Def: Seja f tal que $f^{(k)}(a)$ existe $\forall k \leq n$.

Dizemos que

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

é o polinômio de Taylor de f de grau n.

Dizemos que $R_n(x) = f(x) - P_n(x)$ é erro do polinômio P_n em x.

Teo: (Polinômio de Taylor com resto de Lagrange). Seja f de classe $\mathcal{C}^{n+1}$ num intervalo aberto I tal que $x, a \in I$.

Então existe c entre a e x tal que

$$R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!},$$

aqui é c

Ou seja aqui é a

$$f(x) = \underbrace{\frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \dots \frac{f^{(n)}(a)(x-a)^n}{n!}}_{\text{Polinômio de grau } n} + \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$

Motivação para $\frac{f^{(n)}(a)}{n!}(x-a)^n$

Suponhamos que $a=0$,

$$f(x)=\sum_{n=0}^{+\infty} a_n(x-a)^n$$

$$\boxed{(x-a)^0=1}$$

e suponhamos que

$$\frac{d}{dx}\left(\sum_{n=0}^{+\infty} a_n(x-a)^n\right)=\sum_{n=0}^{+\infty}\frac{d}{dx}\left(a_n(x-a)^n\right)$$

i.e., iremos assumir que a série se comporta como um polinômio.

$$f(x) = \sum_{n=0}^{+\infty} a_n (x-a)^n$$

$$f'(x) = \sum_{n=1}^{+\infty} n a_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{+\infty} n(n-1) a_n (x-a)^{n-2}$$

$\vdots$

$$f^{(k)}(x) = \sum_{n=k}^{+\infty} n \cdots (n+1-k) a_n (x-a)^{n-k}$$

$\vdots$

$$f^{k}(a) = k \cdots (k+1-k) a_k + \sum_{n=k+1}^{+\infty} n \cdots (n+1-k) a_n 0^{n-k}$$

$$= k!\, a_k$$

$\therefore$

$$a_k \text{ "=" } \frac{f^{(k)}(a)}{k!}$$

$$f(x) \text{ "=" } \sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Ex:

$f(x) = e^x \quad f^{(n)}(x) = e^x$

$f^{(n)}(0) = 1$

$$\therefore \quad e^x \text{ "=" } \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

<u>Dem:</u> Seja G uma função derivável com $G'(t) \neq 0$ $\forall t$ entre a e x, contínua em $[a,x]$ se $a<x$ (ou $[x,a]$ se $x<a$)

e $F(t) = f(t) + \frac{f'(t)(x-t)}{1!} + \dots + \frac{f^{(n)}(t)(x-t)^n}{n!}$

F é derivável $\therefore$ pelo TVM de Cauchy, $\exists c$ entre a e x tal que

1) $\frac{F'(c)}{G'(c)} = \frac{F(x)-F(a)}{G(x)-G(a)}$

2) $F(x) - F(a) = (f(x) + 0 + \dots + 0) - \left(f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(n)}(a)(x-a)^n}{n!}\right)$

$= f(x) - P_n(x) = R_n(x)$.

3) $F'(t) = \cancel{f'(t)} + \left[\cancel{f'(t)(-1)} + \frac{f''(t)}{2!}(x-t)\right] + \left[-\frac{f^{(k)}(t)(x-t)^{k-1}}{(k-1)!} + \cancel{\frac{f^{(k+1)}(t)(x-t)^k}{k!}}\right]$

$$+ \left[\cancel{-\frac{f^{(k+1)}(t)(x-t)^k}{k!}} + \frac{f^{(k+2)}(t)(x-t)^{k+1}}{(k+1)!}\right] + \dots$$

$$\boxed{\begin{aligned} &\frac{d}{dt}\left(\frac{f^{(k)}(t)(x-t)^k}{k!}\right) \\ &= \frac{f^{(k+1)}(t)(x-t)^k}{k!} \\ &+ \frac{f^{(k)}(t)(x-t)^{k-1}\cdot(-k)}{k!} \end{aligned}}$$

$$+ \left[-\frac{f^{(n)}(t)(x-t)^{n-1}}{(n-1)!} + \frac{f^{(n+1)}(t)(x-t)^n}{n!}\right]$$

$$= \frac{f^{n+1}(t)(x-t)^n}{n!}$$

$$\boxed{F'(c) = \frac{f^{n+1}(c)(x-c)^n}{n!}}$$

4) $G(t) = -(x-t)^{n+1}$

$$G(x) - G(a) = (x-a)^{n+1}$$

$$G'(t) = (n+1)(x-t)^n \quad \therefore G'(c) = (n+1)(x-c)^n$$

Fazendo as substituições em 1) temos

$$5) \quad \frac{\frac{f^{(n+1)}(c)\cancel{(x-c)^n}}{n!}}{\cancel{(x-c)^n}(n+1)} = \frac{F'(c)}{G'(c)} = \frac{F(x)-F(a)}{G(x)-G(a)}$$

$$= \frac{R_n(x)}{(x-a)^{n+1}}$$

$$\therefore \boxed{R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}}$$

erro de Lagrange

Ex: $f(x) = e^x$ é C^∞.

