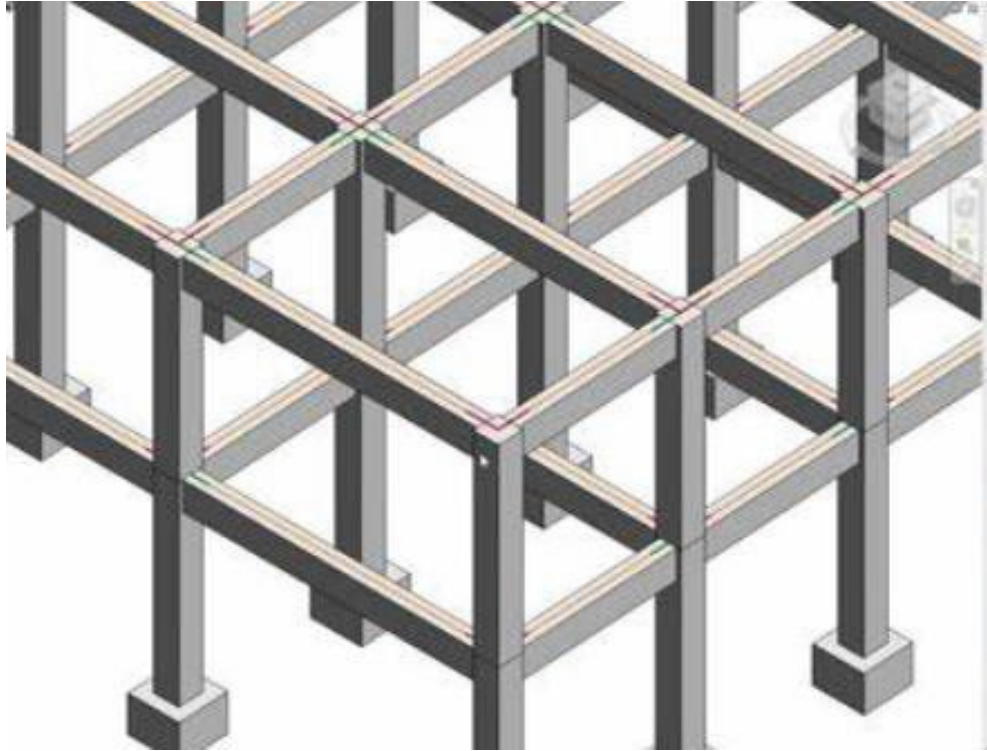
Corte de Ritter



Exemplo 2: Obter N_{FG}



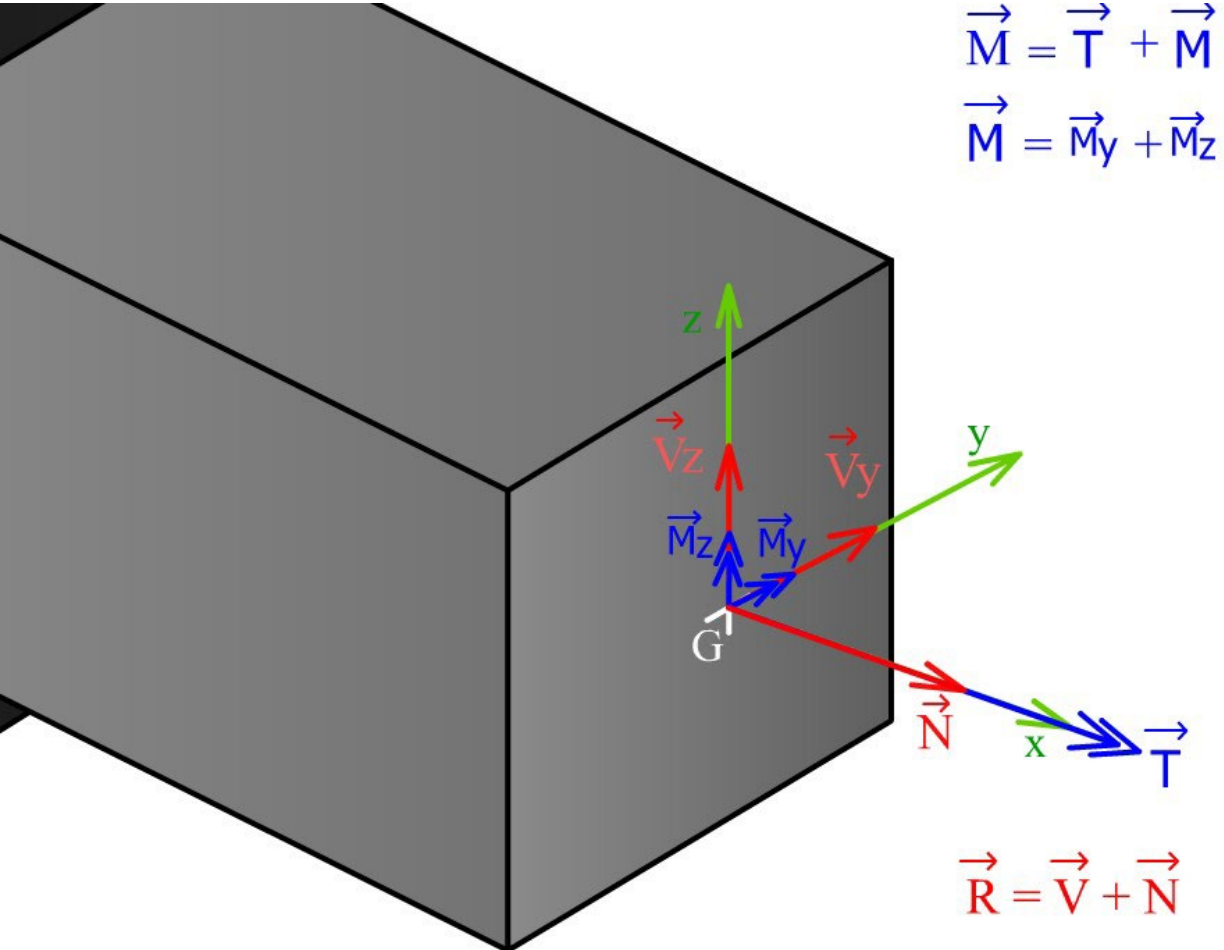
Quadros espaciais isostáticos



TOTAL DE ESFORÇOS (6)

$$\vec{M} = \vec{T} + \vec{M}$$

$$\vec{M} = \vec{M}_y + \vec{M}_z$$



$$\vec{R} = \vec{V} + \vec{N}$$

$$\vec{V} = \vec{V}_y + \vec{V}_z$$

$$N = \int_A \sigma dA$$

$$V_y = \int_A \tau_y dA$$

$$V_z = \int_A \tau_z dA$$

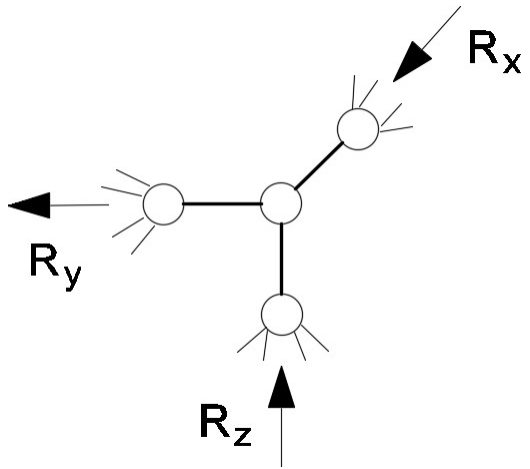
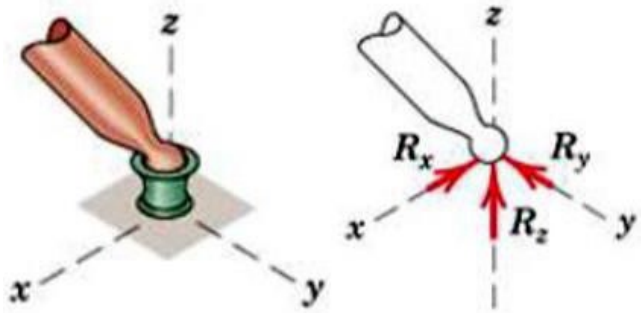
$$M_z = \int_A \sigma \cdot y dA$$

$$M_y = \int_A \sigma \cdot z dA$$

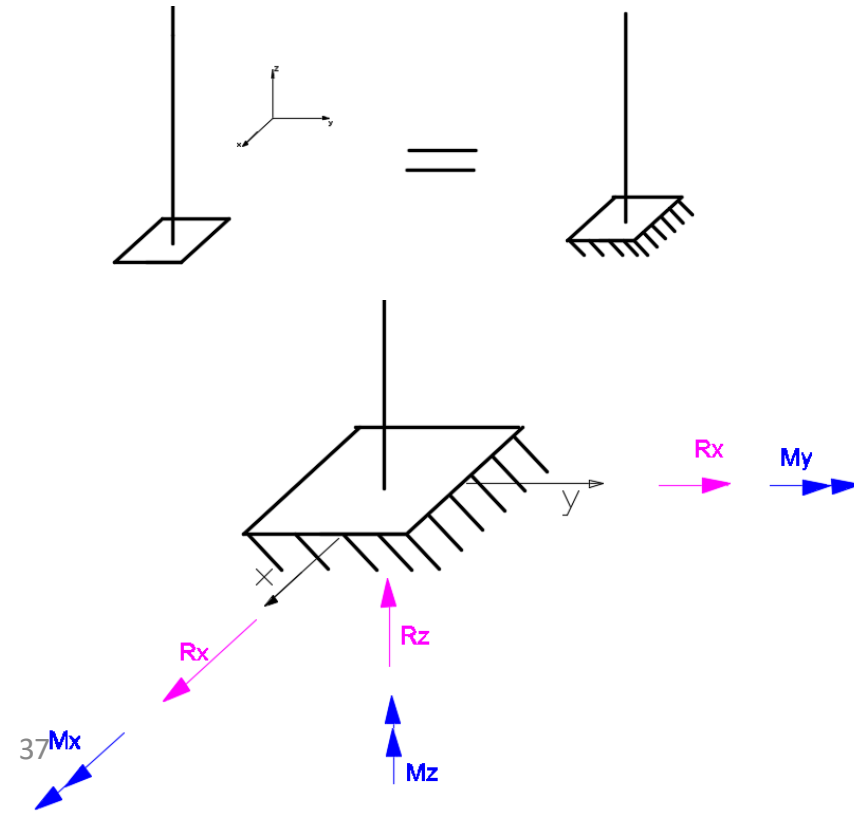
$$T = \int_A (\tau_z y - \tau_y z) dA$$

VÍNCULOS

Apoios fixos



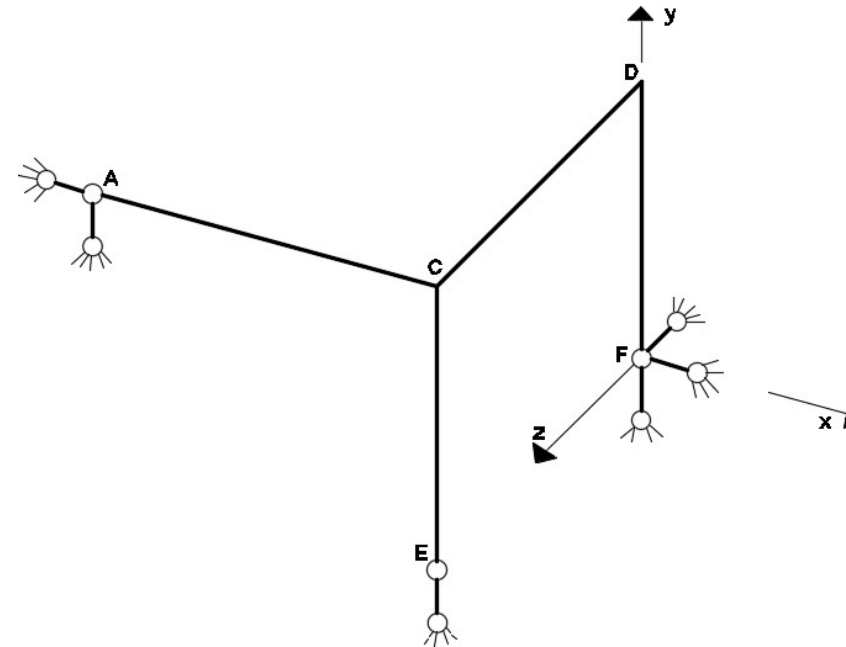
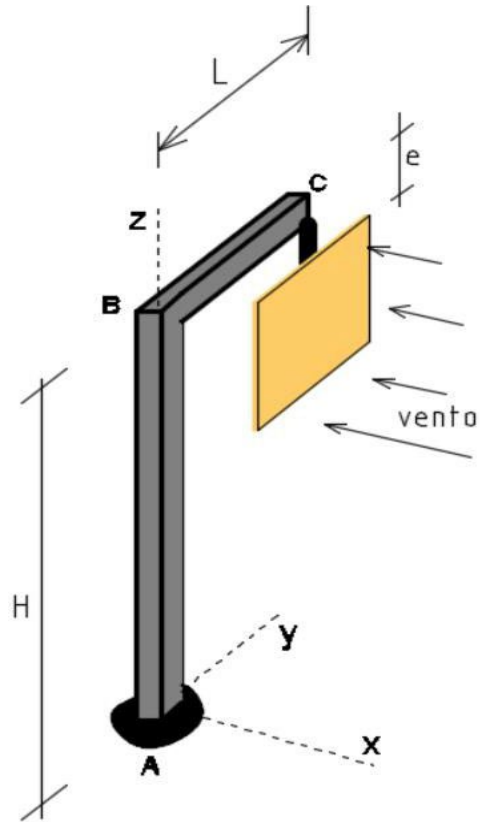
Engaste



