A Simulation Comparison of Normalization Procedures for TOPSIS

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ABSTRACT

Multiattribute decision making (MADM) uses a normalization procedure to transform performance ratings with different data measurement units in a decision matrix into a compatible unit. MADM methods generally use one particular normalization procedure without justifying its suitability. The technique for order preference by similarity to ideal solution (TOPSIS) is one of the most popular and widely applied MADM methods. This study compares four commonly known normalization procedures in terms of their ranking consistency and weight sensitivity when used with TOPSIS to solve the general MADM problem with various decision settings. The comparison study is validated using two performance measures: ranking consistency and weight sensitivity. A large number of MADM problems with varying attributes and alternatives are generated using a new simulation technique. The study results justify the use of the vector normalization procedure for TOPSIS and provide suggestive insights for using other normalization procedures in certain decision settings.

Key-Words: - MADM, Ranking consistency, TOPSIS, Normalization, Weight sensitivity.

1. INTRODUCTION

Multiattribute decision making (MADM) techniques have been used for various ranking and selection problems, with a finite number of alternatives with respect to multiple, often conflicting selection attributes [1,16,18]. Research shows MADM methods based on multiattribute value theory (MAVT) [7], are most widely used in MADM problem solving [13]. MAVT-based MADM methods use the additive value function, which is more attractive to decision makers due to its sound mathematical concept and simplicity [17]. After the decision matrix and weight vector are prepared by the decision maker, MAVT-based MADM methods combine them to obtain a single value for each alternative, which provides the basis for ranking the alternatives.

In MADM problems, each alternative has a performance rating for each attribute, which represents the characteristics of the alternative. It is common that performance ratings for different attributes are measured by different units. To transform performance ratings into a compatible measurement unit, a normalization procedure is used. MADM methods often use one normalization procedure to achieve compatibility between different measurement units. For example, SAW uses linear scale transformation (max method) [5,6,15], TOPSIS uses vector normalization procedure [16,18], ELECTRE uses vector normalization [4,9,16] and AHP uses linear scale transformation (sum) [10,11,12].

Enormous efforts have been made to comparative studies of MADM methods [8,14,15,17], but no significant study is conducted on the suitability of normalization procedures used in those MADM methods. This leaves the effectiveness of various MADM methods in doubt and certainly raises the necessity to examine the effects of various normalization procedures on decision outcome when used with given MADM methods. Recent studies with the simple additive weighting (SAW) method provide useful insight to this problem issue [2, 3].

The purpose of this study is to identify the most suitable normalization procedure for TOPSIS under various problem settings. In subsequent sections, we first explain the general MADM problem setting, normalization procedures, TOPSIS method. Then we present and discuss performance measures, simulation experiments and results.

2. MADM SETTING AND NORMALIZATION

An MADM problem usually involves a set of m alternatives A_i ($i = 1, 2, ..., m$), which are to be evaluated based on a set of n attributes (evaluation attributes) C_j ($j = 1, 2, ..., n$). Assessments are to be made to determine (a) the weighting vector $W = (w_1, w_2, ..., w_j, ..., w_n)$ and (b) the decision matrix $X = (x_{ij}, i=1, 2, ..., m; j=1, 2, ..., n)$. The weighting vector W represents the relative importance of n attributes C_j ($j=1, 2, ..., n$) for the problem. The decision matrix X represents the

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performance ratings x_{ij} of alternatives A_i ($i = 1, 2, ..., m$) with respect to attributes C_j ($j = 1, 2, ..., n$). Given the weighting vector W and decision matrix X, the objective is to rank or select the alternatives by giving each of them an overall preference value with respect to all attributes.

MADM methods generally involve two processes in order to obtain the overall preference value for each alternative: (a) a normalization procedure and (b) an aggregation method. Normalization is first applied to transform performance ratings to a compatible unit scale. An aggregation method is then used to combine normalized decision matrix and attributes' weights W to achieve an overall preference value for each alternative, on which the overall ranking of alternatives is based.

To help present the study, the four well known normalization procedures used in MADM are briefly described below, including: (a) vector normalization, (b) linear scale transformation (max-min), (c) linear scale transformation (max) and (d) linear scale transformation (sum).

