Method of Characteristics

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1 Introduction

Let us recall a simple fact from the theory of ordinary differential equations: the equation

$$\frac{du}{dt} = f(t,u)$$

can be uniquely solved for a given initial condition $u(0) = u_0$, provided that f is a continuous function of t and u. The solution may exists for all time or may blow up at some finite time.

If we now allow the equation and the initial condition to depend on a parameter x, then the solution u also depend on x and may be written as $u(x,t)$. In fact, u become a solution of

$$\frac{\partial u}{\partial t} = f(x,t,u) \qquad \text{with} \qquad u(x,0) = F(x)$$

This may be thought of as an initial value problem for a PDE in which the derivative $\partial u/\partial x$ does not appear. Assuming f and F are continuous functions of x, the solution $u(x,t)$ will also be continuous in x as well as t. Geometrically, the graph $u = u(x,t)$ represents a surface in $\mathbf{R}^3$ that contains the curve (initial data) $F(x)$. This surface may be defined for all $t > 0$, or may blow up at some finite time.

This elementary ideas from ODE theory is the basis of the *Method of Characteristics* (MoC) which applies to general quasilinear PDEs.

1.1 Solution of linear advection equation using MoC

For the purpose of illustration of method of characteristics, let us consider the simple case of a one-dimensional linear advection equation also called wave equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1a}$$

$$u(x,0) = F(x) \tag{1b}$$

where $u(x,t)$ is the unknown function of (x,t) and a the uniform advection speed. $F(x)$ is called the *initial data* (waveform specified along the *initial curve*, $t = 0$) and the equation (1b) is called the *initial condition* of the problem. Let us try to reduce this problem to a ODE along some

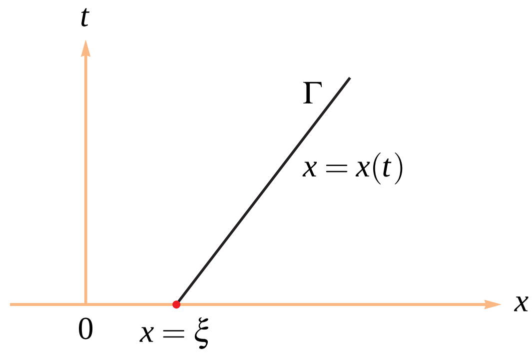

Figure 1: A typical characteristic curve.

curve $x(t)$ in (x,t) plane whose slope is given by dt/dx. That is, we determine the curve $x(t)$ such that

$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{2}$$

From the chain rule of differentiation, we have

$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} \tag{3}$$

A comparison of equations (2) and (3) shows that ODE on the left-hand side of (2) is equivalent to PDE in the same equation if the reciprocal of slope of the curve $x(t)$

$$\frac{dx}{dt} = a$$

It follows from this result that the PDE (1a) can be regarded as the ordinary differential equation

$$\frac{du}{dt} = 0 \tag{4a}$$

along any member of the family of curves $x(t)$ which are the solution curves of the equation given by

$$\frac{dx}{dt} = a \tag{4b}$$

A *characteristic curve* of PDE (1a) is a curve in the (x,t)-plane given by $x = x(t)$, where $x(t)$ is a solution of the differential equation (4b). From (4a) it is clear that the value of u remains a constant along such curves. Thus, the solution of (1a) has been reduced to the solution of a pair of simultaneous ordinary differential equations (4a) and (4b). We now integrate the equation (4b) to obtain the characteristic curves:

$$x(t) = at + \xi$$

where ξ is the x-intercept of the curve (see figure 1). It shows that the characteristic curves are straight lines with slope, $dt/dx = 1/a$. Further, since u is a constant along a given characteristic curve its value can be readily determined from the initial data. That is, along a given characteristic curve we must have

$$u(x,t) = u(\xi,0) = F(\xi)$$

Since $\xi = x - at$, the solution of the PDE (1a) is simply given by

$$u(x,t) = F(x - at) \tag{5}$$

Indeed, if F has a C^1 continuity, it can be easily verified that $u(x,t) = F(x - at)$ satisfies the PDE and the initial condition. The reduction of a PDE to an ODE along its characteristics is called the method of characteristics.

The solution of PDE (1a) corresponds to transporting the initial profile $F(x)$ unaltered (preserving the shape of initial waveform) along the characteristics with a speed $dx/dt = a$ (see figure 1). To prove this, consider the transport of u over a period of time Δt. Let Δx be the displacement in the x-direction during the time Δt. From solution (5) we obtain

$$u(x, t + \Delta t) = F[x - a(t + \Delta t)] = F(x - at - a\Delta t) = F[(x - \Delta x) - at] = u(x - \Delta x, t)$$

or, after a time period of t_1 from the initial time,

$$u(x, t_1) = F(x - at_1) = u(x - at_1, 0)$$

Thus, the solution after a time t_1 is a copy of the initial profile $F(x)$, but displaced to the right to a distance at_1 for $a > 0$ as shown in figure 2.

