TVM de Cauchy

Sejam f e g deriváveis em $]a,b[$ e contínuas em $[a,b]$. Então $\exists c \in]a,b[$ tal que

$$(f(b)-f(a))g'(c) = (g(b)-g(a)).f'(c).$$

Dem: Usaremos o TVM na função

$$h(x) = (f(b)-f(a))g(x) - (g(b)-g(a))f(x).$$

Claramente h é contínua em $[a,b]$ e derivável em $]a,b[$.

Vamos então calcular $h(b)$ e $h(a)$.

$$h(a) = (f(b)-f(a))g(a) - (g(b)-g(a))f(a)$$
$$= f(b)g(a) - \cancel{f(a)g(a)} - g(b)f(a) + \cancel{g(a)f(a)}$$
$$= f(b)g(a) - g(b)f(a).$$

$$h(b) = (f(b)-f(a))g(b) - (g(b)-g(a))f(b)$$
$$= \cancel{f(b)g(b)} - f(a)g(b) - \cancel{g(b)f(b)} + g(a)f(b)$$
$$= -f(a)g(b) + g(a)f(b)$$

$$\therefore \boxed{h(a) = h(b)}$$

Pelo TVM, existe $c \in]a,b[$ tal que

$h'(c) = 0 \therefore$

$$(f(b)-f(a))g'(c) - (g(b)-g(a))f'(c) = 0$$

$$\therefore (f(b)-f(a))g'(c) = (g(b)-g(a))f'(c). \blacksquare$$

L'Hospital

Polinômio de Taylor com resto de Lagrange

Teo: (L'Hospital $\frac{0}{0}$).

Sejam f e g deriváveis em todos os pontos $\neq a$ suficientemente próximos de a com $g'(x) \neq 0$, $\forall x \neq a$ suf. próximos de a. Então se $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

e $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$

então $\lim_{x\to a} \frac{f(x)}{g(x)}$ existe

e é igual a L.

Dem: (1) Fixe $\underline{\varepsilon > 0}$ e suponha $\boxed{\underline{L \in \mathbb{R}}}$

Podemos tomar $\underline{\varepsilon \leq 1}$.

Iremos primeiro fazer

manipulação antes de achamos $\delta > 0$, tal que

$0 < |x-a| < \delta \Rightarrow \left|\frac{f(x)}{g(x)} - L\right| < \varepsilon$.

considere y entre x e a.

$\delta > 0$ será escolhido tal que

$g'(t) \neq 0$ se $0 < |t-a| < \delta$.

então $g(x) - g(y) \neq 0$

(se fosse 0, $\exists\, c$ entre x e y tal que $g'(c) = 0$ $\therefore$

$\therefore$ $c \in \,]x, a[\,$ $\therefore$

$0 < |c-a| < \delta \Rightarrow g'(c) \neq 0$.

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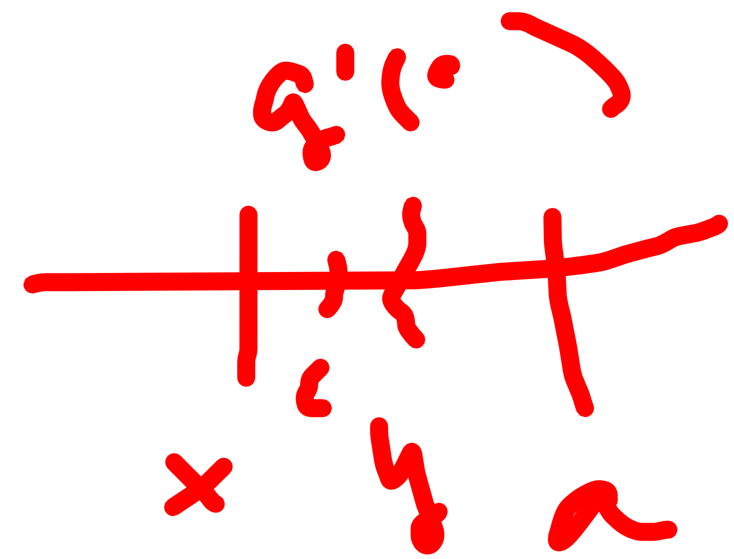
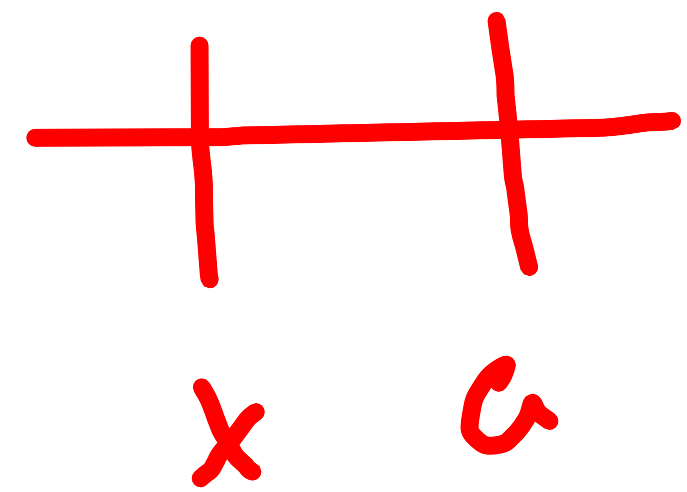


