12. If $AB = BA$, then $e^A e^B = e^B e^A$ and $e^A B = B e^A$.

13. Let an operator $A: \mathbf{R}^n \to \mathbf{R}^n$ leave invariant a subspace $E \subset \mathbf{R}^n$ (that is, $Ax \in E$ for all $x \in E$). Show that e^A also leaves E invariant.

14. Show that if $\| T - I \|$ is sufficiently small, then there is an operator S such that $e^S = T$. (*Hint*: Expand $\log(1 + x)$ in a Taylor series.) To what extent is S unique?

15. Show that there is no real 2×2 matrix S such that $e^S = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$.

§4. Homogeneous Linear Systems

Let A be an operator on $\mathbf{R}^n$. In this section we shall express solutions to the equation:

$$x' = Ax \tag{1}$$

in terms of exponentials of operators.

Consider the map $\mathbf{R} \to L(\mathbf{R}^n)$ which to $t \in \mathbf{R}$ assigns the operator e^{tA}. Since $L(\mathbf{R}^n)$ is identified with $\mathbf{R}^{n^2}$, it makes sense to speak of the derivative of this map.

Proposition

$$\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A.$$

In other words, the derivative of the operator-valued function e^{tA} is another operator-valued function $A e^{tA}$. This means the composition of e^{tA} with A; the order of composition does not matter. One can think of A and e^{tA} as matrices, in which case $A e^{tA}$ is their product.

Proof of the proposition.

$$\begin{aligned} \frac{d}{dt} e^{tA} &= \lim_{h \to 0} \frac{e^{(t+h)A} - e^{tA}}{h} \\ &= \lim_{h \to 0} \frac{e^{tA} e^{hA} - e^{tA}}{h} \\ &= e^{tA} \lim_{h \to 0} \left(\frac{e^{hA} - I}{h} \right) \\ &= e^{tA} A; \end{aligned}$$

that the last limit equals A follows from the series definition of e^{hA}. Note that A commutes with each term of the series for e^{tA}, hence with e^{tA}. This proves the proposition.

We can now solve equation (1). We recall from Chapter 1 that the general solution of the scalar equation

$$x' = ax \qquad (a \in \mathbf{R})$$

is

$$x(t) = ke^{ta}; \qquad k = x(0).$$

The same is true where x, a, and k are allowed to be complex numbers (Chapter 3). These results are special cases of the following, which can be considered as the fundamental theorem of linear differential equations with constant coefficients.

Theorem *Let A be an operator on $\mathbf{R}^n$. Then the solution of the initial value problem*

(1′) $$x' = Ax, \qquad x(0) = K \in \mathbf{R}^n,$$

is

(2) $$e^{tA}K,$$

and there are no other solutions.

Proof. The preceding lemma shows that

$$\frac{d}{dt}(e^{tA}K) = \left(\frac{d}{dt}e^{tA}\right)K$$
$$= Ae^{tA}K;$$

since $e^{0A}K = K$, it follows that (2) is a solution of (1′). To see that there are no other solutions, let $x(t)$ be any solution of (1′) and put

$$y(t) = e^{-tA}x(t).$$

Then

$$y'(t) = \left(\frac{d}{dt}e^{-tA}\right)x(t) + e^{-tA}x'(t)$$
$$= -Ae^{-tA}x(t) + e^{-tA}Ax(t)$$
$$= e^{-tA}(-A + A)x(t)$$
$$\equiv 0.$$

Therefore $y(t)$ is a constant. Setting $t = 0$ shows $y(t) = K$. This completes the proof of the theorem.

As an example we compute the general solution of the two-dimensional system

(3) $$x_1' = ax_1,$$
$$x_2' = bx_1 + ax_2,$$

where a, b are constants. In matrix notation this is

$$x' = Ax; \qquad A = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}; \qquad x = (x_1, x_2).$$

