PRO 5971 - Statistical Process Monitoring

Shewhart control chart: monitoring the variance by S and S^2 chart

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Outline

- $S = \sqrt{\sum_{j=1}^{n} \frac{(X_j - \overline{X}_)^2}{n-1}}$, standard deviation of sample size n observations
- E(S)=σc_4; $c_4 = \left(\frac{2}{n-1}\right)^{1/2} \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}$
- $Var(S) = \sigma_S^2 = (1 - c_4)\sigma^2$
- σ_0 available
  - Central line: $c_4\sigma_0$; Control limits: $c_4\sigma_0 \pm z_{\alpha/2}\sigma_0\sqrt{1 - c^4}$
  - $z_{\alpha/2} = 3$; $B_i = c_4 \pm 3\sqrt{1 - c^4}$, Control limits: $B_i\sigma_0$, i=5;6
- σ_0 unavailable
  - $\widehat{E(S)} = \overline{S} = \frac{S_1 + \ldots + S_m}{m}$, estimator of E(S) $\rightarrow \hat{\sigma} = \frac{\overline{S}}{c_4}$;
  - $\hat{\sigma}_S = \sqrt{1 - c^4}\,\frac{\overline{S}}{c_4}$
  - In case of variable sample size use $\overline{S} = \sqrt{\frac{\sum_{i=1}^{m}(n_i - 1)S_i^2}{\sum_{i=1}^{m}(n_i - 1)}}$
  - Central line: $\overline{S}$; Control limits: $\overline{S} \pm z_{\alpha/2}\sqrt{1 - c^4}\,\frac{\overline{S}}{c_4}$
  - $z_{\alpha/2} = 3$; $B_j = 1 \pm \frac{3\sqrt{1 - c^4}}{c_4}$;
  - Central line: $\overline{S}$; Control Limits: $B_j\overline{S}$, j=3, 4
- See Tables for c_4, B_3, B_4, B_5, B_6

Building $\overline{X}$ chart with $\overline{\overline{X}}$ and $\overline{S}$

- Fixed α
- Central line: $\overline{\overline{X}}=\frac{\overline{X}_1+\ldots+\overline{X}_m}{m}$
- Control limits: $\overline{\overline{X}} \pm z_{\alpha/2}\frac{\overline{S}}{c_4\sqrt{n}}$
- If $z_{\alpha/2}=3$, $A_3 = \frac{3}{c_4\sqrt{n}}$, the control limits are: $\overline{\overline{X}} \pm A_3\overline{S}$
- See Tables for c_4, A_3