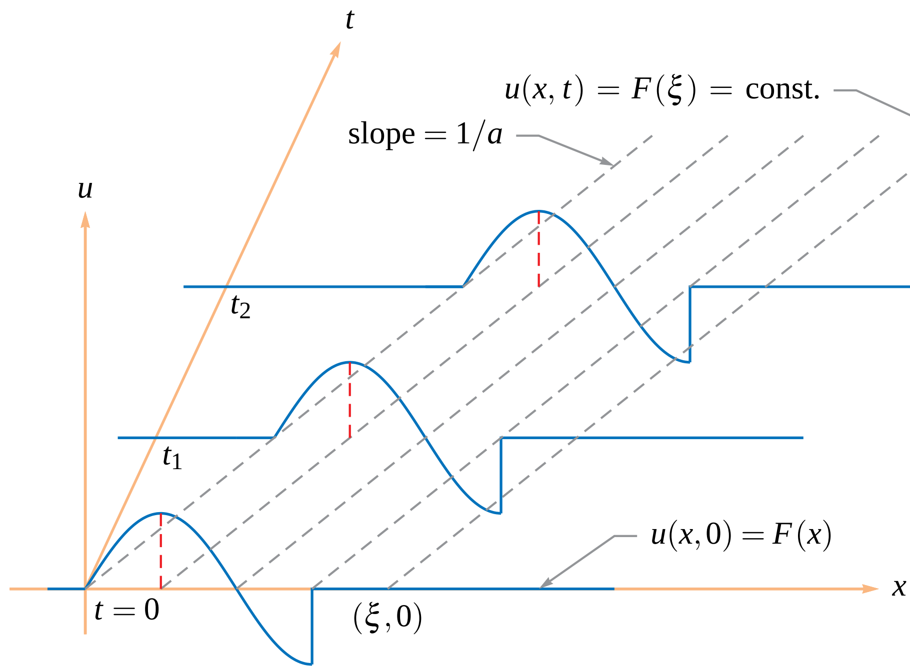

Figure 2: Initial profile, characteristics, and solution at various times.

2 Method of Characteristics for Quasilinear PDE

The method of characteristics is a technique for solving hyperbolic partial differential equations (PDE). Typically the method applies to first-order equations, although it is valid for any hyperbolic-type PDEs. The method involves the determination of special curves, called characteristics curves, along which the PDE becomes a family of ordinary differential equations (ODE). Once the ODEs are found, they can be solved along the characteristics curves to obtain the solutions of ODE and subsequently the solutions of ODE can be related to the solution of original PDE.

MoC can be applied to linear, semilinear, or quasilinear PDEs. For the purpose of illustration of MoC, let us consider a general quasilinear first-order PDE

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u) \qquad \text{in} \quad \mathscr{D} \tag{6}$$

for the variable u. We assume that the coefficients a, b, and c are at least C^1 continuous of variables x, y, u. Further, we assume that the possible solution of (6) in the form $u = u(x,y)$, then the graph $u = u(x,y)$ represents the *solution surface* S embedded in $\mathbf{R}^3$ (just as $y = y(x)$ represents a curve Γ embedded in $\mathbf{R}^2$, i.e., (x,y) plane). Here $u(x,y)$ is a function of x and y, for example, we may have $u(x,y) = x^2 - y^2$. The solution surface S can also be represented implicitly by the equation of the form $f(x,y,u) = 0$. In the present case we have $f(x,y,u) = u(x,y) - u$. Now, recall from vector calculus that the normal vector to the surface $f(x,y,u) = 0$ is given by ∇f. Therefore, the equation of the normal vector to the surface S is given by

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial u}\hat{k} = \frac{\partial u(x,y)}{\partial x}\hat{i} + \frac{\partial u(x,y)}{\partial y}\hat{j} - \frac{\partial u}{\partial u}\hat{k} \equiv \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right) = (u_x, u_y, -1)$$

Thus the vector $(u_x, u_y, -1)$ represents the normal vector at any point on the solution surface $u = u(x,y)$. Further, we introduce the vector field $\overline{A} = a\hat{i} + b\hat{j} + c\hat{k}$, where a, b, and c are the given coefficients of the quasilinear equation (6). Therefore, the PDE given by equation (6) can be written as $\overline{A}\cdot\nabla f = 0$, or

$$\left(a\hat{i} + b\hat{j} + c\hat{k}\right)\cdot\left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} - \hat{k}\right) = 0, \quad \text{or} \tag{7a}$$

$$(a, b, c)\cdot(u_x, u_y, -1) = 0 \tag{7b}$$

Equation (7) shows that the vectors (a,b,c) and $(u_x, u_y, -1)$ are orthogonal, and because $(u_x, u_y, -1)$ is normal to the surface $u = u(x,y)$, the vector $\overline{A} = (a,b,c)$ must be a tangent vector of the solution surface S at every point (x,y,u). Therefore, $\overline{A}$ defines a vector field in (x,y,u) space, that are tangents to the graphs of solutions of (6) at each point (x,y,u). Surfaces that are tangent to a vector field at each point in $\mathbf{R}^3$ are called *integral surfaces* of the vector field. To summarize, we have shown that $f(x,y,u) = u(x,y) - u = 0$, as a surface in the (x,y,u) space, is a solution of (6) if and only if the direction vector field $\overline{A} = (a,b,c)$ lies in the tangent plane of the integral surface $f(x,y,u) = 0$ at each point (x,y,u). The vector (a,b,c) determines a direction, which is tangent to the integral surface at every point (x,y,u), called the *characteristic direction* or *Monge axis*.

Since the vector $\overline{A}$ is tangent to the solution surface at every point (x,y,u), we have the condition

$$\overline{A}\times d\overline{s} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ dx & dy & du \end{vmatrix} = 0$$

where $d\bar{s}\,(= dx\hat{\imath} + dy\hat{\jmath} + du\hat{k})$ is an elemental length along $\overline{A}$ on the solution surface. Expanding the determinant we obtain the following equation:

$$(b\,du - c\,dy)\hat{\imath} + (c\,dx - a\,du)\hat{\jmath} + (a\,dy - b\,dx)\hat{k} = 0$$

which leads to what is called the *Lagrange–Charpit equations*:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \tag{8}$$

The solution of the quasilinear equation can therefore be expressed by the description of the tangent plane in terms of the slope of this surface; so we may write

$$\frac{du}{dx} = \frac{c}{a} \qquad \text{and} \qquad \frac{du}{dy} = \frac{c}{b} \tag{9a}$$

or, equivalently, one of this pair together with

$$\frac{dx}{dy} = \frac{a(x,y,u)}{b(x,y,u)} \tag{9b}$$

This last equation (9b) defines a family of curves (but dependent on u) in (x,y,u) space that sit in the solution surface. These curves are usually called *characteristics* (after Cauchy); and the set of equations (9) is usually called the *characteristic equations* of the quasilinear PDE (6). The projection of a characteristic curve on the (x,y) plane $(u = 0)$ is called a *characteristic base curve*, or *projected characteristic curve*, or simply *characteristic*.

