Vamos assumir

$$\lim_{x\to 0}\frac{\operatorname{sen} x}{x}=1$$

$\cos 0 = 1$ $\operatorname{sen} 0 = 0$

$\cos\frac{\pi}{2}=0$ $\operatorname{sen}\frac{\pi}{2}=1$

$\operatorname{sen}(a+b)=\operatorname{sen} a\cos b+\operatorname{sen} b\cos a$

$\cos(a+b)=\cos a\cos b-\operatorname{sen} a\operatorname{sen} b$

$\operatorname{sen}(-a)=-\operatorname{sen} a$ impar

$\cos(-b)=\cos b$ par

$\operatorname{sen}^2 a+\cos^2 a=1$

$\cos a>0$ se $-\frac{\pi}{2}<a<\frac{\pi}{2}$

$\operatorname{sen} a>0$ se $0<a<\pi$

$\operatorname{sen} x$ e $\cos x$ são periódicas de Período 2π.

Teo: $\operatorname{sen} x$ é contínua em 0.

Dem: $\lim_{h\to 0}\operatorname{sen} h=\lim_{h\to 0}\underbrace{\frac{\operatorname{sen} h}{h}}_{\to 1}\cdot\underbrace{h}_{\to 0}=0$ $=\operatorname{sen} 0$

Teo: $\cos x$ é contínua em 0.

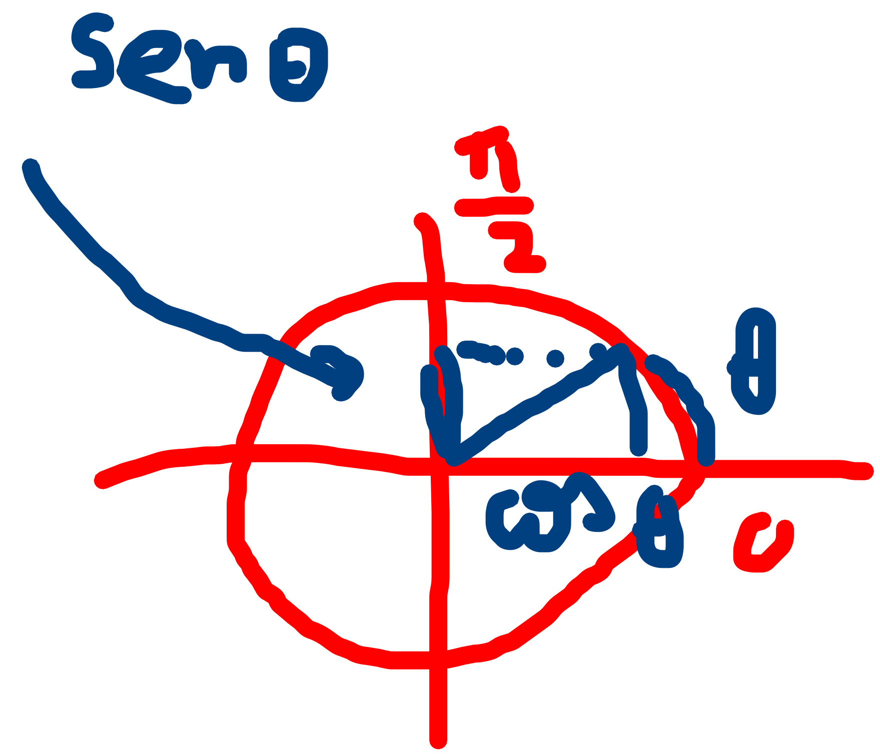
Dem: $\lim_{h\to 0}\cos^2 h=\lim_{h\to 0}1-\operatorname{sen}^2 h=1.$

raiz contínua

$$\lim_{h\to 0}\cos h=\lim_{h\to 0}|\cos h|=\lim_{h\to 0}\sqrt{\cos^2 h}$$

$\cos h\geq 0$ p/ h próximo a 0.

$=\sqrt{1}=1.$



Teor. $\operatorname{sen} x$ e $\cos x$ são contínuas.

