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Linearized Subsonic Potential Flow.

Relevant equations: 2-D case

$$\begin{cases} (1-M_\infty^2)\dfrac{\partial^2\phi'}{\partial x^2} + \dfrac{\partial^2\phi'}{\partial y^2} = 0 & (\vec{U}_\infty \parallel \hat{x}) \\ \vec{u}' = \nabla\phi' \\ p' = -\rho_\infty \vec{U}_\infty \cdot \vec{u}' \Rightarrow C_p = -\dfrac{2u'}{U_\infty} \\ \dfrac{p'}{P_\infty} = \dfrac{\gamma\rho'}{\rho_\infty} = \dfrac{\gamma}{(\gamma-1)}\dfrac{T'}{T_\infty} \end{cases}$$

On making the coordinate transformation:

$$\begin{cases} \xi \equiv \dfrac{x}{\sqrt{1-M_\infty^2}} \Rightarrow d\xi = \dfrac{dx}{\sqrt{1-M_\infty^2}} \\ y = \eta \Rightarrow d\eta = dy \end{cases}$$

We finally get the Laplace's equation, which recovers the results from the incompressible flow, as expected

$$\nabla^2\phi' = 0 \Rightarrow \boxed{\frac{\partial^2\phi'}{\partial\xi^2} + \frac{\partial^2\phi'}{\partial\eta^2} = 0}$$

Relevant Points:

1) The above equation is linear, so any linear combination of solutions is, itself, a solution.

2) The coordinate transformation $(x,y) \rightleftharpoons (\xi,\eta)$ represents a non-isotropic stretching of the domain - it distorts shapes.

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⊛ the rationale behind the minus sign of $\vec{U}$ $(-\vec{u})$ here will be made clear latter on, when we talk about the total derivative.

Potential flow – on the forces the fluid exerts on translating bodies.
(Based on Karamcheti, "Principles of ideal-fluid Aerodynamics" chapter 10, pages 278 – 304

Body surface: $F(\vec{r},t)=0$ ⇒ scalar function

$\vec{V}=\nabla\phi \; ; \; \nabla^2\phi=0$

Where $\vec{V}$ is the flow velocity that is "induced" by the body moving in pure translation with velocity $\vec{U}$.
Note that $\vec{U}$ and $\vec{V}$ are both measured with respect to the lab. (space fixed) inertial reference frame.

Boundary conditions: At infinity $\boxed{\lim_{\vec{r}\to\infty}\vec{V}=0}$

Solid wall: $\underset{\text{Lagrange}}{\frac{DF}{Dt}}=0 \Rightarrow \underset{\text{Euler}}{\frac{\partial F}{\partial t}+\nabla\phi\cdot\nabla F}=0$ on $F(\vec{r},t)=0$

$\nabla F\Big|_{F(\vec{r},t)=0} \parallel \hat{n} \; ; \; \hat{n}=\frac{\nabla F}{\|\nabla F\|}\Big|_{F(\vec{r},t)=0}$ but we can scale $F(\vec{r},t)$ so as to make $\nabla F=\hat{n}$

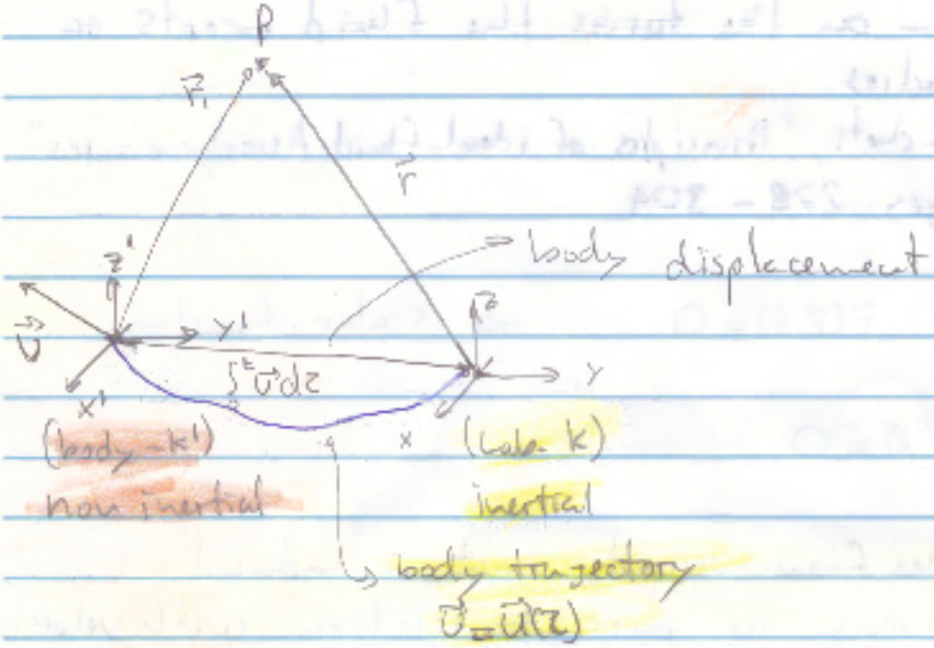
the solid wall boundary condition implies that:

$$\frac{DF}{Dt}\Big|_{F=0}=0 \Rightarrow \left[\underbrace{\frac{\partial F}{\partial x^i}}_{\nabla F}\underbrace{\frac{\partial x^i}{\partial t}}_{-\vec{U}} + \underbrace{\vec{V}}_{\nabla\phi}\cdot\nabla F\right]_{F(\vec{r},t)=0}=0 \Rightarrow \left[\vec{U}\cdot\nabla F\right]_{F=0}=\left[\nabla\phi\cdot\nabla F\right]_{F=0}$$

⊛

$$\boxed{\left[\nabla\phi\cdot\hat{n}\right]_{F(\vec{r},t)=0}=\left[\vec{U}\cdot\hat{n}\right]_{F(\vec{r},t)=0}}$$

Notice that this does not imply by any means that $\nabla\phi=\vec{U}$ at $F(\vec{r},t)=0$


P
r1
r
z'
y'
U
x'
∫U dτ
y
x
body displacement
(body - k')
non inertial
(Lab k)
inertial
body trajectory
U = U(τ)

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Bernoulli equation in the lab reference frame:

$$P(\vec{r},t) = P_{\infty} - \rho\left[\frac{\partial\phi}{\partial t} + \frac{\|\nabla\phi\|^2}{2}\right]$$

Let's assume that $\vec{U} = \vec{U}(t)$ — After all, it's pure translation

(K) → Lab Reference frame (inertial) : $(\vec{r}, t)$

(K$_1$) → body fixed, non inertial reference frame $(\vec{r}^{(1)}, t^1)$

$$\begin{cases} \vec{r}^{(1)} = \vec{r} - \int_0^t U(\tau)d\tau \\ t^1 = t \end{cases}$$

$$\begin{cases} \phi_1(\vec{r}^{(1)}, t^1) = \phi(\vec{r},t) \\ F_1(\vec{r}^{(1)}, t^1) = F(\vec{r},t) \\ P_1(\vec{r}^{(1)}, t^1) = P(\vec{r},t) \end{cases}$$

Scalar functions cannot depend on the reference frame.

Furthermore, let's assume the two reference frames coincide at $t = t_1 = 0$, then the differential operations become:

$\nabla_1(\ldots) = \nabla(\ldots)$; $\nabla^2(\ldots) = \nabla^2(\ldots)$; $\frac{\partial t_1}{\partial t} = 1$

$$\boxed{\frac{\partial(\ldots)}{\partial t_1} = \frac{\partial(\ldots)}{\partial t} + \vec{U}\cdot\nabla(\ldots)} \Rightarrow \boxed{\frac{\partial(\ldots)}{\partial t} = \frac{\partial(\ldots)}{\partial t_1} - \vec{U}\cdot\nabla^{(1)}(\ldots)}$$

Wall: $\frac{DF}{Dt} = 0 \Rightarrow \frac{\partial F}{\partial t} + \nabla\phi\cdot\nabla F = 0$.

$$\frac{\partial F_1}{\partial t_1} - \vec{U}\cdot\nabla^{(1)}F_1 + \nabla\phi\cdot\nabla F = 0$$

And this is why we have a $-\vec{U}$ in the lab frame

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$$C_p \triangleq \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

$$C_p = -\frac{2}{U_\infty^2}\left[\frac{\partial \phi_1}{\partial t} - \vec{U}\cdot\vec{q_1} + \frac{\|\vec{q_1}\|^2}{2}\right]$$

$$\frac{\partial F}{\partial t_1} - \vec{U}\cdot\nabla F + \nabla\phi\cdot\nabla F = 0$$

Wall b.c. $\vec{U}\cdot\nabla F = \nabla\phi\cdot\nabla F$

For a rigid body, in its own reference frame, $F = F(\vec{r}_1)$ independent of time.

therefore, we get: $\boxed{\dfrac{\partial F}{\partial t_1} = 0}$ as expected. $\boxed{F_1 = F_1(\vec{r}_1)}$

Potential equation (Laplace) : $\boxed{\nabla_1^2 \phi_1 = 0}$, $\boxed{\vec{q}_1 = \nabla\phi_1}$

Boundary conditions: infinity: $\boxed{\lim_{\|\vec{r}_1\|\to\infty} \vec{q}_1 = 0}$

$\vec{q}_1$: perturbation velocity

wall: $\boxed{\nabla^{(1)}\phi_1\cdot\hat{n}\Big|_{F_1=0} = \vec{U}(t)\cdot\hat{n}\Big|_{F_1=0}}$

ϕ_1: perturbation Potential.