It may be noted that there are only two independent ordinary differential equations in the system (9) and thus, solving these equations gives a two-parameter family of characteristic curves in (x,y,u) space that can be expressed in the form

$$F(x,y,u,A,B) = 0, \qquad G(x,y,u,A,B) = 0$$

Through each point (x,y,u) in space there is a unique characteristic curve and a tangent vector (a,b,c) to every such point (Figure 1.3). Any smooth surface composed of characteristic curves is a solution of PDE (6). Such surfaces can be found analytically by specifying B as a function of A, $B = f(A)$. This leads a one-parameter family of characteristic curves, a surface,

$$F[x,y,u,A,f(A)] = 0, \qquad G[x,y,u,A,f(A)] = 0$$

The equation of the surface is found implicitly or explicitly by eliminating A between these equations. Here is an example to illustrate these ideas.

Example 1

Find the general solution of the quasilinear PDE

$$a\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial t} = c$$

where

$$b = 1, \qquad c = 0$$

Using equation (9b), we have the reciprocal of the slope of characteristic curves

$$\frac{dx}{dt} = \frac{a}{b} = a$$

Separating the variables and integrating to obtain

$$x = at + A$$

where A is an arbitrary constant. Further, from (9a), we have

$$\frac{du}{dt} = \frac{c}{b} = 0$$

which leads to

$$u = B$$

where B is an arbitrary constant. Thus,

$$x - at = A, \qquad u = B$$

is a two-parameter family of characteristic curves. Specifying B as a function of A defines a one-parameter family of characteristic curves, a solution surface. Thus, the general solution is expressed by writing $B = f(A)$. Therefore, the general solution is

$$u(x,y) = f(x - at)$$

where $f(\cdot)$ is an arbitrary function. By direct substitution, it is east to see that we do indeed have a solution of the equation for arbitrary f.

Example 2

Find the general solution of the quasilinear PDE

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

where

$$a = x, \qquad b = y, \qquad c = u$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$$

Using the first of Lagrange–Charpit equation, we have the reciprocal of the slope of characteristic curves

$$\frac{dx}{dy} = \frac{x}{y}$$

Separating the variables and integrating to obtain

$$\ln x = \ln y + \ln A \qquad \Longrightarrow \qquad x = Ay$$

where A is an arbitrary constant. Further, we have

$$\frac{du}{dy} = \frac{u}{y}$$

Separating the variables and integrating to obtain

$$\ln u = \ln y + \ln B \qquad \Longrightarrow \qquad u = By$$

where B is an arbitrary constant. Thus,

$$\frac{x}{y} = A, \qquad \frac{u}{y} = B$$

is a two-parameter family of characteristic curves. Specifying B as a function of A defines a one-parameter family of characteristic curves, a solution surface. Thus, the general solution is expressed by writing $B = f(A)$. Therefore, the general solution is

$$u(x,y) = yf\left(\frac{x}{y}\right)$$

where $f(\cdot)$ is an arbitrary function. By direct substitution, it is east to see that we do indeed have a solution of the equation for arbitrary f.

It may be noted that had we selected the equations of the form

$$\frac{dy}{dx} = \frac{y}{x}, \qquad \frac{du}{dx} = \frac{u}{x}$$

we would have obtained the general solution as

$$u(x,y) = xg\left(\frac{y}{x}\right)$$

where $g(\cdot)$ is an arbitrary function. This is essentially the same as the previous form of the solution.

Example 3

Find the general solution of the quasilinear PDE

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = (x+y)u$$

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

where

$$a = x^2, \qquad b = y^2, \qquad c = (x+y)u$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u}$$

Using the first of Lagrange–Charpit equation, we have the reciprocal of the slope of characteristic curves

$$\frac{dx}{dy} = \frac{x^2}{y^2}$$

Separating the variables and integrating to obtain

$$x^{-1} = y^{-1} + A'$$

where A' is an arbitrary constant. In addition, when we subtract the equation $dy = y^2\dfrac{du}{(x+y)u}$ from $dx = x^2\dfrac{du}{(x+y)u}$ to obtain

$$dx - dy = (x^2 - y^2)\frac{du}{(x+y)u} \qquad \Longrightarrow \qquad \frac{dx - dy}{x-y} = \frac{du}{u} \qquad \Longrightarrow \qquad u = B(x-y)$$

where B is an arbitrary constant. Thus,

$$\frac{xy}{x-y} = A, \qquad \frac{u}{x-y} = B$$

is a two-parameter family of characteristic curves. Specifying B as a function of A defines a one-parameter family of characteristic curves, a solution surface. Thus, the general solution is expressed by writing $B = f(A)$. Therefore, the general solution is

$$u(x,y) = (x-y)\,f\left(\frac{xy}{x-y}\right)$$

where $f(\cdot)$ is an arbitrary function.