■ APPENDIX VI

Factors for Constructing Variables Control Charts

Observations in Sample, n	Chart for Averages: Factors for Control Limits			Chart for Averages: Factors for Center Line		Chart for Standard Deviations: Factors for Control Limits				Chart for Ranges: Factors for Center Line		Chart for Ranges: Factors for Control Limits				
	A	A_2	A_3	c_4	$1/c_4$	B_3	B_4	B_5	B_6	d_2	$1/d_2$	d_3	D_1	D_2	D_3	D_4
2	2.121	1.880	2.659	0.7979	1.2533	0	3.267	0	2.606	1.128	0.8865	0.853	0	3.686	0	3.267
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276	1.693	0.5907	0.888	0	4.358	0	2.574
4	1.500	0.729	1.628	0.9213	1.0854	0	2.266	0	2.088	2.059	0.4857	0.880	0	4.698	0	2.282
5	1.342	0.577	1.427	0.9400	1.0638	0	2.089	0	1.964	2.326	0.4299	0.864	0	4.918	0	2.114
6	1.225	0.483	1.287	0.9515	1.0510	0.030	1.970	0.029	1.874	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	0.419	1.182	0.9594	1.0423	0.118	1.882	0.113	1.806	2.704	0.3698	0.833	0.204	5.204	0.076	1.924
8	1.061	0.373	1.099	0.9650	1.0363	0.185	1.815	0.179	1.751	2.847	0.3512	0.820	0.388	5.306	0.136	1.864
9	1.000	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707	2.970	0.3367	0.808	0.547	5.393	0.184	1.816
10	0.949	0.308	0.975	0.9727	1.0281	0.284	1.716	0.276	1.669	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	0.905	0.285	0.927	0.9754	1.0252	0.321	1.679	0.313	1.637	3.173	0.3152	0.787	0.811	5.535	0.256	1.744
12	0.866	0.266	0.886	0.9776	1.0229	0.354	1.646	0.346	1.610	3.258	0.3069	0.778	0.922	5.594	0.283	1.717
13	0.832	0.249	0.850	0.9794	1.0210	0.382	1.618	0.374	1.585	3.336	0.2998	0.770	1.025	5.647	0.307	1.693
14	0.802	0.235	0.817	0.9810	1.0194	0.406	1.594	0.399	1.563	3.407	0.2935	0.763	1.118	5.696	0.328	1.672
15	0.775	0.223	0.789	0.9823	1.0180	0.428	1.572	0.421	1.544	3.472	0.2880	0.756	1.203	5.741	0.347	1.653
16	0.750	0.212	0.763	0.9835	1.0168	0.448	1.552	0.440	1.526	3.532	0.2831	0.750	1.282	5.782	0.363	1.637
17	0.728	0.203	0.739	0.9845	1.0157	0.466	1.534	0.458	1.511	3.588	0.2787	0.744	1.356	5.820	0.378	1.622
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496	3.640	0.2747	0.739	1.424	5.856	0.391	1.608
19	0.688	0.187	0.698	0.9862	1.0140	0.497	1.503	0.490	1.483	3.689	0.2711	0.734	1.487	5.891	0.403	1.597
20	0.671	0.180	0.680	0.9869	1.0133	0.510	1.490	0.504	1.470	3.735	0.2677	0.729	1.549	5.921	0.415	1.585
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459	3.778	0.2647	0.724	1.605	5.951	0.425	1.575
22	0.640	0.167	0.647	0.9882	1.0119	0.534	1.466	0.528	1.448	3.819	0.2618	0.720	1.659	5.979	0.434	1.566
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438	3.858	0.2592	0.716	1.710	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429	3.895	0.2567	0.712	1.759	6.031	0.451	1.548
25	0.600	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.420	3.931	0.2544	0.708	1.806	6.056	0.459	1.541

For $n > 25$.

$$A = \frac{3}{\sqrt{n}} \quad A_3 = \frac{3}{c_4\sqrt{n}} \quad c_4 \cong \frac{4(n-1)}{4n-3}$$

$$B_3 = 1 - \frac{3}{c_4\sqrt{2(n-1)}} \quad B_4 = 1 + \frac{3}{c_4\sqrt{2(n-1)}}$$

$$B_5 = c_4 - \frac{3}{\sqrt{2(n-1)}} \quad B_6 = c_4 + \frac{3}{\sqrt{2(n-1)}}$$

Figure 1: Tables of constants

- Sample Variance $S^2 = \sum_{j=1}^{n} \frac{(X_j - \overline{X}_)^2}{n-1}$,
- Sample variance S^2 is an unbiased estimator as $\mathrm{E}(S^2)=\sigma^2$
- To determine the control limits the distribution of S^2 must be known.
- $Z \sim N(0;1)$ then $Z^2 \sim \chi_1^2$ a chi-square distribution with one degree of freedom.
- Let $Z_1, Z_2, \ldots, Z_k$ independent random variables then $\sum_{i=1}^{n} Z_i^2 \sim \chi_k^2$ follows a chi-square distribution with n degrees of freedom.
- Thus $X \sim N(\mu;\sigma^2)$ then $\left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi_1^2$ a chi-square distribution with one degree of freedom.
- And let $X_1, X_2, \ldots, X_k$ be independent identical random normal variables then $\sum_{i=1}^{n} \left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi_k^2$ follows a chi-square distribution with n degrees of freedom.

S^2 chart

- Adding and deducting $\overline{X}$ in $\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$. It can be rewritten as
$$\underbrace{\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2}_{\chi_n^2} =$$
$$\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2 + \underbrace{\sum_{i=1}^{n}\left(\frac{\overline{X}-\mu}{\sigma}\right)^2}_{\chi_1^2} + \underbrace{2\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)\left(\frac{\overline{X}i-\mu}{\sigma}\right)}_{=0}$$
- Thus $\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2}$ follows χ^2_{n-1}
- Fixed α, $\alpha/2 = P(S^2 > UCL_{S^2}|\sigma^2 = \sigma_0^2) = P(S^2 < LCL_{S^2}|\sigma^2 = \sigma_0^2)$
- $LCL_{S^2} : \frac{\sigma_0^2}{n-1}\chi^2_{(\alpha/2;n-1)}; UCL_{S^2} : \frac{\sigma_0^2}{n-1}\chi^2_{(1-\alpha/2;n-1)}$
- See Tables for $\chi^2_{(\alpha/2;n-1)}, \chi^2_{(1-\alpha/2;n-1)}$, the quantiles of chi-square distribution

