Def: f è di classe C^n se $f^{(n)}$ è continua

$$e^x = \sum_{n=0}^{+\infty} a_n x^n \longrightarrow 1 = e^0 = a_0 \longrightarrow a_0 = \frac{1}{0!}$$

deriva $$e^x = \sum_{n=1}^{+\infty} a_n\, n x^{n-1} \longrightarrow 1 = e^0 = a_1 \cdot 1 \longrightarrow a_1 = \frac{1}{1!}$$

deriva di nuovo $$e^x = \sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} \longrightarrow 1 = e^0 = a_2 \cdot 2! \qquad a_2 = \frac{1}{2!}$$

$$e^x = \sum_{n=3}^{+\infty} a_n n(n-1)(n-2) x^{n-3} \longrightarrow 1 = e^0 = a_3 \cdot 3! \qquad a_3 = \frac{1}{3!}$$

$$e^x = \sum_{n=k}^{+\infty} a_n (n) \dots (n-k+1) x^{n-k} \longrightarrow 1 = e^0 = a_k = \frac{1}{k!}$$

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$\boxed{\sum_{n=0}^{k} \frac{x^n}{n!}}$$

Teorema do Polinómio de Taylor com resto de Lagrange)

f de classe C^{k+1} num intervalo I em torno de 0.

Se $x \in I$ existe $\bar{x}$ entre 0 e x [$0 < \bar{x} < x$ se $0 < x$ ou $x < \bar{x} < 0$ se $x < 0$]

$$f(x) = \underbrace{\sum_{n=0}^{k} \frac{f^{(n)}(0)x^n}{n!}} + \frac{f^{(k+1)}(\bar{x})}{(k+1)!} x^{k+1}$$

Ex: e^x aplicar o teorema anterior

p/ $k=3$ $x=1$

$\exists \bar{x}$ entre 0 e 1 tal que

$$e = \sum_{n=0}^{3} \frac{1^n}{n!} + \frac{e^{\bar{x}}}{4!} 1^4$$

$$e = \sum_{n=0}^{3} \frac{1}{n!} + \frac{e^{\bar{x}}}{4!}$$

e^x é crescente

$\bar{x} < 1$ $\quad e^{\bar{x}} < e < 3$

$$\left| e - \left(\sum_{n=0}^{3} \frac{1}{n!} \right) \right| = \frac{e^{\bar{x}}}{4!} < \frac{3}{4!}$$

Encontre K tal que

$$\left| e - \sum_{n=0}^{K} \frac{1}{n!} \right| < 10^{-5}.$$

Fixado x existe $\bar{x}$ entre 0 e x

$$e^{x} = \sum_{n=0}^{K} \frac{x^n}{n!} + \frac{e^{\bar{x}}\, x^{K+1}}{(K+1)!}$$

$x = 1$

$$\left| e - \sum_{n=0}^{K} \frac{1}{n!} \right| \leq \frac{e^{\bar{x}}}{(K+1)!} < \frac{3}{(K+1)!} < 10^{-5}$$

$$\frac{3}{(K+1)!} < 10^{-5} \Leftrightarrow 3 \cdot 10^{5} < (K+1)!$$

$K = 5 \rightarrow 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$K = 10 \rightarrow 11! = 11 \cdot 10 \cdot \overbrace{9 \cdot 8}^{>10} \cdot \underbrace{8 \cdot 7}_{>50} \cdot 6 \cdot \underbrace{5 \cdot 4}_{20} \cdot 3 \cdot 2 > 10 \cdot 10 \cdot 10^{3} \cdot 3 = 3 \cdot 10^{5}$

Erro $< 10^{-10}$

$$\frac{3}{(k+1)!} < 10^{-10}$$

$$3 \cdot 10^{10} < (k+1)!$$

$k = 14$ $$15! = \underbrace{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}_{> 10^{5}} \cdot \underbrace{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}_{> 3 \cdot 10^{5}} > 3 \cdot 10^{10}$$

$$\left| e - \sum_{n=0}^{14} \frac{1}{n!} \right| < 10^{-10}$$

x qualquer

$$\left| e^{x} - \sum_{n=0}^{k} \frac{x^n}{n!} \right| = \frac{e^{\bar{x}}\, x^{k+1}}{(k+1)!} < \max\left\{ \frac{1}{(k+1)!}\, x^{k+1},\ \frac{3^{x}\, x^{k+1}}{(k+1)!} \right\}$$

$$e^{x} = \lim_{k \to +\infty} \sum_{n=0}^{k} \frac{x^n}{n!} = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