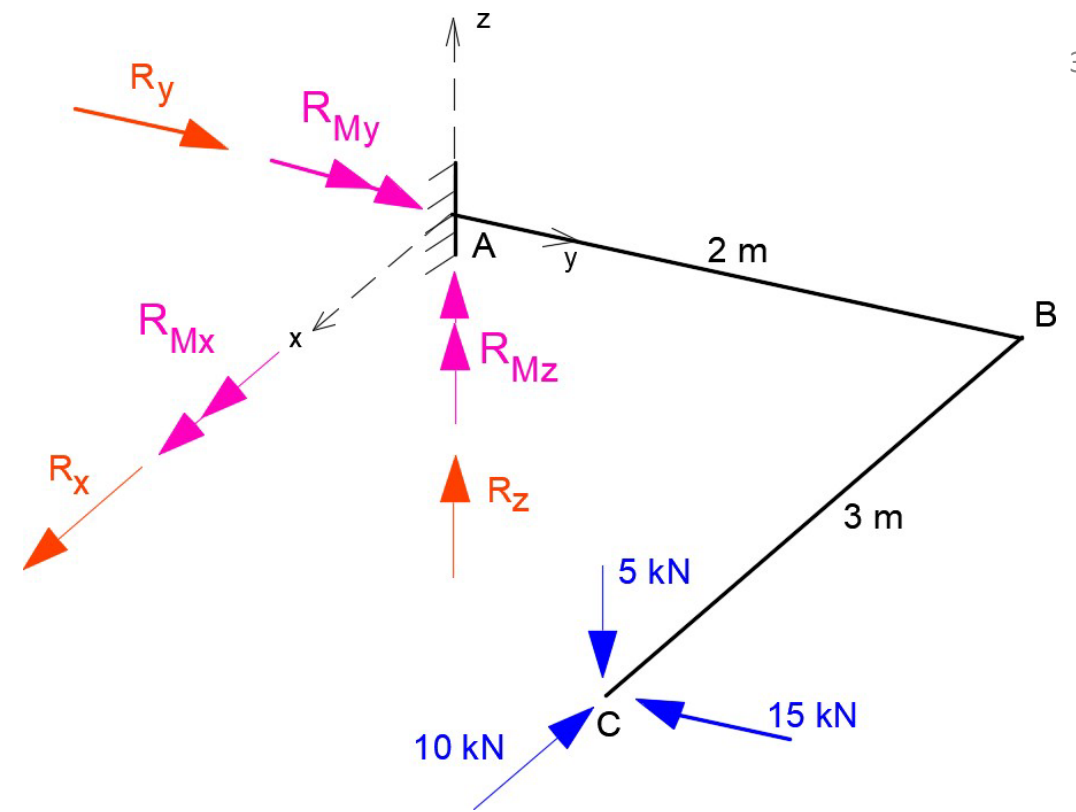
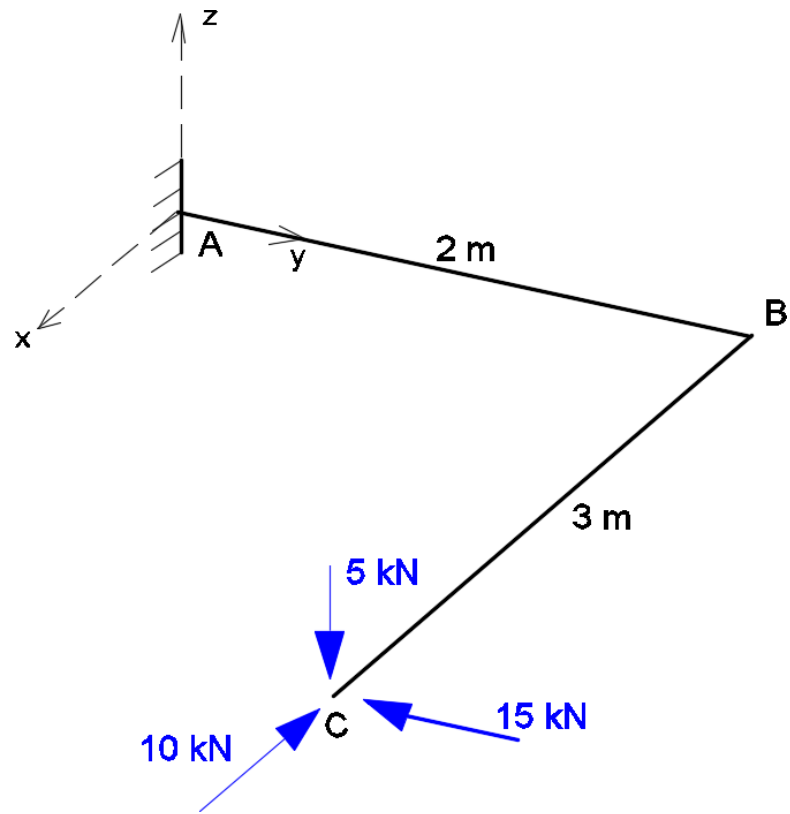
Quadros isostáticos

Vínculos suficiente para manter estrutura em equilíbrio

Obtendo reações e esforços apenas com as equações da estática



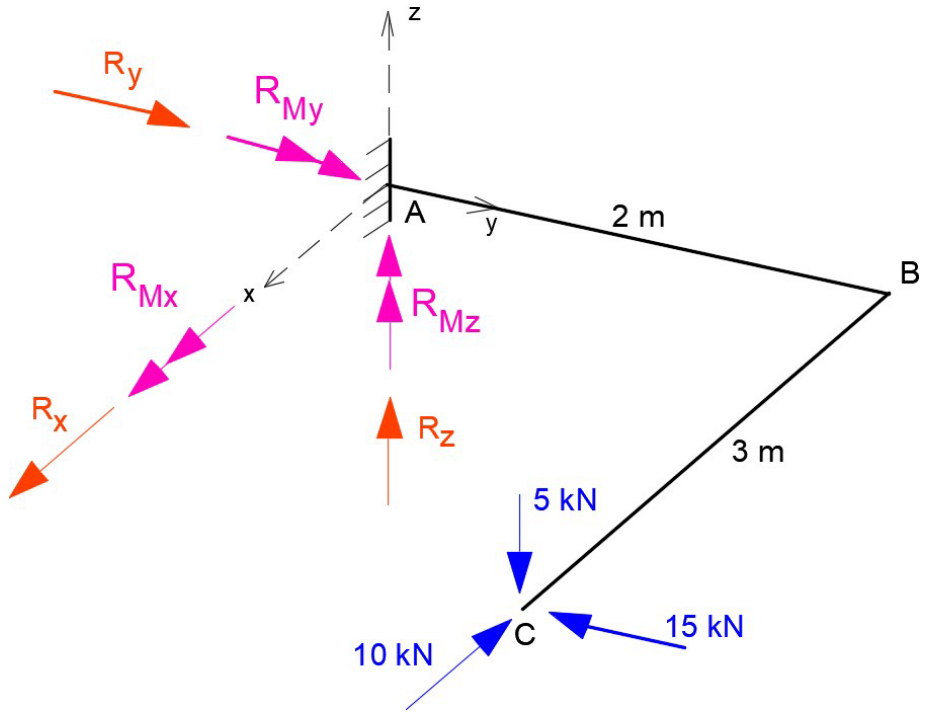
Exemplo 3: Determinar as reações da estrutura



$$\text{? } F_x = 0: R_x - 10 = 0 \rightarrow R_x = 10 \text{ kN}$$

$$\text{? } F_y = 0: R_y - 15 = 0 \rightarrow R_y = 15 \text{ kN}$$

$$\text{? } F_z = 0: R_z - 5 = 0 \rightarrow R_z = 5 \text{ kN}$$



nr. forças

$$R_{Mx} + \sum_{i=1}^{\text{nr. forças}} (M_x)_i = 0$$

$$R_{Mx} + (2\text{m})(-5\text{kN}) - (0\text{m})(-15\text{kN}) = 0$$

$$R_{Mx} = 10 \text{ kNm}$$

$$M_o = M_x i + M_y j + M_z k$$

$$M_x = (y_p - y_o)F_z - (z_p - z_o)F_y$$

$$M_y = (z_p - z_o)F_x - (x_p - x_o)F_z$$

$$M_z = (x_p - x_o)F_y - (y_p - y_o)F_x$$

$$F_x = -10 \text{ kN}; F_y = -15 \text{ kN}; F_z = -5 \text{ kN}$$

$$x_p - x_o = 3 \text{ m} \quad y_p - y_o = 2 \text{ m} \quad z_p - z_o = 0$$

nr. forças

$$R_{My} + \sum_{i=1}^{\text{nr. forças}} (M_y)_i = 0$$

$$R_{My} = -15 \text{ kNm}$$

$$R_{My} + (0\text{m})(-10\text{kN}) - (3\text{m})(-5\text{kN}) = 0$$

nr. forças

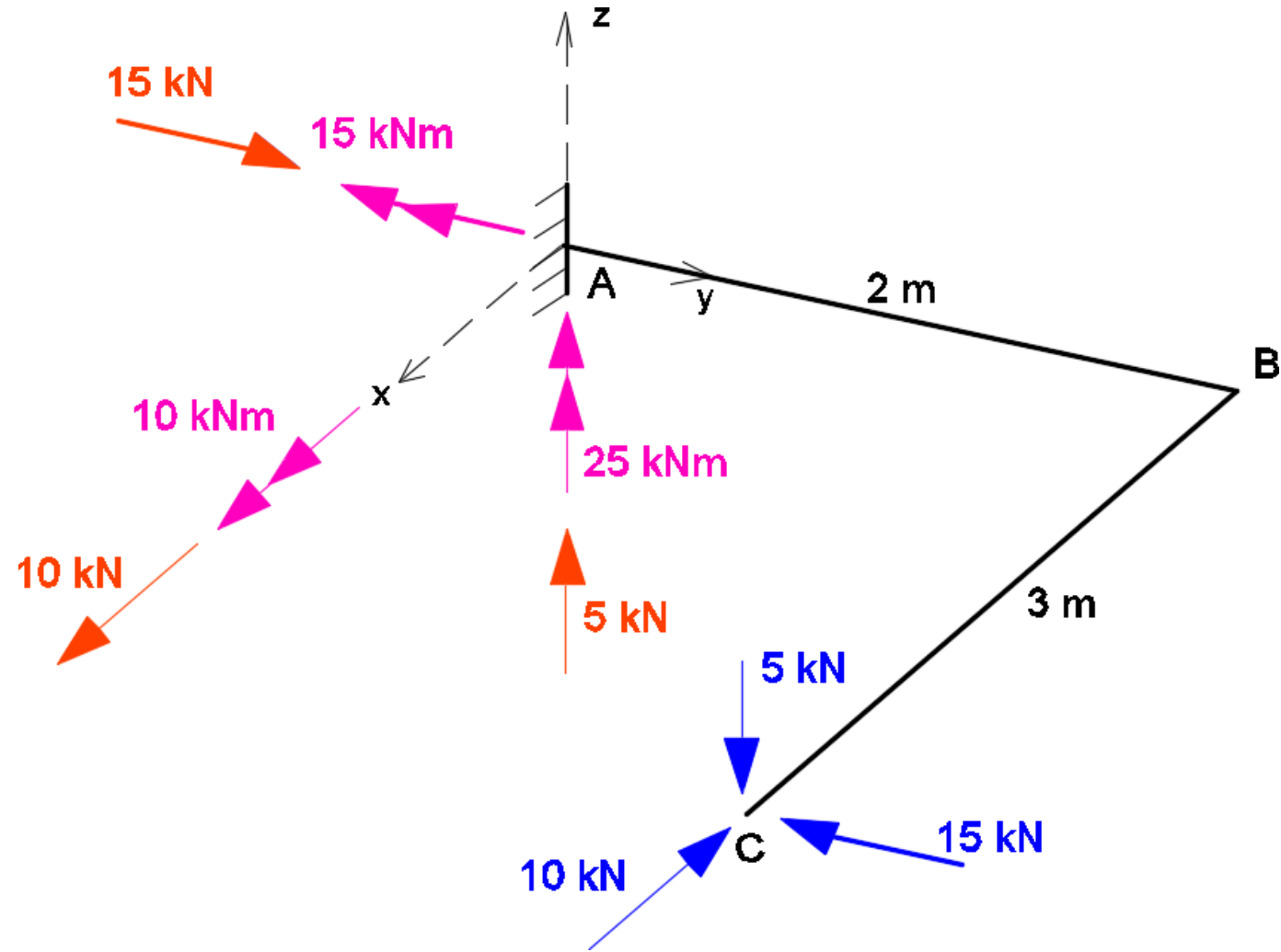
$$R_{Mz} + \sum_{i=1}^{\text{nr. forças}} (M_z)_i = 0$$

$$R_{Mz} = 25 \text{ kNm}$$

$$R_{Mz} + (3\text{m})(-15\text{kN}) - (2\text{m})(-10\text{kN}) = 0$$

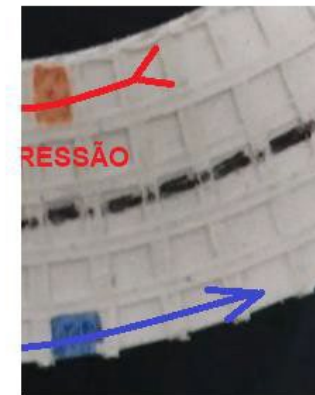
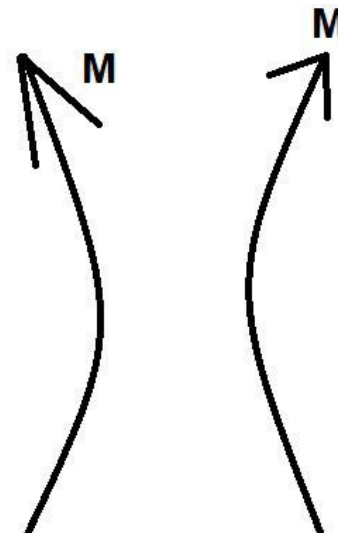
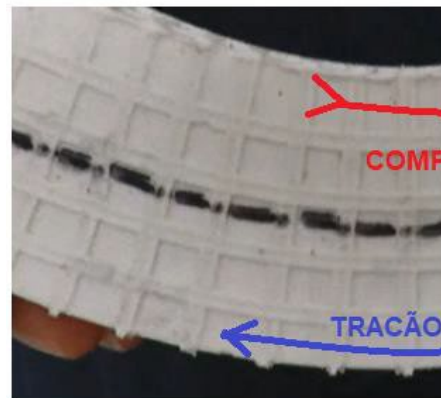
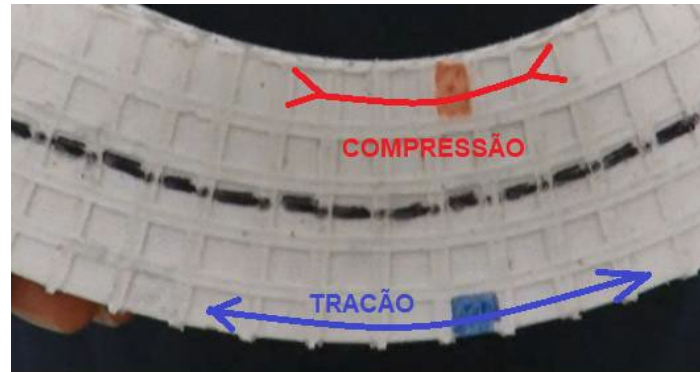
Exemplo 3: Determinar as reações da estrutura