Example 4

Find the general solution of the Cauchy problem governed by the quasilinear PDE

$$2y\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 2yu^2$$

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

where

$$a = 2y, \qquad b = u, \qquad c = 2yu^2$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{2y} = \frac{dy}{u} = \frac{du}{2yu^2}$$

Using the first of Lagrange–Charpit equation, we have the reciprocal of the slope of characteristic curves

$$\frac{dx}{dy} = \frac{2y}{u}$$

Further, we have

$$\frac{du}{dy} = \frac{2yu^2}{u} = 2yu \quad (\text{for } u \neq 0)$$

Separating the variables and integrating to obtain

$$\int \frac{du}{u} = 2\int y dy + \ln B \qquad \Longrightarrow \qquad \ln u = y^2 + \ln B$$

where B is an arbitrary constant. The above equation can be rewritten as

$$u = Be^{y^2}$$

Plugging the value of u in the expression for dx/dy yields

$$\frac{dx}{dy} = \frac{2y}{Be^{y^2}}$$

which on integration

$$B\int dx = \int 2ye^{-y^2} dy + A \qquad \Longrightarrow \qquad Bx = -e^{-y^2} + A$$

where A is the second arbitrary constant. Thus,

$$Bx + e^{-y^2} = A, \qquad u = Be^{y^2}$$

is a two-parameter family of characteristic curves. Specifying B as a function of A defines a one-parameter family of characteristic curves, a solution surface. Thus, the general solution is expressed by writing $B = f(A)$, where

$$A = Bx + e^{-y^2} = ue^{-y^2}x + e^{-y^2} = (xu+1)e^{-y^2}$$

which gives

$$u(x,y) = e^{y^2} f\left[(1+xu)e^{-y^2}\right]$$

an implicit relation for $u(x,y)$, where $f(\cdot)$ is an arbitrary function.

Remark: That we have obtained an implicit, rather than explicit, representation of the solution is to be expected: the original PDE is nonlinear. In the light of this complication, it is a useful exercise to confirm, by direct substitution, that we do indeed have a solution of the equation for arbitrary f; this requires a little care.

We summarize these ideas in the following theorem: *Every one-parameter family of characteristic curves generates a solution surface to PDE* (6)*. Conversely, every solution surface may be considered as a one-parameter family of characteristic curves.*

2.1 Parametric form of characteristic equations

It is often convenient to solve the characteristic equations (9) and the associated Cauchy problem if various curves in the problem are represented in parametric form. Let

$$x = x(s), \qquad y = y(s), \qquad u = u(s)$$

be the parametric equations of C in terms of the variable s. With the introduction of the parameter s, we may write the Lagrange–Charpit equations (8) as

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} = ds$$

Then the set of equations may be written as a system of ODE for $x(s)$, $y(s)$, and $u(s)$:

$$\frac{dx}{ds} = a \tag{10a}$$

$$\frac{dy}{ds} = b \tag{10b}$$

$$\frac{du}{ds} = c \tag{10c}$$

where s actually gives a measure of the distance (arc length) along the curve. The system of ODE (10) is called the characteristic equations of the PDE (6) in parametric form. The family of curves $x = x(s)$, $y = y(s)$, $u = u(s)$ determined by solving the system of ODEs (10), are called the *characteristic curves* of the original PDE (6).

The existence and uniqueness theory of ODEs, assuming that certain smoothness conditions on the coefficients a, b, and c, guarantees that exactly one solution curve $(x(s),y(s),u(s))$ of

(10) (i.e., a characteristic curve) passes through a given point (x_0, y_0, u_0) in (x, y, u) space. Often, we are not interested in the determination of a general solution of the PDE (6) but rather a specific solution $u = u(x,y)$ along a given curve Γ that passes through a given point (x,y), typically (x_0, y_0) when $s = 0$.

2.2 A general case of first-order quasilinear PDE

Consider a general quasilinear partial differential equation of the form:

$$\sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n, u) \frac{\partial u}{\partial x_i} = c(x_1, x_2, \ldots, x_n, u) \tag{11}$$

The parametrized form of characteristics curves of (11) are obtained by solving the following system of ODEs

$$\frac{dx_i}{ds} = a_i(x_1, x_2, \ldots, x_n, u) \tag{12a}$$

$$\frac{du}{ds} = c(x_1, x_2, \ldots, x_n, u) \tag{12b}$$

where s is the parametric variable. Equations (12a) and (12b) are the characteristic equations of the PDE (11).

3 One-dimensional linear advection equation

We will now illustrate the solution of one-dimensional linear advection equation also called one-way (or unidirectional) wave equation

$$a\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \tag{13a}$$

using method of characteristics written in parametric form. Let the initial data (waveform) of u is given in the form:

$$u(x,0) = F(x) \tag{13b}$$

The value of function u at an arbitrary point ξ on the initial curve ($t = 0$ in this case) is then given by $u(\xi, 0) = F(\xi)$. The curve starts at the initial point $(x = \xi, t = 0)$, when $s = 0$, where the parameter s gives a measure of the distance along the curve.

From equation (12), we see that the characteristic equations of (13a) is given by

$$\frac{dt}{ds} = 1 \quad \text{with} \quad t(0) = 0 \tag{14a}$$

$$\frac{dx}{ds} = a \quad \text{with} \quad x(0) = \xi \tag{14b}$$

$$\frac{du}{ds} = 0 \quad \text{with} \quad u(0) = F(\xi) \tag{14c}$$

These equations can be solved to get the equation of characteristic curves as explained below. Solving equation (14a) produces

$$t = s + c_1$$

where c_1 is the integration constant. Using the initial condition $t(s=0)=0$, we find that $c_1 = 0$. Therefore, we have

$$s = t$$

The solution of (14b) is given by

$$x = as + c_2$$

where c_2 is a constant. Using initial condition $x(s=0) = \xi$, we get $c_2 = \xi$. Therefore, the characteristic curve along which the PDE simplifies to an ODE is given by

$$x = at + \xi \tag{15}$$

where we have used the result that $s = t$. Equation (15) represents a straight line in (x,t) plane with a slope

$$\frac{dt}{dx} = \frac{1}{a}$$

Since the characteristics are straight lines with constant slope, they do not intersect in (x,t) plane.