2.1 Vector Normalization (N1)

In this procedure, each performance rating of the decision matrix is divided by its norm. The normalized value r_{ij} is obtained by

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x^2_{ij}}} \quad (1)$$

This procedure has the advantage of converting all attributes into dimensionless measurement unit, thus making inter-attribute comparison easier. But it has the drawback of having non-equal scale length leading to difficulties in straightforward comparison [6, 16].

2.2 Linear Scale Transformation (Max-Min) [N2]

This method considers both the maximum and minimum performance ratings of attributes during calculation.

For benefit attributes, the normalized value r_{ij} is obtained by

$$r_{ij} = \frac{x_{ij} - x_j^{\min}}{x_j^{\max} - x_j^{\min}} \quad (2)$$

For cost attributes, r_{ij} is computed as

$$r_{ij} = \frac{x_j^{\max} - x_{ij}}{x_j^{\max} - x_j^{\min}} \quad (3)$$

where $x_j^{\max}$ is the maximum performance rating among alternatives for attribute C_j ($j = 1, 2, ..., n$) and $x_j^{\min}$ is the minimum performance rating among alternatives for attribute C_j ($j = 1, 2, ..., n$).

This procedure has the advantage that the scale measurement is precisely between 0 and 1 for each attribute. The drawback is that the scale transformation is not proportional to outcome [6].

2.3 Linear Scale Transformation (Max) [N3]

This method divides the performance ratings of each attribute by the maximum performance rating for that attribute.

For benefit attributes, the normalized value r_{ij} is obtained by

$$r_{ij} = \frac{x_{ij}}{x_j^{\max}} \quad (4)$$

For cost attributes, r_{ij} is computed as

$$r_{ij} = 1 - \frac{x_{ij}}{x_j^{\max}} \quad (5)$$

where $x_j^{\max}$ is the maximum performance rating among alternatives for attribute C_j ($j = 1, 2, ..., n$).
Advantage of this procedure is that outcomes are transformed in a linear way [6, 16]

2.4 Linear Scale Transformation (Sum) [N4]

This method divides the performance ratings of each attribute by the sum of performance ratings for that attribute as follows

$$r_{ij} = \frac{x_{ij}}{\sum_{j=1}^{n} x_j} \quad (6)$$

where x_j is performance rating for each alternative for attribute C_j ($j = 1, 2, ..., n$) [16].

In order to obtain the overall preference value, the normalized decision matrix generated by a normalization procedure needs to be aggregated by an MADM method. The aggregation method of TOPSIS used in this study is explained in the next section.

3. THE TOPSIS METHOD

TOPSIS has been used extensively to solve various practical MADM problems for the following reasons:

(a) Comprehensive mathematical concept,
(b) Easy usability and simplicity,
(c) Computational efficiency,
(d) Ability to measure alternative performances in simple mathematical form [15].

In TOPSIS, the overall preference value index known as similarity to positive-ideal solution is defined by combining the closeness to positive-ideal solution and remoteness to negative-ideal solution. This index is used to rank the competing alternatives [3,6]. The TOPSIS method involves the following steps:

Step1: Calculate normalized performance ratings r_{ij} by using the vector normalization procedure in Equation (1). The normalized decision matrix can be denoted as

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} \quad (7)$$

Step 2: Calculate weighted Normalized value for each performance rating by Equation (7)

$$v_{ij} = w_j r_{ij} \; ; i = 1, 2, \ldots, m \quad (8)$$

where w_j ($j = 1, 2, \ldots, n$) is the attribute weight.

The weighted normalized decision matrix is generated using Equations (7) and (8) and denoted as

$$V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} \quad (9)$$

Step 3: Identify the positive-ideal and negative-ideal solutions A^* and A^- in terms of weighted normalized values by

$$A^* = \{v_1^*, v_2^*, \ldots, v_n^*\} \quad (10)$$

$$A^- = \{v_1^-, v_2^-, \ldots, v_n^-\} \quad (11)$$

where $v_j^* = \text{Max } v_{ij} ; \; \forall i = 1,2,\ldots,m; j = 1,2,\ldots,n$

$v_j^- = \text{Min } v_{ij} ; \; \forall i = 1,2,\ldots,m; j = 1,2,\ldots,n$

Step 4: Calculate the separation measure for alternatives using n-dimensional Euclidean distance. The separation (distance) of each alternative from the positive-ideal solution, A^*, can be obtained as

$$S_i^* = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^*)^2} \; ; i = 1,2,\ldots,m \quad (12)$$

Similarly the separation from the negative-ideal solution, A^-, is given by

$$S_i^- = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2} \; ; i = 1,2,\ldots,m \quad (13)$$

Step 5: Obtain the overall preference value (P_i) for alternatives A_i ($i = 1, 2, \ldots, m$) by

$$P_i = S_i^- / (S_i^* + S_i^-) ; i = 1,2,\ldots,m \quad (14)$$

Ranking of the alternatives is done according to C^*_i in descending order.