Dem: $\lim_{h\to 0} \operatorname{sen}(x+h) = \lim_{h\to 0} \operatorname{sen} x \cos h + \underbrace{\cos x \operatorname{sen} h}_{\to 0} = \operatorname{sen} x$

$\lim_{h\to 0} \cos(x+h) = \lim_{h\to 0} \cos x \cos h - \underbrace{\operatorname{sen} x \operatorname{sen} h}_{\to 0} = \cos x$

Prop. $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$

Dem: $\lim_{h\to 0} \frac{(\cos h - 1)}{h} \cdot \frac{(\cos h + 1)}{(\cos h + 1)} = \lim_{h\to 0} \frac{-\operatorname{sen}^2 h}{h \cdot (\cos h + 1)}$

$= \lim_{h\to 0} -\operatorname{sen} h \cdot \frac{\operatorname{sen} h}{h} \cdot \frac{1}{(\cos h+1)} = 0$

$\downarrow 0$ $\quad \downarrow 1$ $\quad$ limitado para h próximos a 0.

(Na aula ainda não tínhamos visto que $\cos x$ é contínua)

Teor: $\lim_{h\to 0} \frac{\operatorname{sen}(x+h) - \operatorname{sen} x}{h} = \cos x$

$$\lim_{y\to x} f(y) = \lim_{h\to 0} f(x+h)$$

$\cos^2 h = (\cos h)^2$

$(a - b)(a + b)$

$(\cos h - 1)(\cos h + 1)$

$a^2 - b^2$

$\cos^2 h - 1 = -\operatorname{sen}^2 h$

$\cos^2 h + \operatorname{sen}^2 h = 1$

$\operatorname{sen}^2 h = 1 - \cos^2 h = -(\cos^2 h - 1)$

Dem: $\lim_{h\to 0} \frac{\operatorname{sen}(x+h)-\operatorname{sen}x}{h} = \lim_{h\to 0} \frac{\operatorname{sen}x\cos h + \operatorname{sen}h\cos x - \operatorname{sen}x}{h}$

$= \lim_{h\to 0} \frac{\operatorname{sen}x\cos h - \operatorname{sen}x}{h} + \frac{\operatorname{sen}h\cos x}{h}$

$= \lim_{h\to 0} \operatorname{sen}x \underbrace{\frac{(\cos h - 1)}{h}}_{\to 0} + \underbrace{\frac{\operatorname{sen}h}{h}}_{\to 1}\cdot\cos x = 0 + \cos x$

lim fund.

$= \cos x$

Teo: $\lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} = -\operatorname{sen} x$

Dem: $\lim_{h\to 0} \frac{\cos x\cos h - \operatorname{sen}x\operatorname{sen}h - \cos x}{h}$

$= \lim_{h\to 0} \frac{\cos x\cos h - \cos x - \operatorname{sen}x\operatorname{sen}h}{h}$

$= \lim_{h\to 0} \cos x \underbrace{\frac{(\cos h - 1)}{h}}_{\to 0} - \operatorname{sen}x \cdot \underbrace{\frac{\operatorname{sen}h}{h}}_{\to 1} =$