Reações



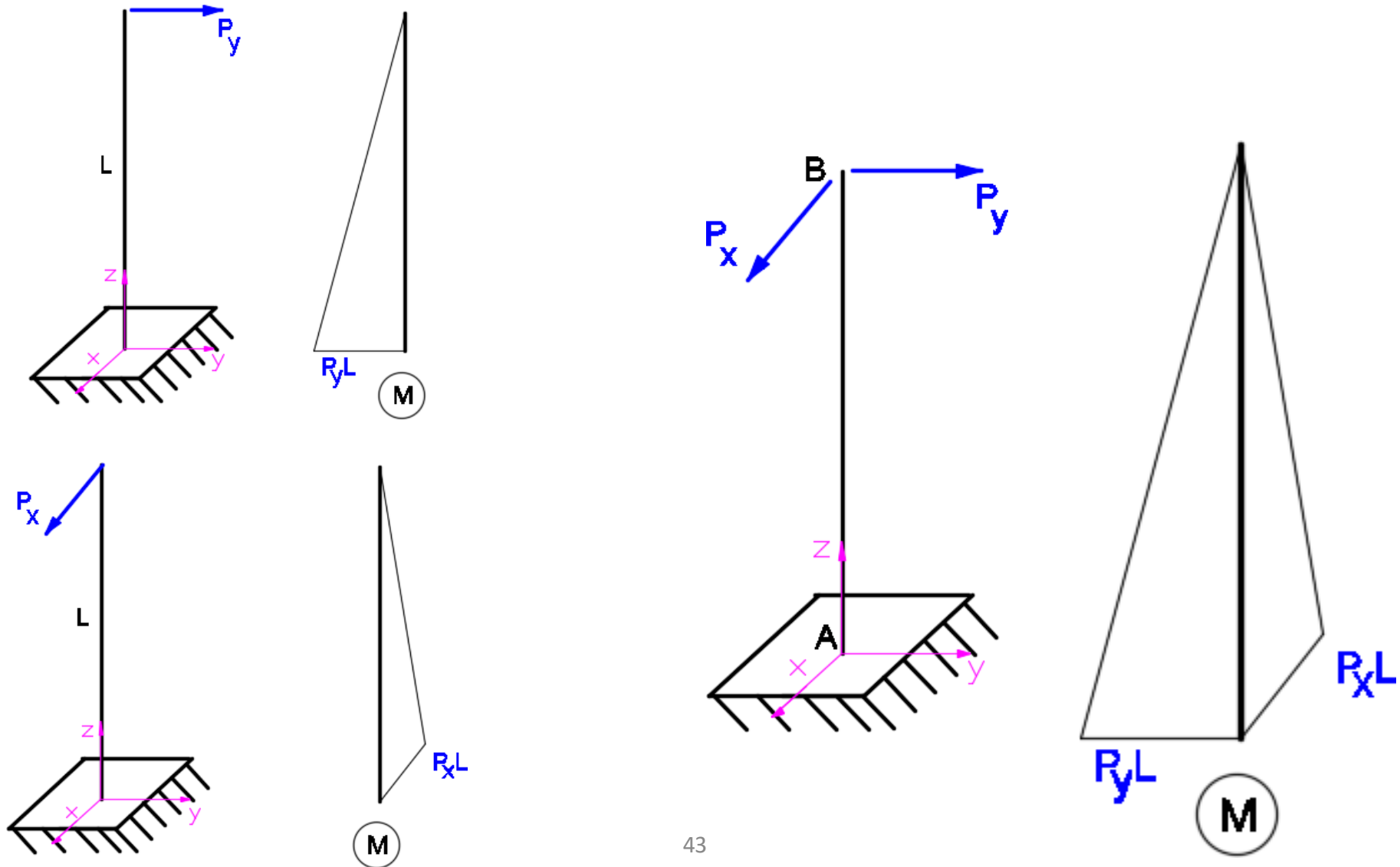
Convenção de sinais para diagramas

Momentos fletores: desenhar lado que traciona



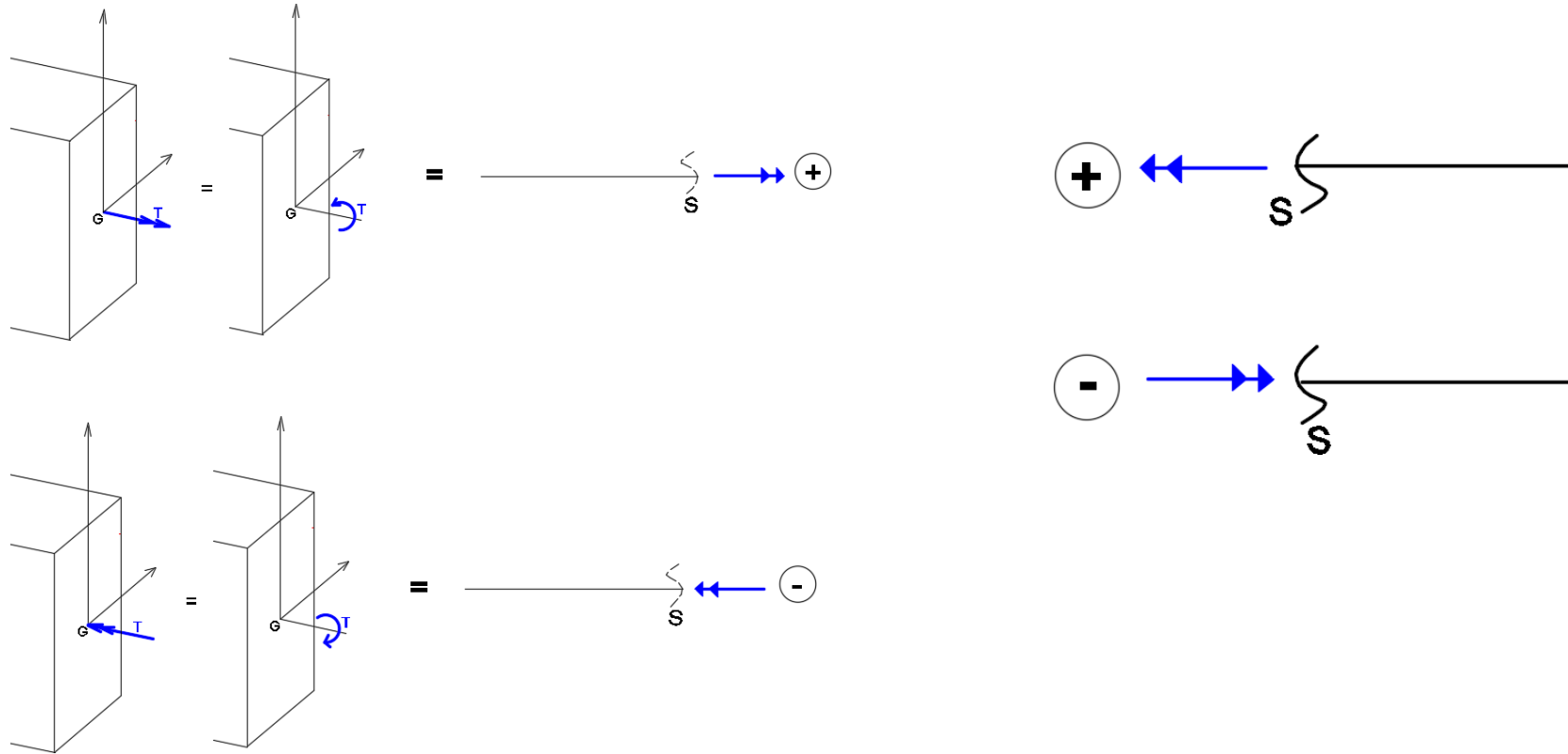
Convenção de sinais para diagramas

Momentos fletores: desenhar lado que traciona



Convenção de sinais para diagramas

Momento torçor (T) > 0 se vetor sai da seção



Momento de torção²

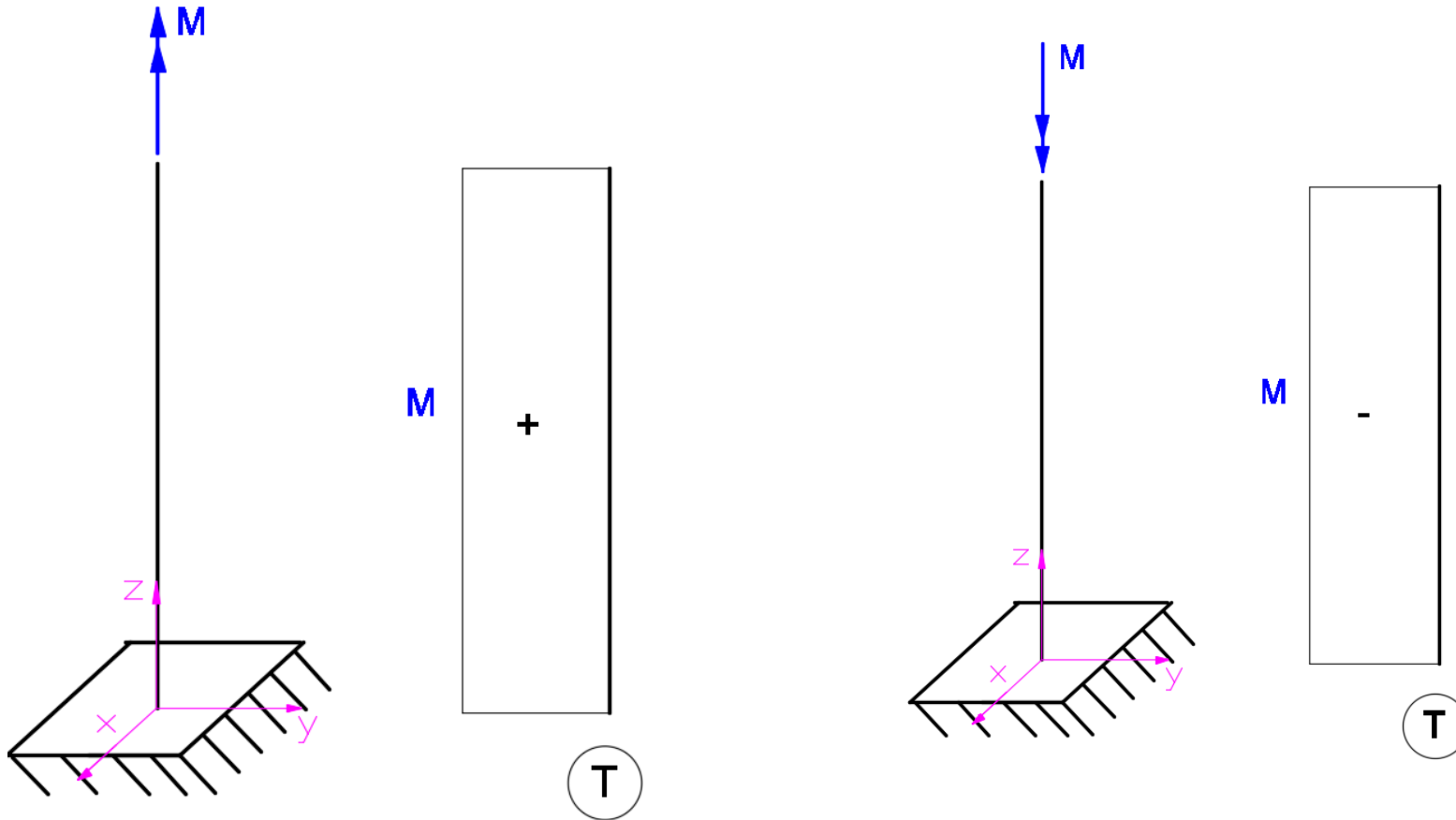
O vetor momento tem o sentido da normal externa à seção transversal em que atua

O vetor momento tem sentido contrário ao da normal externa à seção transversal em que atua

Rotacionar a seção no sentido anti-horário ($T > 0$)

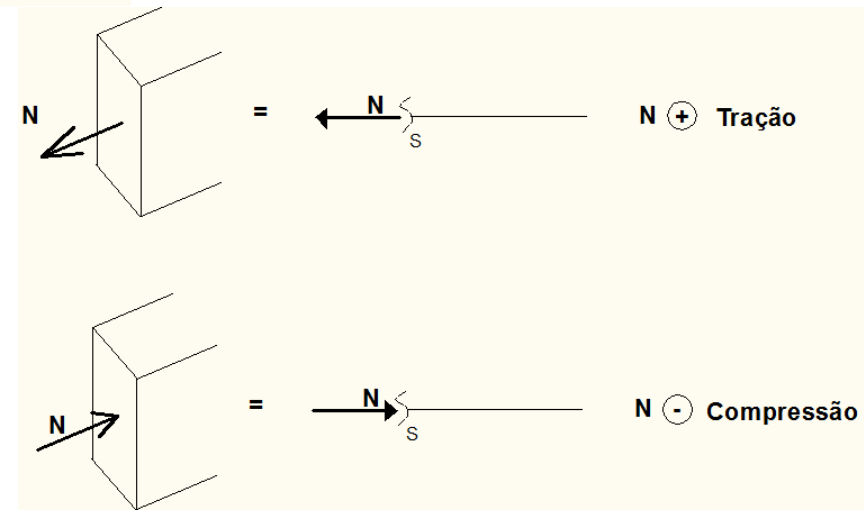
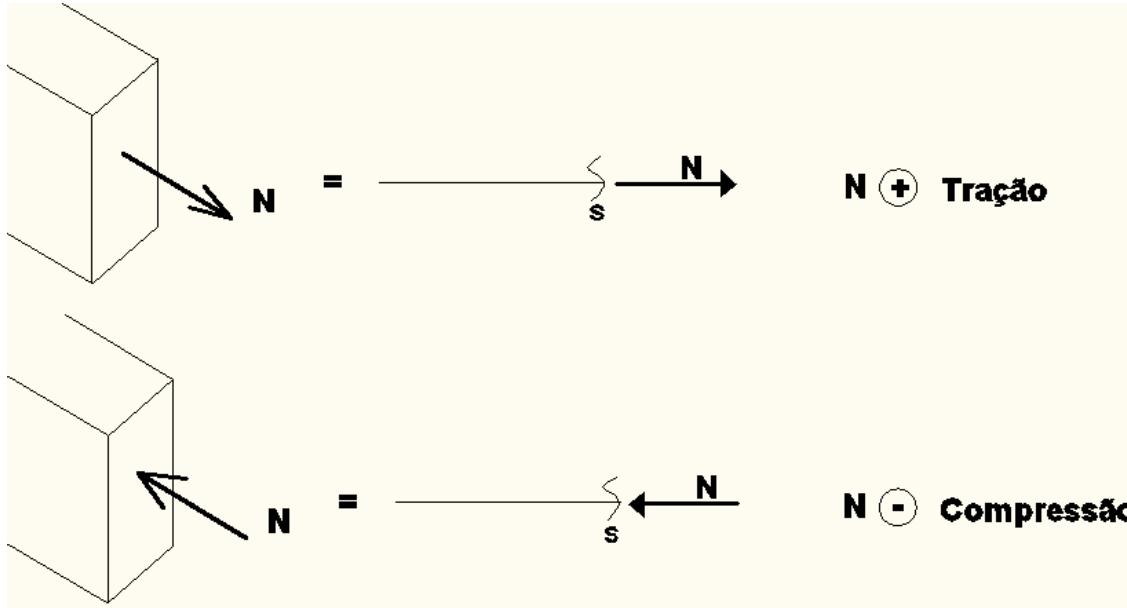
Convenção de sinais para diagramas