The solution with initial value $K = (K_1, K_2) \in \mathbf{R}^2$ is

$$e^{tA}K.$$

In Section 3 we saw that

$$e^{tA} = e^{ta}\begin{bmatrix} 1 & 0 \\ tb & 1 \end{bmatrix}.$$

Thus

$$e^{tA}K = (e^{ta}K_1, e^{ta}(tbK_1 + K_2)).$$

Thus the solution to (3) satisfying

$$x_1(0) = K_1, \qquad x_2(0) = K_2$$

is

$$x_1(t) = e^{ta}K_1,$$
$$x_2(t) = e^{ta}(tbK_1 + K_2).$$

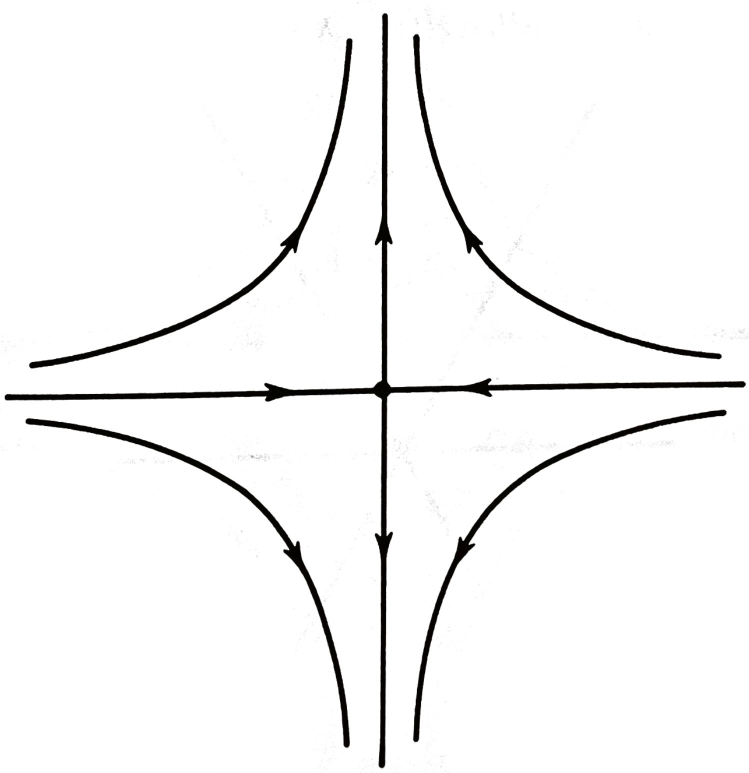

FIG. A. Saddle: $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, $\lambda < 0 < \mu$.

Since we know how to compute the exponential of any 2×2 matrix (Section 3), we can explicitly solve any two-dimensional system of the form $x' = Ax$, $A \in L(\mathbf{R}^2)$. Without finding explicit solutions, we can also obtain important qualitative information about the solutions from the eigenvalues of A. We consider the most important special cases.

Case I. *A has real eigenvalues of opposite signs.* In this case the origin (or sometimes the differential equation) is called a *saddle*. As we saw in Chapter 3, after a suitable change of coordinates $x = Py$, the equation becomes

$$y' = By,$$

$$B = PAP^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \qquad \lambda < 0 < \mu.$$

In the (y_1, y_2) plane the phase portrait looks like Fig. A on p. 91.

Case II. *All eigenvalues have negative real parts.* This important case is called a *sink*. It has the characteristic property that

$$\lim_{t \to \infty} x(t) = 0$$

for every solution $x(t)$. If A is diagonal, this is obvious, for the solutions are

$$y(t) = (c_1 e^{\lambda t}, c_2 e^{\mu t}); \qquad \lambda < 0, \quad \mu < 0.$$

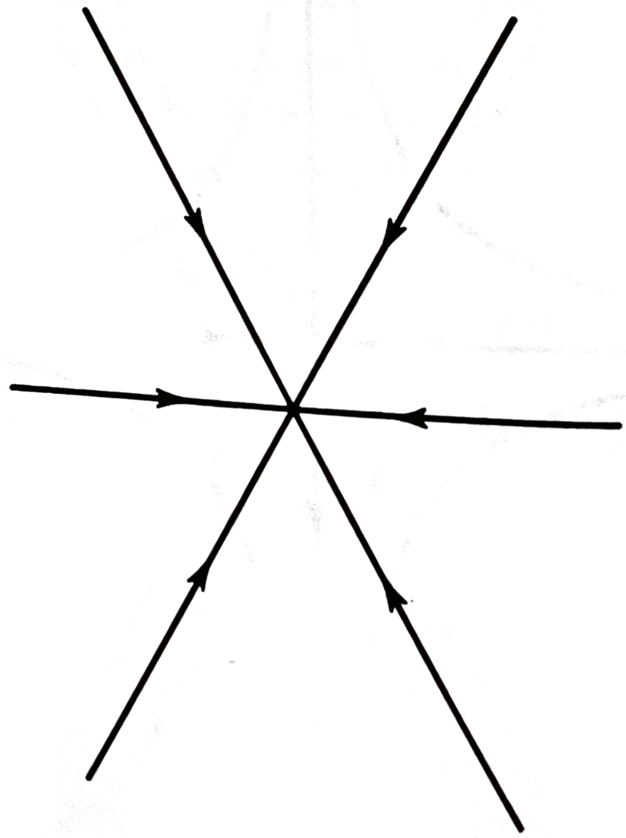

FIG. B. Focus: $B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $\lambda < 0$.