Bernoulli equation

$$P_\infty - P = \rho\left[\frac{\partial\phi}{\partial t} + \frac{\|\nabla\phi\|^2}{2}\right]$$

$$P_\infty - P = \rho\left[\frac{\partial\phi_1}{\partial t} - \vec{U}\cdot\nabla^{(1)}\phi_1 + \frac{\|\nabla^{(1)}\phi_1\|^2}{2}\right]$$

$$\boxed{\frac{P_\infty - P}{\rho} = \frac{\partial\phi_1}{\partial t} - \vec{U}\cdot\vec{q}_1 + \frac{\|\vec{q}_1\|^2}{2}}$$

$\vec{U}$ and $\nabla^{(1)}\phi_1$ are not necessarily equal, but their normal projection are the same.

We shall hereafter assume the body-fixed reference frame and drop altogether the index (1).

The surface force acting on the body is given by:

$$\vec{F} = -\oiint_S P\hat{n}\,ds = \oiint_S \rho\frac{\partial\phi}{\partial t}\hat{n}\,ds + \oiint_S \rho\left[\frac{q^2}{2} - \vec{U}\cdot\vec{q}\right]\hat{n}\,ds$$

Since the body is assumed to be rigid, we can write:

$$\vec{F} = \frac{\partial}{\partial t}\oiint_S \rho\phi\hat{n}\,ds + \oiint_S \rho\left[\frac{q^2}{2} - \vec{U}\cdot\vec{q}\right]\hat{n}\,ds$$

vector identity: $\vec{U}\times(\hat{n}\times\vec{q}) = (\vec{U}\cdot\vec{q})\hat{n} - (\vec{U}\cdot\hat{n})\vec{q}$

But on the body surface we also have: $\vec{U}\cdot\hat{n} = \vec{q}\cdot\hat{n}$

Hence, we can write:

$$\vec{U}\times(\hat{n}\times\vec{q}) = (\vec{U}\cdot\vec{q})\hat{n} - (\vec{q}\cdot\hat{n})\vec{q}$$

which only holds on the body surface

And the expression for the force may be cast in the form:

$$\vec{F} = \frac{\partial}{\partial t}\oiint_S \rho\phi\hat{n}\,ds + \underbrace{\oiint_S \rho\left[\frac{q^2}{2}\hat{n} - \cancel{(\vec{U}\cdot\hat{n})\vec{q}}^{\,(\vec{q}\cdot\hat{n})\vec{q}}\right]ds} - \rho\vec{U}\times\oiint_S(\hat{n}\times\vec{q})\,ds$$

$$\oiint_{S_0}\left[\frac{\vec{q}\cdot\vec{q}}{2}\hat{n} - (\vec{q}\cdot\hat{n})\vec{q}\right]ds + \oiint_{\Sigma\,\text{(far-field)}}\left[\frac{\vec{q}\cdot\vec{q}}{2}\hat{n} - (\vec{q}\cdot\hat{n})\vec{q}\right]ds = \iiint_{\forall}\left[\vec{q}\cdot\nabla\vec{q} - \vec{q}\cdot\nabla\vec{q}\right]d\forall = 0$$

(the far-field integral can be shown to vanish as $\vec{r}\to\infty$)

Hence, we get:

$$\oiint_{S_0} \left[\frac{q^2}{2}\hat{n} - (\vec{q}.\hat{n})\vec{q} \right] ds = 0$$

And the expression for the force boils down to:

$$\boxed{\vec{F} = \frac{\partial}{\partial t} \oiint_S \rho \phi \hat{n} \, ds - \rho \vec{U} \times \oiint_S (\hat{n} \times \vec{q}) \, ds}$$

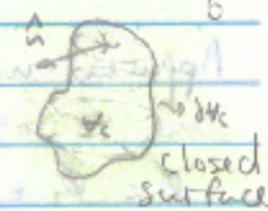
Where the first integral depends on accelerations and it gives rise to the added mass coefficients. Whereas the second integral depends on the presence of circulation, it appears in stationary flows, as well, and it gives rise to lift.