Momento torçor (T) > 0 se vetor sai da seção



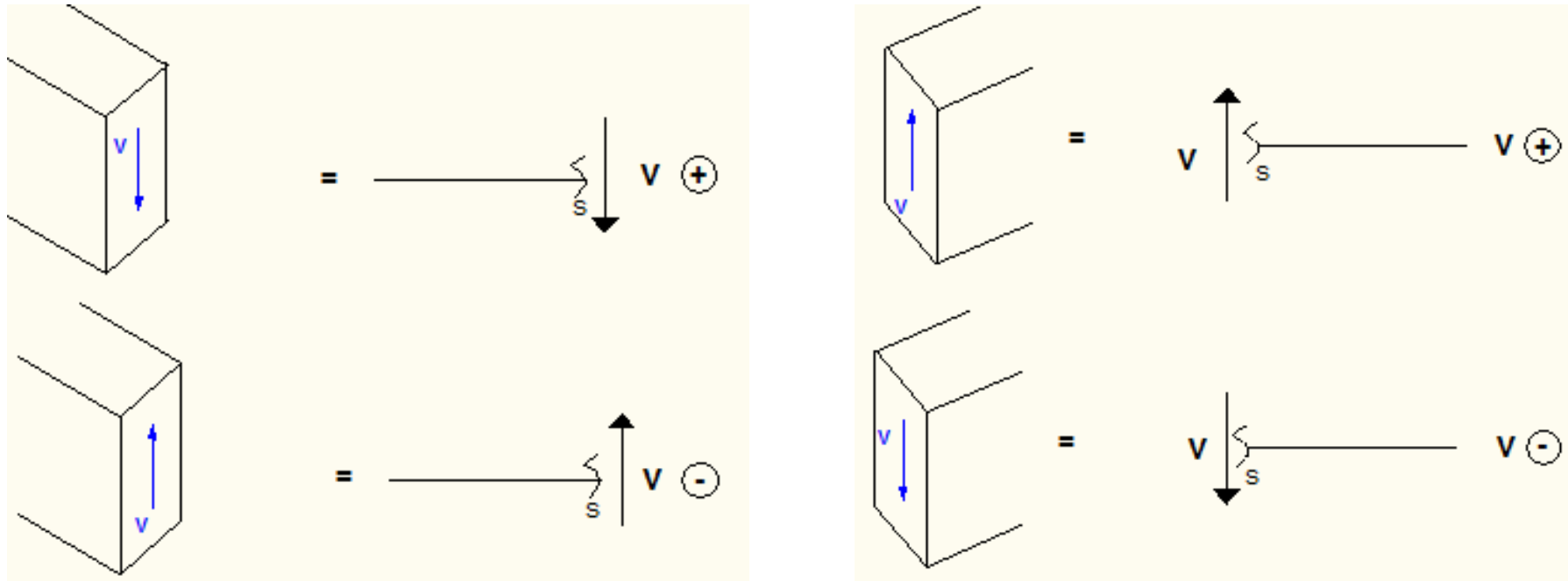
Convenção de sinais para diagramas

Normal (N) > 0 se vetor sai da seção



Convenção de sinais para diagramas

Cortantes > 0 se gira a barra no sentido horário



Força cortante

Gira o trecho de barra em que atua no sentido horário

Gira o trecho de barra em que atua no sentido anti-horário

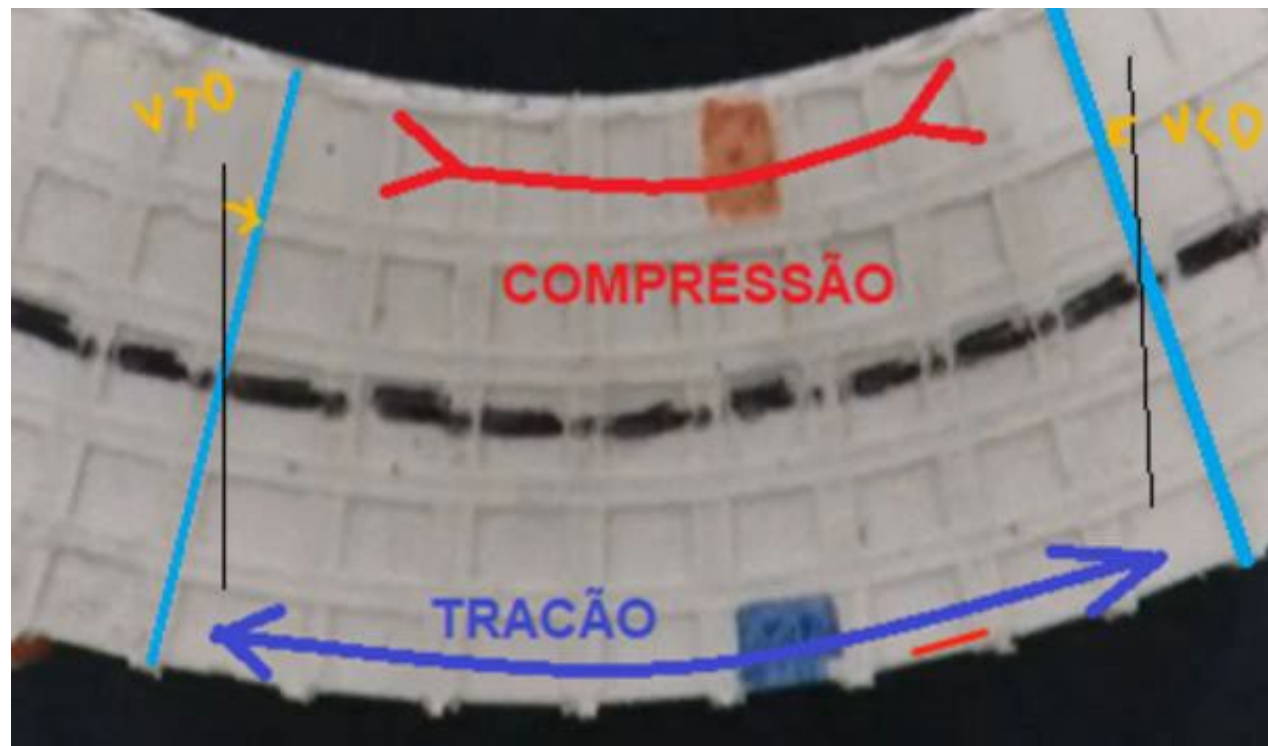
Convenção de sinais para diagramas

Cortantes > 0 se gira a barra no sentido horário

$V > 0$



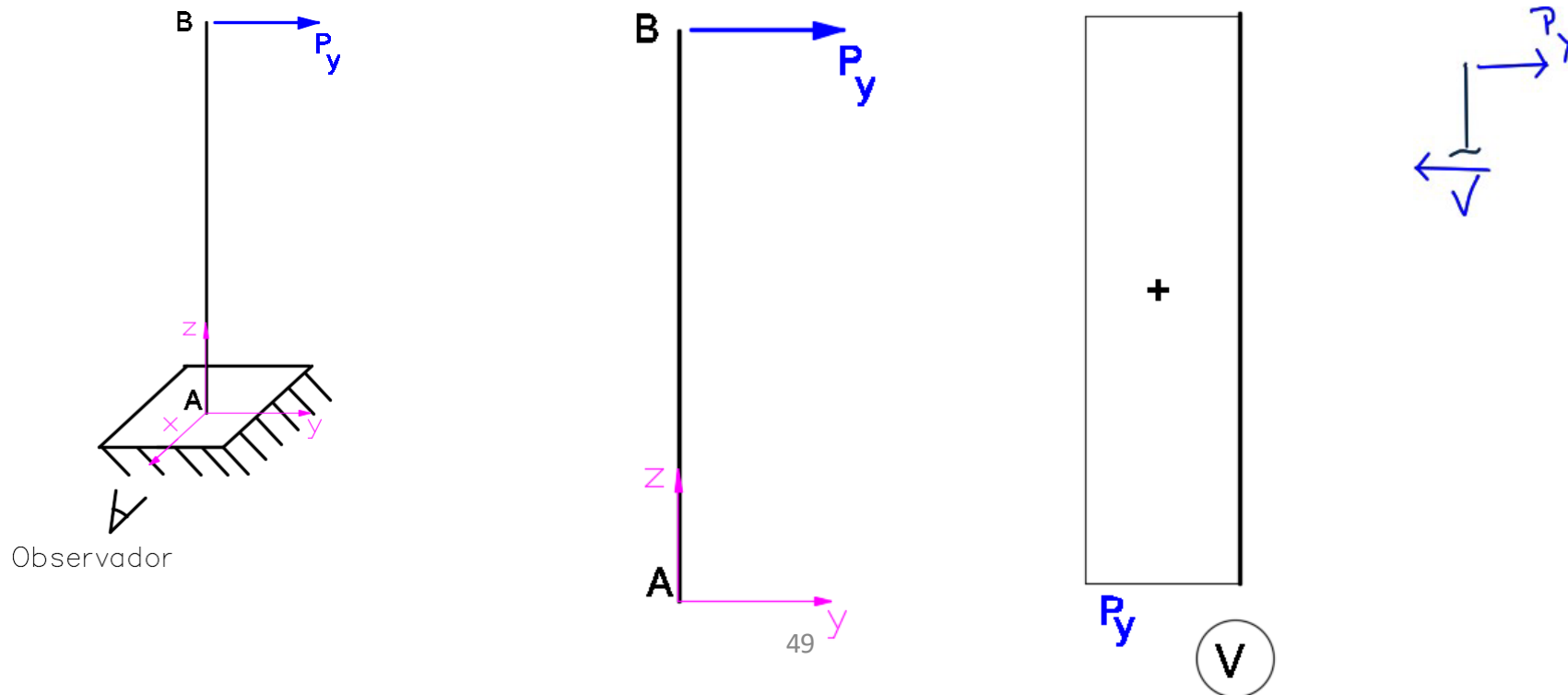
$V < 0$



Convenção de sinais para diagramas

Cortantes > 0 se gira a barra no sentido horário

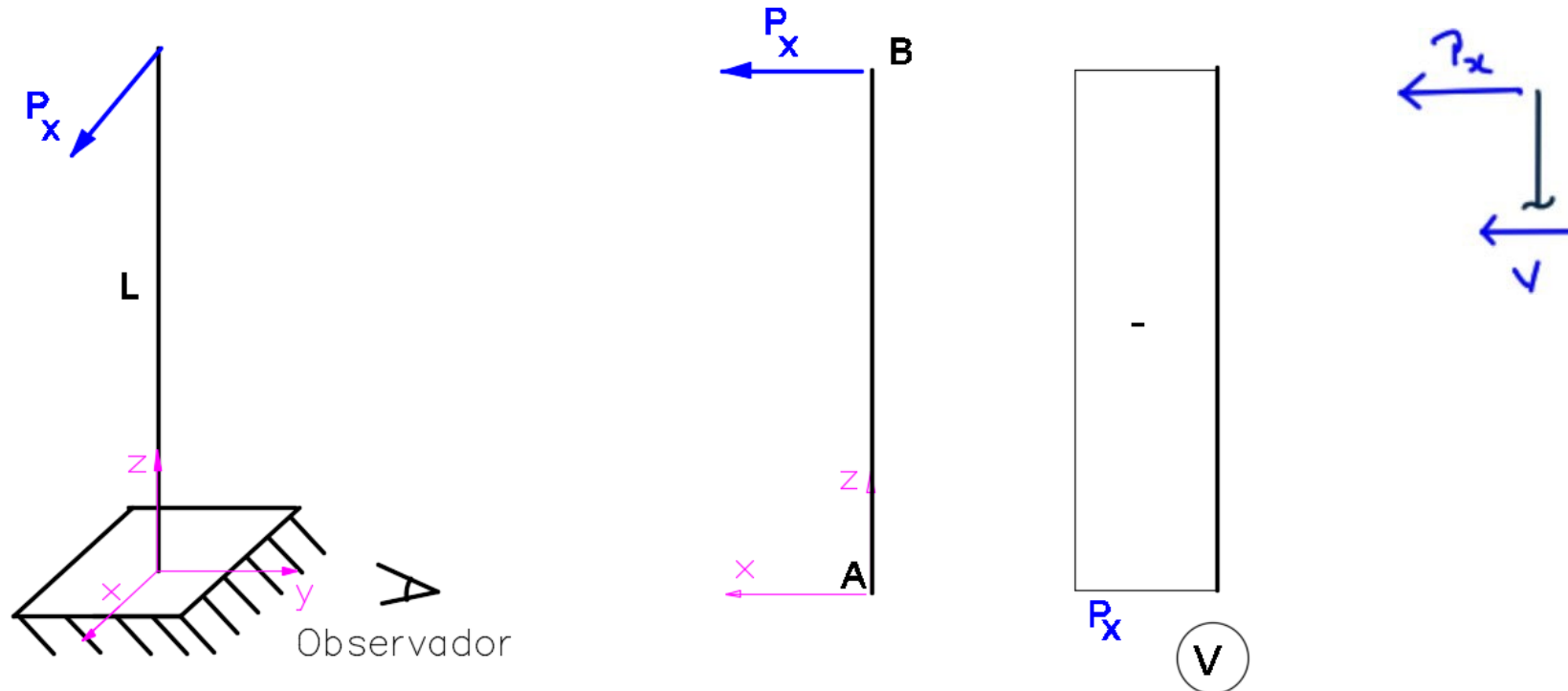
Visualizar a barra no plano pelo lado que está o eixo ortogonal positivo



Convenção de sinais para diagramas

Cortantes > 0 se gira a barra no sentido horário

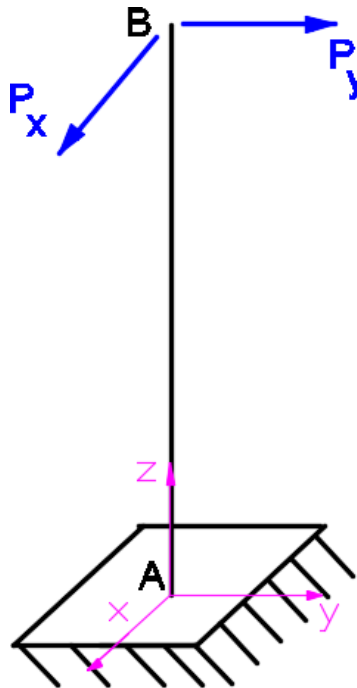
Visualizar a barra no plano pelo lado que está o eixo ortogonal positivo