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polinômio de Taylor de sen x

$f(x) = \operatorname{sen} x$ $\qquad f^{(0)}(0) = 0$

$f'(x) = \cos x$ $\qquad f^{(1)}(0) = 1$

$f''(x) = -\operatorname{sen} x$ $\qquad f^{(2)}(0) = 0$

$f'''(x) = -\cos x$ $\qquad f^{(3)}(0) = -1$

$f^{(4)}(x) = \operatorname{sen} x$ $\qquad f^{(4)}(0) = 0$

$f^{(5)}(0) = 1$

$f^{(6)}(0) = 0$

$f^{(7)}(0) = -1$

classe C^{2k+2}

$$\underbrace{\operatorname{sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots \frac{(-1)^k x^{2k+1}}{(2k+1)!}} + \frac{f^{(2k+2)}(\bar{x})}{(2k+2)!} x^{2k+2}$$

$$\left| \operatorname{sen} x - \left(x - \frac{x^3}{3!} \cdots \frac{(-1)^k x^{2k+1}}{(2k+1)!}\right) \right| = \frac{|f^{(2k+2)}(\bar{x})| \, |x|^{2k+2}}{(2k+2)!}$$

Fixando x

$$\leq \frac{|x|^{2k+2}}{(2k+2)!} \xrightarrow[k \to +\infty]{} 0$$

$$\therefore \operatorname{sen} x = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$g(x) = \cos x$ $\qquad g(0) = 1$

$g'(x) = -\sin x$ $\qquad g'(0) = 0$

$g''(x) = -\cos x$ $\qquad g''(0) = -1$

$g'''(x) = \sin x$ $\qquad g'''(0) = 0$

$g''''(x) = \cos x$ $\qquad g''''(0) = 1$

$\vdots$ $\qquad g^{(5)}(0) = 0$

$\bar{x}$ entre 0 e x

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \frac{x^{2k}}{(2k)!} + \frac{g^{(2k+1)}(\bar{x})}{(2k+1)!} x^{2k+1}$$

$$\left| \cos x - \left(1 - \frac{x^2}{2!} \dots \frac{(-1)^k x^{2k}}{(2k)!}\right) \right| < \frac{\left| g^{(2k+1)}(\bar{x}) \right| |x|^{2k+1}}{(2k+1)!} < \frac{|x|^{2k+1}}{(2k+1)!}$$

$$\boxed{\cos x = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k}}{(2k)!}}$$

$\pi = ?$

$$\boxed{\operatorname{arctg} 1}$$

$$\left| \operatorname{sen} 2 - \left(2 - \frac{2^3}{3!} + \frac{2^5}{5!} \cdots + \frac{(-1)^8 2^{17}}{17!} \right.\right|$$

encontre sen 2 com erro menor que 10^{-3}.

$$\left| \operatorname{sen} 2 - \left(2 - \frac{2^3}{3!} + \frac{2^5}{5!} + \cdots \frac{(-1)^k 2^{2k+1}}{(2k+1)!} \right.\right| \leq \frac{2^{2k+2}}{(2k+2)!} < 10^{-3}$$

$2^{2k+2} \cdot 10^3 < (2k+2)!$ $\Big\{ k = 8$? $2^{18} \, 10^3 < 18!$

$18! = 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot \underline{10 \cdot 9 \cdot 8 \cdot 7} \cdot \underline{6 \cdot 5 \cdot 4 \cdot 3} \cdot 2 \cdot > 2^{18} \, 10^3$

$>2^4 \quad >2^4 \quad >2^4 \quad >2^{3} \quad 2^2 \quad 2^4 \quad 2^1 \quad >10^3$

Série alternada

$b = \sum_{n=0}^{+\infty} a_n$ alterna

$\begin{cases} a_n \cdot a_{n+1} < 0 \\ |a_n| \longrightarrow 0 \\ |a_{n+1}| < |a_n| \end{cases}$

$\boxed{\left| b - \sum_{n=0}^{k} a_n \right| \leq |a_{k+1}|}$

$\arctg \frac{1}{2}$

$\arctg \frac{1}{3}$

N

b

Exercício

Encontre K tal que

$$\left| e - \sum_{n=0}^{K} \frac{1}{n!} \right| < 10^{-20}$$

$$a = \sum_{n=0}^{+\infty} a_n$$

$$\left| a - \sum_{n=0}^{K} a_n \right| < ?$$

Encontre as 5 primeiras casas decimais depois da vírgula de e usando as operações básicas.

$\boxed{3{,}28\,\underline{7}}$