D'allambert's Paradox | If the circulation is completely absent and the flow body motion is steady, then there should absolutely no force (no drag) acting upon the body. Well, everybody knows this doesn't hold in reality.

$$\oiint_{\partial\forall_c} \vec{q}\cdot\hat{n}\,ds = \iiint_{\forall_c} \nabla\cdot\vec{q}\,d\forall$$

Gauss' Theorem : (vector product)

$$\oiint_{\partial\forall_c} \hat{n}\times\vec{q}\,ds = \iiint_{\forall_c} \nabla\times\vec{q}\,d\forall$$



Stokes' Theorem :

$$\oint_{\mathcal{C}} \vec{q}\cdot d\vec{l} = \iint_{S_c} (\nabla\times\vec{q})\cdot\hat{n}\,ds$$

Circulation definition :

$$\Gamma \equiv \int_{\mathcal{C}} \vec{q}\cdot d\vec{l} = \iint_{S_c} (\nabla\times\vec{q})\cdot\hat{n}\,ds$$

Hence, the circulation about a closed contour $\mathcal{C}$ is defined so as to match the flux of vorticity through any control surface (regular) that is supported by that contour $\mathcal{C}$.

It can be shown that the second integral in the expression for the force :

$$\vec{F} = \frac{\partial}{\partial t}\oiint_S \rho\phi\hat{n}\,ds - \rho\vec{U}\times\overbrace{\oiint_S (\hat{n}\times\vec{q})\,ds}^{\vec{I}}$$

Can be cast in an alternative form, when it is projected onto a given direction $\hat{l}$ (where $\hat{l}$ is a unitary vector.

$$\hat{l}\cdot\vec{I} = \hat{l}\cdot\oiint_S (\hat{n}\times\vec{q})\,ds = \int_{h_1}^{h_2}\left(\oint_{\mathcal{C}} \vec{q}\cdot d\vec{l}\right)dh = \int_{h_1}^{h_2} \Gamma_l'(h)\,dh$$

$$\vec{I} \equiv \oiint_S (\hat{n}\times\vec{q})\,ds$$

$\nabla^2\phi = 0$, $\nabla\phi \cdot n\big|_S = \vec{U}\cdot\hat{n}\big|_S$

Apparent mass tensor (basics): $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, $\vec{U} = (u_1, u_2, u_3)$

one ϕ_i for each direction

$\phi = \phi_1 + \phi_2 + \phi_3$, $\nabla^2\phi_i = 0$ $\quad i = 1, 2, 3$, (n_1, n_2, n_3)

$$\nabla^2\phi_i = 0 \; , \; \nabla\phi_i \cdot \hat{n} = u_i n_i \quad \text{on } S$$

$\phi_i \equiv u_i \varphi_i$ $\quad i = 1, 2, 3$ $\Rightarrow$ Unitary motions

$\nabla^2\varphi_i = 0$ $\qquad \nabla\varphi_i \cdot \hat{n} = \dfrac{\partial \varphi_i}{\partial n} = n_i$ on S

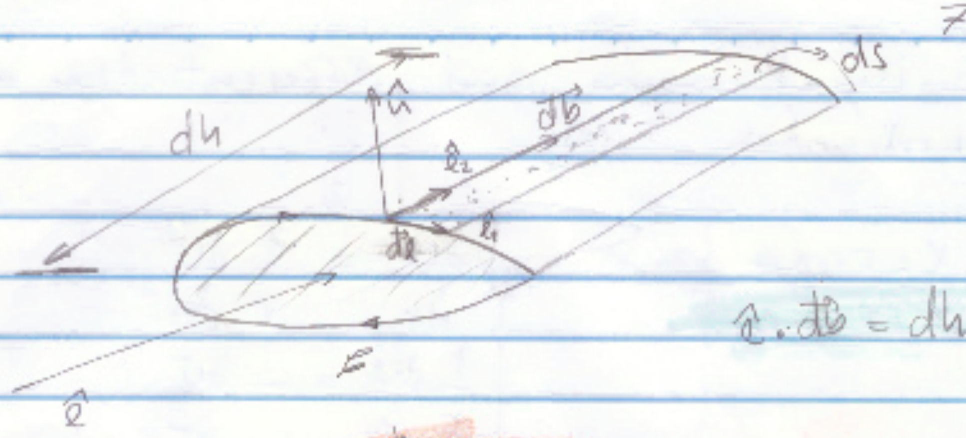
$\phi = u_1\varphi_1 + u_2\varphi_2 + u_3\varphi_3 = \vec{U}\cdot\hat{n}$ $\qquad$ on S

$$\vec{F}_e = \frac{d}{dt}\left(m\vec{U} - \oiint \rho\phi\hat{n}\, ds \right)$$

force applied to
the body (externally)
to move it through the
fluid.

$$F_{ei} = \left(m\delta_{ik} + m_{ik} \right)\frac{du_k}{dt}$$

$$m_{ik} = m_{ki} = -\oiint_S \rho\varphi_k n_i\, dS = -\iint \rho\varphi_k \frac{\partial\varphi_i}{\partial n}\, dS$$



$\hat{e} \cdot \vec{db} = dh$

$$\rho \vec{U} \times \oiint_S \hat{n} \times \vec{q}\, ds = \rho \vec{U} \times \int_{b_1}^{b_2} \Gamma(b)\, \vec{db} \Rightarrow$$ Lift is related to circulation and Vorticity!

