# PRO 5971 - Statistical Process Monitoring

## Shewhart control chart: join monitoring of the mean and variance

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# Outline

# Joint monitoring the mean and variance

- Sometimes there is interest to joint monitoring the mean and the variance
- Thus two types of charts are used simultaneously as $\overline{X}$ for the mean and $S^2$ (or $S$ or $R$) for the variance in the process monitoring
- The stability of variability is firstly verified. If it is in-control, then the monitoring of the mean starts
- Whenever any chart signals, an action is taken.
- Thus the global type one error $\alpha$ is $1-(1-\alpha_{\overline{X}})\times(1-\alpha_i)$=$\alpha_{\overline{X}}+\alpha_i-\alpha_{\overline{X}}\times\alpha_i$, $i=S^2,S,R$
- And the overall power is $power_{\overline{X}}+power_i-power_{\overline{X}}\times power_i$, $i=S^2,S,R$

# Exercise

Consider that $\overline{X}$ and $S^2$ control chart are used to monitor jointly the mean and variance

1 - Show that the global type I error $\alpha$ of is $\alpha_{\overline{X}} + \alpha_{S^2} - \alpha_{\overline{X}} \times \alpha_{S^2}$,

2 - Demonstrate that the overall power is $power_{\overline{X}} + power_{S^2} - power_{\overline{X}} \times power_{S^2}$,

## Exercises

To solve the next items, consider samples of size $n = 5$ with $\mu_0 = 5$ and $\sigma_0 = 4$

a)determine the control limits for $\overline{X}$ and $\sigma^2$ to have an overall $\alpha = 0.0027$. Choose $\alpha_{\overline{X}}$ and $\alpha_{S^2}$ adequately

b) (consider the standard deviation stable) If the mean shifts to 7.5, what is the probability of the $\overline{X}$ control chart signals such shift in the sample after the shift? And what about to signal before the 4th sample?

c) (consider mean stable) If the standard deviation shifts to 6.0, what is the probability of the $S^2$ control chart to signal at first sample after the shift ? And what is the probability of the $\overline{X}$ control chart signals at first sample after the shift?

d) Consider that mean and standard deviation shift to 6 and is the probability that such shifts being signaled by any chart?

# Exercise

**Table 1:** Volumes of soft drink in $cm^3$ taken at every 30 min in 15 hours of production

| Sample | X1 | X2 | X3 | Sample | X1 | X2 | X3 |
|---|---|---|---|---|---|---|---|
| 1 | 252.16 | 250.34 | 249.7 | 16 | 248.29 | 249.6 | 249.15 |
| 2 | 248.34 | 248.61 | 250.63 | 17 | 249.59 | 249.89 | 248.51 |
| 3 | 249.19 | 250.02 | 250.84 | 18 | 248.03 | 249.11 | 249.81 |
| 4 | 251.29 | 249.93 | 250.24 | 19 | 250.99 | 251.5 | 249.92 |
| 5 | 248.16 | 250.41 | 251.19 | 20 | 247.62 | 250.43 | 250.39 |
| 6 | 250.37 | 251.98 | 248.44 | 21 | 250.6 | 250.54 | 250.2 |
| 7 | 250.31 | 248.71 | 251.13 | 22 | 250.44 | 251.17 | 250.01 |
| 8 | 250.27 | 249.64 | 249.92 | 23 | 249.35 | 249.16 | 250.2 |
| 9 | 250.72 | 250.8 | 249.35 | 24 | 248.17 | 249.94 | 248.15 |
| 10 | 250.45 | 249.18 | 250.04 | 25 | 249.98 | 251.57 | 249.79 |
| 11 | 251.76 | 252.01 | 251.9 | 26 | 250.1 | 249.57 | 249.11 |
| 12 | 249.33 | 251.21 | 250.58 | 27 | 248.82 | 251.01 | 248.9 |
| 13 | 249.26 | 247.67 | 249.99 | 28 | 248.39 | 248.26 | 250.57 |
| 14 | 249.41 | 249.01 | 249.51 | 29 | 251.43 | 250.92 | 250.12 |
| 15 | 249.9 | 249.07 | 250.32 | 30 | 248.82 | 249.28 | 248.57 |

- Use the data set from Table 1 to answer these queries
- if each chart (one for the mean and other for the variability) is set to have each $\alpha$=0.0027, what happens with the global error of type I?
- if a global error of type I equal to 0.0027 is desirable, adjust the two control charts in order to achieve this target.
- Using the above control limits, If the standard deviation increases in 30%, What is probability of the x-bar to give a signal? (Assumes that the mean does not shift and its type I error is equal to 0.1%).

