PRO 5971

Statistical Process Monitoring - About process capability

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Outline

Process Capability

Statistical methods - useful in the whole cycle of product as:

- in development of the product
- quantify the variability of process
- analyze the variability in relation the requirement or specifications of the product
- to help the development and the manufacturing in reduction of the variability

Process capability - verify if the process is able to produce (conforming) items such that met the specifications

Defining many limits

- Natural tolerance limits (TL):
  - $\mu_0 \pm 3\sigma$
  - Meaning: If the random variable $X \sim N(\mu_0, \sigma)$ then it is expected to have averagely 27 items out of these limits per 10 thousand produced items
  - Which should be the tolerance limits for a process following $X \sim N(1000\ ml, 100\ ml)$?
- Control limits (CL)
  - To judge if the process is stable related to a parameter of interest like the mean (for example)
  - $\mu_0 \pm 3\sigma/\sqrt{n}$ is known as 3 sigma control limits to monitor the process mean

Specification limits

- Externally set by management, engineers, customer, product developers to minimize risk, to avoid penalty, etc
- It is used to evaluate the items, units or individual observations. Units that do not meet the specification limits are said to be non-conforming.
- NO MATHEMATICAL/STATISTICAL RELATIONSHIP BETWEEN THE CL AND SL
- NO SENSE IN COMPARING SL AND CL
- Plot of SL on $\overline{X}$ chart is completely **incorrect**
- Dealing with plots of **individual observations** is helpful to plot the SL on that chart

cont Specification limits

Process stable (μ_0)

- it is desirable that almost "all values" of X are within the specification limits
- That is, $p_{L_0} = P(X > USL|\mu_0) + P(X < LSL|\mu_0)$ should be low.

Process unstable (μ_1)

- The proportion of values out of specification limits $p_{L_1} = P(X > USL|\mu_1) + P(X < LSL|\mu_1)$ must increase.
- The probability of the control chart in signaling such shift is

$$power = P(\overline{X} > UCL|\mu_1) + P(\overline{X} < LCL|\mu_1)$$

Exercise

Consider a filling coffee package process. The individual weight $X \sim N(1000g; 10g)$. Samples of 5 packages are collected. Use as the specification limits: [985; 1015] and 3 sigma control limits for the mean monitoring. Due to special causes the mean shifts to $\mu_1 = \mu_0 + \delta\sigma$, $\delta = 0.25, 0.5, 1, 1.25, 1.5, 2, 3$. Calculate the power and non-conforming fraction p_L

δ	p_L	power
0.25		
0.5		
1.0		
1.25		
1.5		
2.0		
3.0		

Exercises

- Find papers discussing the most used transformations to get approximately a normal distribution? What are the cautions and consequences?
- The following parameters for $\overline{X}$ chart are: UCL=710; Center line=700 and LCL=690; with $\sigma^2 = 64$
  - a)If the specification limits are set at 705 $\pm$15, give the nonconforming fraction;
  - b)For the $\overline{X}$, find the probability of type I error, assuming σ constant for a sample of 5 units.
  - c)Suppose the process mean shifts to 705 and the standard deviation simultaneously shifts to 12. Find the probability of detecting this shift on the $\overline{X}$ on the second subsequent sample.
  - d)For the shift of the previous item, find the average run length

- One pound coffee cans are filled by a machine, sealed and then weighed automatically. After adjusting for the weight of the can, any package that weighs less than 16 oz is cut out of the conveyor. The weights of 25 successive cans are show in Table 1. Set up a moving average range control and a control chart individual observations.
  - Estimate the mean and the s.d.
  - Is it reasonable to assume that can weigh is normally distributed
  - If the process remains in control at this level, what percentage of cans will be under filled

Data

Table 1: Can weight

# can	weight	# can	weight
1	16.11	14	16.12
2	16.08	15	16.10
3	16.12	16	16.08
4	16.10	17	16.13
5	16.10	18	16.15
6	16.11	19	16.12
7	16.12	20	16.10
8	16.09	21	16.08
9	16.12	22	16.07
10	16.10	23	16.11
11	16.09	24	16.13
12	16.07	25	16.10
13	16.13		

Signal of control chart x non-conforming items

In practical situations, it is common to relate a point of the monitored statistic out of control limits with the production of non-conforming items. However one thing does not imply another.

Let us consider an example.

Control limits: [3.64; 15.76]; Specification Limits: [−5; 15]

Sample #	Values					$\overline{X}$
1:	9.10	6.17	10.73	13.83	13.60	10.68
2:	15.20	3.45	9.30	13.28	6.74	9.59
3:	2.11	−4.09	10.08	1086	−3.98	3.00
4:	13.75	12.44	11.38	10.47	9.63	11.53

All values are within the specification limits but $\overline{X} = 3.00$ indicates an out-of-control situation.

Capability index

However, the process may be said to be in-control and has a certain quantity of items which does not meet the specification limits. In this case it is said **the process is not capable** to meet the specification even being stable the process.

Capability Index -

- Used to measure if the process meets the specification
- There is not relationship between % of item able to meet the specifications and its value; such relation depends on the probability density function
- In general, larger index higher % of item meeting the specifications.