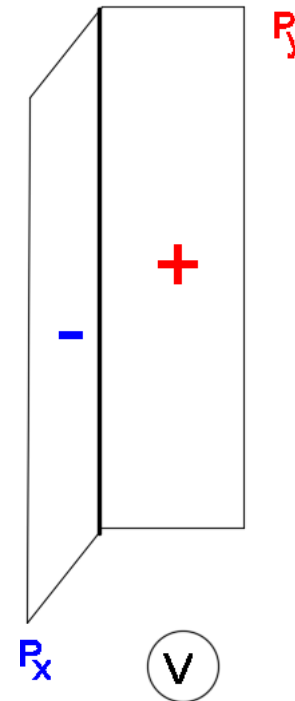
Convenção de sinais para diagramas

Cortantes > 0 se gira a barra no sentido horário

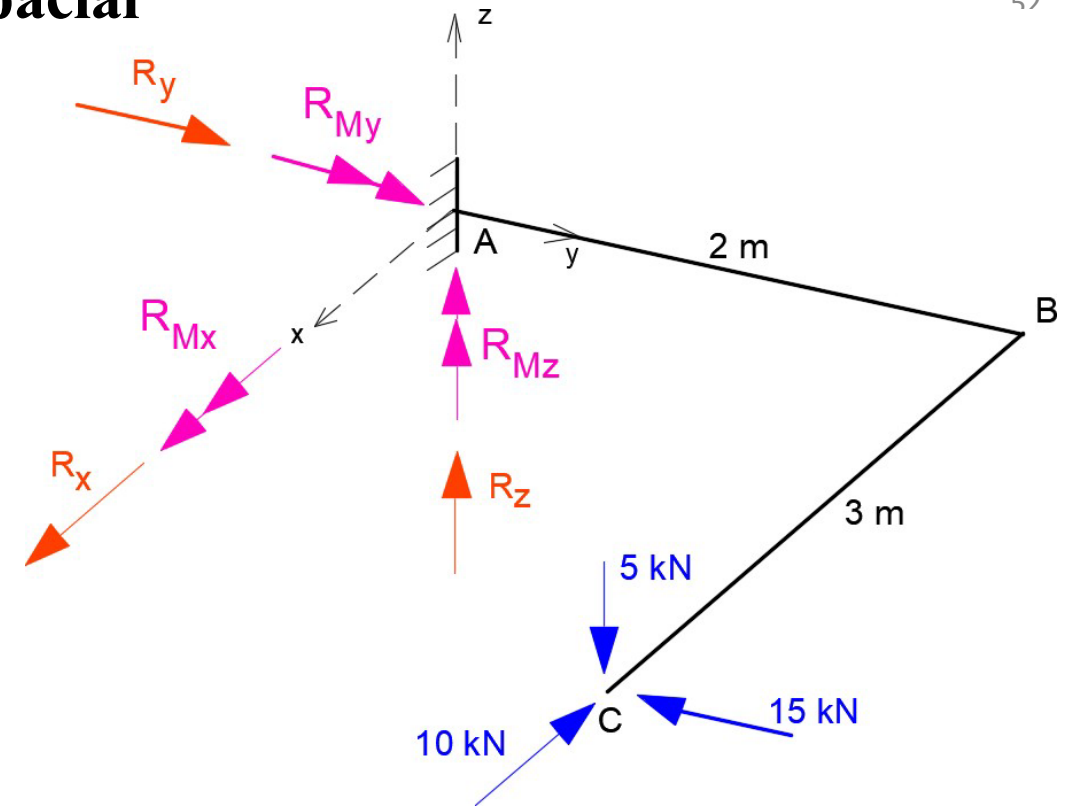
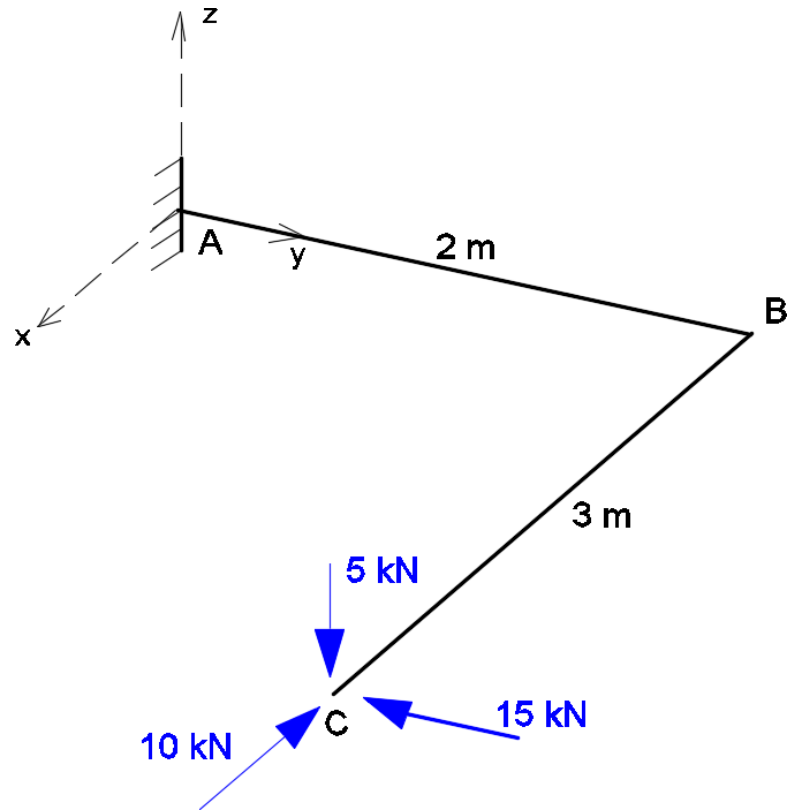
Visualizar a barra no plano pelo lado que está o eixo ortogonal positivo



51



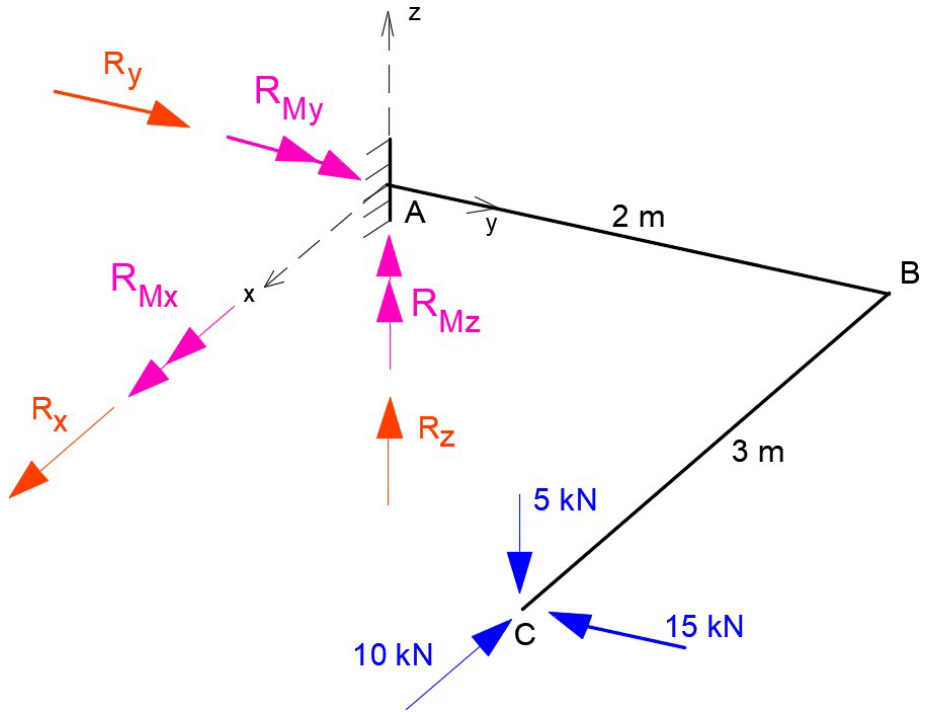
Exemplo 4: Determinar os esforços na estrutura espacial



$$\boxed{?} \quad F_x = 0: R_x - 10 = 0 \rightarrow R_x = 10 \text{ kN}$$

$$\boxed{?} \quad F_y = 0: R_y - 15 = 0 \rightarrow R_y = 15 \text{ kN}$$

$$\boxed{?} \quad F_z = 0: R_z - 5 = 0 \rightarrow R_z = 5 \text{ kN}$$



nr. forças

$$R_{Mx} + \sum_{i=1}^{\text{nr. forças}} (M_x)_i = 0$$

$$R_{Mx} + (2\text{m})(-5\text{kN}) - (0\text{m})(-15\text{kN}) = 0$$

$$R_{Mx} = 10 \text{ kNm}$$

$$M_o = M_x i + M_y j + M_z k$$

$$M_x = (y_p - y_o)F_z - (z_p - z_o)F_y$$

$$M_y = (z_p - z_o)F_x - (x_p - x_o)F_z$$

$$M_z = (x_p - x_o)F_y - (y_p - y_o)F_x$$

$$F_x = -10 \text{ kN}; F_y = -15 \text{ kN}; F_z = -5 \text{ kN}$$

$$x_p - x_o = 3 \text{ m} \quad y_p - y_o = 2 \text{ m} \quad z_p - z_o = 0$$

nr. forças

$$R_{My} + \sum_{i=1}^{\text{nr. forças}} (M_y)_i = 0$$

$$R_{My} = -15 \text{ kNm}$$

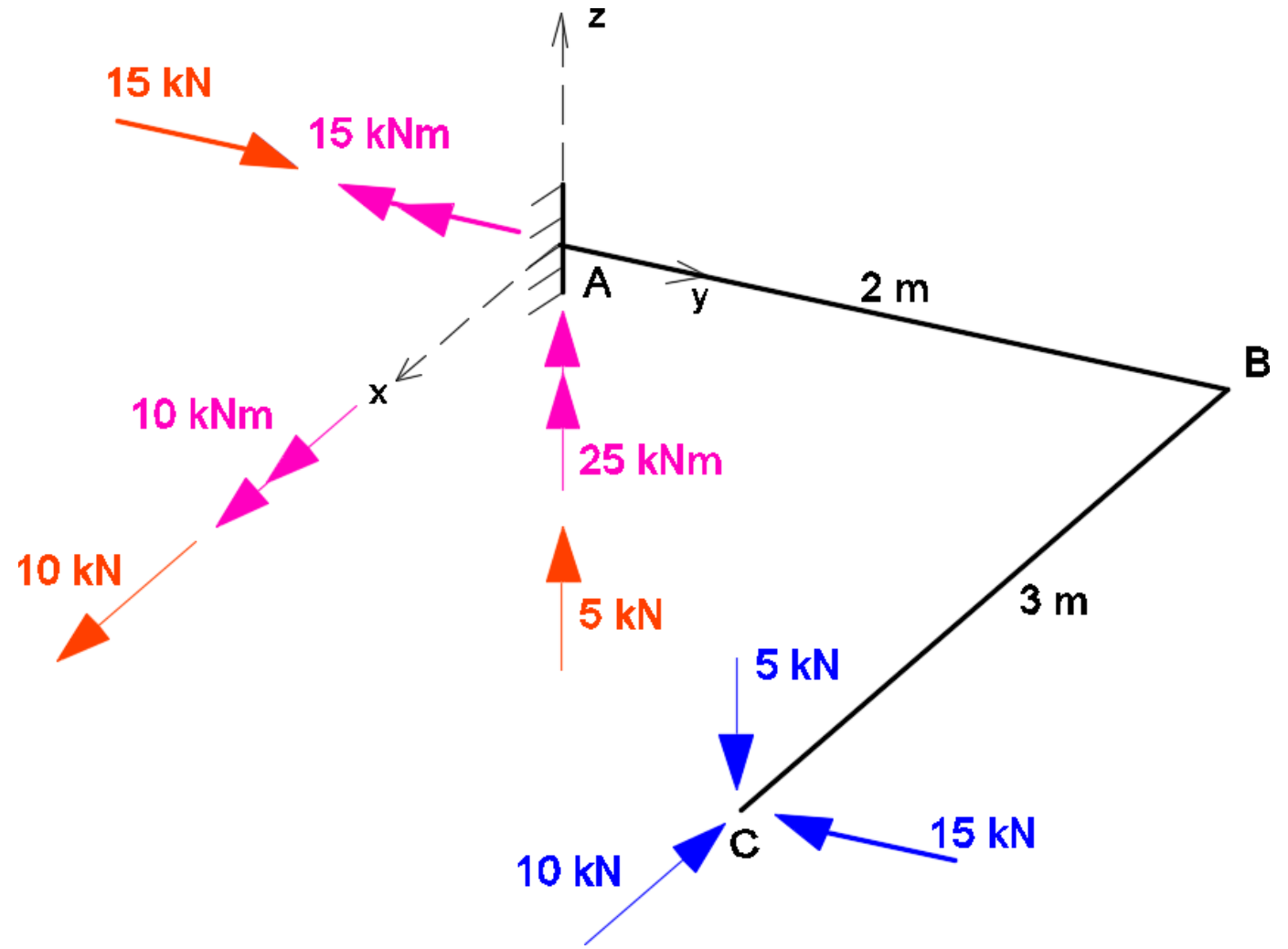
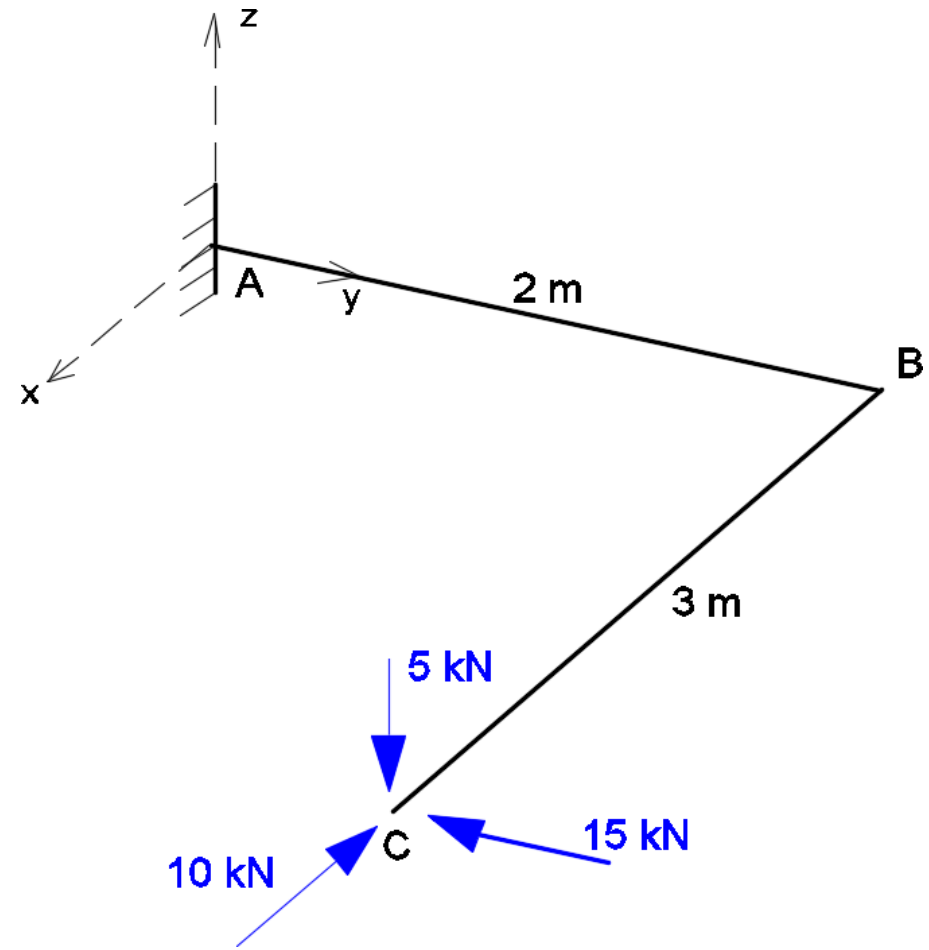
$$R_{My} + (0\text{m})(-10\text{kN}) - (3\text{m})(-5\text{kN}) = 0$$

