$$\boxed{} \le \left(1+\frac{1}{x}\right)^x \le \{\ \}$$

$e \qquad\qquad e$

Vimos que

$$e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = \sum_{n=0}^{+\infty}\frac{1}{n!} = \lim_{k\to+\infty}\sum_{n=0}^{k}\frac{1}{n!}$$

$n \in \mathbb{N}$

$$e^x = \sum_{n=0}^{+\infty}\frac{x^n}{n!},\ x\in\mathbb{R}$$

polinômio de Taylor p/ estimar e

Teo: $e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$.

Dem: Para cada x, seja $[x]$ o maior inteiro $n \in \mathbb{Z}$ tal que $n \le x < n+1$.

Então $[x] \le x < [x]+1$.

$$\left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{[x]}\right)^x \le \left(1+\frac{1}{[x]}\right)^{[x]+1}$$

$$= \underbrace{\left(1+\frac{1}{[x]}\right)^{[x]}}_{\to e}\cdot\underbrace{\left(1+\frac{1}{[x]}\right)^1}_{\to 1}$$

$[5,5] = 5$

$[-7,3] = -8$

$-8 \quad -7,3 \quad -7$

$[x] \le x$

$$\frac{1}{[x]} \ge \frac{1}{x}$$

$$\left(1+\frac{1}{[x]}\right)^x \ge \left(1+\frac{1}{x}\right)^x$$

Dado $\varepsilon > 0$ $\exists$ N_0 tal que

$$\left|\left(1+\frac{1}{n}\right)^n - e\right| < \varepsilon, \ \forall n \geq N_0.$$

assim

$\forall x > N_0 + 1$ $\therefore$ $[x] \geq N_0 + 1$.

$\therefore [x] > N_0 \therefore$

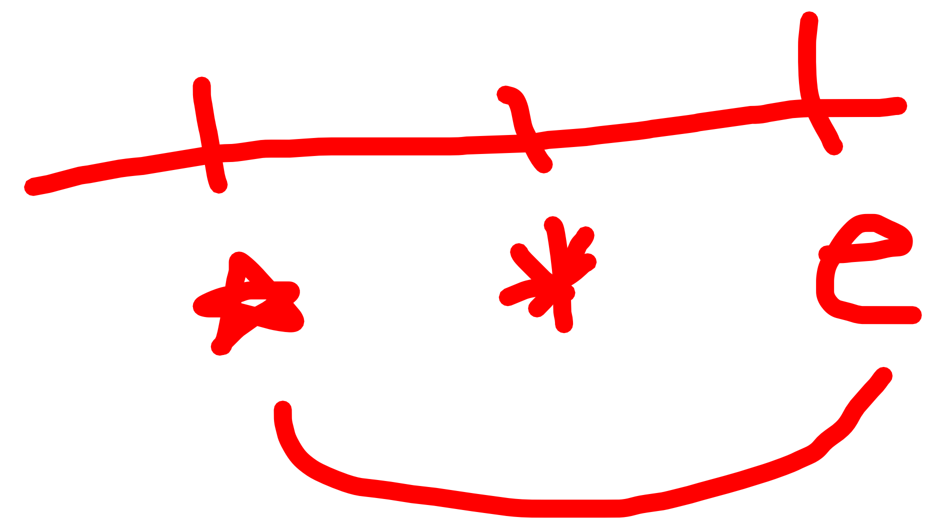
$$e > \left(1+\frac{1}{[x]}\right)^{[x]} \geq \left(1+\frac{1}{N_0}\right)^{N_0} \therefore$$

$$\left|e - \left(1+\frac{1}{[x]}\right)^{[x]}\right|^{*} \leq \left|e - \left(1+\frac{1}{N_0}\right)^{N_0}\right|^{*} < \varepsilon$$

$$\therefore \lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]} = e.$$

$\forall \varepsilon > 0$ $\exists N_1 \in \mathbb{N}$ tal que ($N_1 > 1$)

$$\left|\frac{1}{N_1}\right| < \varepsilon.$$



$\left(1+\frac{1}{n}\right)^n$ é crescente assim $\forall x > N_1+1$

$[x] \geq N_1+1$

$\therefore [x] > N_1$

$$\therefore \left|1-\left(1+\frac{1}{[x]}\right)\right|$$

$$= \frac{1}{[x]} < \frac{1}{N_1} < \varepsilon$$

$$\lim_{x\to+\infty} 1+\frac{1}{[x]} = 1$$

$$\therefore \lim_{x\to+\infty} \left(1+\frac{1}{[x]}\right)^{[x]} \left(1+\frac{1}{[x]}\right)$$

$$= e \cdot 1$$

Por outro lado

$$\left(1+\frac{1}{x}\right)^x \geq \left(1+\frac{1}{[x]+1}\right)^{x} \geq \left(1+\frac{1}{[x]+1}\right)^{[x]}$$

$$= \left(1+\frac{1}{[x]+1}\right)^{[x]+1} \cdot \left(1+\frac{1}{[x]+1}\right)^{-1}$$