lim. fund

$= -\operatorname{sen}x$

Teo: $(\operatorname{sen} x)' = \cos x$

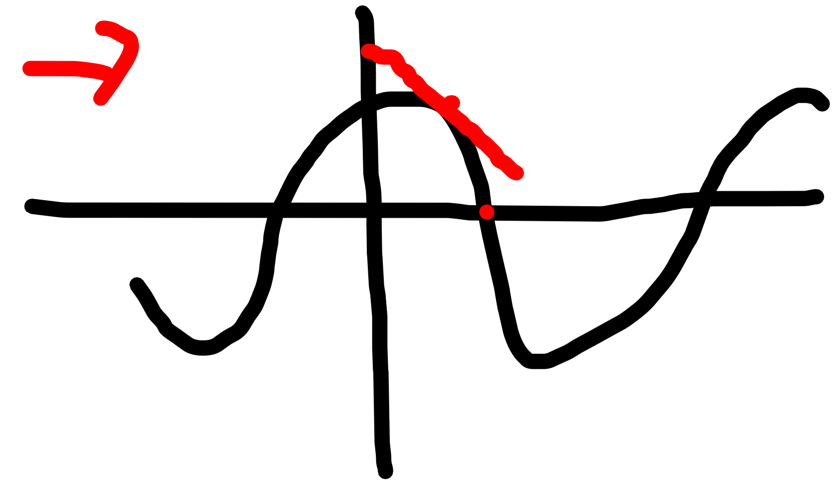
$(\cos x)' = -\operatorname{sen} x$

Ex: $\cos x = \operatorname{sen}\left(x + \frac{\pi}{2}\right)$

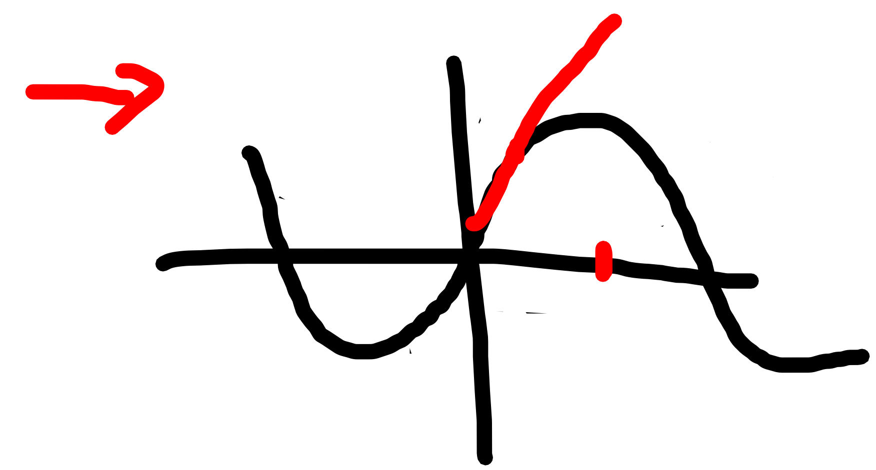
De fato,

$$\operatorname{sen}\left(x+\frac{\pi}{2}\right) = \operatorname{sen} x \underbrace{\cos\frac{\pi}{2}}_{=0} + \underbrace{\operatorname{sen}\frac{\pi}{2}}_{=1} \cos x$$

$$= \cos x$$



sen x



$$\operatorname{sen} x = \cos\left(x - \frac{\pi}{2}\right)$$

$$= \cos x \underbrace{\cos\left(-\frac{\pi}{2}\right)}_{=0} - \operatorname{sen} x \underbrace{\operatorname{sen}\left(-\frac{\pi}{2}\right)}_{=-1}$$

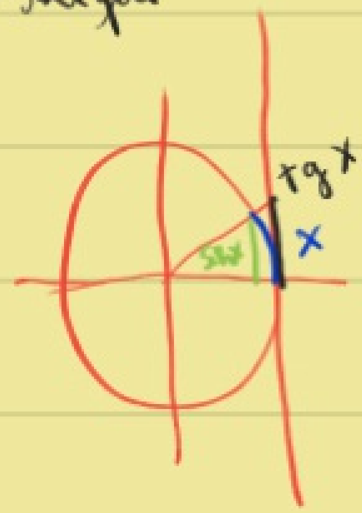
$$= \operatorname{sen} x$$

O limite fundamental

$$\lim_{x\to 0}\frac{\operatorname{sen} x}{x} = 1.$$

Assumimos o seguinte fato: $\exists\,\delta>0$ tal que

$0<\operatorname{sen} x<x<\operatorname{tg} x$, $\forall\; 0<x<\delta$.



Então

$$0<\operatorname{sen} x<x \Rightarrow 0<\frac{\operatorname{sen} x}{x}<1 \quad \forall\, x\in\,]0,\delta[.$$

$$0<x<\operatorname{tg} x \Rightarrow 0<x<\frac{\operatorname{sen} x}{\cos x} \Rightarrow 0<\cos x<\frac{\operatorname{sen} x}{x}$$

($\cos x>0$, já que $\delta<\frac{\pi}{2}$)