nr. forças

$$R_{Mz} + \sum_{i=1}^{\text{nr. forças}} (M_z)_i = 0$$

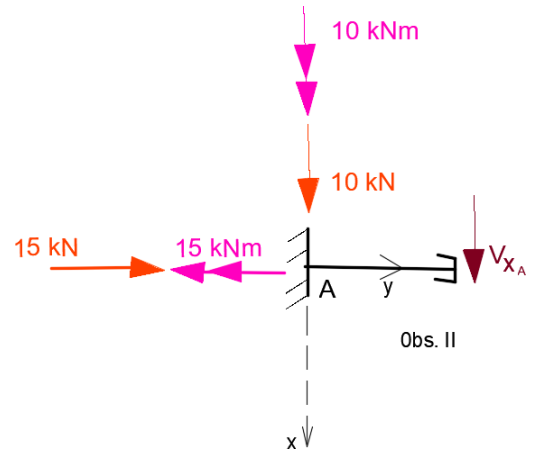
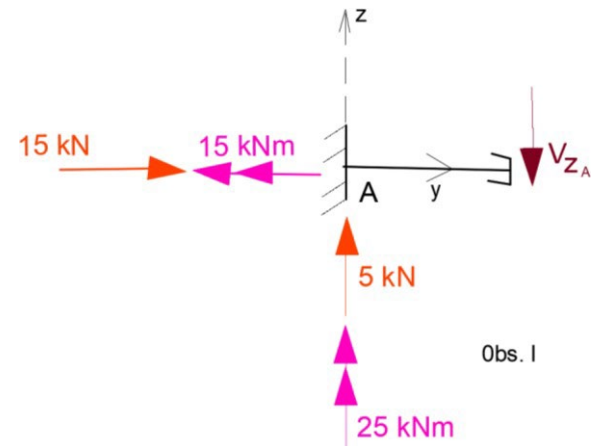
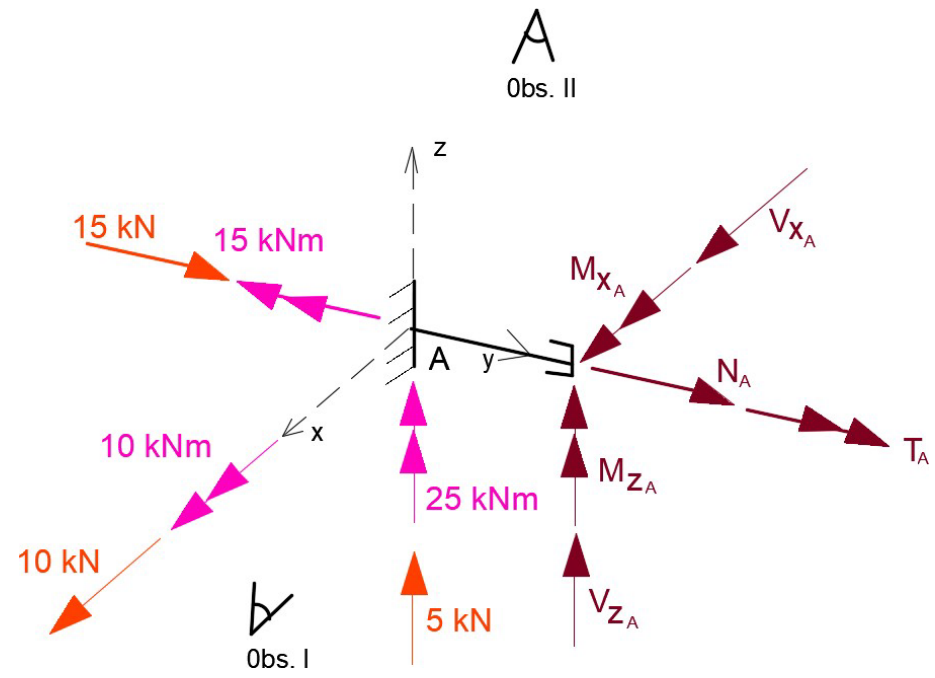
$$R_{Mz} = 25 \text{ kNm}$$

$$R_{Mz} + (3\text{m})(-15\text{kN}) - (2\text{m})(-10\text{kN}) = 0$$



Esforços

a) Corte junto ao engaste em A



$$\sum M_{xA} = 0$$

$$M_{xA} + 10 = 0$$

$$M_{xA} = -10 \text{ kNm}$$

$$\sum M_{yA} = 0$$

$$T_A - 15 = 0$$

$$T_A = 15 \text{ kNm}$$

$$\sum M_{zA} = 0$$

$$M_{zA} + 25 = 0$$

$$M_{zA} = -25 \text{ kNm}$$

$$\sum F_z = 0$$

$$V_{zA} - 5 = 0$$

$$V_{zA} = 5 \text{ kN}$$

$$\sum F_x = 0$$

$$V_{xA} + 10 = 0$$

$$V_{xA} = -10$$

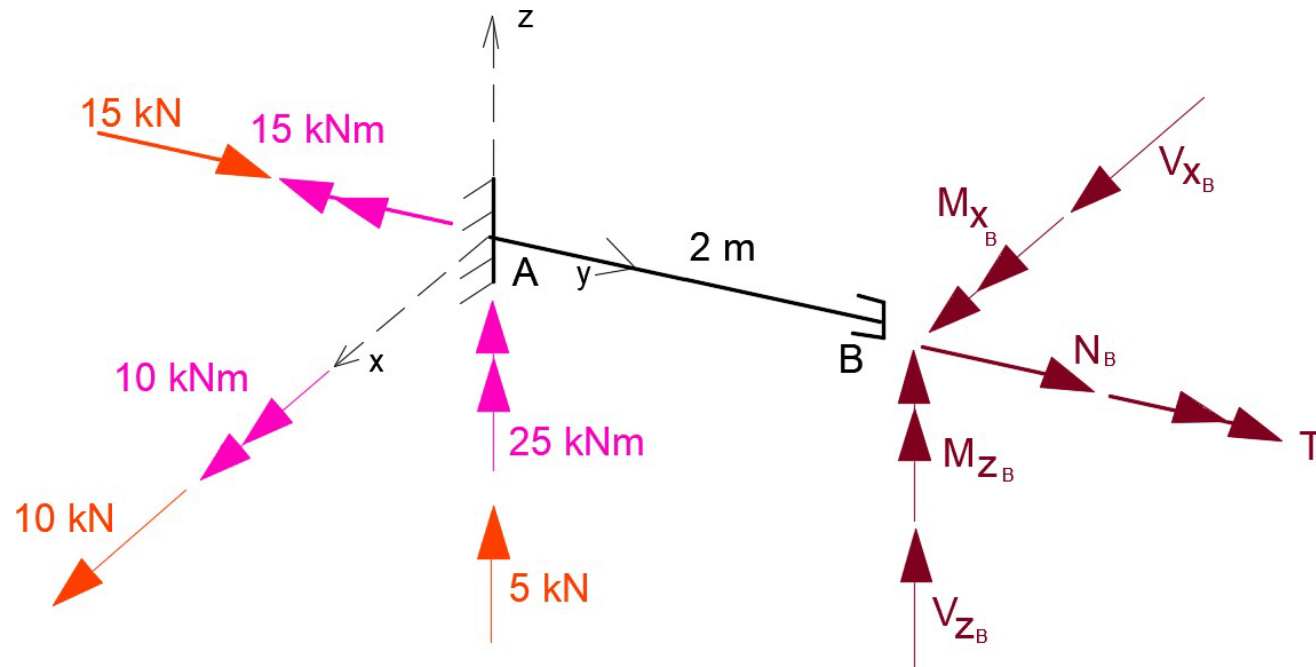
$$\sum F_{yA} = 0$$

$$N_A + 15 = 0$$

$$N_A = -15 \text{ kN}$$

Exemplo 4

b) Corte junto a B



N_b, V_{zb}, V_{xb} :
idem a A

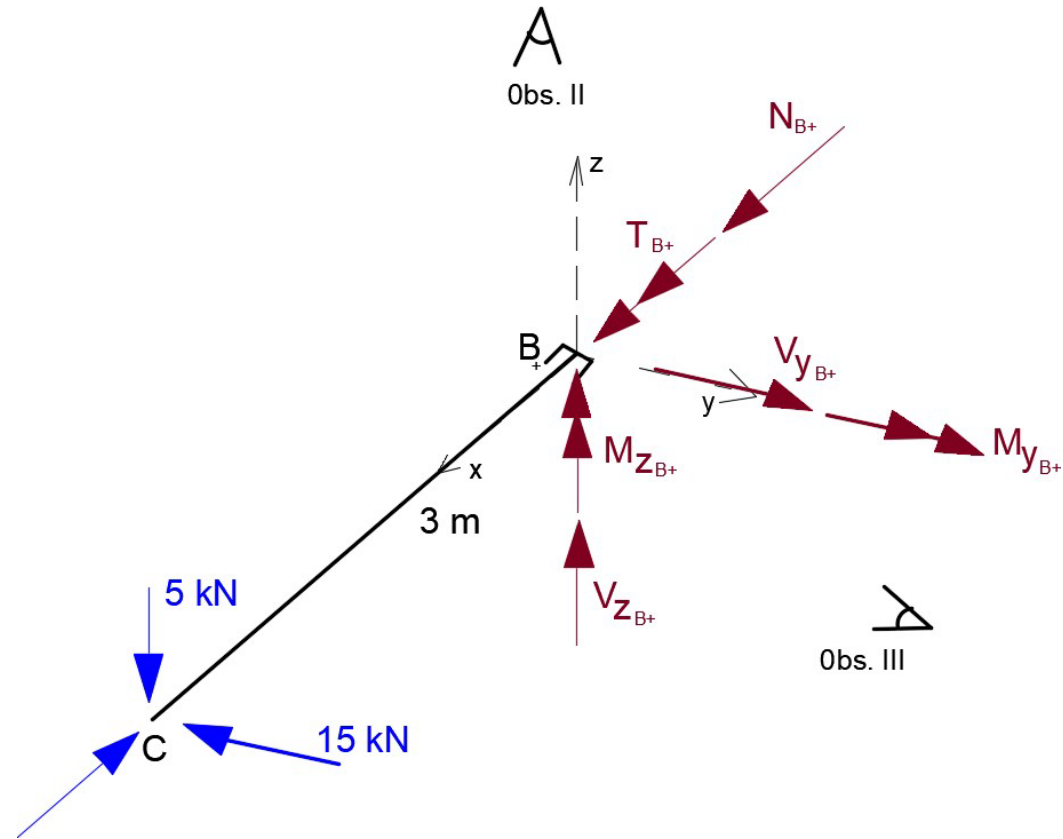
$$\begin{aligned}\sum M_y &= 0 \\ T_B - 15 &= 0 \\ T_B &= 15 \text{ kNm}\end{aligned}$$

$$\begin{aligned}\sum M_x &= 0 \\ M_{x_B} + 10 - 5 \cdot 2 &= 0 \\ M_{x_B} &= 0\end{aligned}$$

$$\begin{aligned}\sum M_{z_B} &= 0 : \\ M_{z_B} + 25 + (10) \cdot 2 &= 0 \\ M_{z_B} &= -45 \text{ kNm}\end{aligned}$$

Exemplo 4

c) Corte junto a B+



$$\sum M_{y_{B+}} = 0$$

$$M_{y_{B+}} + 5 \cdot 3 = 0$$

$$M_{y_{B+}} = -15 \text{ kNm}$$

$$\sum M_{z_{B+}} = 0$$

$$M_{z_{B+}} - 15 \cdot 3 = 0$$

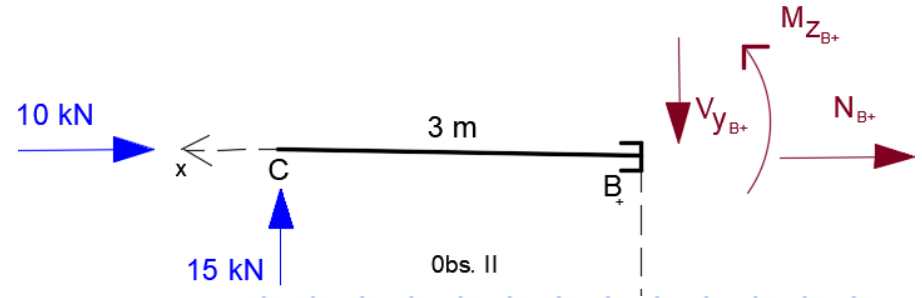
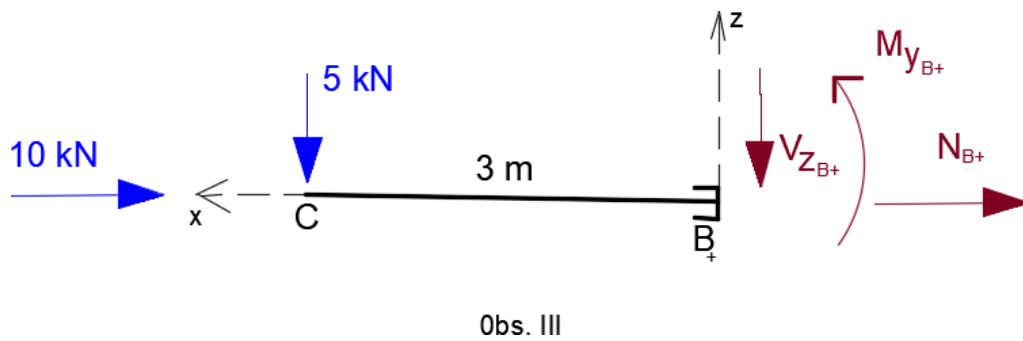
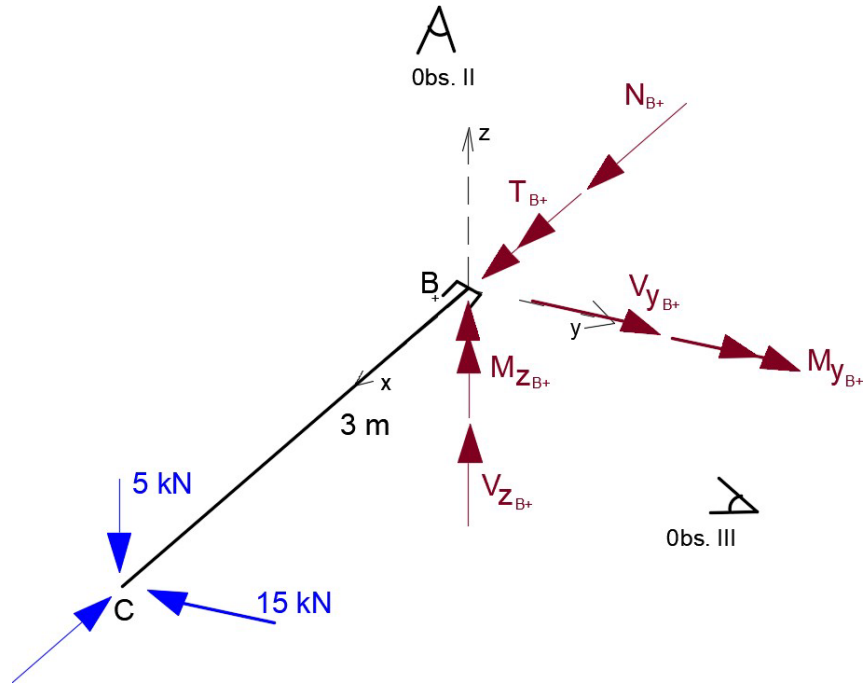
$$M_{z_{B+}} = 45 \text{ kNm}$$