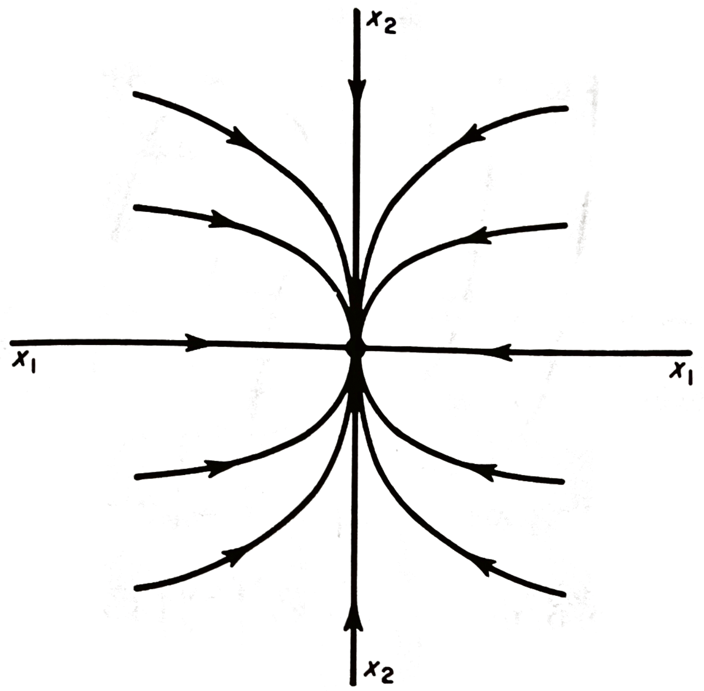

FIG. C. Node: $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, $\lambda < \mu < 0$.

If A is diagonalizable, the solutions

$$x(t) = Py(t)$$

are of the form with $y(t)$ as above and $P \in L(\mathbf{R}^2)$; clearly, $x(t) \to 0$ as $t \to \infty$.

The phase portrait for these subcases looks like Fig. B if the eigenvalues are equal (a *focus*) and like Fig. C if they are unequal (a *node*).

If the eigenvalues are negative but A is not diagonalizable, there is a change of coordinates $x = Py$ (see Chapter 6) giving the equivalent equation

$$y' = By,$$

where

$$B = P^{-1}AP = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}, \qquad \lambda < 0.$$

We have already solved such an equation; the solutions are

$$y_1(t) = K_1 e^{t\lambda},$$

$$y_2(t) = K_2 e^{t\lambda} + K_1 t e^{t\lambda},$$

which tend to 0 as t tends to ∞. The phase portrait looks like Fig. D (an *improper node*).

If the eigenvalues are $a \pm ib$, $a < 0$ we can change coordinates as in Chapter 4

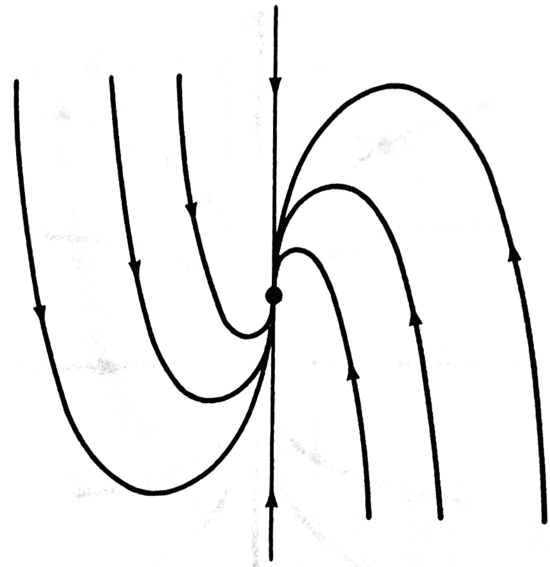

FIG. D. Improper node: $B = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$, $\lambda < 0$.