A few useful results from Complex variables:

$$z = x + iy \quad , \quad i = \sqrt{-1}$$

$$e^x = \sum_{k=0}^{\infty} \frac{(x)^k}{k!} \Rightarrow e^{iy} = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!}$$

$$e^{iy} = \sum_{k=0}^{\infty} \frac{i^{2k} y^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{(2k+1)} y^{(2k+1)}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i \sum \frac{(-1)^k y^{2k+1}}{(2k+1)!}$$

$$e^{iy} = \cos(y) + i \sin(y)$$

$$z = r e^{i\theta} = r(\cos\theta + i \sin\theta) \quad \text{where } r = |z| = x^2 + y^2$$
$$\theta = \arg(z)$$

$$f(z) = u(x,y) + i\, v(x,y)$$

Cauchy-Rimann Conditions for analytic functions (Analytic functions are those that satisfy C-R.

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

$f'(z) = u_x + i v_x$

$f'(z) = v_y - i u_y$

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Cauchy-Riemann and differentiation in polar coordinates

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \qquad \left\{\begin{array}{l} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial v}{\partial r} \end{array}\right.$$

$$\frac{df}{dz} = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$\exp(z) \equiv \exp(z \pm 2\pi i)$ Periodic function

$\log(z) \equiv \ln(r) + i(\theta \pm 2k\pi)$ Multi-valued function

$$\left\{\begin{array}{l} |z_1 + z_2| \leq |z_1| + |z_2| \\ |z_1 - z_2| \geq \big|\, |z_1| - |z_2| \,\big| \end{array}\right. \qquad \left|\begin{array}{l} \exp[\log(z)] = z \\ \log[\exp(z)] = z \pm i2k\pi \end{array}\right.$$

Cylindrical:

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\nabla(\dots) = \frac{\partial(\dots)}{\partial r}\hat{\lambda}_r + \frac{1}{r}\frac{\partial(\dots)}{\partial \theta}\hat{\lambda}_\theta + \frac{\partial(\dots)}{\partial z}\hat{\lambda}_z$$

Spherical:

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0$$

$$\nabla(\dots) = \frac{\partial(\dots)}{\partial r}\hat{\lambda}_r + \frac{1}{r}\frac{\partial(\dots)}{\partial \theta}\hat{\lambda}_\theta + \frac{1}{r\sin\theta}\frac{\partial(\dots)}{\partial \varphi}

Cross differentiation of the C.R. conditions leads to :

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 v}{\partial y \partial x}\\ \frac{\partial^2 u}{\partial y^2} &= -\frac{\partial^2 v}{\partial x \partial y}\end{aligned}\right\} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \nabla^2 u = 0$$

and similarly, we get : $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow \nabla^2 v = 0$

Whence it comes that the analytic functions real and imaginary parts both satisfy Laplace's equation. Because of that, they are termed harmonic conjugate functions. These few results are going to be extremely useful to us pretty soon.

For the time being, though, let's turn our attention to a few simple 3-D potential flows:

(Talk about singlevalued and Multivalued Φ → from Personal Notes)

Simple Potential Solutions (3-D): Based on Katz and Plotkin, chapter 3, p 44

Point Source : $\phi = \frac{-\sigma}{4\pi r}$; $\vec{q} = \frac{-\sigma}{4\pi} \nabla\left(\frac{1}{r}\right) = \frac{\sigma \vec{r}}{4\pi r^3}$

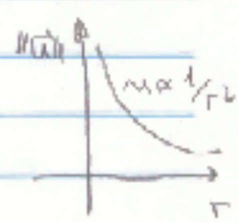
$\vec{r} = (x, y, z)^T$

$r = \|\vec{r}\|$

$\sigma \Rightarrow$ source/sink intensity

$$(q_r, q_\theta, q_\varphi) = \left(\frac{\sigma}{4\pi r^2}, 0, 0\right)$$

($\vec{r}$ refers to a point source at the origin. Whenever the source is placed anywhere else, we replace $\vec{r}$ by the vector $(\vec{r} - \vec{r}_0)$ where $\vec{r}_0$ indicates the actual position of the source



Laplace's eq: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2 \sigma}{4\pi r^2}\right) = 0$ okay