$$\sum M_{x_{B+}} = 0$$

$$T_{B+} = 0$$

Exemplo 4

c) Corte junto a B+



$$\sum F_x = 0$$

$$N_{B+} = -10 \text{ kN}$$

$$\sum F_y = 0$$

$$V_{y_{B+}} = 15 \text{ kN}$$

$$\sum M_{B+} = 0$$

$$M_{z_{B+}} = 15 \cdot 3 = 45 \text{ kNm}$$

$$\sum F_z = 0$$

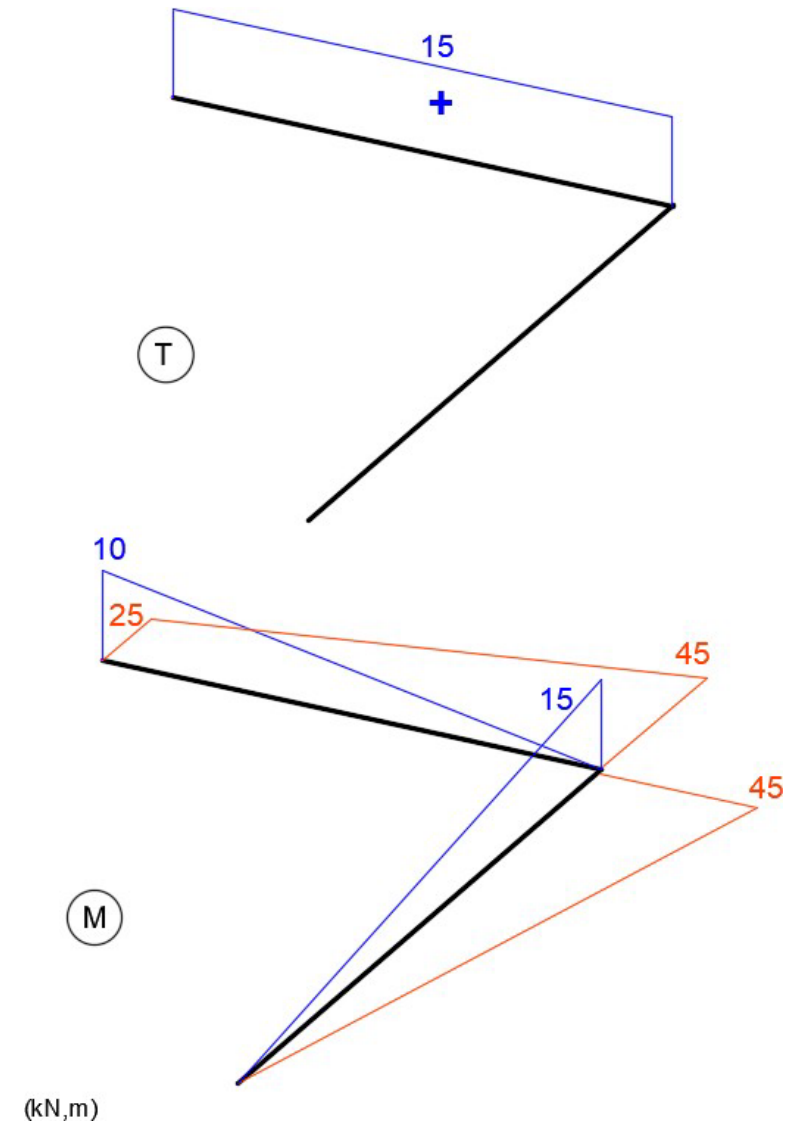
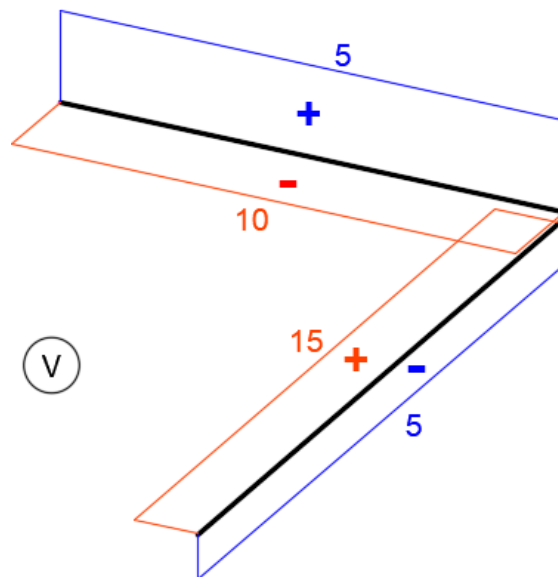
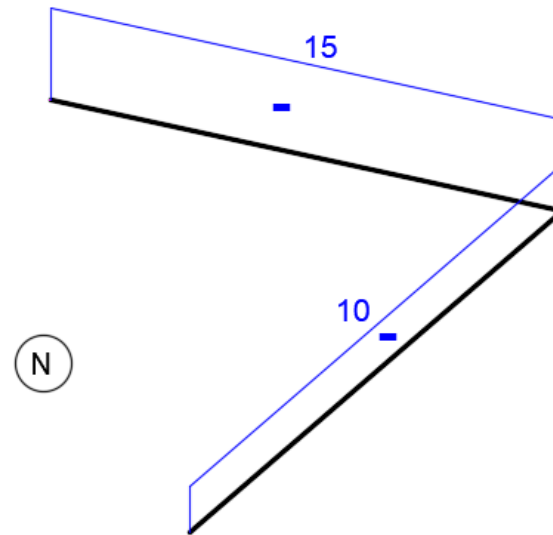
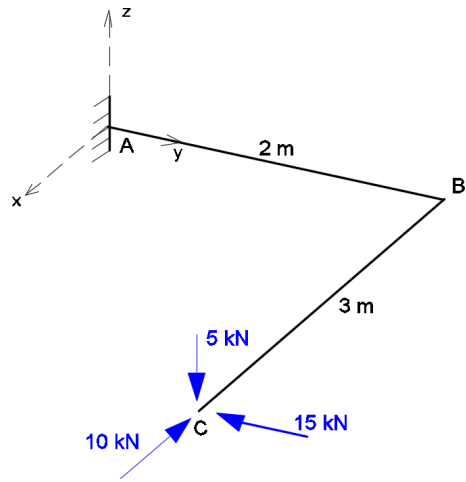
$$V_{z_{B+}} = -5 \text{ kN}$$

$$\sum M_{y_{B+}} = 0$$

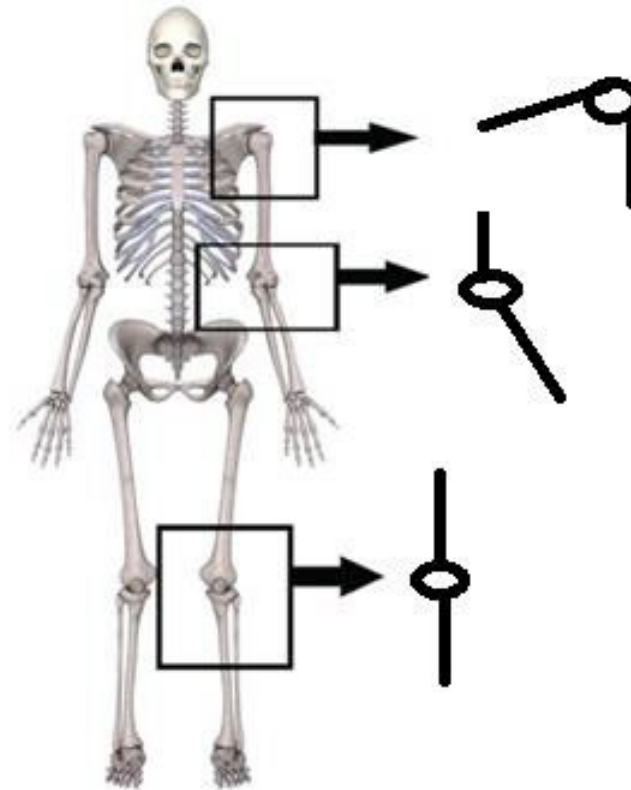
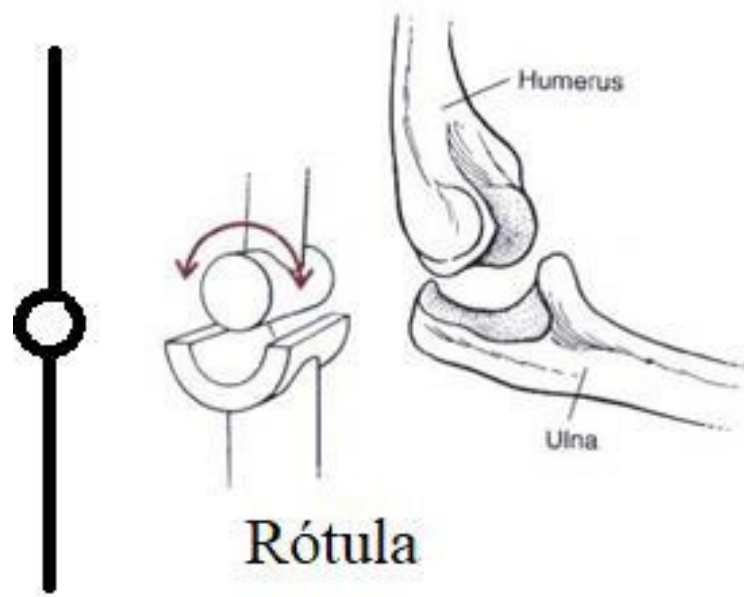
$$M_{y_{B+}} = -3 \cdot 5 = -15 \text{ kNm}$$

