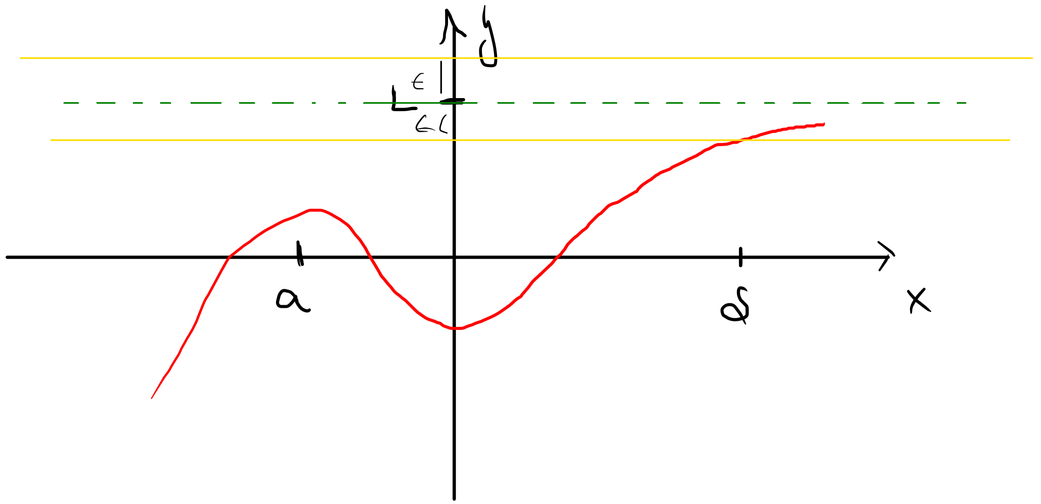
Monitoria - 17/05

- limites no infinito / limites infinitos
- limites laterais
- sequências

Seção 4.1 - Limites no infinito

Definição 1. Seja f uma função e suponhamos que exista a tal que $]a, +\infty[\subset D_f$. Definimos

$$\lim_{x \to +\infty} f(x) = L \Leftrightarrow \begin{cases} \forall \epsilon > 0, \exists \delta > 0, \text{ com } \delta > a, \text{ tal que} \\ x > \delta \Rightarrow L - \epsilon < f(x) < L + \epsilon. \end{cases}$$



Definição 2. Seja f uma função e suponhamos que exista a tal que $]-\infty, \alpha[\subset D_f$. Definimos

$$\lim_{x \to -\infty} f(x) = L \Leftrightarrow \begin{cases} \forall\, \epsilon > 0, \exists\, \delta > 0, \text{ com } -\delta < a, \text{ tal que} \\ x < -\delta \Rightarrow L - \epsilon < f(x) < L + \epsilon. \end{cases}$$

Teorema 1. Sejam f e g duas funções tais que $\operatorname{Im} f \subset D_g$ e $\lim\limits_{x \to +\infty} f(x) = a$

a) Se g for contínua em a, então

$$\lim_{x \to +\infty} g(f(x)) = \lim_{u \to a} g(u).$$

b) Se g não estiver definida em a e se $\lim\limits_{u \to a} g(u)$ existir, então

$$\lim_{x \to +\infty} g(f(x)) = \lim_{u \to a} g(u).$$

limite da fç composta

Teorema 2. Seja k uma constante e suponhamos que $\lim\limits_{x \to +\infty} f(x) = L$ e $\lim\limits_{x \to +\infty} g(x) = L_1$. Então

a) $\lim\limits_{x \to +\infty} [f(x) + g(x)] = L + L_1$.

b) $\lim\limits_{x \to +\infty} kf(x) = k \lim\limits_{x \to +\infty} f(x) = kL$.

c) $\lim\limits_{x \to +\infty} f(x)\, g(x) = LL_1$.

d) $\lim\limits_{x \to +\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{L_1}$, desde que $L_1 \neq 0$.

Observamos que os teoremas acima continuam válidos se substituirmos "$x \to +\infty$" por "$x \to -\infty$".