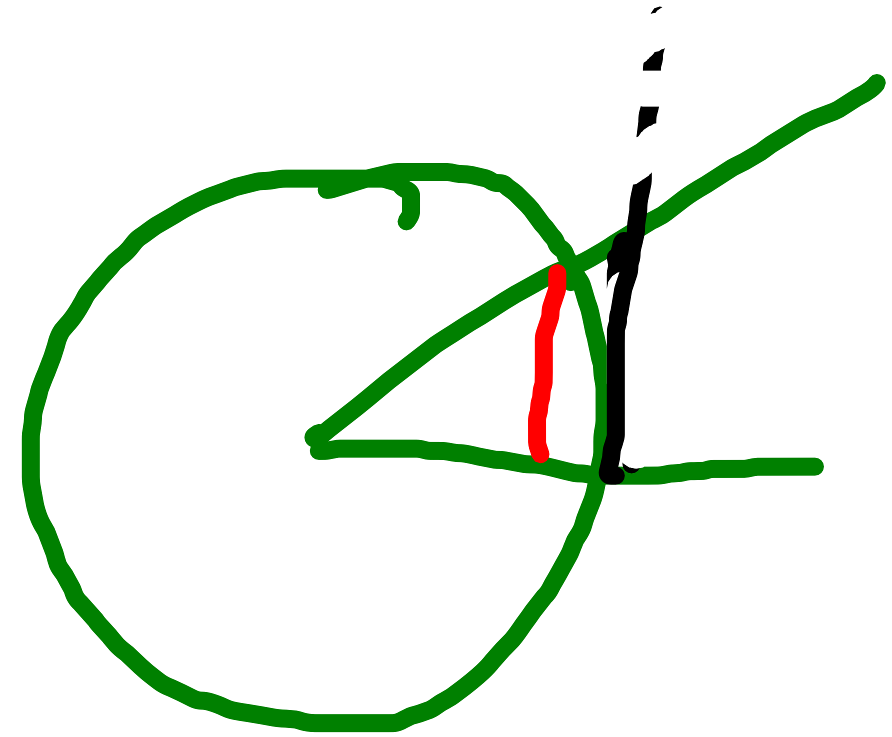
Assim para $0<x<\delta$ temos

$$\cos x<\frac{\operatorname{sen} x}{x}<1$$

$x\to 0^+ \downarrow 1$ $\qquad$ $\downarrow x\to 0^+$ 1

$\therefore$ pelo teorema do confronto

$$\lim_{x\to 0^+}\frac{\operatorname{sen} x}{x}=1.$$



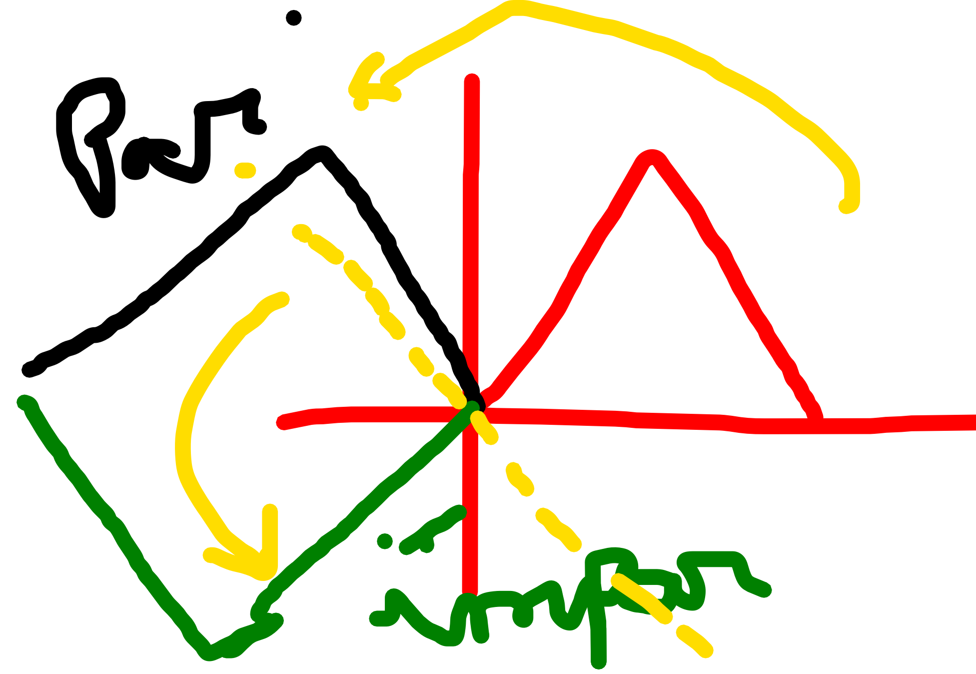
$$\lim_{x\to 0^-}\frac{\operatorname{sen} x}{x} = \lim_{t\to 0^+}\frac{\operatorname{sen}(-t)}{-t} = \lim_{t\to 0^+}\frac{-\operatorname{sen} t}{-t} = \lim_{t\to 0^+}\frac{\operatorname{sen} t}{t} = 1$$