Então pelo TVM de Cauchy, existe c_y entre x e y tal que

$$\boxed{\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(c_y)}{g'(c_y)}}$$

$$\frac{f(x)}{g(x)} \left(\frac{1 - \frac{f(y)}{f(x)}}{1 - \frac{g(y)}{g(x)}} \right) = \frac{f'(c_y)}{g'(c_y)}$$

$$\therefore \frac{f(x)}{g(x)} = \frac{f'(c_y)}{g'(c_y)} \cdot \left(\frac{1 - \frac{g(y)}{g(x)}}{1 - \frac{f(y)}{f(x)}} \right)$$



Vamos começar a demonstração. Seja $\delta > 0$ tal que

$$0<|t-a|<\delta \Rightarrow \left|\frac{f'(t)}{g'(t)} - L\right| < \frac{\varepsilon}{2}$$

Fixe x tal que

$0<|x-a|<\delta$.

Como x está fixado

$$\lim_{t\to a} \frac{1-\frac{g(t)}{g(x)}}{1-\frac{f(t)}{f(x)}} = 1$$

Existe $\gamma > 0$, $\gamma < \delta$ tal que

$0 < |t - a| < \gamma \Rightarrow$

$$\left| 1 - \frac{1 - \frac{g(t)}{g(x)}}{1 - \frac{f(t)}{f(x)}} \right| < \frac{\varepsilon}{2(|L|+1)}$$

Fixe y tal que (basta algum y)

$0 < |y - a| < \gamma$

e y entre x e a.

Pelo cálculo já feito,

$$\frac{f(x)}{g(x)} = \frac{f'(c_y)}{g'(c_y)} \left(\frac{1 - \frac{g(y)}{g(x)}}{1 - \frac{f(y)}{f(x)}} \right)$$

Se $\dfrac{f'(c_y)}{g'(c_y)} = 0$ então

$$\frac{f(x)}{g(x)} = 0 \in \left] \overset{\overset{0}{\|}}{\frac{f'(c_y)}{g'(c_y)}} - \frac{\varepsilon}{2}, \frac{f'(c_y)}{g'(c_y)} + \frac{\varepsilon}{2} \right[$$

$$|c_y - a| < \delta \Rightarrow \frac{f'(c_y)}{g'(c_y)} \in \left]L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right[\Rightarrow \subseteq \left] L - \varepsilon, L + \varepsilon \right[$$

Vamos supor que

$\dfrac{f'(c_y)}{g'(c_y)} \neq 0$. Primeiro,

assumimos que

$$\frac{f'(c_y)}{g'(c_y)} > 0.$$

Então

$$0 \ (\varepsilon \leq 1)$$

$$\frac{f(x)}{g(x)} \in \left] \frac{f'(c_y)}{g'(c_y)} \cdot \left(\overset{\wedge}{1} - \frac{\varepsilon}{2(|L|+1)} \right), \frac{f'(c_y)}{g'(c_y)} \left(1 + \frac{\varepsilon}{2(|L|+1)} \right) \right[$$