The power of S^2 chart

The power: the determination of the probability $1 - \beta$

- $1 - \beta = P(S^2 > UCL_{S^2} | \sigma^2 = \sigma_1^2) + P(S^2 < LCL_{S^2} | \sigma^2 = \sigma_1^2)$
$= P\left(\frac{S^2(n-1)}{\sigma_1^2} > \frac{UCL_{S^2}(n-1)}{\sigma_1^2} | \sigma^2 = \sigma_1^2\right) + P\left(\frac{S^2(n-1)}{\sigma_1^2} < \frac{LCL_{S^2}(n-1)}{\sigma_1^2} | \sigma^2 = \sigma_1^2\right)$
- Replacing the control limits, assuming that $\sigma_1^2 = \delta\sigma_0^2$ and after some algebraic manipulations, the power is
$P\left(\chi^2_{n-1} > \frac{\chi^2_{(1-\alpha/2;n-1)}}{\delta}\right) + \left(\chi^2_{n-1} < \frac{\chi^2_{(\alpha/2;n-1)}}{\delta}\right)$

Exercise

Use de functions dchisq, pchisq, etc to answer these items:

1. Determine the control limits for S^2 chart for sample sizes $n = 5, 10, 15$, $\alpha = 0.05, 0.01$ and $\sigma_0^2 = 1, 10$
2. What do you observe about the control limits as n increases? And as α decreases?
3. Calculate the power of S^2 chart if the variance shifts to $\sigma_1^2 = \delta\sigma_0^2$, $\delta = 1.25; 1.5; 2; 3$
4. Compare these results with those obtained with R chart. Which chart is more powerful?

Get the asymptotic control limits for S^2 control chart.

$\overline{X}$ Control Chart: μ_0 and σ, both unknown - Estimators for μ_0 and σ_0^2

- σ unavailable
  - Replace σ^2 by $\overline{S}^2$ in control limits
- Take samples when the process is in-control
- Take m (20 to 25) samples; each with n (4 to 6) units
- Let $\overline{X}_i$, S_i^2, be the average and variance sample of the ith sample
- Best estimator for μ_0: $\overline{\overline{X}} = \dfrac{\overline{X}_1 + \ldots + \overline{X}_m}{m}$
- Best estimator for σ_0^2: $\overline{S}^2 = \dfrac{S_1^2 + \ldots + S_m^2}{m}$

$\overline{X}$ Control Chart: μ_0 and σ, both unknown - Estimators for μ_0 and σ_0^2

- When the sample sizes are not equal, use

$$\overline{\overline{X}} = \frac{n_1\overline{X}_1 + \ldots + n_m\overline{X}_m}{n_1 + \ldots + n_m}$$

$$\overline{S}^2 = \frac{(n_1 - 1)S_1^2 + \ldots + (n_m - 1)S_m^2}{n_1 - 1 + \ldots + n_m - 1}$$

Exercise

Table 1: Volumes of soft drink in cm^3 taken at every 30 min in 15 hours of production

Sample	X1	X2	X3	Sample	X1	X2	X3
1	252.16	250.34	249.7	16	248.29	249.6	249.15
2	248.34	248.61	250.63	17	249.59	249.89	248.51
3	249.19	250.02	250.84	18	248.03	249.11	249.81
4	251.29	249.93	250.24	19	250.99	251.5	249.92
5	248.16	250.41	251.19	20	247.62	250.43	250.39
6	250.37	251.98	248.44	21	250.6	250.54	250.2
7	250.31	248.71	251.13	22	250.44	251.17	250.01
8	250.27	249.64	249.92	23	249.35	249.16	250.2
9	250.72	250.8	249.35	24	248.17	249.94	248.15
10	250.45	249.18	250.04	25	249.98	251.57	249.79
11	251.76	252.01	251.9	26	250.1	249.57	249.11
12	249.33	251.21	250.58	27	248.82	251.01	248.9
13	249.26	247.67	249.99	28	248.39	248.26	250.57
14	249.41	249.01	249.51	29	251.43	250.92	250.12
15	249.9	249.07	250.32	30	248.82	249.28	248.57

Use data of Table 1 determine the control limits for S and S^2 chart considering $\alpha = 0.0027$

Is stable the variability of the volume of soft drink?

References