$f^{(n)}(x) = e^x$

$\therefore f^{(n)}(0) = 1 \ \forall n \in \mathbb{N}.$

$$R_n(x) = \frac{e^c x^{n+1}}{(n+1)!}$$

$m > 2|x|$.

$$\left|\frac{e^c x^{n+1}}{(n+1)!}\right| \leq \frac{e^c |x|^m}{m!} \cdot \frac{|x|^{(n+1)-m}}{m \cdots (n+1)} \leq \frac{\max\{e^0, e^x\}|x|^m}{m!} \cdot \left(\frac{1}{2}\right)^{n+1-m} \xrightarrow{} 0$$

Fixo

$\therefore \; R_n(x) \xrightarrow[n \to +\infty]{} 0$

Assim $\quad e^x = \lim\limits_{n \to +\infty} \sum\limits_{n=0}^{n} \frac{x^k}{k!} = \sum\limits_{k=0}^{+\infty} \frac{x^k}{k!}, \forall x \in \mathbb{R}$

<u>Ex:</u> $f(x) = \sin x$ é C^∞ ($f^{(n)}(x)$ é contínua, $\forall n$).

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f''''(x) = \sin x$

$f^{(2n)}(0) = 0$

$f^{(2n+1)}(0) = (-1)^n$

$\exists \, c$ entre 0 e x tal que

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)\, x^{n+1}}{(n+1)!}\right| \leq \frac{|x|^{n+1}}{n!}$$

$R_n(x) \xrightarrow[n \to +\infty]{} 0$

$$P_{2n+1}(x) = x - \frac{x^3}{3!} + \dots (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$P_{2n+2}(x) = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\operatorname{sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \forall x \in \mathbb{R}$$

Ex: Estime e usando uma soma de frações de racionais com erro menor que 10^{-10}.

$$f(x) = e^x \qquad P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

$$R_n(1) = \frac{e^c}{n+1!} 1^{n+1} \leq \frac{e^1}{(n+1)!}, \text{ com } 0 < c < 1.$$

$$\therefore |R_n(1)| \leq \frac{3}{(n+1)!}$$

Para que $|R_n(1)| < 10^{-10}$

Basta que $\frac{3}{(n+1)!} < 10^{-10}$

$$\Leftrightarrow \quad 3\cdot 10^{10} < (n+1)!$$

$$14! = \underbrace{14\cdot 13\cdot 12\cdot 11\cdot 10}_{>10^5}\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2$$

$>\frac{10^2}{2}$ $\quad 10$

$>2\cdot 10^2$ $\quad 3$

$> 3\cdot 10^{10}$

$$\therefore \left|e - \left(1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots \frac{1}{13!}\right)\right| < 10^{-10}$$

Ex: $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f''''(x) = \cos x$

$\vdots$

$f^{(2n+1)}(0) = 0$

$f^{(2n)}(0) = (-1)^n$

$$P_{2m}(x) = \sum_{k=0}^{m} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$P_{2m+1}(x) = \sum_{k=0}^{m} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$R_n(x) = \frac{f^{(n+1)}(c)\, x^{n+1}}{(n+1)!}$$

$n = 2m$
ou $2m+1$

Um c entre 0 e x.

Como $|f^{(n+1)}(c)| \leq 1$,

é análogo à conta de $\operatorname{sen} x$ que

$$R_n(x) \xrightarrow[n \to +\infty]{} 0$$

$$\therefore \cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$\forall x \in \mathbb{R}$

Ex: Arcotangente e aproximação de π.

$$\operatorname{arctg} x = \int_0^x \frac{1}{1+t^2}\,dt$$

$$1-t^n = (1-t)(1+t \dots + t^{n-1})$$

$\therefore$

$$1+t+\dots+t^{n-1} = \frac{1}{1-t} - \frac{t^n}{1-t} \qquad t \neq 1$$

$$\frac{1}{1-t} = \underbrace{1+t+\dots+t^{n-1}}_{\text{Polinômio}} + \underbrace{\frac{t^n}{1-t}}_{\text{Resto.}}$$

trocando t por $-t^2$ temos

$$\frac{1}{1+t^2} = 1 - t^2 + \dots + (-1)^k t^{2k} + \dots + (-1)^{n-1} t^{2n-2} + \frac{(-1)^n t^{2n}}{1+t^2}$$

Para $|t|<1$.

assim se $|x|<1$ temos

$$\int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \left(1 - t^2 + \dots + (-1)^k t^{2k} + \dots + (-1)^{n-1} t^{2n-2}\right) dt + \int_0^x \frac{(-1)^n t^{2n}}{1+t^2}\,dt$$

não provamos mas esto é o polinômio de Taylor de grau $2n-1$.