Now the solution of (14c) is given by

$$u = c_3$$

where c_3 is a constant. Using initial condition $u(0) = F(\xi)$, we get $c_3 = F(\xi)$. Thus we see that u is a constant along the characteristic line. The value of the constant is $F(\xi)$. The constancy of u along the characteristics combined with (15) shows that the characteristics propagate with constant speed a. Since u is constant along the characteristics, we have

$$u(x,t) = u(\xi,0) = F(\xi) \tag{16}$$

It means that the value of $F(\xi)$ on the initial data line ($t = 0$ line) uniquely determine the value of u at every point of every characteristic curve that issues from $t = 0$ line. So, one can obtain the value of $u(x,t)$ at any point $P(x,t)$ by tracing it back to the x-axis along the characteristics (see figure 3):

$$u(x,t) = u(x-at, t-t) = u(\xi,0) = F(\xi)$$

Since $\xi = x - at$, the solution of the PDE (6) along the characteristic curve (straight line in this case) is given by

$$u(x,t) = F(x-at) \tag{17}$$

As mentioned earlier, the initial profile $F(x)$ translates unaltered with speed a along the characteristics. Let us now consider the solution of equation (13) for various initial conditions.

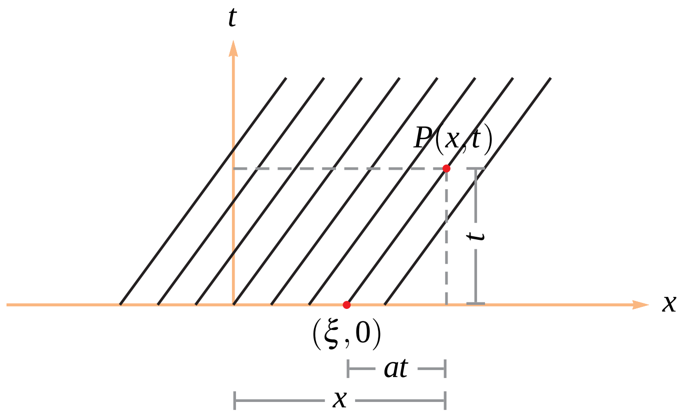

Figure 3: Characteristics solution of linear advection equation (13a).

Example 1

Find the solution of the Cauchy problem governed by the linear PDE

$$a\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

that takes on the values

$$F(x) = u(x,0) = \begin{cases} -\dfrac{x}{3} & \text{if } x \leq 0 \\ 2x+3 & \text{if } x > 0 \end{cases}$$

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial t} = c$$

where

$$b = 1, \qquad c = 0$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{a} = \frac{dt}{1} = \frac{du}{0}$$

Using the Lagrange–Charpit equation, we have the reciprocal of the slope of characteristic curves

$$\frac{dx}{dt} = a$$

Separating the variables and integrating to obtain

$$x = at + A$$

where A is an arbitrary constant. Further, we have

$$\frac{du}{dt} = 0$$

which leads to

$$u = B$$

where B is an arbitrary constant. Note that for selecting the second equation above, we have two choices; they are du/dt and du/dx. We select du/dt if the Cauchy data is specified on $t = 0$ line as a function of x. On the other hand, du/dx is selected if the Cauchy data is specified on $x = 0$ line as a function of t.

Thus,

$$x - at = A, \qquad u = B$$

is a two-parameter family of characteristic curves. For solution curves to pass through the initial data, $F(x) = u(x,0) = -x/3$ for $x \leq 0$, we set

$$\xi = A, \qquad -\frac{\xi}{3} = B \qquad \Longrightarrow \qquad A = \xi, \qquad B = -\frac{\xi}{3}$$

where ξ is a constant (x-intercept, in this case) that identifies a characteristic curve. Thus, the characteristic and solution curves through this part of the initial curve are

$$x = at + \xi, \qquad u = -\frac{\xi}{3}$$

Eliminating ξ from the second equation using the first yields

$$u(x,t) = -\frac{1}{3}(x - at) = \frac{1}{3}(at - x)$$

We could also obtain the general solution first and then use the Cauchy data to obtain the particular solution. The general solution is expressed by writing $B = f(A)$ as follows

$$u(x,t) = f(x - at)$$

where $f(\cdot)$ is an arbitrary function. For solution curves to pass through the initial data, $u(x,0) = -x/3$ for $x \leq 0$, we set

$$-\frac{x}{3} = f(x) \qquad \Longrightarrow \qquad f(x - at) = -\frac{x - at}{3}$$

Therefore the solution of the PDE for $x \leq 0$ is

$$u(x,t) = f(x - at) = \frac{1}{3}(at - x)$$

For solution curves to pass through the initial data, $F(x) = u(x,0) = 2x + 3$ for $x > 0$, we set

$$\xi = A, \qquad 2\xi + 3 = B \qquad \Longrightarrow \qquad A = \xi, \qquad B = 2\xi + 3$$

Thus, the characteristic and solution curves through this part of the initial curve are

$$x = at + \xi, \qquad u = 2\xi + 3$$

Eliminating ξ from the second equation using the first yields

$$u(x,t) = 2(x-at)+3$$

Again, we could obtain the general solution first and then use the Cauchy data to obtain the particular solution. The general solution is expressed by writing $B = f(A)$ as follows

$$u(x,t) = f(x-at)$$

where $f(\cdot)$ is an arbitrary function. For solution curves to pass through the initial data, $u(x,0) = 2x+3$ for $x > 0$, we set

$$2x+3 = f(x) \qquad \Longrightarrow \qquad f(x-at) = 2(x-at)+3$$

Therefore the solution of the PDE for $x > 0$ is

$$u(x,t) = f(x-at) = 2(x-at)+3$$

The solution surface is composed of two planes, and to determine regions in the (x,t) plane onto which these planes project, we draw base characteristic curves. They are the lines $x = at + \xi$ shown in Figure 1.10a. Below the characteristic curve $x = at$ are characteristic curves along which $u = -\xi/3$; along characteristic curves above $x = at$, $u = 2\xi + 1$. The solution surface in Figure 1.10b, consists of two planes above the regions corresponding to these two sets of characteristic curves. It is discontinuous along the base characteristic curve $x = at$ through the point $(0,0)$ where the initial data is discontinuous.