# Exercise

1. Diameters of shafts are measured in 30 samples of 5 units each - Averages and ranges values of each sample are in Table 2
2. Use a global $\alpha = 0.0027$, determine the control limits for $\overline{X}$ and $R$ charts considering unknown $\mu_0$ cm and $\sigma = 5$ and known $\mu_0 = 5$ cm $\sigma = 5$
3. If the process mean shifts to $\mu_1 = 7.50$ what is the probability to detect such change immediately at the first sample after the shift using the $\overline{X}$ chart? And to detect such shift before than the fourth sample after the change?
4. If the standard deviation shifts to $\sigma_1 = 3.6$, what is the probability to detect such event by R chart at the first sample after the change? And what is the probability to detect such event by $\overline{X}$ chart at the first sample after shift?
5. Beyond the change in the variability, consider that the process mean also shifts to $\mu_1 = 6$. Recalculate the probability of the last item

# Data

**Table 2:** Exercise

| # of sample | AVG | Range | # of sample | AVG | Range |
|---|---|---|---|---|---|
| 1 | 5.00 | 4.12 | 16 | 7.10 | 2.00 |
| 2 | 7.05 | 6.18 | 17 | 4.90 | 0.12 |
| 3 | 3.10 | 4.00 | 18 | 5.00 | 2.24 |
| 4 | 6.15 | 7.04 | 19 | 4.00 | 4.12 |
| 5 | 2.90 | 4.12 | 20 | 5.20 | 6.00 |
| 6 | 5.05 | 0.08 | 21 | 3.85 | 2.12 |
| 7 | 6.00 | 4.12 | 22 | 3.90 | 4.12 |
| 8 | 3.25 | 6.12 | 23 | 6.00 | 1.19 |
| 9 | 4.90 | 10.20 | 24 | 6.15 | 1.20 |
| 10 | 5.00 | 2.06 | 25 | 4.90 | 5.24 |
| 11 | 6.10 | 8.16 | 26 | 5.00 | 4.09 |
| 12 | 3.75 | 4.12 | 27 | 4.90 | 4.24 |
| 13 | 5.00 | 7.91 | 28 | 6.55 | 4.15 |
| 14 | 2.95 | 3.00 | 29 | 5.00 | 4.12 |
| 15 | 5.00 | 4.24 | 30 | 3.45 | 7.67 |

# Exercise

- Consider that $\overline{X}$ and $R$ charts are jointly used. And an action is taken if at least one chart signals. Discuss how to evaluate the joint performance of the two charts
- If an overall $\alpha$=0.001 is desired, discuss how to choose the control limits for the two charts to assure good performance for each chart individually? Choose two sets of control charts
- Using the control limits determined in the previous item, obtain the power for $\mu_1 = \mu_0 + k\sigma$, $\sigma_1 = \delta\sigma_0$, $k = 1.0; 1.5$ combined with $\delta = 0.5; 1$. Which set has better performance?

# Exercise

Consider that $\overline{X}$ and $S^2$ charts are jointly used to monitor the mean and the variance.

In-control parameters: $\mu_0 = 0; \sigma_0 = 1$.

Out-of-control parameters: $\mu_1 = \mu_0 + k\sigma_0$; $k = 0, 0.5, 1, 2$; and $\sigma_1 = a\sigma_0$; $a = 1, 1.5, 2$

Use $\alpha_{\overline{X}} = \alpha_{S^2} 0.00135$ and $n = 10$

Find the overall power and individual powers of $\overline{X}$, $\sigma^2$,

# Variable Control Chart: Departures from Normality

- $\overline{X}$ chart
  - Known the form of underlying distribution: derive the sampling distribution and get the exact control limits - Difficult in some cases
  - Otherwise use the normal theory results
  - In general: control limits based on normal theory are robust (sample size $n = 4, 5$ are enough to ensure the robustness), unless for extremely non-normal
  - Worst results observed for Gamma distribution for small values of r [ as r=0.5 and r=1 (exponential)]
- R chart: Very sensitive to departures from normality

# Variable Control Chart: for individual observations

- Assumption: X normally distributed
- $\mu$ and $\sigma$, both available
  - The control limits: $\mu_0 \pm z_{\alpha/2}\sigma$
- $\mu$ and $\sigma$, unavailable, assuming absent assignable causes, use
  - $\overline{X}$ and $\hat{\sigma_1} = \frac{\overline{MR}}{d_2}$, $\overline{MR} = \frac{\sum_{i=1}^{n-1} MR_i}{n-1}$, $MR_i = |X_i - X_{i+1}|$, $d_2 = 1.128$
  - Central line: $\overline{X}$; Control limit: $\overline{X} \pm z_{\alpha/2}\hat{\sigma}$
- Or use $\hat{\sigma}_2 = \frac{S}{c_4}$,
- When a sustained shift in the mean is present, an estimator based on the median of $MR_i$ ($\widetilde{MR}$) can be used

$$\hat{\sigma}_3 = \frac{\widetilde{MR}}{d_4},\ d_4 = 0.955$$

# Variable Control Chart: Departures from Normality

- Individual Observations: Very sensitive to departures from normality
    - 1) Use the percentiles of the underlying distribution (from histogram, empirical distribution) as the control limits
    - 2) Transform the original variable in a new variable that is approximately normal and then apply control chart to the new variable