Importante:

$$\lim_{x \to +\infty} \frac{1}{x^n} = 0 \quad , \; n > 0 \text{ natural}$$

$$\lim_{x \to -\infty} \frac{1}{x^n} = 0$$

- $\lim_{x \to \pm\infty} \dfrac{1}{\sqrt[n]{x}} = 0$, $n > 0$ natural

$$\frac{1}{\sqrt[3]{x^2}} = \frac{1}{\sqrt[3]{x}} \cdot \frac{1}{\sqrt[3]{x}}$$

$$\frac{1}{\sqrt[3]{x}} \to 0 \qquad \frac{1}{\sqrt[3]{x}} \to 0$$

- $\lim_{x \to \pm\infty} \dfrac{1}{\sqrt[p]{x^q}} = 0$, $p, q > 0$ naturais

Exercícios (4.1)

1. Calcule

h) $$\lim_{x \to +\infty} \frac{5x^4 - 2x + 1}{4x^4 + 3x + 2} = \lim_{x \to +\infty} \frac{\cancel{x^4}}{\cancel{x^4}} \frac{\left(5 - \overbrace{\frac{2}{x^3}}^{\to 0} + \overbrace{\frac{1}{x^4}}^{\to 0}\right)}{\left(4 + \overbrace{3/x^3}^{\to 0} + \overbrace{2/x^4}^{\to 0}\right)} = \frac{5}{4}.$$

j) $$\lim_{x \to -\infty} \frac{2x^3 + 1}{x^4 + 2x + 3} = \lim_{x \to -\infty} \underbrace{\frac{\cancel{x^3}}{\cancel{x^4}}}_{} \left(\frac{2 + \overbrace{1/x^3}^{\to 0}}{1 + \overbrace{\frac{2}{x^3}}^{\to 0} + \overbrace{\frac{3}{x^4}}^{\to 0}} \right) = 0$$

$$\overbrace{\frac{1}{x}}^{\to 0}.$$

m) $\lim_{x \to -\infty} \sqrt[3]{\frac{x}{x^2+3}} = \lim_{x \to -\infty} \frac{x^{1/3}}{\sqrt[3]{x^2+3}} = \lim_{x \to -\infty} \frac{\cancel{x}^{1/3}}{x^{2/3}\sqrt[3]{1+\frac{3}{x^2}}} =$

$= \lim_{x \to -\infty} \frac{1}{x^{1/3}\sqrt[3]{1+3/x^2}} = 0$

p) $\lim_{x \to +\infty} \frac{\sqrt{x}+\sqrt[3]{x}}{x^2+3} = \lim_{x \to +\infty} \frac{x^{1/2}(x^{-1/6}+1)}{x^2(1+3/x^2)} = 0$.

$x^{-3/2} = \frac{1}{\sqrt{x^3}}$

Possível solução

- Quando quer fazer $\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)}$,

$P(x)$ e $Q(x)$ funções polinomiais ou escritas apenas com raízes, tirar "maior expoente" do numerador e do denominador.

Ex:

$$p(x) = \sqrt[3]{x} + \sqrt{x} = x^{1/2}\left(x^{-1/6} + 1 \right)$$

$$x^{1/2} \cdot (\quad) = x^{1/3}$$

$$(\quad) = x^{\frac{1}{3} - \frac{1}{2}} = x^{-1/6}$$

$$\frac{x^p}{x^q} = x^{p-q}$$

$\sqrt[3]{x^{2}+3}$: "maior expoente" : $x^{2/3}$

$\sqrt[5]{x^{7}+x^{2}+x-1}$: $x^{7/5}$

$$\lim_{x \to \pm\infty} \frac{x^5 + x^4 + \cdots + 3}{x^7 + 2x^6 - x^5 + \cdots + c} = 0$$

"x^2" "3" $$\frac{x^5 + 3x^4}{x^5 - x}$$

$$\lim_{x \to -\infty} \frac{x^4 + x}{x^2 - x}$$

s) $\lim\limits_{x \to +\infty} [\sqrt{x+1} - \sqrt{x+3}] = \lim\limits_{x \to +\infty} (\sqrt{x+1} - \sqrt{x+3}) \dfrac{(\sqrt{x+1} + \sqrt{x+3})}{(\sqrt{x+1} + \sqrt{x+3})}$