$$\frac{f(x)}{g(x)} \in \left] \frac{f'(c_y)}{g'(c_y)} - \frac{\varepsilon}{2}, \frac{f'(c_y)}{g'(c_y)} + \frac{\varepsilon}{2} \right[$$

$$\left| \frac{f'(c_y)}{g'(c_y)} \right| \cdot \frac{\varepsilon}{2(|L|+1)} \leq |L|+1 \cdot \frac{\varepsilon}{2(|L|+1)}$$

$$= \frac{\varepsilon}{2}$$

$0 < |c_y - a| < \delta$

$$L - \varepsilon \leq L - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} < \frac{f'(c_y)}{g'(c_y)} - \frac{\varepsilon}{2}$$

$$< \frac{f(x)}{g(x)} < \frac{f'(c_y)}{g'(c_y)} + \frac{\varepsilon}{2} < L + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = L + \frac{\varepsilon}{2}$$

Se $\dfrac{f'(c_y)}{g'(c_y)} < 0$

$$\frac{f(x)}{g(x)} \in \left] \frac{f'(c_y)}{g'(c_y)} \cdot \left(1 + \frac{\varepsilon}{2(|L|+1)}\right), \frac{f'(c_y)}{g'(c_y)} \cdot \left(1 - \frac{\varepsilon}{2(|L|+1)}\right) \right[$$

$$\subseteq \left] \frac{f'(c_y)}{g'(c_y)} - \frac{\varepsilon}{2}, \frac{f'(c_y)}{g'(c_y)} + \frac{\varepsilon}{2} \right[$$

e o resto é como antes.

e $\quad L - \varepsilon < \dfrac{f(x)}{g(x)} < L + \varepsilon.$

Se $L = +\infty$ e $N > 0$

$\exists\, \delta > 0 \quad \forall t$

$$0 < |t - a| < \delta \Rightarrow \left|\frac{f'(t)}{g'(t)}\right| > 2N$$

$$\frac{f(x)}{g(x)} = \frac{f'(c_y)}{g'(c_y)} \cdot \left(\frac{1 - \frac{g(y)}{g(x)}}{1 - \frac{f(y)}{f(x)}} \right)$$

Queremos

$$> \frac{f'(c_y)}{g'(c_y)} \cdot \frac{1}{2} > N$$

$\exists\, \gamma > 0$

$0 < \gamma < \delta$

$$0 < |t - a| < \gamma \Rightarrow \left| \frac{1 - \frac{g(t)}{g(x)}}{1 - \frac{f(t)}{f(x)}} - 1 \right| < \frac{1}{2}$$

Então tome y
tal que

$0<|y-a|<\gamma$

e y entre x e a.

Então

$$\frac{f(x)}{g(x)} \geq \frac{f'(c_y)}{g'(c_y)} \cdot \frac{1-\frac{g(y)}{g(x)}}{1-\frac{f(y)}{f(x)}}$$

$$\geq \frac{f'(c_y)}{g'(c_y)} \cdot \frac{1}{2} > \frac{2N}{2} = N$$

$\therefore \lim_{x \to +\infty} \frac{f(x)}{g(x)} = +\infty.$

$\frac{0}{0}$ $\frac{\infty}{\infty}$

Se $L = -\infty$

basta aplicar L'Hospital em $-f(x)$ e $g(x)$.

Então $\lim_{x \to a} -f(x)$

$= \lim_{x \to a} g(x) = 0$

e $\lim_{x \to a} \frac{-f(x)}{g(x)} = \lim_{x \to a} \frac{-f'(x)}{g'(x)} = +\infty$