Example 2

Find the solution surface for the linear PDE

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

subject to the Cauchy condition that $u = \sin x$ on $y = 3x+1$.

Solution The Cauchy condition in this case is prescribed on an oblique straight line C. The characteristic equations of the PDE in nonparametric form is given by

$$\frac{dx}{dy} = \frac{1}{2}$$

$$\frac{du}{dy} = 0$$

These equations are now solved to get the equation of characteristic curves. Integrating gives

$$x = \frac{y}{2} + A, \qquad u = B$$

where A and B are arbitrary constants that identifies the characteristics. In fact, A is the x-intercept (denoted by ξ) of the characteristics. The y-intercept of the characteristic line η is then equal to 2ξ. Therefore,

$$x = \frac{y}{2} + \xi, \qquad u = B$$

For solution curves to pass through the initial data $u = \sin x$ on $y = 3x+1$, we set

$$x = \frac{3x+1}{2} + \xi, \qquad \sin x = B$$

which leads to

$$x = -2\xi - 1, \qquad B = \sin(-2\xi - 1)$$

on $y = 3x+1$ line. Thus the characteristic and solution curves through the initial curve are

$$x = \frac{y}{2} + \xi, \qquad u = \sin(-2\xi - 1)$$

Eliminating ξ from the second equation using the first yields

$$u(x,y) = \sin(y - 2x - 1)$$

Differentiation confirms that $u(x,y)$ satisfies the PDE, and it also satisfies the Cauchy condition on the line $y = 3x+1$, so it is the required solution.

We could have also obtained the general solution first and then use the Cauchy data to obtain the particular solution. The general solution is expressed by writing $B = f(A)$ as follows

$$u(x,y) = f\left(x - \frac{y}{2}\right)$$

where $f(\cdot)$ is an arbitrary function. For solution curves to pass through the initial data $u = \sin x$ on $y = 3x+1$, we set

$$\sin x = f\left(\frac{-x-1}{2}\right) \quad \Longrightarrow \quad \sin(x-1) = f(-x/2) \quad \Longrightarrow \quad f(x) = \sin(-2x-1)$$

Therefore the solution of the PDE is

$$u(x,t) = f\left(x - \frac{y}{2}\right) = \sin\left[-2\left(x - \frac{y}{2}\right) - 1\right] = \sin(y - 2x - 1)$$

3.1 Inhomogeneous one-dimensional linear advection equation

Next we consider the case of a one-dimensional inhomogeneous linear advection equation, which has a nonzero right-hand side:

$$a\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = c \tag{18a}$$

The initial profile of u is

$$u(x,0) = F(x) \tag{18b}$$

The value of u at a point ξ on the initial profile is given by $u(\xi, 0) = F(\xi)$. The curve starts at the initial point $(x = \xi)$, when $t = 0$.

The characteristic equations of (18a) in nonparametric form is given by

$$\frac{dx}{dt} = a \quad \text{with} \quad x(0) = \xi \tag{19a}$$

$$\frac{du}{dt} = c \quad \text{with} \quad u(0) = F(\xi) \tag{19b}$$

These equations are now solved to get the equation of characteristic curves. The solution of (19a) using initial condition $x(t=0) = \xi$ is given by

$$x = at + \xi \tag{20}$$

Equation (20) represents a straight line in (x,t) plane. Hence, characteristics from a family of non-intersecting straight lines with same slopes. Finally, the solution of (19b) using initial condition $u(t=0) = F(\xi)$ is given by

$$u = ct + F(\xi)$$

Since $\xi = x - at$, the solution of the PDE (18a) along the characteristic lines is given by

$$u(x,t) = F(x - at) + ct \tag{21}$$

Note that the value of u does not remain constant along the characteristics. In fact, it increases linearly with time from the initial value of $F(\xi)$.

Example 3

Find the solution surface for the linear PDE

$$3\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} = 10$$

that contains the lines $y = 2x$, $u = 2x/5$. Show that the projection of the initial curve in the (x,y) plane is nowhere tangent to a base characteristic curve.

Solution The Lagrange–Charpit equations (8) for the PDE are

$$\frac{dx}{3} = \frac{dy}{4} = \frac{du}{10}$$

From this, we have

$$\frac{dx}{dy} = \frac{3}{4}, \qquad \frac{du}{dy} = \frac{10}{4}$$

Integration of these gives characteristic curves

$$x = \frac{3}{4}y + A, \qquad u = \frac{5}{2}y + B$$

where A and B are arbitrary constants. From the first of the above equation, we have

$$x - \frac{3}{4}y = A$$

Specifying B as a function of A gives a solution surface,

$$u = \frac{5}{2}y + f\left(x - \frac{3}{4}y\right)$$

where $f(\cdot)$ is an arbitrary function. For solution curves to pass through the initial data, $y = 2x$, $u = 2x/5$, we set

$$\frac{2}{5}x = 5x + f\left(x - \frac{3}{4}2x\right) \quad \Longrightarrow \quad -\frac{23}{5}x = f\left(-\frac{x}{2}\right) \quad \Longrightarrow \quad f(x) = \frac{46}{5}x$$

Therefore the solution of the PDE is

$$u(x,t) = \frac{5}{2}y + f\left(x - \frac{3}{4}y\right) = \frac{5}{2}y + \frac{46}{5}\left(x - \frac{3}{4}y\right) = \frac{46}{5}x - \frac{22}{5}y$$

The solution surface is a plane defined for all x and y. Base characteristic curves are straight lines $y = 4x/3 + A$ with slope $4/3$. Since the projection of the initial curve in the (x,y) plane is the line $y = 2x$ with slope 2, it is nowhere tangent to a base characteristic curve.

Example 4

Find the solution of the Cauchy problem governed by the linear PDE

$$\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial y} = 2xu$$

subject to the conditions (a) $u(x,0) = x^2$ for all x and (b) $u(0,y) = y^2$ for all y.