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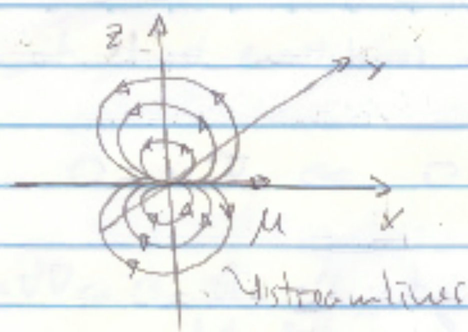


it looks like a small jet engine blowing in the $\vec{\mu}$ direction

⊛ From the cosine law:

$$\|\vec{r}-\vec{\ell}\|^2 = \|\vec{r}\|^2 + \|\vec{\ell}\|^2 - 2\|\vec{r}\|\|\vec{\ell}\|\cos\theta$$

$\|\vec{r}\| = r$; $\|\vec{\ell}\| = \ell$

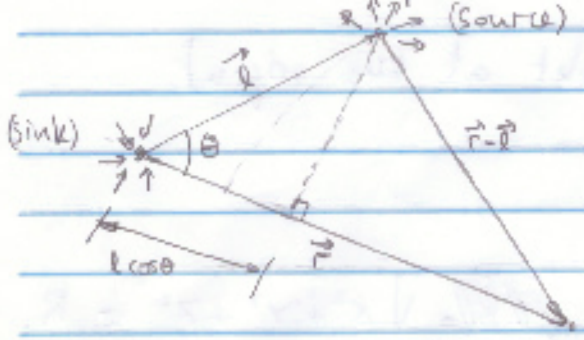
$$\|\vec{r}-\vec{\ell}\|^2 - \|\vec{r}\|^2 = \ell^2 - 2r\ell\cos\theta$$

$$(\|\vec{r}-\vec{\ell}\| - \|\vec{r}\|) = \frac{\ell^2 - 2r\ell\cos\theta}{\|\vec{r}-\vec{\ell}\| + \|\vec{r}\|}$$

then, for $\ell \ll 1$, we can write

$$\|\vec{r}-\vec{\ell}\| - \|\vec{r}\| \cong \frac{-2r\ell\cos\theta}{2r} = -\ell\cos\theta$$

Doublet: Consider a point sink at the origin and a source at a position $\vec{l}$, as shown below:



$$\Phi_p = \frac{\sigma}{4\pi}\left(\frac{1}{|\vec{r}|} - \frac{1}{|\vec{r}-\vec{l}|}\right)$$

Now let's have $\vec{l} \to 0$ while $\sigma \to \infty$, while at the same time we force their product (σl) to remain finite $\sigma l \to \mu$

both the source and the sink have the same intensity σ, only the sign changes.

$$\Phi = \lim_{\substack{l\to 0\\ \sigma\to\infty\\ \sigma l\to\mu}} \frac{\sigma}{4\pi}\left(\frac{\|\vec{r}-\vec{l}\| - \|\vec{r}\|}{\|\vec{r}\|\|\vec{r}-\vec{l}\|}\right)$$

$\lim_{l\to 0} \|\vec{r}\|\|\vec{r}-\vec{l}\| = r^2$ and $\|\vec{r}-\vec{l}\| - \|\vec{r}\| = -l\cos\theta$ ⊕

$$\Phi = \lim_{\substack{l\to 0\\ \sigma\to\infty\\ \sigma l\to\mu}} -\frac{\sigma l\cos\theta}{4\pi r^2} \Rightarrow \Phi = -\frac{\mu\cos\theta}{4\pi r^2} = -\frac{\vec{\mu}\cdot\vec{r}}{4\pi r^3}$$

For $\vec{\mu}\cdot\vec{r} = \mu r\cos\theta$

Hence, a doublet has a directional property as it is given by the vector $\vec{\mu}$. At this point it is worth adding that although a single doublet doesn't have any circulation ($\Gamma = 0$) a distribution of doublets may indeed have circulation.

$$q_r = \frac{\partial\Phi}{\partial r} = \frac{\mu\cos\theta}{2\pi r^3} \; ; \; q_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{\mu\sin\theta}{4\pi r^3} \; ; \; q_\varphi = \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi} = 0$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\mu\cos\theta}{2\pi r^3}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta\,\mu}{4\pi r^2}\right) = -\frac{\mu\cos\theta}{2\pi r^4} + \frac{\mu\cos\theta}{2\pi r^4} = 0$$