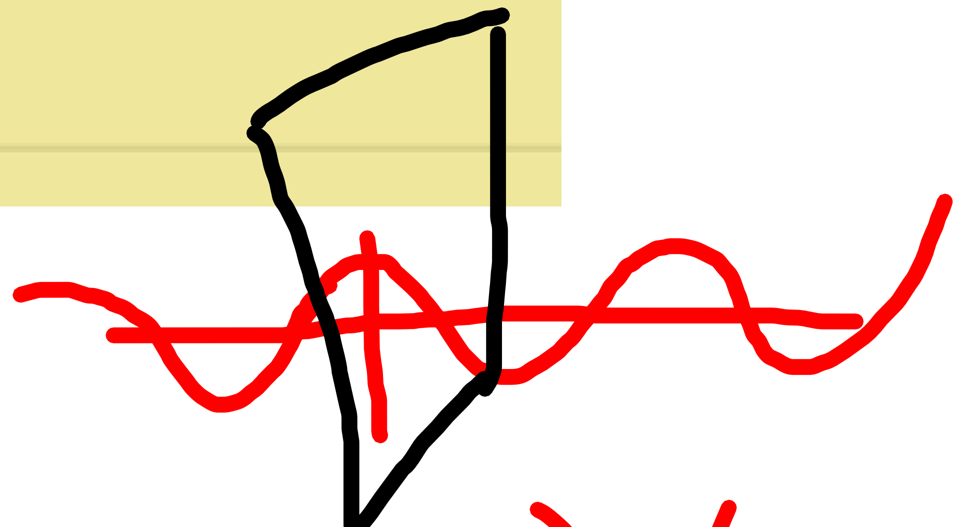
$t = -x$

← seno ímpar

visto anteriormente

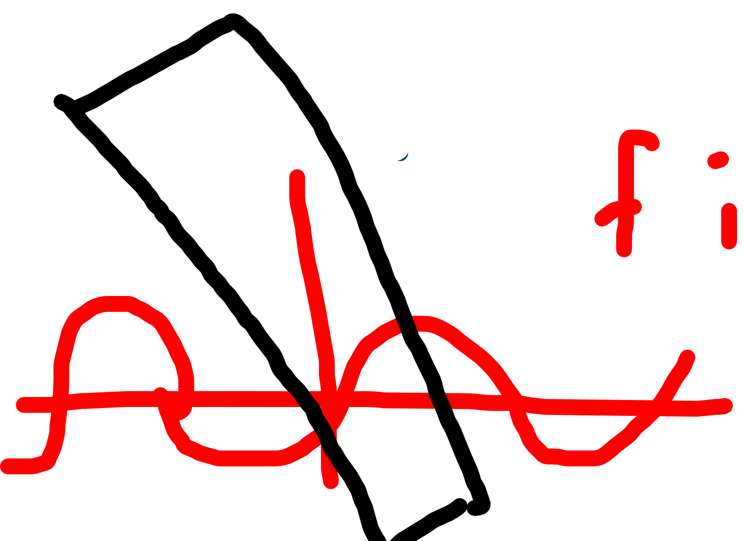
Como os limites laterais dão 1, segue que $\lim_{x\to 0}\frac{\operatorname{sen} x}{x} = 1$





Def. f par

$f(x) = f(-x)\ \forall x \in \mathbb{R}$



f impar

$f(x) = -f(-x)\ \forall x \in \mathbb{R}$

$f(-x) = -f(x)$

Função módulo é contínua

$\forall \varepsilon > 0 \ \exists \delta > 0$

$|h| < \underset{\varepsilon}{\underset{\shortparallel}{\delta}} \Rightarrow \big||x+h| - |x|\big| < \varepsilon$