$\therefore \lim_{x \to a} \frac{f(x)}{g(x)} = -\infty.$

$f^{(4)}(x) = e^x > 0$

$g^{(4)}(x) =$

$\underbrace{e(e-1)(e-2)}_{>0}\underbrace{(e-3)}_{<0}\underbrace{x^{e-4}}_{>0}$

< 0

$\therefore$ $f^{(4)}(x) > g^{(4)}(x)$
$\forall x > e$

$f(\pi) > g(\pi)$

$e^\pi = f(\pi)$ $\quad \pi^e = g(\pi)$, $\quad \pi > e$

$\therefore e^\pi > \pi^e$

$f(x) = e^x \qquad g(x) = x^e$

$f(e) = e^e = g(e)$

$\Downarrow$ TVM $f(x) > g(x)$
$\Downarrow$
$x > e$

$f'(x) = e^x$

$g'(x) = e\,x^{e-1}$

$f'(e) = e^e = e\,e^{e-1} = g'(e)$

$\Downarrow$ TVM

$f'(x) > g'(x), \forall x > e$

$f''(x) = e^x \qquad e > e-1$

$g''(x) = e(e-1)\,x^{e-2}$

$f''(e) = e^e = e^1 \cdot e^{e-2}$

$= e\,e\,e^{e-2}$

$\geq e(e-1)e^{e-2}$

$= g''(e)$

$\Leftarrow$ TVM $f''(x) > g''(x)$
$\forall x > e$

$f'''(x) = e^x \xRightarrow{\text{TVM}} f'''(x) > g'''(x) \forall x > e$

$g'''(x) = e(e-1)(e-2)\,x^{e-3}$

$f'''(e) = e^e \geq g'''(e)$

Consequência TVM

$h(x) = f(x) - g(x)$

$b > a$

$\exists c \in]a,b[$ $\to 0$

$h'(c) = \dfrac{h(b) - h(a)}{b-a} > 0$

$\parallel$

$f'(c) - g'(c) > 0$

$f(a) = g(a)$

$\underline{f'(x) > g'(x)}$ $\forall x > a \Rightarrow$ $\boxed{f(x) > g(x) \ \forall x > a}$