Dado $\varepsilon > 0$ $\exists N_2$ tal que

$$\left|\left(1+\frac{1}{n}\right)^n - e\right| < \varepsilon \quad \forall n \geq N_2$$

Se $x > N_2$ $\quad [x] \geq N_2 \therefore [x]+1 > N_2$

e $\quad e > \left(1+\frac{1}{[x]+1}\right)^{[x]+1} > \left(1+\frac{1}{N_2}\right)^{N_2}$

assim

$$\left|e - \left(1+\frac{1}{[x]+1}\right)^{[x]+1}\right| \leq \left|e - \left(1+\frac{1}{N_2}\right)^{N_2}\right|$$

$$< \varepsilon \;\therefore\; \lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]+1} = e.$$

$$x < [x]+1 \qquad x > 0$$

$$\frac{1}{x} > \frac{1}{[x]+1}$$

$$\left(1+\frac{1}{x}\right) > \left(1+\frac{1}{[x]+1}\right)$$

$$\left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{[x]+1}\right)^x$$

Dado $\varepsilon > 0$ $\exists N_3 > 0$

$\frac{1}{N_3} < \varepsilon$.

$\forall x > N_3$ $[x]+1 > N_3$

$$\therefore \left| 1 - \left(1 + \frac{1}{[x]+1}\right) \right| = \frac{1}{[x]+1} < \frac{1}{N_3} < \varepsilon$$

$$\therefore \lim_{x \to +\infty} 1 + \frac{1}{[x]+1} = 1$$

como $\lim_{x \to +\infty} 1 + \frac{1}{[x]+1} \neq 0$

segue que

$$\lim_{x \to +\infty} \left(1 + \frac{1}{[x]+1}\right)^{-1} = 1^{-1} = 1$$

Assim $\lim_{x \to +} \left(1 + \frac{1}{[x]+1}\right)^{[x]+1} \cdot \left(1 + \frac{1}{[x]+1}\right)^{-1}$

$= e$

assim

$$\underbrace{\left(1+\frac{1}{[x]+1}\right)^{[x]+1} \cdot \left(1+\frac{1}{[x]+1}\right)^{-1}}_{\downarrow \atop e} < \left(1+\frac{1}{x}\right)^{x} \leq \underbrace{\left(1+\frac{1}{[x]}\right)^{[x]} \cdot \left(1+\frac{1}{[x]}\right)^{1}}_{\downarrow \atop e}$$

Pelo confronto

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x} = e.$$

Teor: $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$

Dem: $x=-y$

$$\lim_{y\to+\infty}\left(1-\frac{1}{y}\right)^{-y} = \lim_{y\to+\infty}\left(\frac{y-1}{y}\right)^{-y}$$

$$= \lim_{y\to+\infty}\left(\frac{y}{y-1}\right)^{y} = \lim_{z\to+\infty}\left(\frac{z+1}{z}\right)^{z+1} = \lim_{z\to+\infty}\left(1+\frac{1}{z}\right)^{z+1}$$

$z=y-1 \quad y=z+1$

$$= \lim_{z\to+\infty}\underbrace{\left(1+\frac{1}{z}\right)^{z}}_{\to e}\cdot\underbrace{\left(1+\frac{1}{z}\right)}_{\to 1} = e \cdot 1 = e$$

Teor: $\lim_{u\to 0}(1+u)^{\frac{1}{u}} = e$

Dem: Vamos mostrar que

$$\lim_{u\to 0^+}(1+u)^{\frac{1}{u}} = e = \lim_{u\to 0^-}(1+u)^{\frac{1}{u}}$$

$$\lim_{u \to 0} \frac{e^u - 1}{u}!$$

$$e^u - 1 = x$$

$$e^u = 1 + x$$

$$u = \ln(1+x)$$

$$= \lim_{x \to 0} \frac{x}{\ln(1+x)}$$

$$= \lim_{x \to 0} \frac{1}{\frac{1}{x}\ln(1+x)}$$

$$\lim_{u \to 0^+} (1+u)^{\frac{1}{u}} = \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

$$x = \frac{1}{u}$$

$$u = \frac{1}{x}$$

$$\lim_{u \to 0^-} (1+u)^{\frac{1}{u}} = \lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$x = \frac{1}{u} \quad u = \frac{1}{x}$$

aqui usamos que ln x é contínua

Teor: $$\lim_{u \to 0} \frac{e^u - 1}{u} = 1$$

Dem: $$\lim_{u \to 0} \frac{e^u - 1}{u} = \lim_{x \to 0} \frac{x}{\ln(x+1)} = \lim_{x \to 0} \frac{1}{\frac{1}{x}\ln(x+1)} = \lim_{x \to 0} \frac{1}{\ln(1+x)^{\frac{1}{x}}} = \frac{1}{\ln e} = 1$$