$$\operatorname{arctg} x - \left(x - \frac{x^3}{3} + \dots + \frac{(-1)^k x^{2k+1}}{2k+1} + \dots + \frac{(-1)^{n-1} x^{2n-1}}{2n-1} \right)$$

$$= \int_0^x \frac{(-1)^n t^{2n}}{1+t^2}\, dt$$

$$\therefore R_{2n-1}(x) = \int_0^x \frac{(-1)^n t^{2n}}{1+t^2}\, dt$$

$$|R_{2n-1}(x)| \leq \int_0^{|x|} t^{2n}\, dt = \frac{|x|^{2n+1}}{2n+1}$$

como $|x|<1$, segue

que $R_n(x) \xrightarrow[n \to +\infty]{} 0$

$R_{2n-1}(x) = R_{2n}(x)$

$$\boxed{\operatorname{arctg} x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad , |x| < 1}$$

Dá para mostrar que ↑ vale p/ $x = 1$

$$\frac{\pi}{4} = \operatorname{arctg} 1 = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}$$

essa série converge

Por ser uma série alternada

e pode se estimar π usando

$$\left|\frac{\pi}{4} - \sum_{n=0}^{k} \frac{(-1)^n}{2n+1}\right| < \frac{1}{2k+3} \qquad \left|\pi - 4\sum_{n=0}^{k} \frac{(-1)^n}{2n+1}\right| < \frac{4}{2k+3}$$

Mas é uma aproximação lenta até para ver que

$\pi \approx 3,14$.

Uma aproximação muito mais rápida para π:

$$tg(a+b) = \frac{sen(a+b)}{cos(a+b)} = \frac{sen\,a\,cos\,b + sen\,b\,cos\,a}{cos\,a\,cos\,b - sen\,a\,sen\,b}$$

dividindo o numerador e o denominador por cos a cos b:

$$= \frac{tg\,a + tg\,b}{1 - tg\,a\,tg\,b}$$

$$\therefore a+b = arctg\left(\frac{tg\,a + tg\,b}{1 - tg\,a\,tg\,b}\right)$$

$a = arctg\,\frac{1}{2}$ e $b = arctg\,\frac{1}{3}$

temos

$$arctg\,\frac{1}{2} + arctg\,\frac{1}{3} = arctg\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) =$$

$$= arctg\left(\frac{\frac{5}{6}}{1-\frac{1}{6}}\right)$$

$$= arctg\,1 = \frac{\pi}{4}$$

$\therefore \quad \pi = 4\left(\operatorname{arctg} \frac{1}{2} + \operatorname{arctg}\frac{1}{3}\right)$

assim

$$\left|\pi - 4\left(P_{2k_1+1}\left(\tfrac{1}{2}\right) + P_{2k_2+1}\left(\tfrac{1}{3}\right)\right)\right| =$$

$$\Big|\underbrace{\pi - 4\big(P_{2k_1+1}(\tfrac{1}{2}) + R_{2k_1+1}(\tfrac{1}{2})}_{}\overset{\operatorname{arctg}\frac{1}{2}}{} + \underbrace{P_{2k_2+1}(\tfrac{1}{3}) + R_{2k_2+1}(\tfrac{1}{3})\big)}_{=0}\overset{\operatorname{arctg}\frac{1}{3}}{} + 4R_{2k_1+1}(\tfrac{1}{2}) + 4R_{2k_2+1}(\tfrac{1}{3})\Big|$$

$$\leq \left|4R_{2k_1+1}\left(\tfrac{1}{2}\right)\right| + \left|4R_{2k_2+1}\left(\tfrac{1}{3}\right)\right|$$

$$\leq 4\frac{1}{2k_1+3}\cdot\frac{1}{2^{2k_1+3}} + \frac{4}{2k_2+3}\cdot\frac{1}{3^{2k_2+3}}$$

EX: Use a série acima para estimar π com erro menor que 10^{-5}.

$$\begin{cases} ① \ \dfrac{4}{2k_1+3}\cdot\dfrac{1}{2^{2k_1+3}} < 9\cdot 10^{-6} \\[2ex] ② \ \dfrac{4}{2k_2+3}\cdot\dfrac{1}{3^{2k_2+3}} < 10^{-6} \end{cases}$$

$2^{10} > 10^3$

$① \Leftrightarrow \quad 4\cdot 10^6 < 9(2k_1+3)\,2^{2k_1+3}$

$k_1 = 7$

$9\cdot 17\cdot 2^{17} > 9\cdot 2^4\cdot 2^{17} > 4\cdot 2^{20} > 4\cdot 10^6$

$② \Leftrightarrow 4 \cdot 10^6 < (2K_2+3) \cdot 3^{2K_2+3}$ $\quad 3^5 > 2 \cdot 10^2$

$K_2 = 5$

$13 \cdot 3^{13} = 13 \cdot 3^5 \cdot 3^5 \cdot 3^3$

$> (3^5)^3 > (2 \cdot 10^2)^3 > 4 \cdot 10^6$

assim

$$\left| \pi - \left(4 \sum_{n=0}^{7} \frac{(-1)^n \frac{1}{2^{2n+1}}}{2n+1} + 4 \sum_{n=0}^{5} \frac{(-1)^n \frac{1}{3^{2n+1}}}{2n+1} \right) \right| < 10^{-5}$$