$f(a) > g(a)$

$f'(x) \geq g'(x) \Rightarrow f(x) > g(x)$ $\forall x > a$

> 0

$h(b) - h(a) = h(b)$

$= f(b) - g(b) > 0$

$\Rightarrow f(b) > g(b)$

$f(a) \geq g(a)$

$f'(x) > g'(x)$ $\Rightarrow f(x) > g(x)$ $\forall x > a$

$f(a) > g(a)$

$f'(x) \geq g'(x)$ $\Rightarrow$ $f(x) > g(x) \ \forall x > a$

$h(x) = f(x) - g(x)$ $\quad h'(x) = f'(x) - g'(x) \geq 0$

Fix and $b > a$

$\exists\, c \in]a, b[$

$$h'(c) = \frac{h(b) - h(a)}{b - a} \geq 0$$

$\Rightarrow h(b) - h(a) \geq 0$

$\Rightarrow h(b) \geq h(a) \Rightarrow f(b) - g(b) \geq f(a) - g(a) > 0$

$\Rightarrow f(b) - g(b) > 0 \Rightarrow f(b) > g(b)$

$f(x) = x^3 + 8x^2 - 7$

$f'(x) = 3x^2 + 16x = 0$

$\iff x(3x+16) = 0$

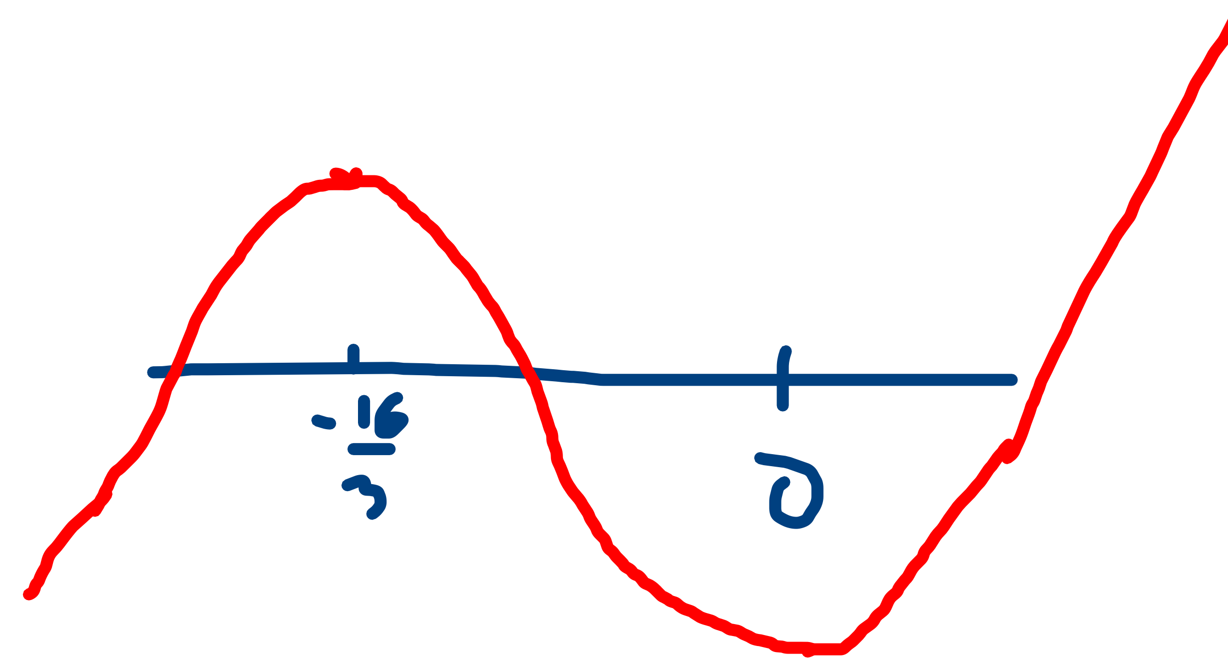
$\iff x = 0$ ou $x = -\frac{16}{3}$

		$-\frac{16}{3}$		0	
x	$-$	$-$	$-$	0	$+$
$3x+16$	$-$	0	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$\rightarrow$	$\searrow$	$\rightarrow$	$\nearrow$

$f\left(-\frac{16}{3}\right)$

$\left(-\frac{16}{3}\right)^2\left[-\frac{16}{3} + 8\right] - 7 > 0$

$f(0) = -7$



$2x+2x^2-x^2-2$
$= x^2+2x-2$

Eshore

$$f(x)=\frac{x^2+2}{1+x} \qquad \mathrm{Dom}\, f = \mathbb{R}\setminus\{-1\}$$

$$f'(x)=\frac{2x(1+x)-(x^2+2)1}{(1+x)^2}=\frac{x^2+2x-2}{(1+x)^2}$$

$x^2+2x-2=0 \iff x=\sqrt{3}-1$ or $x=-\sqrt{3}-1$

		$-\sqrt{3}-1$		-1		$\sqrt{3}-1$	
x^2+2x-2	+	0	−	·	−	0	+
→ $f'(x)$	+	0	−	∄	−	0	+
cr/dec f	↗	→	↘	∄	↘	→	↗

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x^2+2}{1+x} = -\infty$$

$x^2+2 \to 3$

$1+x \to 0^-$

$x < -1$

$\therefore x+1 < 0$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{x^2+2}{1+x} = +\infty$$