$(a+b)(a-b) = a^2 - b^2$

$= (x+1) - (x+3) = -2$

$$= \lim_{x \to +\infty} \frac{-2}{(\sqrt{x+1} + \sqrt{x+3})} = \lim_{x \to +\infty} \frac{1}{x^{1/2}} \cdot \left(\frac{-2}{\sqrt{1 + \frac{1}{x} \to 0} + \sqrt{1 + \frac{3}{x} \to 0}} \right)$$

$(*)$

$$= 0 \cdot (-1) = 0$$

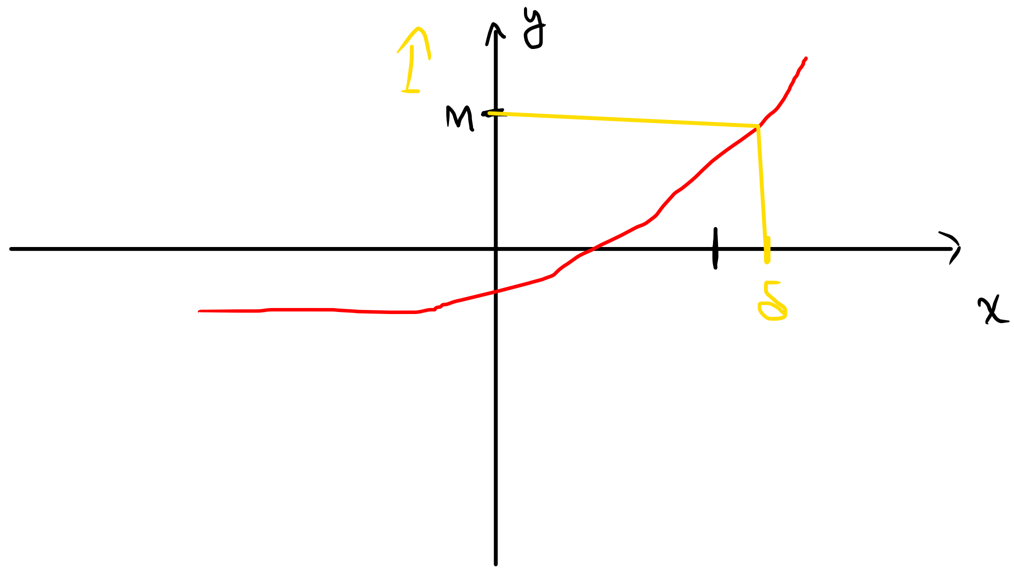
$(*)$ $\lim\limits_{x \to +\infty} \dfrac{1}{\sqrt{x}} = 0$ e $\lim\limits_{x \to \infty} \left(\dfrac{-2}{\sqrt{1 + \frac{1}{x} \to 0} + \sqrt{1 + \frac{3}{x} \to 0}} \right) = -1$

Seção 4.2 - Limites Infinitos

Definição 1. Suponhamos que exista a tal que $]a, +\infty[\subset D_f$. Definimos

$$(a) \quad \lim_{x \to +\infty} f(x) = +\infty \Leftrightarrow \begin{cases} \forall\, \epsilon > 0, \exists\, \delta > 0, \text{ com } \delta > a, \text{ tal que} \\ x > \delta \Rightarrow f(x) > \epsilon. \end{cases}$$

$$(b) \quad \lim_{x \to +\infty} f(x) = -\infty \Leftrightarrow \begin{cases} \forall\, \epsilon > 0, \exists\, \delta > 0, \text{ com } \delta > a, \text{ tal que} \\ x > \delta \Rightarrow f(x) < -\epsilon. \end{cases}$$



y
x
p

a) $\begin{cases} \lim\limits_{x \to +\infty} f(x) = +\infty \\ \lim\limits_{x \to +\infty} g(x) = +\infty \end{cases} \Rightarrow \begin{cases} \lim\limits_{x \to +\infty} [f(x) + g(x)] = +\infty \\ \lim\limits_{x \to +\infty} f(x)\, g(x) = +\infty \end{cases}$

b) $\begin{cases} \lim\limits_{x \to +\infty} f(x) = L,\ L \text{ real}, \\ \lim\limits_{x \to +\infty} g(x) = +\infty \end{cases} \Rightarrow \begin{cases} \lim\limits_{x \to +\infty} f(x)\, g(x) = +\infty & \text{se } L > 0 \\ \lim\limits_{x \to +\infty} f(x)\, g(x) = -\infty & \text{se } L < 0 \end{cases}$

c) $\begin{cases} \lim\limits_{x \to +\infty} f(x) = -\infty \\ \lim\limits_{x \to +\infty} g(x) = +\infty \end{cases} \Rightarrow \lim\limits_{x \to +\infty} f(x)\, g(x) = -\infty$