$$(1+x)^{\frac{1}{x}} \to e$$

$$e^u - 1 = x$$

$$e^u = x + 1$$

$$u = \ln(x+1)$$

Teor: $$\frac{de^x}{dx} = e^x$$

Dem: $$\frac{de^x}{dx} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} e^x \frac{(e^h - 1)}{h}.$$

$$= e^x \cdot 1 = e^x.$$

Teor: $$\frac{d \ln x}{dx} = \frac{1}{x}$$

Dem: $$\frac{d \ln x}{dx} = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} =$$

$$\lim_{u \to 0^+} (1+u)^{\frac{1}{u}} = \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$x = \frac{1}{u}$$

$$u \to 0^+ \Rightarrow x \to +\infty$$

$$\lim_{u \to 0^-} (1+u)^{\frac{1}{u}} = \lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$x = \frac{1}{u} \quad u = \frac{1}{x}$$

$$u \to 0^- \Rightarrow x \to -\infty$$

$$\lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right)$$

$$= \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \lim_{t\to 0}\ln(1+t)^{\frac{1}{t}\cdot\frac{1}{x}}$$

$t=\frac{h}{x}$ $\quad \frac{1}{h}=\frac{1}{tx}$

$$= \lim_{t\to 0}\frac{1}{x}\underbrace{\ln(1+t)^{\frac{1}{t}}}_{\to 1} = \frac{1}{x}$$

$$e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}$$

$$= \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}$$

$$= \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

$$e^{t} = \lim_{x\to+\infty}\left(1+\frac{t}{x}\right)^{x} = \lim_{y\to 0}(1+y)^{\frac{t}{y}} = \lim_{y\to 0}\Big[\underbrace{(1+y)^{\frac{1}{y}}}_{\to e}\Big]^{t} = e^{t}$$

$\frac{t}{x}=y$

$$\left(1+\frac{1}{2x}\right)^{x}$$

$$\left(1+\frac{\frac{1}{2}}{x}\right)^{x}$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^{x} = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{\frac{y}{2}}$$

$y = 2x$

$x = \frac{y}{2}$

$$= \lim_{y\to+\infty}\left[\underbrace{\left(1+\frac{1}{y}\right)^{y}}_{e}\right]^{\frac{1}{2}} = e^{\frac{1}{2}}$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{x^2}\right)^x = \lim_{x\to+\infty}\left(1+\frac{1}{x^2}\right)^{x^2\cdot\frac{1}{x}}$$

$y = x^2$

$$= \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y\cdot\frac{1}{\sqrt{y}}}$$

$\cancel{\left(\frac{1}{\sqrt{y}}\right)}$

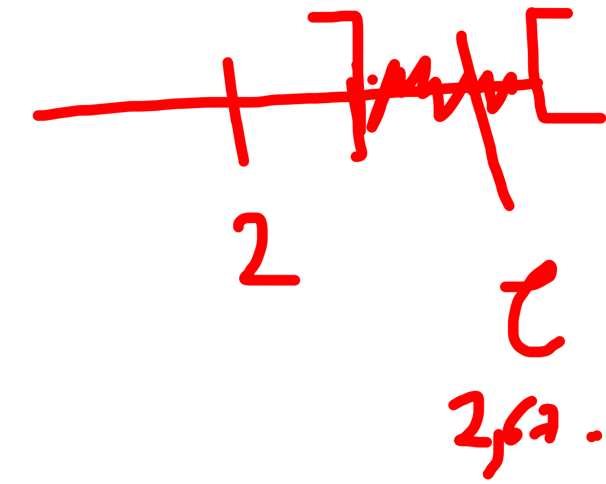
$\left(1+\frac{1}{x}\right)^x$

$1 < 1+\frac{1}{y} < 3 \implies 1^{\frac{1}{\sqrt{y}}} \le \left(1+\frac{1}{y}\right)^{y\cdot\frac{1}{\sqrt{y}}} \le 3^{\frac{1}{\sqrt{y}}}$

$\to 1$ $\qquad \to 1$ confronto $\qquad \to 1$

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x^2} = \lim_{x\to+\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{x}$$

$1+1+\frac{1}{2}+\dots \geq 2$



$\lim \left(1+\frac{1}{x}\right)^{x} = e > 2$

$\cancel{e^x \geq \left(1+\frac{1}{x}\right)^{x^2}}$ $+\infty$

$\exists$ N tal que $\left(1+\frac{1}{x}\right)^{x} \geq 2 \quad \forall x > N$

$$2^x \leq \left(1+\frac{1}{x}\right)^{x^2} \therefore \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x^2} = +\infty$$

$2^x \to +\infty$

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\underbrace{(\ln x)}_{\to -\infty}\cdot \underbrace{x}_{\to 0^+}}$$

→ L'Hospital $\quad \dfrac{0}{0} \quad \dfrac{\infty}{\infty}$