The TOPSIS method applies the vector normalization procedure by default. In the following sections we will examine the appropriateness of using this method in terms of ranking consistency and weight sensitivity.

4. PERFORMANCE MEASURES

4.1 Ranking Consistency

Ranking consistency is used to indicate how well a particular normalization procedure produces rankings similar to other procedures. To measure the ranking consistency index (RCI) of a particular normalization procedure, the total number of times the procedure showed similarities/dissimilarities in various extents with other procedures applied is calculated, over 10,000 simulation runs and then its ratio with total number of simulation runs is calculated. The higher the RCI, the better the procedure is [2, 3].

In calculating RCI, a consistency weight (*CW*) is used as follows:

1) If a method is consistent with all 3 of other 3 methods, then $CW = 3/3 = 1$.
2) If a method is consistent with any 2 of other 3 methods, then $CW = 2/3$.
3) If a method is consistent with any 2 other methods, then $CW = 1/3$.
4) If a method is not consistent with any other methods, then $CW = 0/3 = 0$.

The ranking consistency index of N1 is calculated as

$$RCI\,(N1) = [(T_{1234} * (CW{=}1) + T_{123} * (CW{=}2/3) + T_{124} * (CW{=}2/3) + T_{134} * (CW{=}2/3) + T_{12} * (CW{=}1/3) + T_{13} * (CW{=}1/3) + T_{14} * (CW{=}1/3) + TD_{1234} * (CW{=}0)) / TS] \quad (15)$$

The ranking consistency index of N2 is calculated as

$$RCI\,(N2) = [(T_{1234} * (CW{=}1) + T_{123} * (CW{=}2/3) + T_{124} * (CW{=}2/3) + T_{234} * (CW{=}2/3) + T_{12} * (CW{=}1/3) + T_{23} * (CW{=}1/3) + T_{24} * (CW{=}1/3) + TD_{1234} * (CW{=}0)) / TS] \quad (16)$$

The ranking consistency index of N3 is calculated as

$$RCI\,(N3) = [(T_{1234} * (CW{=}1) + T_{123} * (CW{=}2/3) + T_{134} * (CW{=}2/3) + T_{234} * (CW{=}2/3) + T_{13} * (CW{=}1/3) + T_{23} * (CW{=}1/3) + T_{34} * (CW{=}1/3) + TD_{1234} * (CW{=}0)) / TS] \quad (17)$$

The ranking consistency of N4 is calculated as

$$RCI\,(N4) = [(T_{1234} * (CW{=}1) + T_{124} * (CW{=}2/3) + T_{134} * (CW{=}2/3) + T_{234} * (CW{=}2/3) + T_{14} * (CW{=}1/3) + T_{24} * (CW{=}1/3) + T_{34} * (CW{=}1/3) + TD_{1234} * (CW{=}0)) / TS] \quad (18)$$

where *RCI (X)* = Ranking consistency index for normalization procedure (*X* = N1, N2, …, N4)

TS = Total number of times the simulation was run (10,000 in this experiment which generate 10,000 random matrices).
T_{1234} = Total number of times N1, N2, N3 and N4 produced the same ranking.
T_{123} = Total number of times N1, N2 and N3 produced the same ranking.
T_{124} = Total number of times N1, N2 and N4 produced the same ranking.
T_{134} = Total number of times N1, N3 and N4 produced the same ranking.
T_{234} = Total number of times N2, N3 and N4 produced the same ranking.
T_{12} = Total number of times N1and N2 produced the same ranking.
T_{13} = Total number of times N1and N3 produced the same ranking.
T_{14} = Total number of times N1 and N produced the same ranking.
T_{23} = Total number of times N2 and N3 produced the same ranking.
T_{24} = Total number of times N2 and N4 produced the same ranking.
T_{34} = Total number of times N3 and N4 produced the same ranking.
TD_{1234} = Total number of times N1, N2, N3 and N4 produced different rankings.