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

where

$$a = 1, \qquad b = 2x, \qquad c = 2xu$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{1} = \frac{dy}{2x} = \frac{du}{2xu}$$

(a) Using the Lagrange–Charpit equation, we have the reciprocal of the slope of characteristic curves

$$\frac{dx}{dy} = \frac{1}{2x}$$

Separating the variables and integrating to obtain

$$x^2 = y + A$$

where A is an arbitrary constant. Further, we have

$$\frac{du}{dy} = u$$

which leads to

$$u = Be^y$$

where B is an arbitrary constant. Thus,

$$x^2 - y = A, \qquad ue^{-y} = B$$

is a two-parameter family of characteristic curves. For solution curves to pass through the initial data, $F(x) = u(x,0) = x^2$, we set

$$\xi^2 = A, \qquad \xi^2 = B \qquad \Longrightarrow \qquad A = \xi^2, \qquad B = \xi^2$$

where ξ is a constant (x-intercept, in this case) that identifies a characteristic curve. Thus, the characteristic and solution curves through this part of the initial curve are

$$x^2 = y + \xi^2, \qquad u = \xi^2 e^y$$

Eliminating ξ from the second equation using the first yields

$$u(x,y) = (x^2 - y)e^y$$

We could have also obtained the general solution first and then use the Cauchy data to obtain the particular solution. The general solution is expressed by writing $B = f(A)$ as follows

$$u(x,y) = e^y f(x^2 - y)$$

where $f(\cdot)$ is an arbitrary function. For solution curves to pass through the initial data, $u(x,0) = x^2$, we set

$$x^2 = f(x^2) \qquad \Longrightarrow \qquad f(x^2) = x^2$$

Therefore the solution of the PDE is

$$u(x,t) = e^y f(x^2 - y) = e^y(x^2 - y)$$

(b) Using the Lagrange–Charpit equation, we have the reciprocal of the slope of characteristic curves

$$\frac{dy}{dx} = 2x$$

Separating the variables and integrating to obtain

$$y = x^2 + A$$

where A is an arbitrary constant. Further, we have

$$\frac{du}{dx} = 2xu$$

which leads to

$$u = Be^{x^2}$$

where B is an arbitrary constant. Thus,

$$y - x^2 = A, \qquad ue^{-x^2} = B$$

is a two-parameter family of characteristic curves. For solution curves to pass through the initial data, $F(y) = u(0,y) = y^2$, we set

$$\eta = A, \qquad \eta^2 = B \qquad \Longrightarrow \qquad A = \eta, \qquad B = \eta^2$$

where η is a constant (y-intercept, in this case) that identifies a characteristic curve. Thus, the characteristic and solution curves through this part of the initial curve are

$$y = x^2 + \eta, \qquad u = \eta^2 e^{x^2}$$

Eliminating η from the second equation using the first yields

$$u(x,y) = (y-x^2)^2 e^{x^2}$$

It is a simple matter to verify that this is the required solution.

Example 5

Find the solution of the Cauchy problem governed by the semilinear PDE

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xe^{-u}$$

subject to the Cauchy data $u = 0$ on $y = x^2$.

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

where

$$a = x, \qquad b = y, \qquad c = xe^{-u}$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{xe^{-u}}$$

Using the Lagrange–Charpit equation, we have the reciprocal of the slope of characteristic curves

$$\frac{dy}{dx} = \frac{y}{x}$$

Separating the variables and integrating to obtain

$$y = Ax$$

where A is an arbitrary constant. Further, we have

$$\frac{du}{dx} = e^{-u}$$

which leads to

$$e^u = x + B$$

where B is an arbitrary constant. Thus,

$$\frac{y}{x} = A, \qquad e^u - x = B$$

is a two-parameter family of characteristic curves. The general solution is expressed by writing $B = f(A)$ as follows

$$e^u - x = f\left(\frac{y}{x}\right)$$

where $f(\cdot)$ is an arbitrary function. Applying the Cauchy data $u = 0$ on $y = x^2$

$$1 - x = f(x) \qquad \Longrightarrow \qquad f\left(\frac{y}{x}\right) = 1 - \frac{y}{x}$$

Therefore the solution of the PDE is

$$e^u - x = 1 - \frac{y}{x}$$

or

$$u(x,y) = \ln\left(x + 1 - \frac{y}{x}\right)$$

Example 6

Find the solution surface for the semilinear PDE

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = u$$

in the first quadrant that takes on values x^3 along the positive x-axis and values y^3 along the positive y-axis.

Solution The Cauchy conditions are given by $u(x,0) = x^3$ and $u(0,y) = y^3$. Characteristic equations (9) for the PDE are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{u}$$

The first two of these give

$$\frac{dy}{dx} = \frac{x}{y} \qquad \Longrightarrow \qquad y^2 = x^2 + A$$

In addition, when we add the equations $dx = y\dfrac{du}{u}$ and $dy = x\dfrac{du}{u}$ to obtain

$$dx + dy = (x+y)\frac{du}{u} \qquad \Longrightarrow \qquad \frac{dx+dy}{x+y} = \frac{du}{u} \qquad \Longrightarrow \qquad u = B(x+y)$$