Af. $\big||a| - |b|\big| \leq |a-b|$

$|a| = |a-b+b| \leq |a-b| + |b|$

$|a| - |b| \leq |a-b|$

$|b| = |b-a+a| \leq |b-a| + |a|$

$|b| - |a| \leq |b-a| = |a-b|$ $\therefore$

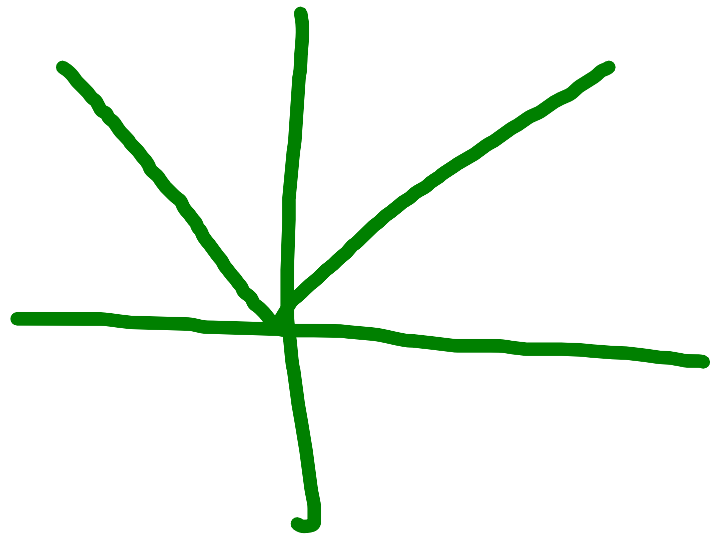
$$||a|-|b|| \leq |a-b|.$$

$$||x+h|-|x|| \leq |x+h-x| = |h| < \varepsilon$$

$|x|$

não é

derivável



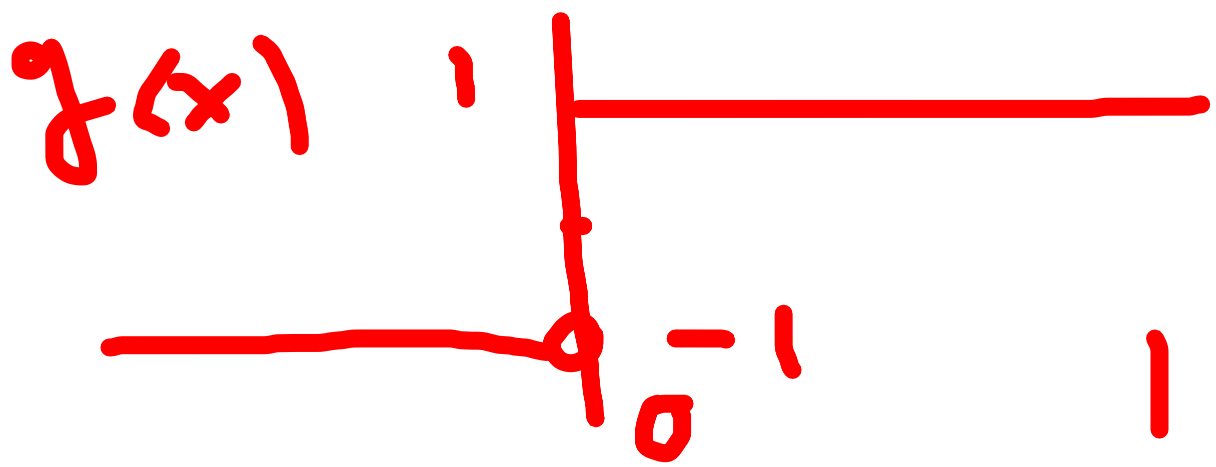
$|x^2|$ é derivável

em 0

Prop: $\lim_{x \to a} |f(x)| = 0$ (só vale p/ 0!!)