4.2 Weight Sensitivity

Relative sensitivity to attribute weight change is regarded as a performance measure in this study for the selection of most suitable normalization procedure for TOPSIS. Weight sensitivity is estimated as follows:

Step 1: A decision problem is solved by applying each normalization procedure with TOPSIS considering equal weights for all attributes. The ranking outcome for each procedure is recorded and is used as the initial ranking state for the corresponding procedure.

Step 2: The weight of the first attribute is increased gradually until any ranking produced by the procedures changes from the initial state. Note down the amount of weight change required for the normalization procedure.

Step 3: Continue incrementing the weight until all the rankings are changed and note down the amount of weight change required for each ranking change.

Step 4: The corresponding normalization procedure of the ranking that is changed with minimal weight change is the most sensitive. The least sensitive procedure to weight change is more likely to better handle weight induced anomalies. The procedures are given a sensitivity score for the attribute based on their required weight change amount. The procedure with the lowest amount given sensitivity score 1 and so on.

Step 5: Similarly weight of each attribute is increased (one at a time) to find the amount of weight change required for each normalization procedure and the procedures are given a sensitivity score.

Step 6: Average sensitivity score of all attributes for each procedure is calculated which indicates the sensitivity level of the corresponding procedure.
The weight sensitivity measure is more suitable to find the general sensitivity trend in different problem settings. The trend can be identified by applying the measure on a large number of problems with certain settings.

5. SIMULATION BASED STUDY

Three different simulation based experiments are conducted to provide results applicable to the general MADM problem rather than a particular MADM problem under different problem settings.

5.1 Ranking Consistency for Various Problem Sizes

This experiment has been conducted to find out the consistency level of each normalization procedure under different problem settings with a different number of attributes and alternatives. This will help us identify the most consistent normalization procedure/s for different problem sizes in terms of attributes and alternatives.

The experiment is conducted as follows:

- Random decision matrices of different sizes are created by increasing the number of attributes and alternatives in an increment of 2.
- The initial matrix size is with 4 attributes and 4 alternatives. This is small enough size to provide significant outcomes.
- The maximum size is with 20 attributes and 20 alternatives. This particular size is selected for its capability to provide significant results representing even larger matrices.
- 10,000 non-dominant decision matrices are generated for each problem size in each simulation run.
- The lower data range for each attribute is 1 and upper data range is between 10 and 10,000 and selected evenly based on a number of attributes.
- Each decision matrix is then normalized using each of the four normalization procedures. Then TOPSIS is applied for aggregation to generate an overall preference value for each alternative, used for ranking the alternatives.
- The ranking consistency index (RCI) is then calculated for each normalization procedure.
- All the attributes are assigned equal weights to remove any weight related error and bias.
- In order to achieve error free and unbiased results the complete experiment are repeated 3 times.

This experiment does not consider the diversity in performance ratings. Hence complementary experiments are required.

5.2 Ranking Consistency for Various Data Ranges

This experiment enables us to find the most consistent normalization procedure with a diverse range of data (performance ratings) for each attribute.

The experiment is conducted as follows:

- The matrix size is decided as 8 attributes and 8 alternatives. This size represents the moderate size group and its results can be related to smaller or larger sizes. Hence it is selected to represent the general sized matrices.
- The initial data ranges for the 8 attributes are selected as 1-10, 1-100, 1-500, 1-1,000, 1-2,500, 1-5000, 1-7,500, 1-10,000.
- The lower limit of each data range is increased by 10% of its upper limits until 90% increment is achieved. This will decrease the divergence in data for each attribute.
- 10,000 non-dominant decision matrices are generated for each set of data range in each simulation run.
- The RCI for each normalization procedure is then calculated similar to the previous experiment.
- Similar to the previous experiment, equal attribute weights are used and the experiment is repeated 3 times for correctness.

Both the above mentioned ranking consistency experiments will help us greatly choose the most consistent normalization procedure. These experiments are conducted in a controlled manner by applying equal weights to all the attributes. In practical decision settings we often work with non-equal weights. Hence sensitivity analysis of ranking results from different normalization procedures with respect to attribute weights needs to be conducted.

