Exemplo 1 Partícula sob ação da força $\vec{F}$

(teste de sanidade :-)

11/4/23

Neste caso $q_1 = x$ $q_2 = y$ $q_3 = z$

$$Q_k = \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_k} \Rightarrow Q_1 = \vec{F} \cdot \frac{\partial \vec{r}}{\partial x} = F_x$$

$$Q_2 = \vec{F} \cdot \frac{\partial \vec{r}}{\partial y} = F_y$$

$$Q_3 = \vec{F} \cdot \frac{\partial \vec{r}}{\partial z} = F_z .$$

$$T = \frac{1}{2} m \vec{v}^2 = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 \right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = Q_k \Rightarrow$$

$k=1$ $\frac{d}{dt}\left(m\dot{q}_1 \right) = F_x \Rightarrow m\ddot{x} = F_x$!

e

$m\ddot{y} = F_y$!

$m\ddot{z} = F_z$

OK ✓

Exemplo 2: 1 partícula no plano usando coordenadas polares: $x = r\cos\theta$ $y = r \operatorname{sen}\theta$

$q_1 = r$ $q_2 = \theta$

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 \right) = \frac{m}{2}\left(\dot{q}_1^2 + q_1^2 \dot{q}_2^2 \right)$$

Mais: $Q_1 = \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_1} = \vec{F} \cdot \frac{\partial}{\partial q_1}\left(q_1 \vec{e}_r \right) \Rightarrow Q_1 = \vec{e}_r \cdot \vec{F} = F_r$

$$Q_2 = \vec F\cdot\frac{\partial \vec r}{\partial q_2} = \vec F\cdot\frac{\partial}{\partial\theta}(r\vec e_r) = \vec F\cdot r\vec e_\theta = F_\theta\, r$$

$$\left.\begin{aligned}\vec e_\theta &= \cos\theta\,\vec j - \sin\theta\,\vec i\\ \vec e_r &= \cos\theta\,\vec i + \sin\theta\,\vec j\end{aligned}\right\}$$

Euler $\overset{k=1}{\Rightarrow}$ $$\frac{d}{dt}\frac{\partial}{\partial \dot q_1}\left[\frac{m}{2}\left(\dot q_1^2 + q_1^2\dot q_2^2\right)\right] - \frac{\partial}{\partial q_1}\left[\frac{m}{2}\left(\dot q_1^2 + q_1^2\dot q_2^2\right)\right] = Q_1$$

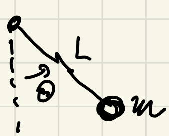
$$\Rightarrow m\ddot q_1 - m q_1\dot q_2^2 = Q_1 \Rightarrow m\left(\ddot r - r\dot\theta^2\right) = F_r$$

Euler $k=2 \Rightarrow$ $$\frac{d}{dt}\frac{\partial}{\partial \dot q_2}\left[\frac{m}{2}\left(\dot q_1^2 + q_1^2\dot q_2^2\right)\right] - \overbrace{\frac{\partial}{\partial q_2}\frac{m}{2}\left(\dot q_1^2 + q_1^2\dot q_2^2\right)}^{0} = Q_2$$

$$\Rightarrow m\frac{d}{dt}\left(q_1^2\dot q_2\right) = Q_2 \Rightarrow m\left(r^2\ddot\theta + 2\dot r\dot\theta\right) = rF_\theta$$

OK ✓

Exemplo 3 Pêndulo plano: $q_1 = \theta$



$$Q_1 = \vec F\cdot\frac{\partial \vec r}{\partial\theta} = L\, m\vec g\cdot\vec e_\theta = -mgL\,\mathrm{sen}\,\theta$$

$$T = \frac{1}{2}mL^2\dot q_1^2$$

Euler $\Rightarrow$ $$\frac{d}{dt}\left[\frac{\partial}{\partial\dot q_1}\left(\frac{1}{2}mL^2\dot q_1^2\right) - \frac{\partial}{\partial q_1}\left(\frac{1}{2}mL^2\dot q_1^2\right)\right] = -mgL\,\mathrm{sen}\,\theta \Rightarrow$$

$$mL^2\ddot{\theta} = -mgL\,\text{sen}\,\theta$$ ✓ eq. para o torque!