Some capability indexes

There are several capability indexes, the most used are:

$$C_p = \frac{USL - LSL}{6\sigma}$$

$$C_{pk} = min\left(\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right)$$

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (d - \mu)^2}}, d = \frac{USL + LSL}{2}$$

For one sided specification, the Process Capability index can be expressed as:

$$C_{pu} = \frac{USL - \mu}{3\sigma}$$

$$C_{pl} = \frac{\mu - LSL}{3\sigma}$$

About process capability C_p

$$C_p = \frac{USL - LSL}{6\sigma}$$

- C_p measures the relationship between the specification and tolerance limits
- For a correct accuracy and validity of C_p, some assumptions are considered:
  - The quality characteristic has a normal distribution
  - The process must be **stable, in statistical control**
  - In a case of two-sided specification, the process mean is centered between the LSL and USL
- Some reference values of C_p=1; 1.33; 1.67

Estimators for C_p

C_p can be estimated as

$$\widehat{C}_p = \frac{USL - LSL}{6\widehat{\sigma}}, \ \widehat{\sigma} = R, S$$

Let us consider $\widehat{\sigma} = S$ then

$$S = \frac{USL - LSL}{6\widehat{C}_p} \rightarrow S^2 = \left(\frac{USL - LSL}{6\widehat{C}_p}\right)^2$$

As $\dfrac{(n-1)S^2}{\sigma^2} = \dfrac{n-1}{\sigma^2} \times \left(\dfrac{USL - LSL}{6\widehat{C}_p}\right)^2$ follows a Chi-square distribution with (n-1) degrees of freedom, then a decision rule can be stated for C_p

Decision rule for $\widehat{C}_p$

Assume there is interest to verify if $H_0 : C_P$ is stable at level c_0 or $H_1 : C_P < c_0$ (capability of process deteriorated/decreased)

Note that

$$\frac{(n-1)S^2}{\sigma^2} = \frac{n-1}{\sigma^2} \times \left(\frac{USL - LSL}{6\widehat{C}_p}\right)^2 = \frac{n-1}{\widehat{C}_p^2} \times \left(\frac{USL - LSL}{6\sigma}\right)^2 = \frac{(n-1)C_p^2}{\widehat{C}_p^2}$$

If the process is stable then $\dfrac{(n-1)c_0^2}{\widehat{C}_p^2}$. Fixed α then

$$P\left(\frac{(n-1)c_0^2}{\widehat{C}_p^2} > \chi^2_{(n-1,1-\alpha)}\right) = \alpha \rightarrow P\left(\frac{\widehat{C}_p^2}{(n-1)c_0^2} < \frac{1}{\chi^2_{(n-1,1-\alpha)}}\right)$$

Decision rule: if $\widehat{C}_p^2 < \dfrac{(n-1)c_0^2}{\chi^2_{(n-1,1-\alpha)}}$ signals that C_p may be deteriorated.

$\chi^2_{(n-1,1-\alpha)}$, a quantile at $(1-\alpha)$ of chi-square distribution

And the power is ....

Letting $k = \dfrac{(n-1)c_0^2}{\chi^2_{(n-1,1-\alpha)}}$.

As $\dfrac{(n-1)c_0^2}{\widehat{C}_p^2}$ follows a chi-square distribution then the power is equal to

$$P(\widehat{C}_p^2 < k | C_P^2 = \delta C_{p_0}^2) = P\left(\frac{1}{\widehat{C}_p^2} > \frac{1}{k} | C_P^2 = \delta C_{p_0}^2\right) =$$

$$P\left(\frac{(n-1)\delta C_{p_0}^2}{\widehat{C}_p^2} > \frac{(n-1)\delta C_{p_0}^2}{k} | C_P^2 = \delta C_{p_0}^2\right) = P\left(\chi^2_{n-1} > \frac{(n-1)\delta C_{p_0}^2}{k} | C_P^2 = \delta C_{p_0}^2\right)$$

Replacing k by $\dfrac{(n-1)c_0^2}{\chi^2_{(n-1,1-\alpha)}}$ and some manipulations

$$P\left(\chi^2_{(n-1)} > \delta\chi^2_{(n-1,1-\alpha)} | C_P^2 = \delta C_{p_0}^2\right)$$

- But C_P does not take account where the process mean is located relatively to the specifications
- To deal with, the index C_{pk} may be used

$$C_{pk} = min(C_{pu}, C_{pl})$$

$$C_{pk} = min\left(\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right)$$

- Like C_p, we have also an estimate of C_{pk}
- The decision rule is based on its confidence interval expressed as

$$\hat{C}_{pk}\left[1 - z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}}\,\right] \leq C_{pk} \leq$$

$$\hat{C}_{pk}\left[1 + z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}}\,\right]$$

Exercise

1. - Consider that when the process is stable. The specification limits are [985;1005]. Use $\alpha = 2.5\%$ and sample size $n = 30$ to obtain a decision rule for $H_0 : C_p = 1.33$ versus $H_1 : C_p < 1.33$
2. - Obtain the power if the C_p has decreased to 1.1, 1.0; 0.9.
3. - Find the asymptotic control limits for C_p. Are they close to the probability control limits?
4. - By a Monte Carlo simulation (10 thousand runs), indicate a decision rule based on the empirical distribution of $\widehat{C}_{pk}$ for the C_{pk} index considering: a sample size $n = 10$, a process normally distributed $N \sim (\mu = 900, \sigma = 10)$, specification limits [840;936] and $\alpha = 0.01$. Use as estimators of σ, the sample standard deviation S and sample range R.
5. - Using the rules stated in previous item, what are the powers a) if only the mean has shift to $\mu_1 = 910$ b) if only σ shifts to 15; c) and when both shift.

References