5.3 Sensitivity to Weights

The sensitivity analysis experiment helps us find the sensitivity level of different normalization procedures under different problem settings.

The experiment is conducted as follows:

- A problem setting is chosen with the same ranking produced by all the normalization procedures. This is done for easy comparison of the result of weight change.
- The size of the problem is chosen to be with 8 attributes and 8 alternatives.
- Initially each attribute is given equal weight (0.125).
- The weight of each attribute is increased by 0.0001 to find the weight change amount.
- Decision problems are generated to cover different problem settings.
- The weight sensitivity measure is applied to find the trend for each procedure under different settings. The sensitivity trends identified are rated as "best, good, average and poor" respectively.

6. STUDY RESULTS ANALYSIS

6.1 Ranking Consistency and Problem Size

As the problem size increases, the ranking consistency of all four normalization drops significantly. As shown in Figure1, vector normalization (N1) is the most consistent procedure for all problem sizes. The linear scale max-min (N2) has been least consistent. Linear scale max (N3) and linear scale sum (N4) show good consistency throughout. With smaller problem sizes, N4 and N3 have the same consistency level. However, as the problem size increases, N3 and N2 show similar consistency performance with N1 and N4.

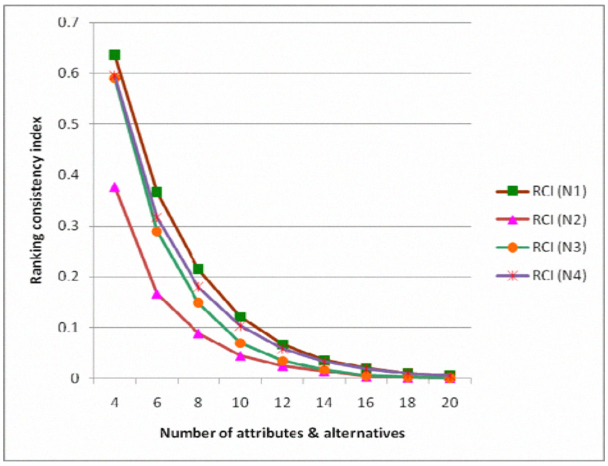

Figure1. Consistency results for problem size

6.2 Ranking Consistency and Data Range

The data range compression reduces the possible differences between performance ratings for any attribute.

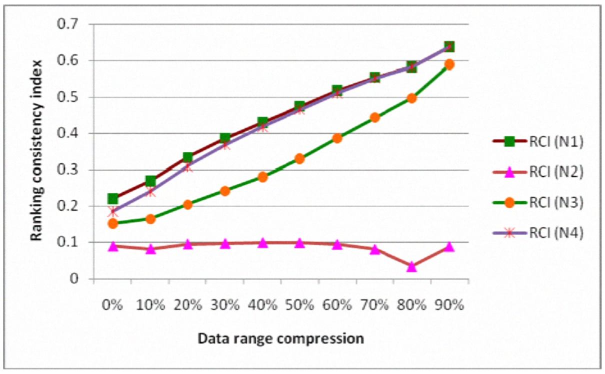

Figure2. Consistency results for data range

As shown in Figure2, the vector normalization (N1) procedure outperforms all other procedures over all data ranges. The procedure N2 has almost no impact of range compression on its ranking consistency which is the lowest among all. Procedure N3 is a steady performer and gradually increases with range compression. N4 is almost as consistent as N1 with larger data ranges and even closes the gap with narrow data ranges.

6.3 Sensitivity to Weights

Table1. Weight sensitivity trend for different settings

Decision Settings	N1	N2	N3	N4
Small problem size	best	poor	average	good
Medium Problem size	best	poor	average	good
Large problem size	best	poor	average	good
Wide data range	good	poor	good	average
Moderate data range	best	poor	average	good
Narrow data range	best	poor	average	good

From Table1, we can understand that vector normalization (N1) is overall the least sensitive to weight under different settings. Although N1 is not the best for all settings, it is most consistent for all settings.

7. CONCLUSION

The experiment results justify the use of vector normalization with the TOPSIS method. It is the most consistent in ranking and is able to handle weight sensitivity quite well. The study identifies possible alternatives to the vector normalization procedure under different problem settings. It also helps the decision maker choose the best normalization procedure if the weight is a very important factor in certain decision settings.

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