to obtain the equivalent system

$$y' = By, \qquad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

From Section 3 we find

$$e^{tB} = e^{ta} \begin{bmatrix} \cos tb & -\sin tb \\ \sin tb & \cos tb \end{bmatrix}.$$

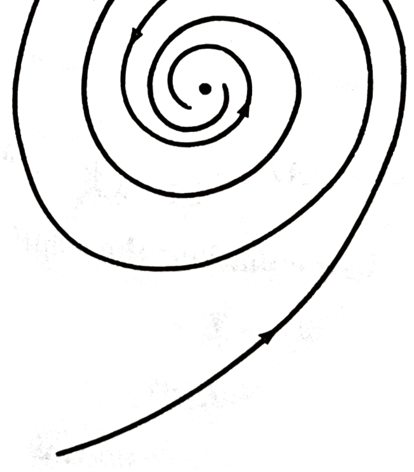

FIG. E. Spiral sink: $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $b > 0 > a$.

Therefore the general solution is expressed in y-coordinates as

$$y(t) = e^{ta}(K_1 \cos tb - K_2 \sin tb,\ K_2 \cos tb + K_1 \sin tb).$$

Since $|\cos tb| \leq 1$ and $|\sin tb| \leq 1$, and $a < 0$, it follows that

$$\lim_{t \to \infty} y(t) = 0.$$

If $b > 0$, the phase portrait consists of counterclockwise spirals tending to 0 (Fig. E), and clockwise spirals tending to 0 if $b < 0$.

Case III. *All eigenvalues have positive real part.* In this case, called a *source,* we have

$$\lim_{t \to \infty} |x(t)| = \infty \quad \text{and} \quad \lim_{t \to -\infty} |x(t)| = 0.$$

A proof similar to that of Case II can be given; the details are left to the reader. The phase portraits are like Figs. B–E with the arrows reversed.

Case IV. *The eigenvalues are pure imaginary.* This is called a *center.* It is characterized by the property that all solutions are *periodic* with the same period. To see this, change coordinates to obtain the equivalent equation

$$y' = By, \qquad B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}.$$

We know that

$$e^{tB} = \begin{bmatrix} \cos tb & -\sin tb \\ \sin tb & \cos tb \end{bmatrix}.$$

Therefore if $y(t)$ is any solution,

$$y\left(t + \frac{2\pi}{b}\right) \equiv y(t).$$

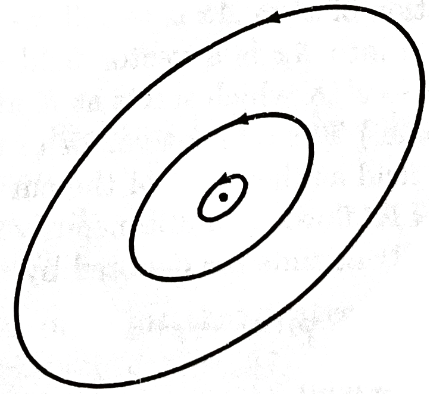

FIG. F. Center: $B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$, $b > 0$.