$x^2+2 \to 3$

$1+x \to 0^+$

$x > -1$

$\therefore x+1 > 0$

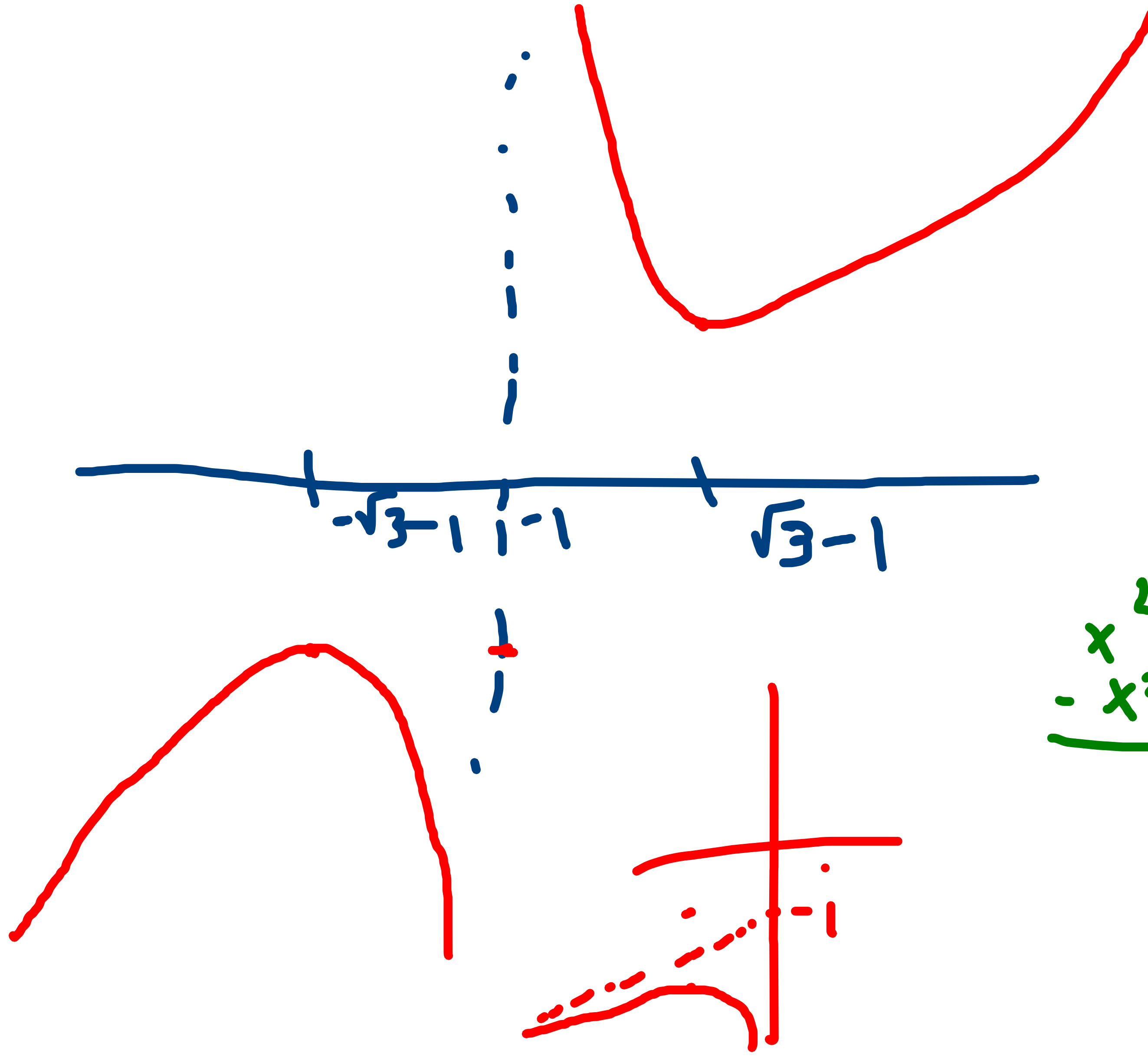
$$f(-\sqrt{3}-1) = \frac{(-\sqrt{3}-1)^2+2}{1+(-\sqrt{3}-1)}$$

$$= \frac{(\sqrt{3}+1)^2+2}{-\sqrt{3}} < 0$$

$$f(\sqrt{3}-1) = \frac{(\sqrt{3}-1)^2+2}{1+(\sqrt{3}-1)} = \frac{(\sqrt{3}-1)^2+2}{\sqrt{3}} > 0$$

$$\lim_{x\to-\infty}\frac{x^2+2}{1+x} = \lim_{x\to-\infty}\frac{\cancel{x}\cdot \overbrace{x}^{-\infty}\overbrace{\left(1+\frac{2}{x^2}\right)}^{1}}{\cancel{x}\underbrace{\left(\frac{1}{x}+1\right)}_{1}} = -\infty$$

$$\lim_{x\to+\infty}\frac{x^2+2}{1+x} = \lim_{x\to+\infty}\frac{\cancel{x}\cdot \overbrace{x}^{+\infty}\overbrace{\left(1+\frac{2}{x^2}\right)}^{1}}{\cancel{x}\underbrace{\left(\frac{1}{x}+1\right)}_{1}} = +\infty$$



$$\frac{x^2+2}{1+x} = \frac{(x-1)(x+1)+3}{x+1} = x-1+\frac{3}{x+1}$$

$$\begin{array}{r|l} x^2+0x+2 & x+1 \\ \hline -x^2-x & x-1 \\ \hline -x+2 & \\ x+1 & \\ \hline 3 & \end{array}$$

$$\lim_{x\to-\infty}\frac{1-x^2}{1+x^2}$$

$$=\lim_{x\to-\infty}\cancel{\frac{x^2}{x^2}}\cdot\frac{\left(\frac{1}{x^2}-1\right)}{\left(\frac{1}{x^2}+1\right)}$$