Exemplo 4

d) Diagramas

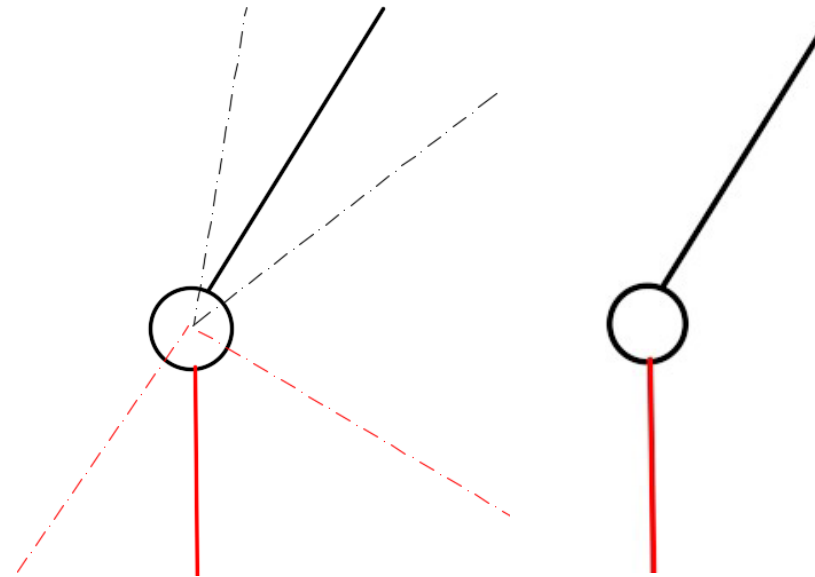


RÓTULAS

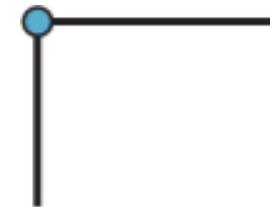


Conecta barras
Permite giro livre das barras

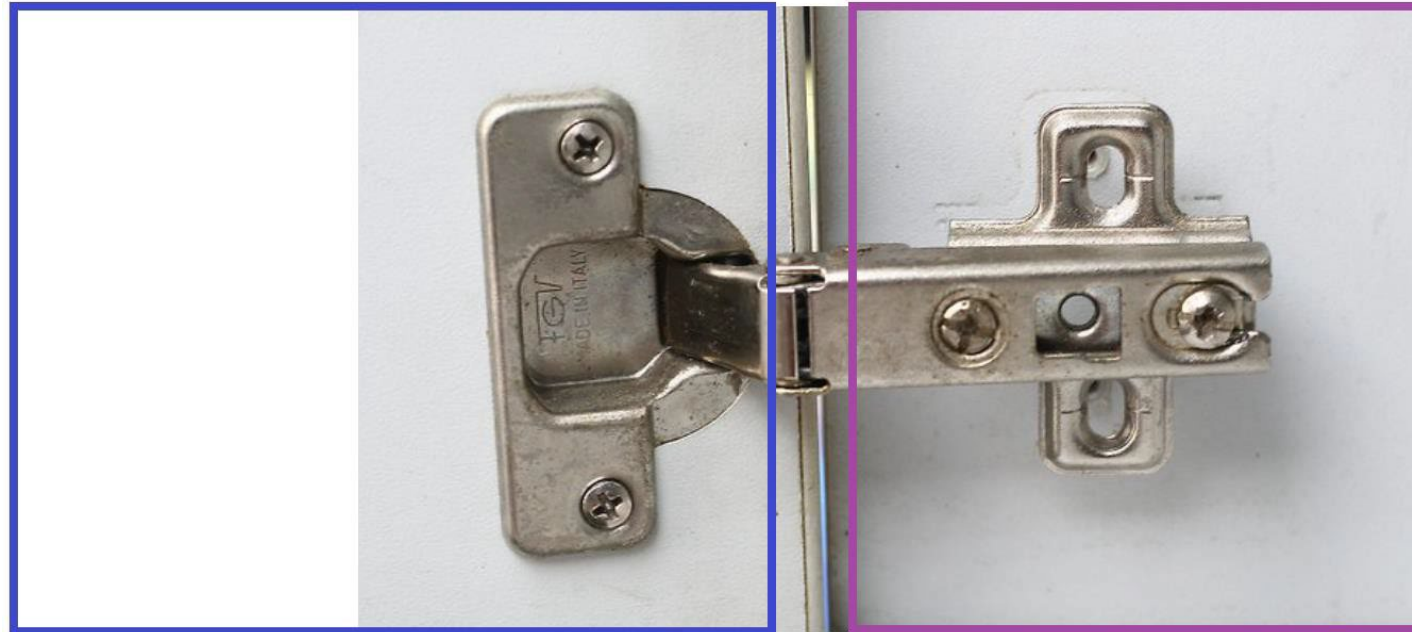
Rótulas



Conecta barras
Permite giro livre das barras

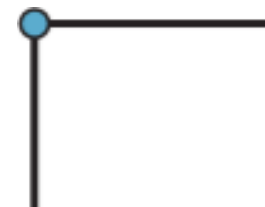


Rótulas



DOBRADIÇA

Conecta barras
Permite giro livre das barras

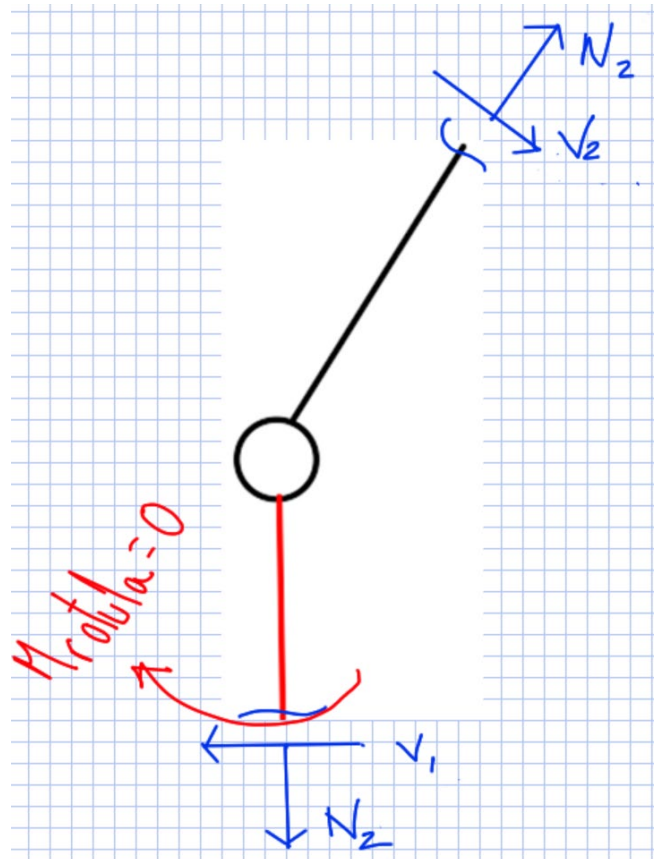


Rótulas

Conecta barras
Permite giro livre das barras



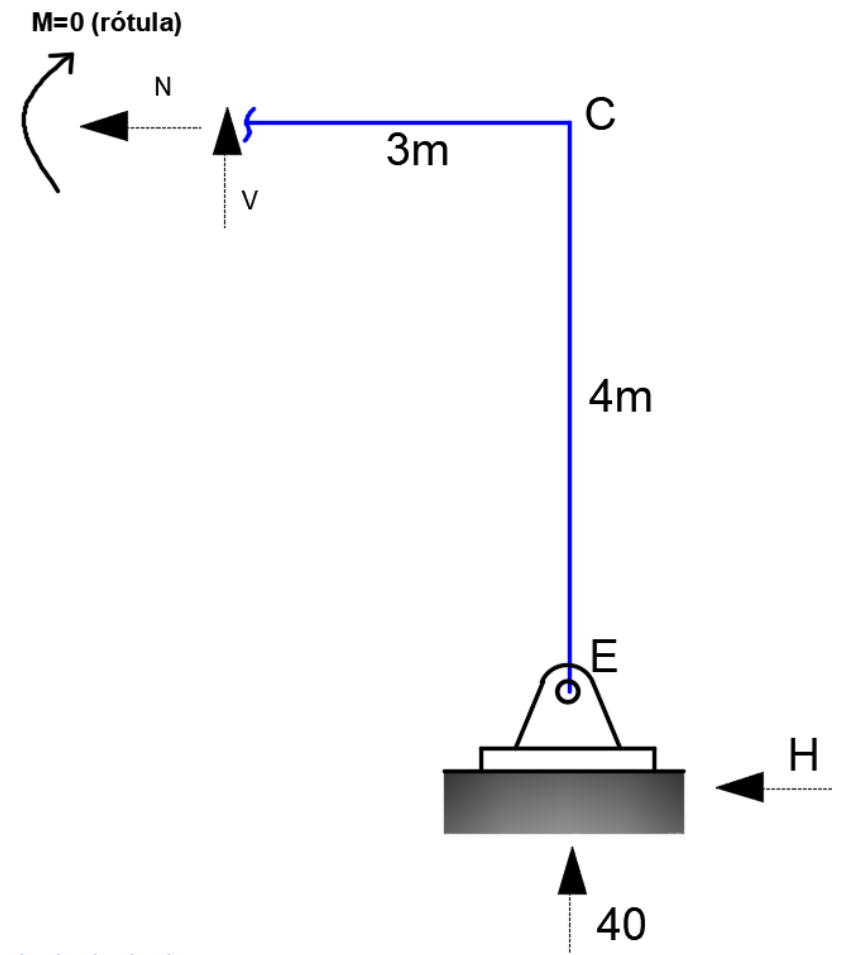
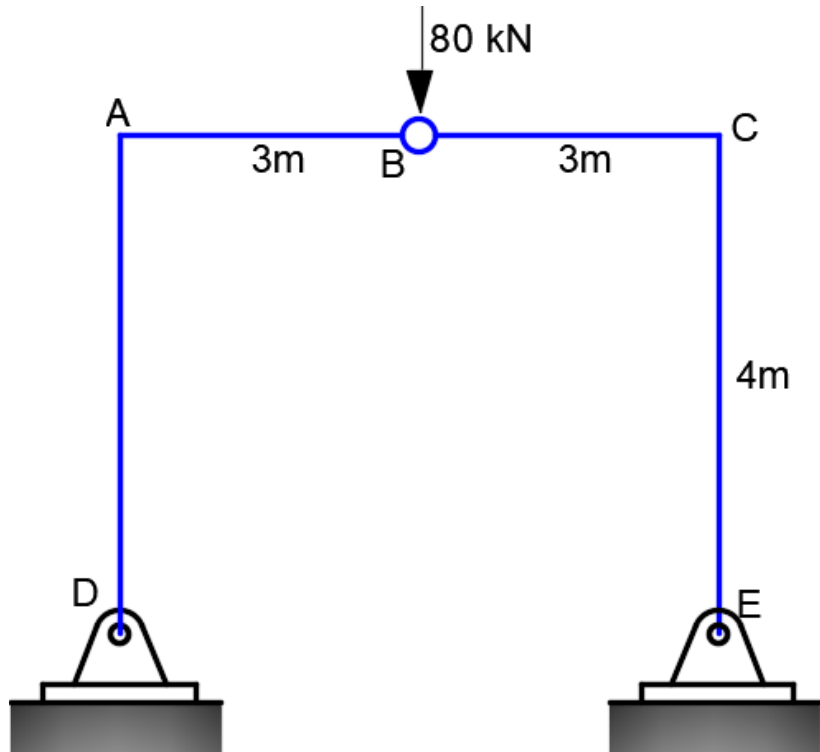
Não há momento fletor das barras junto à seção que liga à rotula



Corta-se a estrutura e usa-se a equação adicional:

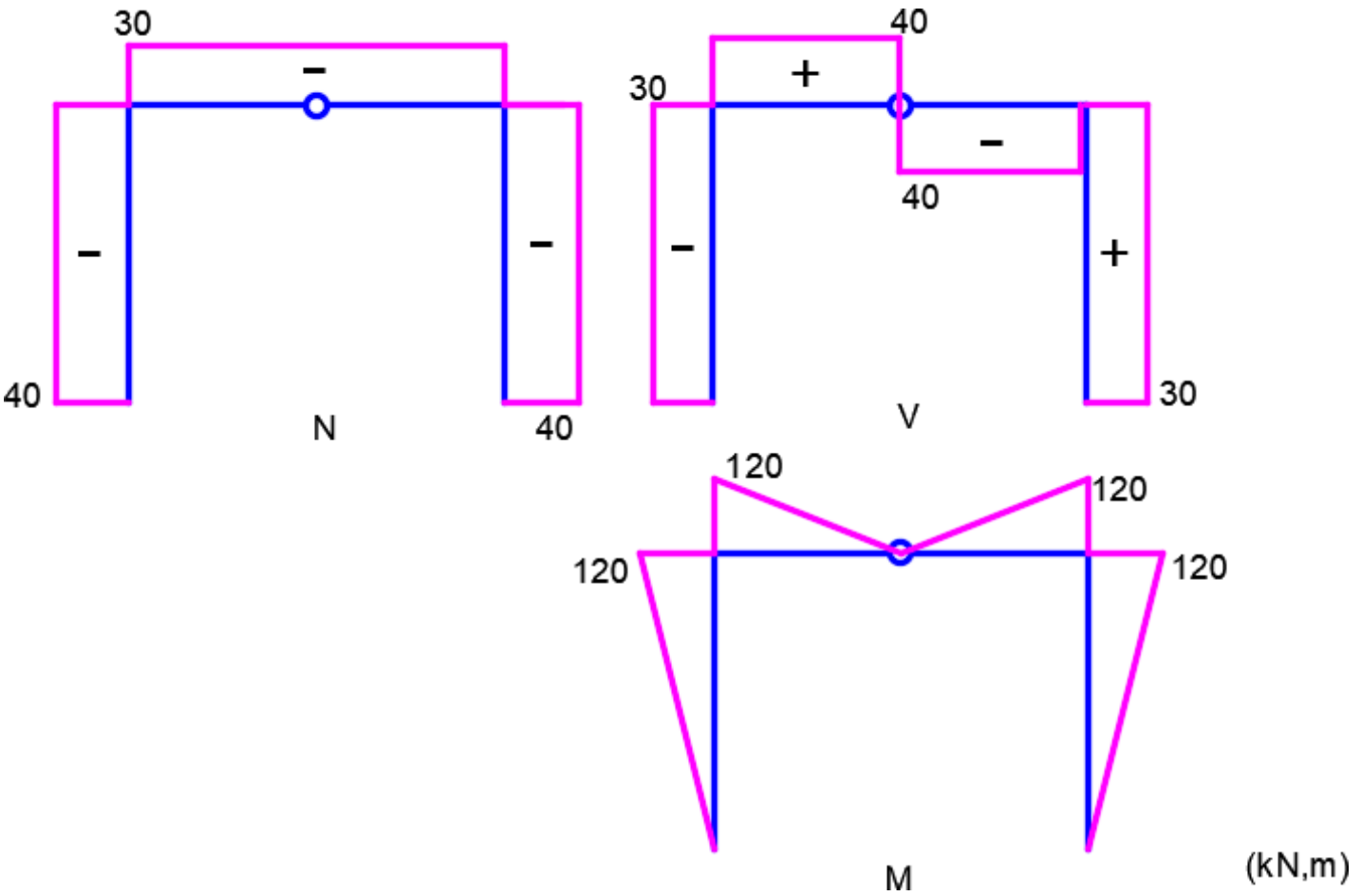
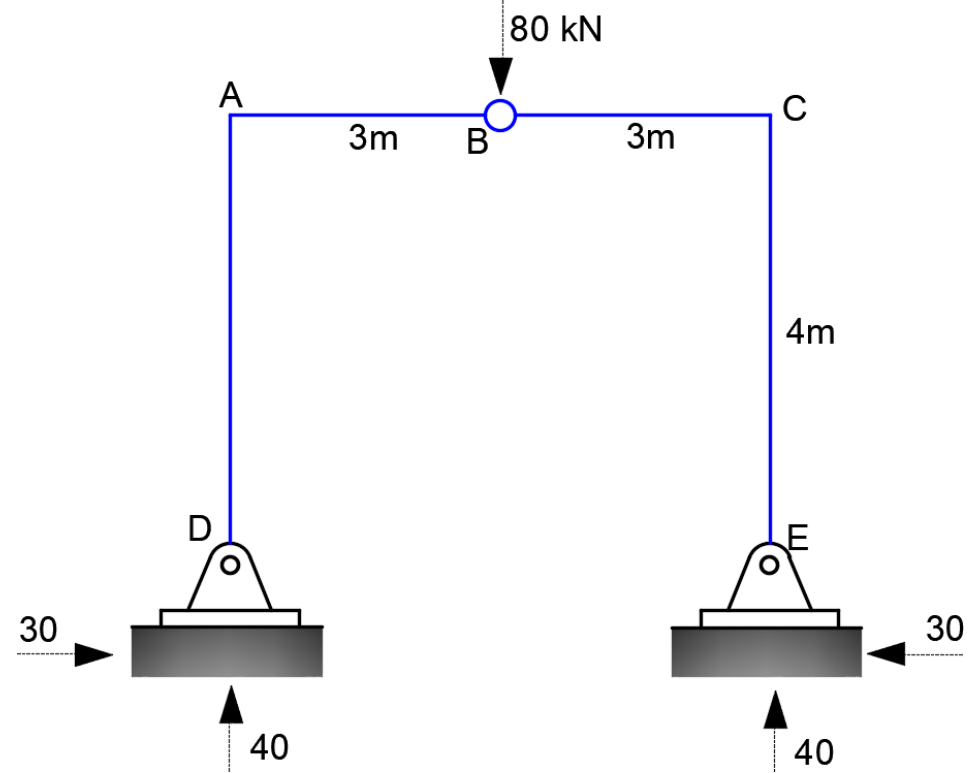
$$M_{rotula} = 0$$

Exemplo 5 Obter esforços no pórtico



$$\begin{aligned} \sum M_B = 0 \\ 40 \cdot 3 + 4 \cdot H = 0 \\ H = -30 \text{ kN} \end{aligned}$$

Diagramas



(kN,m)