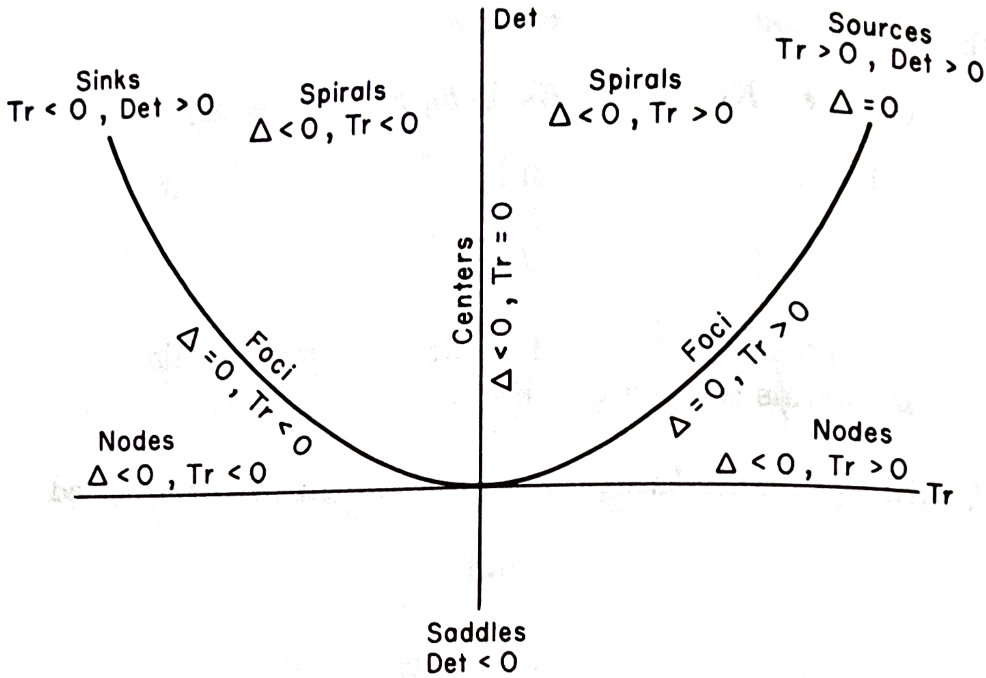

FIG. G

The phase portrait in the y-coordinates consists of concentric circles. In the origina[l] x-coordinates the orbits may be ellipses as in Fig. F. (If $b < 0$, the arrows point clockwise.)

Figure G summarizes the geometric information about the phase portrait of $x' = Ax$ that can be deduced from the characteristic polynomial of A. We write this polynomial as

$$\lambda^2 - (\operatorname{Tr} A)\lambda + \operatorname{Det} A.$$

The *discriminant* Δ is defined to be

$$\Delta = (\operatorname{Tr} A)^2 - 4 \operatorname{Det} A.$$

The eigenvalues are

$$\tfrac{1}{2}(\operatorname{Tr} A \pm \sqrt{\Delta}).$$

Thus real eigenvalues correspond to the case $\Delta \geq 0$; the eigenvalues have negative real part when $\operatorname{Tr} A < 0$; and so on.

The geometric interpretation of $x' = Ax$ is as follows (compare Chapter 1). The map $\mathbf{R}^n \to \mathbf{R}^n$ which sends x into Ax is a vector field on $\mathbf{R}^n$. Given a point K of $\mathbf{R}^n$, there is a unique curve $t \to e^{tA}K$ which starts at K at time zero, and is a solution of (1). (We interpret t as time.) The tangent vector to this curve at a time t_0 is the vector $Ax(t_0)$ of the vector field at the point of the curve $x(t_0)$.

We may think of points of $\mathbf{R}^n$ flowing simultaneously along these solution curves. The position of a point $x \in \mathbf{R}^n$ at time t is denoted by

$$\phi_t(x) = e^{tA}x.$$

Thus for each $t \in \mathbf{R}$ we have a map

$$\phi_t\colon \mathbf{R}^n \to \mathbf{R}^n \qquad (t \in \mathbf{R})$$

given by

$$\phi_t(x) = e^{tA}x.$$

The collection of maps $\{\phi_t\}_{t\in\mathbf{R}}$ is called the *flow* corresponding to the differential equation (1). This flow has the basic property

$$\phi_{s+t} = \phi_s \circ \phi_t,$$

which is just another way of writing

$$e^{(s+t)A} = e^{sA}e^{tA};$$

this is proved in the proposition in Section 2. The flow is called *linear* because each map $\phi_t: \mathbf{R}^n \to \mathbf{R}^n$ is a linear map. In Chapter 8 we shall define more general *nonlinear* flows.

The phase portraits discussed above give a good visualization of the corresponding flows. Imagine points of the plane all moving at once along the curves in the direction of the arrows. (The origin stays put.)

PROBLEMS

1. Find the general solution to each of the following systems:

(a) $\begin{cases} x' = 2x - y \\ y' = 2y \end{cases}$ (b) $\begin{cases} x' = 2x - y \\ y' = x + 2y \end{cases}$

(c) $\begin{cases} x' = y \\ y' = x \end{cases}$ (d) $\begin{cases} x' = -2x \\ y' = x - 2y \\ z' = y - 2z \end{cases}$

(e) $\begin{cases} x' = y + z \\ y' = z \\ z' = 0 \end{cases}$

2. In (a), (b), and (c) of Problem 1, find the solutions satisfying each of the following initial conditions:
(a) $x(0) = 1, y(0) = -2$; (b) $x(0) = 0, y(0) = -2$;
(c) $x(0) = 0, y(0) = 0$.