$[\sin^2\theta]_\theta = 2\sin\theta\cos\theta$

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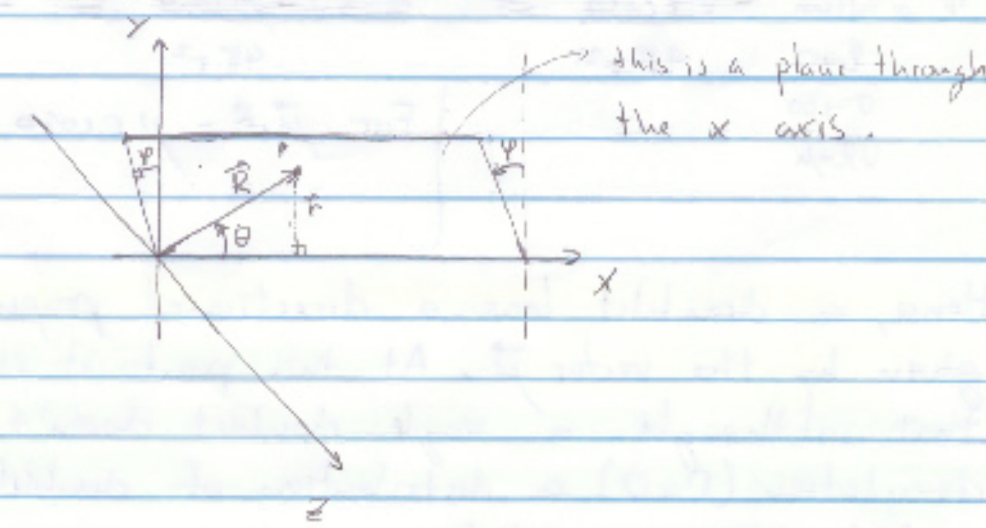
Karamcheti. P 336 (Doublet in 3-D)
eqs. (11.66) – (11.68)

3-D doublet Potential (Doublet at the origin)
$\vec{\mu} \parallel \hat{x}$ in the same direction
Cartesian Coordinates:

$$\Phi(x,y,z) = -\frac{\mu}{4\pi}\frac{x}{\|\vec{R}\|^3} \quad ; \|\vec{R}\| = \sqrt{x^2+y^2+z^2} = R$$

$$\Phi(R,\theta,\varphi) = -\frac{\mu}{4\pi}\frac{\cos\theta}{R^2} \Rightarrow \text{Spherical}$$

$$\Phi(r,\varphi,x) = -\frac{\mu}{4\pi}\frac{x}{(x^2+r^2)^{3/2}} \Rightarrow \text{Cylindrical}$$



Polynomials:

$\Phi = Ax + By + Cz \Rightarrow u = \frac{\partial \Phi}{\partial x} = A = U_\infty$, $v = \frac{\partial \Phi}{\partial y} = B = V_\infty$

and $w = \frac{\partial \Phi}{\partial z} = C = W_\infty$

$\Phi = U_\infty x \Rightarrow$ uniform flow (both cases)

$\Phi = Ax^2 + By^2 + Cz^2 \Rightarrow \nabla^2 \Phi = A + B + C = 0$

Let's consider, for instance, $B = 0 \Rightarrow A = -C$, then we get:

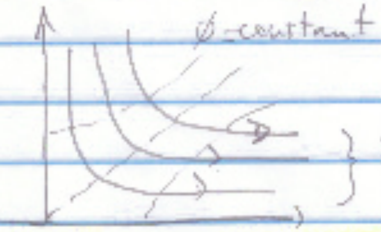
$$\Phi = A(x^2 - z^2)$$

$u = 2Ax$, $v = 0$ and $w = -2Az$

Streamline element $d\vec{l}$; definition of stream line $\vec{q} \times d\vec{l} = 0$

Stream line equation: $\frac{dx}{u} = \frac{dy}{v} = \frac{dw}{w}$

in this case: $\frac{dx}{2Ax} = \frac{dz}{-2Az} \Rightarrow xz = \text{constant}$ along any stream-line.



corner flows

Two dimensional basic solutions (2-D) K P. Page 56

⇒ **2-D Source**: Requirement that the flow be irrotational implies that: ($q_\theta = 0$ for a source)

Vorticity:

$$\omega_y = -\frac{1}{r}\left[\frac{\partial}{\partial r}(r q_\theta) - \frac{\partial}{\partial \theta}(q_r)\right] = \frac{1}{r}\frac{\partial (q_r)}{\partial \theta} = 0 \Rightarrow q_r = q_r(r)$$

Continuity implies that $\nabla \cdot \vec{q} = \frac{1}{r}\frac{\partial (r q_r)}{\partial r} = 0$