Caso de forças conservativas

Consideremos o caso de forças conservativas

$$\vec{F}_i = -\nabla_i U(\vec{r}_1 \dots \vec{r}_N)$$

$$Q_k = \sum_{i=1}^{N} -\nabla_i U \cdot \frac{\partial \vec{r}_i}{\partial q_k} = -\sum_{i=1}^{N}\left[\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_k} + \frac{\partial U}{\partial y_i}\frac{\partial y_i}{\partial q_k} + \frac{\partial U}{\partial z_i}\frac{\partial z_i}{\partial q_k}\right]$$

$$\Rightarrow Q_k = -\frac{\partial}{\partial q_k} U$$

Com isso, $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_k}\right) - \dfrac{\partial T}{\partial q_k} = -\dfrac{\partial U}{\partial q_k}$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{\partial}{\partial q_k}(T-U)$$

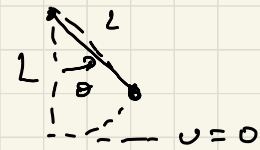
Se U não depende de $\dot{q}_k$ podemos escrever

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_k}(T-U)\right) = \frac{\partial}{\partial q_k}(T-U)$$

definindo a Lagrangiana $\boxed{L = T - U}$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}}$$ $k = 1, \dots, n$

Exemplo: pêndulo plano. $q = \theta$



$$T = \frac{m}{2} L^2 \dot{q}^2$$

$$U = mgL(1 - \cos q)$$

$$L = \frac{m}{2} L^2 \dot{q}^2 - mgL(1 - \cos q)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}\left(mL^2 \dot{q}\right) = mL^2 \ddot{q} = \frac{\partial}{\partial q}(L) = -mgL \,\mathrm{sen}\, q$$

ou seja, $mL^2 \ddot{\theta} = -mgL \,\mathrm{sen}\, \theta$!! Funciona! ☺

Observações:

1) Usamos as leis de Newton p/ obter d'Alembert e Euler-Lagrange ⇒ tenho que usar sistema de coordenadas inercial

2) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \qquad k = 1 \dots n$$

são EDO's de 2º grau ⇒ preciso p/ definir o problema

n $q_k(t_0)$ e n $\dot{q}_k(t_0)$

3) Note que adicionar uma derivada total do tempo não muda as eq's de movimento. De fato.

$$L \to L' = L + \frac{d}{dt} f(q_k, t) = L + \sum_k \frac{\partial f}{\partial q_k}\dot q_k + \frac{\partial f}{\partial t}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot q_k}\right) - \frac{\partial}{\partial q_k} L' = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{\partial L}{\partial q_k} +$$

$$+ \frac{d}{dt}\left[\frac{\partial}{\partial \dot q_k}\left[\sum_j \frac{\partial f}{\partial q_j}\dot q_j + \frac{\partial f}{\partial t}\right]\right] - \frac{\partial}{\partial q_k}\left[\frac{\partial f}{\partial t} + \sum_j \frac{\partial f}{\partial q_j}\dot q_j\right]$$

$$\delta_{jk}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{\partial L}{\partial q_k} + \frac{d}{dt}\frac{\partial f}{\partial q_k} - \frac{\partial}{\partial q_k}\frac{\partial f}{\partial t} - \frac{\partial}{\partial q_k}\sum_j \frac{\partial f}{\partial q_j}\dot q_j$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{\partial L}{\partial q_k} + \sum_j \cancel{\frac{\partial}{\partial q_j}\frac{\partial f}{\partial q_k}}\dot q_j + \cancel{\frac{\partial}{\partial t}\frac{\partial f}{\partial q_k}} - \cancel{\frac{\partial}{\partial q_k}\frac{\partial f}{\partial t}}$$

$$- \sum_j \cancel{\frac{\partial}{\partial q_k}\frac{\partial f}{\partial q_j}}\dot q_j$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot q_k}\right) - \frac{\partial}{\partial q_k} L' = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{\partial}{\partial q_k} L = 0 \checkmark$$

<u>Existe uma liberdade (restrita) na escolha de L.</u>