Characteristic curves in the first quadrant are the hyperbolas $y^2 - x^2 = A$ in Figure 1.11. The characteristic curve $y = x(A = 0)$ separates the first quadrant into two regions R_1 and R_2 corresponding to base characteristic curves that have $A < 0$ and $A > 0$, respectively. Solution surfaces to the PDE are obtained by specifying B as a function of A, i.e., $B = f(A)$,

$$y^2 - x^2 = A, \qquad u = f(A)(x+y)$$

To find $B = f(A)$ so that the solution surface contains the initial curve in region R_1, we use the initial condition $u(x,0) = x^3$,

$$-x^2 = A, \qquad x^3 = f(A)x \qquad \Longrightarrow \qquad f(A) = -A$$

In region R_1 then, the solution surface is given by

$$y^2 - x^2 = A, \qquad u = -A(x+y)$$

Eliminating A from the second equation using the first yields

$$u(x,y) = (x^2 - y^2)(x+y)$$

To find $B = f(A)$ so that the solution surface contains the initial curve in region R_2, we use the initial condition $u(0,y) = y^3$,

$$y^2 = A, \qquad y^3 = f(A)y \qquad \Longrightarrow \qquad f(A) = A$$

In region R_2 then, the solution surface is given by

$$y^2 - x^2 = A, \qquad u = A(x+y)$$

Eliminating A from the second equation using the first yields

$$u(x,y) = (y^2 - x^2)(x+y)$$

The solution is continuous for all x and y, even across the characteristic curve $y = x$ that separates regions R_1 and R_2. However, derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are discontinuous across the characteristic curve $y = x$ through $(0,0)$ as shown below. In region R_1,

$$\frac{\partial u}{\partial x} = (x^2 - y^2) + 2x(x+y) = (x+y)(3x - y)$$

$$\frac{\partial u}{\partial y} = (x^2 - y^2) - 2y(x+y) = (x+y)(x-3y)$$

whereas in region R_2,

$$\frac{\partial u}{\partial x} = (y^2 - x^2) - 2x(x+y) = (x+y)(y-3x)$$

$$\frac{\partial u}{\partial y} = (y^2 - x^2) + 2y(x+y) = (x+y)(3y-x)$$

If points (x,y) on $y = x$ are approached from region R_1,

$$\frac{\partial u}{\partial x} = 2x(2x) = 4x^2, \qquad \frac{\partial u}{\partial y} = 2x(-2x) = -4x^2$$

If points (x,y) on $y = x$ are approached from region R_2,

$$\frac{\partial u}{\partial x} = 2x(-2x) = -4x^2, \qquad \frac{\partial u}{\partial y} = 2x(2x) = 4x^2$$

Thus, although $u(x,y)$ itself is continuous, its derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are discontinuous across the characteristic curve $y = x$ through $(0,0)$.

Example 7

Solve the quasilinear PDE

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = u$$

subject to the Cauchy condition $u(x,0) = 2x$ for $1 \le x \le 2$.

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial t} = c$$

where

$$a = u, \qquad b = 1, \qquad c = u$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{u} = \frac{dt}{1} = \frac{du}{u}$$

The last two of these give

$$\frac{du}{dt} = u$$

which leads to

$$u = Be^t$$

where B is an arbitrary constant. Using the initial condition $u(x,0) = 2x$ for $1 \le x \le 2$, we get

$$2\xi = B$$

Therefore, the solution may be written as

$$u = 2\xi e^t$$

We also have

$$\frac{dx}{dt} = u$$

Using the result $u = 2\xi e^t$ in the above equation becomes

$$\frac{dx}{dt} = 2\xi e^t$$

which leads to

$$x = 2\xi e^t + B$$

where B is an arbitrary constant. Using the initial condition $u(x,0) = 2x$ for $1 \leq x \leq 2$ again to obtain

$$\xi = B$$

Therefore, the equation of the characteristic curve is given by

$$x = \xi\left(2e^t + 1\right)$$

Eliminate ξ from the equation for u using the above equation to obtain

$$u(x,t) = \frac{2xe^t}{(2e^t+1)} \qquad \text{for} \qquad 1 \leq x \leq 2$$

It is a simple matter to verify that this is the required solution.

Example 8

Find the solution of the Cauchy problem governed by the quasilinear PDE

$$(u+y)\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x - y$$

passing through the initial curve $x = y - 1$, $u = y^2 + 1$.

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

where

$$a = u + y, \qquad y = 1, \qquad c = x - y$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{u+y} = \frac{dy}{y} = \frac{du}{x-y}$$

The first two of these give

$$\frac{dx}{dy} = \frac{u+y}{y} = \frac{u}{y} + 1$$

In addition, when we subtract the equation $dy = \dfrac{y}{x-y}du$ from $dx = \dfrac{u+y}{x-y}du$ to obtain

$$dx - dy = \frac{u}{x-y}du \qquad \Longrightarrow \qquad (x-y)d(x-y) = udu \qquad \Longrightarrow \qquad u^2 = (x-y)^2 + B$$

where B is an arbitrary constant. Using the initial condition $x = y-1$, $u = y^2+1$, we get

$$(\eta^2+1)^2 = 1+B \qquad \Longrightarrow \qquad \eta = C$$

where $C^2 = \sqrt{B+1}-1$ is an arbitrary constant. Therefore, we have

$$u^2 = (x-y)^2 + (\eta^2+1)^2 - 1$$

Plugging the value of u in the expression for dx/dy yields

$$\frac{dx}{dy} = \frac{\sqrt{(x-y)^2+B}}{y} + 1$$

This is a nonlinear nonstandard differential equation whose solution can be obtained only numerically.

References

1. Bleecker, D. D. and G. Csordas, *Basic Partial Differential Equations*, Van Nostrand Reinhold (1992).

2. Jeffrey, A., *Applied Partial Differential Equations: An Introduction*, Academic Press (2003).

3. Kevorkian, J., *Partial Differential Equations: Analytical Solutions Techniques*, 2nd ed., Springer (1999).

4. Myint-U, T. and L. Debnath, *Linear Partial Differential Equations for Scientists and Engineers*, 4th ed., Birkhauser (2006).

5. Olver, P. J., *Introduction to Partial Differential Equations*, Springer (2014).

6. Pinchover, Y. and J. Rubinstein, *An Introduction to Partial Differential Equations*, Cambridge Univ. Press (2005).

7. Prasad, P. and R. Ravindran, *Linear Partial Differential Equations*, Wiley Eastern (1985).