3. Let $A: \mathbf{R}^n \to \mathbf{R}^n$ be an operator that leaves a subspace $E \subset \mathbf{R}^n$ invariant. Let $x: \mathbf{R} \to \mathbf{R}^n$ be a solution of $x' = Ax$. If $x(t_0) \in E$ for some $t_0 \in \mathbf{R}$, show that $x(t) \in E$ for all $t \in \mathbf{R}$.

4. Suppose $A \in L(\mathbf{R}^n)$ has a real eigenvalue $\lambda < 0$. Then the equation $x' = Ax$

has at least one nontrivial solution $x(t)$ such that

$$\lim_{t\to\infty} x(t) = 0.$$

5. Let $A \in L(\mathbf{R}^2)$ and suppose $x' = Ax$ has a nontrivial *periodic solution*, $u(t)$: this means $u(t+p) \equiv u(t)$ for some $p > 0$. Prove that every solution is periodic, with the same period p.

6. If $u: \mathbf{R} \to \mathbf{R}^n$ is a nontrivial solution of $x' = Ax$, then

$$\frac{d}{dt}(|u|) = \frac{1}{|u|}\langle u, Au\rangle.$$

7. Supply the details of Case II in the text.

8. Classify and sketch the phase portraits of planar differential equations $x' = Ax$, $A \in L(\mathbf{R}^2)$, where A has zero as an eigenvalue.

9. For each of the following matrices A consider the corresponding differential equation $x' = Ax$. Decide whether the origin is a sink, source, saddle, or none of these. Identify in each case those vectors u such that $\lim_{t\to\infty} x(t) = 0$, where $x(t)$ is the solution with $x(0) = u$:

(a) $\begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

10. Which values (if any) of the parameter k in the following matrices makes the origin a sink for the corresponding differential equation $x' = Ax$?

(a) $\begin{bmatrix} a & -k \\ k & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ k & -4 \end{bmatrix}$

(c) $\begin{bmatrix} k^2 & 1 \\ 0 & k \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & k \end{bmatrix}$

11. Let $\phi_t: \mathbf{R}^2 \to \mathbf{R}^2$ be the flow corresponding to the equation $x' = Ax$. (That is, $t \to \phi_t(x)$ is the solution passing through x at $t = 0$.) Fix $\tau > 0$, and show that ϕ_τ is a linear map of $\mathbf{R}^2 \to \mathbf{R}^2$. Then show that ϕ_t preserves area if and only

if $\operatorname{Tr} A = 0$, and that in this case the origin is not a sink or a source. (*Hint*: An operator is area-preserving if and only if the determinant is ± 1.)

12. Describe in words the phase portraits of $x' = Ax$ for

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (d) $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

13. Suppose A is an $n \times n$ matrix with n distinct eigenvalues and the real part of every eigenvalue is less than some negative number α. Show that for every solution to $x' = Ax$, there exists $t_0 > 0$ such that

$$|x(t)| < e^{t\alpha} \quad \text{if} \quad t \geq t_0.$$

14. Let T be an invertible operator on $\mathbf{R}^n$, n odd. Then $x' = Tx$ has a nonperiodic solution.

15. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ have nonreal eigenvalues. Then $b \neq 0$. The nontrivial solutions curves to $x' = Ax$ are spirals or ellipses that are oriented clockwise if $b > 0$ and counterclockwise if $b < 0$. (*Hint*: Consider the sign of

$$\frac{d}{dt} \arctan(x_2(t)/x_1(t)).)$$

§5. A Nonhomogeneous Equation

We consider a nonhomogeneous nonautonomous linear differential equation

$$x' = Ax + B(t). \tag{1}$$

Here A is an operator on $\mathbf{R}^n$ and $B\colon \mathbf{R} \to \mathbf{R}^n$ is a continuous map. This equation is called *nonhomogeneous* because of the term $B(t)$ which prevents (1) from being strictly linear; the fact that the right side of (1) depends explicitly on t makes it *nonautonomous*. It is difficult to interpret solutions geometrically.

We look for a solution having the form

$$x(t) = e^{tA}f(t), \tag{2}$$

where $f\colon \mathbf{R} \to \mathbf{R}^n$ is some differentiable curve. (This method of solution is called "variation of constants," perhaps because if $B(t) \equiv 0$, $f(t)$ is a constant.) Every solution can in fact be written in this form since e^{tA} is invertible.

Differentiation of (2) using the Leibniz rule yields

$$x'(t) = Ae^{tA}f(t) + e^{tA}f'(t).$$