$+\infty + \infty = +\infty$

$\infty \cdot \infty = \infty$

$L + \infty = \infty$

$L \cdot (+\infty) = +\infty$, si $L > 0$

$-\infty$ _ se $L < 0$

$c > 0$

$x \to \infty$

$c\,\cancel{x}^{\nearrow +\infty} \cdot \cancel{\frac{1}{x}}^{\nearrow 0}$

$\cancel{x^2}^{\nearrow \infty} \cdot \cancel{\frac{1}{x}}^{\nearrow 0}$

$\sim$

$\cancel{x}^{\nearrow \infty}$

$x \cdot \frac{1}{x} = 1$

$$\lim_{x \to +\infty} \left(x \cdot \frac{1}{x} \right) = 1$$

$$\lim_{x \to +\infty} \left(cx \cdot \frac{1}{x} \right) = c$$

$$\cancel{x}^{\infty} \cdot \frac{1}{\cancel{x}^{2}}^{\,0}$$

$$\frac{1}{x} \rightarrow 0$$

d) $\begin{cases} \lim\limits_{x \to +\infty} f(x) = L, \; L \text{ real}, \\ \\ \lim\limits_{x \to +\infty} g(x) = +\infty \end{cases} \Rightarrow \lim\limits_{x \to +\infty} [f(x) + g(x)] = +\infty$

e) $\begin{cases} \lim\limits_{x \to +\infty} f(x) = L, \; L \text{ real}, \\ \\ \lim\limits_{x \to +\infty} g(x) = -\infty \end{cases} \Rightarrow \lim\limits_{x \to +\infty} [f(x) + g(x)] = -\infty$

f) $\begin{cases} \lim\limits_{x \to +\infty} f(x) = -\infty \\ \\ \lim\limits_{x \to +\infty} g(x) = -\infty \end{cases} \Rightarrow \begin{cases} \lim\limits_{x \to +\infty} [f(x) + g(x)] = -\infty \\ \\ \lim\limits_{x \to +\infty} f(x)\, g(x) = +\infty \end{cases}$

g) $\begin{cases} \lim\limits_{x \to +\infty} f(x) = L, \; L \text{ real}, \\ \\ \lim\limits_{x \to +\infty} g(x) = -\infty \end{cases} \Rightarrow \begin{cases} \lim\limits_{x \to +\infty} f(x)\, g(x) = -\infty & \text{se } L > 0 \\ \\ \lim\limits_{x \to +\infty} f(x)\, g(x) = +\infty & \text{se } L < 0. \end{cases}$

As propriedades continuam válidas substituindo "$x \to +\infty$" por "$x \to -\infty$".

<u>Obs</u>: $\lim\limits_{x \to +\infty} x^n = +\infty$, $\forall n > 0$ natural.

Exercícios (4.2)

1. Calcule.

b) $\lim_{x \to +\infty} (5 - 4x + x^2 - x^5)$

i) $\lim_{x \to -\infty} \dfrac{x^4 - 2x + 3}{3x^4 + 7x - 1}$

2.

b) $\lim_{x \to +\infty} \dfrac{x + \sqrt{x + 3}}{2x - 1}$

d) $\lim_{x \to +\infty} (x - \sqrt{3x^3 + 2})$

g) $\lim_{x \to +\infty} (\sqrt{x + \sqrt{x}} - \sqrt{x - 1})$

6. Dê exemplo de funções f e g tais que $\lim_{x \to +\infty} f(x) = +\infty$, $\lim_{x \to +\infty} g(x) = +\infty$ e $\lim_{x \to +\infty} [f(x) - g(x)] \neq 0$.

7. Dê exemplo de funções f e g tais que $\lim_{x \to +\infty} f(x) = +\infty$, $\lim_{x \to +\infty} g(x) = +\infty$ e $\lim_{x \to +\infty} \dfrac{f(x)}{g(x)} \neq 1$.