-1

1

$$=-1$$

$$\lim_{x\to+\infty}\frac{1-x^2}{1+x^2}=-1$$

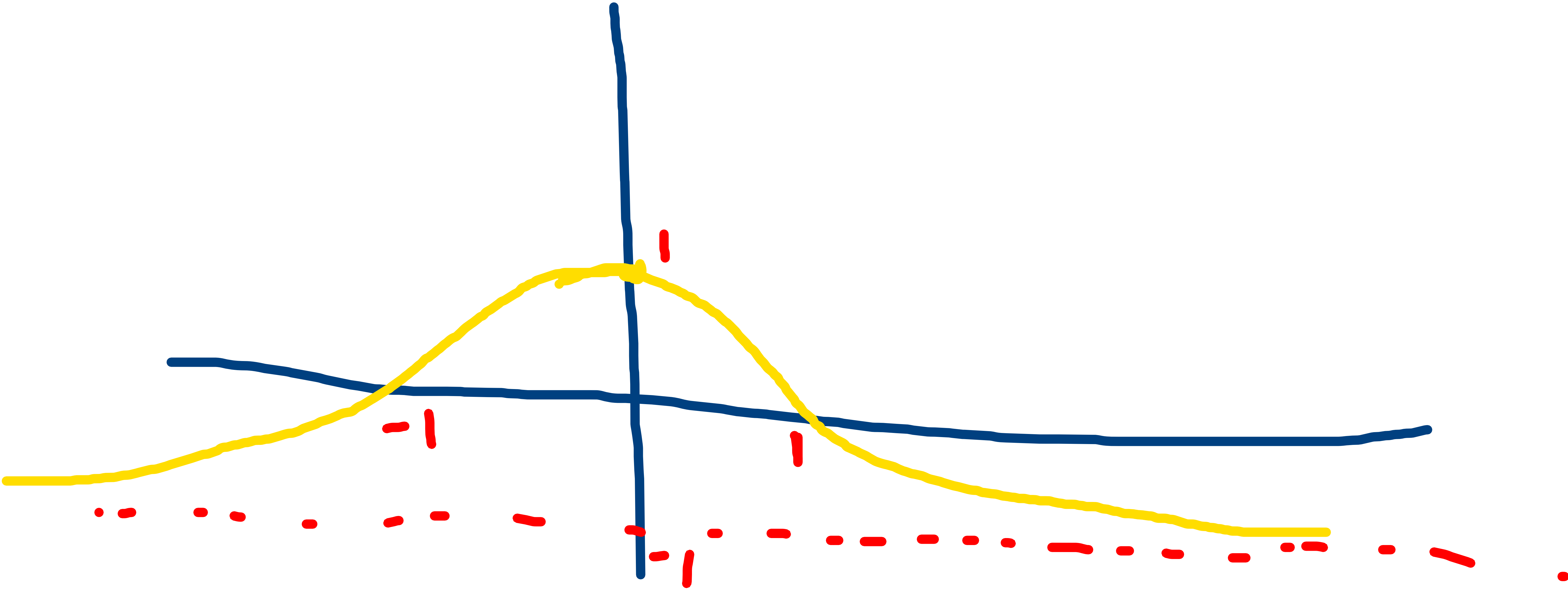
$$f(x)=\frac{1-x^2}{1+x^2}=\underbrace{\frac{-1-x^2}{1+x^2}+\frac{2}{1+x^2}}=-1+\frac{2}{1+x^2}$$