$$\Rightarrow \lim_{x \to a} f(x) = 0$$

Ex



$|g(x)| = 1$

$$\lim_{x \to 0} |g(x)| = 1$$

$$\lim_{x \to 0^+} g(x) = 1 \quad \lim_{x \to 0^-} g(x) = -1$$

Ex

$$f(x) = \begin{cases} 1 & \text{se } x \in \mathbb{Q} \\ -1 & \text{se } x \notin \mathbb{Q} \end{cases}$$

$$|f(x)| = 1$$

$$\therefore \lim_{x \to a} |f(x)| = 1, \quad \forall a \in \mathbb{R}$$

$\lim_{x \to a} f(x)$ não existe $\forall a \in \mathbb{R}$

$\nexists$

$S, \lim_{x \to a} |f(x)| = 0$

$\Rightarrow \forall \varepsilon > 0 \ \exists \delta > 0$

$0 < |x - a| < \delta \Rightarrow ||f(x)| - 0| < \varepsilon$

$\Rightarrow ||f(x)|| < \varepsilon$

$\Rightarrow |f(x)| < \varepsilon$

$\Rightarrow |f(x) - 0| < \varepsilon$

$\forall \varepsilon > 0 \ \exists \delta > 0$
$0 < |x - a| < \delta \Rightarrow |f(x) - 0| < \varepsilon$

$\therefore \lim_{x \to a} f(x) = 0$

$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n \qquad (n \in \mathbb{N})$$

$$\boxed{e = \lim_{u \to 0} (1+u)^{\frac{1}{u}}}$$

	2	-1	-1
1		2	1
	2	1	0

$u = x^3 - 1$

$x \to 1 \Rightarrow u \to 0$

$u \to 0 \Rightarrow \frac{\operatorname{sen} u}{u} \to 1$

$$\lim_{x \to 1} \frac{\operatorname{tg}(x^3-1)}{2x^2 - x - 1} = \lim_{x \to 1} \frac{\operatorname{sen}(x^3-1)}{(x-1)\cdot(2x+1)} \cdot \frac{1}{\cos(x^3-1)}$$

$$= \lim_{x \to 1} \underbrace{\frac{\operatorname{sen}(x^3-1)}{x^3-1}}_{\to 1} \cdot \frac{\overbrace{(x^2+x+1)}^{\to 3}}{\underbrace{(2x+1)}_{\to 3}} \cdot \underbrace{\frac{1}{\cos(x^3-1)}}_{\to 1} = 1$$

$$\lim_{x \to 0} \frac{\cos(\pi x) - 1}{x^2} \cdot \frac{(\cos(\pi x) + 1)}{(\cos(\pi x) + 1)}$$

$\to 0$ $\to 0$

$$= \lim_{x \to 0} \frac{\cos^2(\pi x) - 1}{x^2(\cos \pi x + 1)} = \lim_{x \to 0} \frac{-\operatorname{sen}^2 \pi x}{x^2} \cdot \frac{1}{\cos \pi x + 1}$$

$*$

$$= \lim_{x \to 0} -\frac{\operatorname{sen}^2 \pi x}{(\pi x)^2} \cdot \frac{\pi^2}{(\cos \pi x + 1)} = -\frac{\pi^2}{2}$$

$\to 1^2 = 1$ $\quad 1$ $\quad \to 2$

$$* = \left(\frac{\operatorname{sen} \pi x}{\pi x}\right)^2$$

$u = \pi x \to 0$ se $x \to 0$

vantim, $\frac{\operatorname{sen} u}{u} \to 1$ se $u \to 0$

$$\lim_{x \to -\infty} \frac{x^7 - 18x^3}{8x^5 - 6x^4}$$

$$= \lim_{x \to -\infty} \frac{x^7\left(1 - 18/x^4\right)}{x^5\left(8 - 6/x\right)}$$

$$= \lim_{x \to -\infty} \frac{x^2\left(1 - 18/x^4\right)}{\left(8 - 6/x\right)}$$

$\to +\infty$ $\to 0$ $\to 1$ $\to 8$

$= +\infty$

$+ \,,\; - \,,\; \cdot \,,\; \div \,,\; \circ$

- $(f+g)' = f' + g'$
- $(f \cdot g)' \neq f' \cdot g'$
- $\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$

L'Hospital

$$\frac{f'}{g'}$$