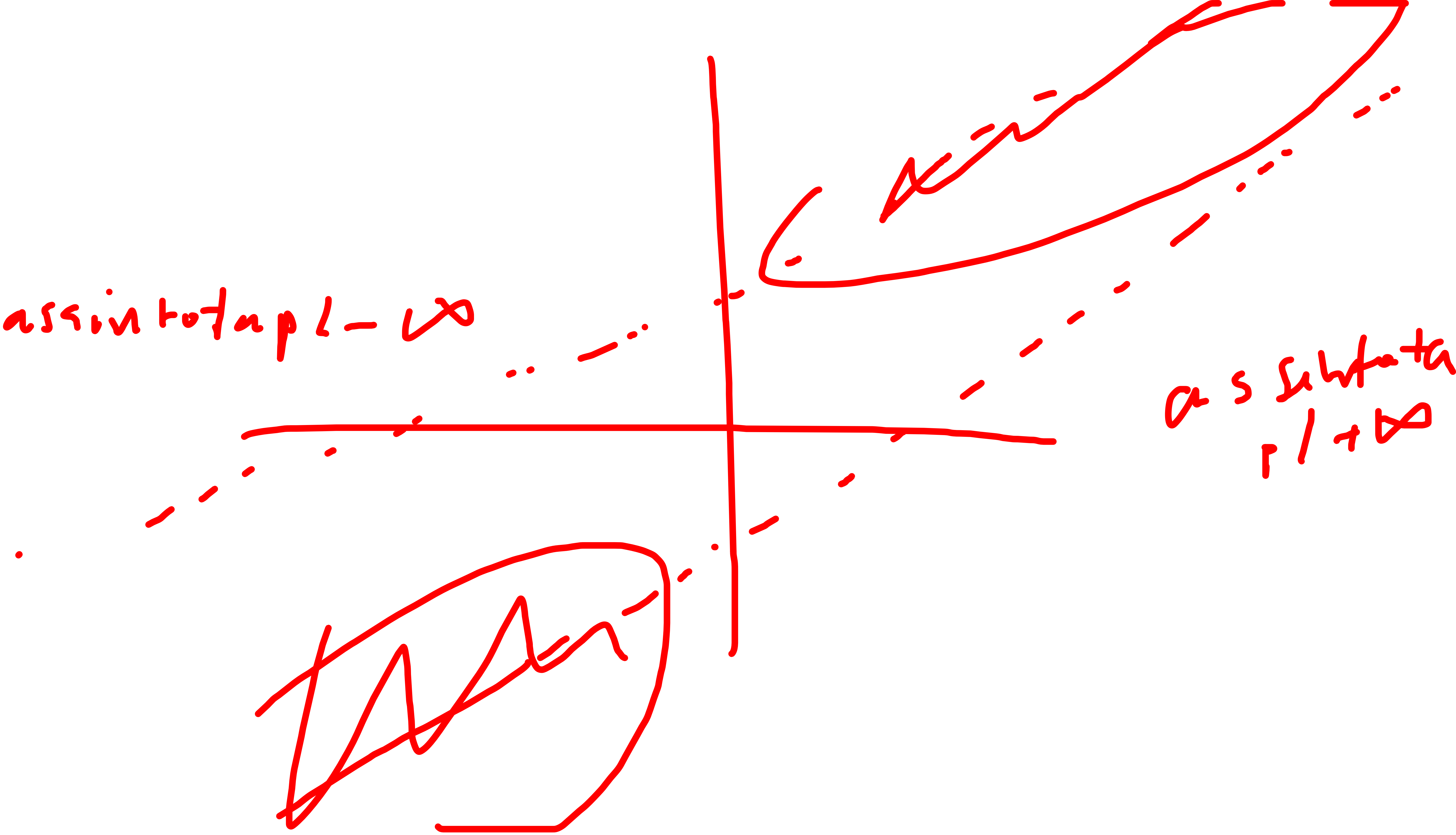
$$f'(x)=\frac{-2x(1+x^2)-(1-x^2)\,2x}{(1+x^2)^2}$$

$$=\frac{-2x-\cancel{2x^3}-2x+\cancel{2x^3}}{(1+x^2)^2}=\frac{-4x}{(1+x^2)^2}$$

		0	
$-4x$	+	0	−
$f'(x)$	+	0	−
inc/dec f	↗	1	↘

$f(0)=1$




assintota p/ -∞
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$$f(x) = \sqrt{x^2 + 1}$$

$$= |x| \sqrt{1 + \frac{1}{x^2}}$$