$r q_r = \text{constant} = \sigma / 2\pi \Rightarrow Q = \sigma = \int_0^{2\pi} q_r r d\theta = 2\pi q_r r$

$$\left.\begin{aligned} q_r &= \frac{\partial \Phi}{\partial r} = \frac{\sigma}{2\pi r} \\ q_\theta &= \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = 0 \end{aligned}\right\} \Rightarrow \Phi = \frac{\sigma}{2\pi}\ln(r) + \cancelto{0}{C}$$

point Source at the origin $\boxed{\Phi = \frac{\sigma}{2\pi}\ln(r)}$

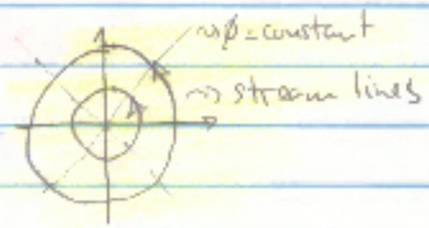
2-D Doublet: A reasoning that is similar to the one pursued in the 3-D case leads to:

$\vec{\mu} = (\mu, 0) \Rightarrow$
$$\left\{\begin{aligned} \Phi(r,\theta) &= -\frac{\mu \cos\theta}{2\pi r} \quad \text{for } \vec{\mu} \parallel \hat{1}_x \\ q_r &= \frac{\partial \Phi}{\partial r} = \frac{\mu \cos\theta}{2\pi r^2} \\ q_\theta &= \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = \frac{\mu \sin\theta}{2\pi r^2} \end{aligned}\right.$$

$$\boxed{\Phi = -\frac{\vec{\mu} \cdot \vec{r}}{2\pi r^2}}$$

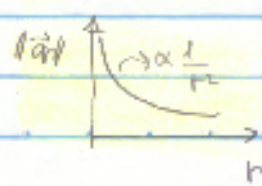
Vortex: Singularity element with only a tangential velocity:

$$\left\{\begin{aligned} q_r &= 0 \\ q_\theta &= q_\theta(r,\theta) \end{aligned}\right\} \text{continuity} \Rightarrow \left\{\begin{aligned} q_r &= 0 \\ q_\theta &= q_\theta(r) \end{aligned}\right.$$



The vorticity everywhere, except possibly at the origin, must be zero

$$\omega_y = -\frac{1}{r}\left[\frac{\partial (r q_\theta)}{\partial r} - \frac{\partial (q_r)}{\partial \theta}\right] = -\frac{1}{r}\frac{\partial (r q_\theta)}{\partial r} = 0$$



$r q_\theta = A = \text{constant}$

this equation fails at the origin ($r=0$) and both q_θ and ω_y blow up $q_\theta \to \infty, \omega_y \to \infty$

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On the orthogonality between $\nabla\Phi$ and $\nabla\Psi$

$$\left.\begin{array}{l} \nabla\Phi = \vec{u} \parallel \Psi\Big|_{constant} \\ \nabla\Phi = \vec{u} \perp \Phi\Big|_{constant} \end{array}\right\} \Longrightarrow \Psi\Big|_{constant} \perp \Phi\Big|_{constant}$$

$\therefore \nabla\Phi \perp \nabla\Psi$

Circulation: $\Gamma = \oint \vec{q} \cdot d\vec{l} = \int_0^{2\pi} q_\theta \cdot r \, d\theta = 2\pi A$

$A = \frac{\Gamma}{2\pi}$ (positive counter-clockwise)
the book by K and P defines positive as the clockwise rotation.

$$\left\{ \begin{array}{l} q_r = 0 \\ q_\theta = \frac{\Gamma}{2\pi r} \end{array} \right\} \Rightarrow \Phi = \int q_\theta r \, d\theta + C$$

$$\Phi = \frac{\Gamma \theta}{2\pi} + \cancel{C}^{\,0}$$

Point Vortex: $\boxed{\Phi = \frac{\Gamma \theta}{2\pi}}$

2-D Flows in terms of Complex Variables:

Complex Potential: $F(z) \equiv \Phi(x,y) + i \Psi(x,y)$

$$\left. \begin{array}{l} \Phi(x,y) \Rightarrow \text{Potential function} \\ \Psi(x,y) \Rightarrow \text{Stream function} \end{array} \right\} \text{C.R.} \left\{ \begin{array}{l} \Phi_x = \Psi_y = u \\ \Phi_y = -\Psi_x = v \end{array} \right.$$

$$F'(z) = \frac{dF}{dz} = \Phi_x + i\Psi_x = \Psi_y - i\Phi_y = u - iv$$

$W(z) = u - iv = \overline{U}(z)$: The complex velocity $W(z)$ is the complex conjugate of the flow velocity $U(z)$

Simple potential solutions can be derived by pursuing a reasoning regarding the physics that is quite similar to the one above. Except for the fact that, now, we make use of analytic functions which we know beforehand that satisfy Laplace's eq.

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Homework set

use Matlab to illustrate examples of each solution from the table

$$\begin{cases} u = \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y} \\ v = \dfrac{\partial \Phi}{\partial y} = -\dfrac{\partial \psi}{x} \end{cases}$$

$$\oint_C W(z)\,dz = \oint_C (u - iv)(dx + idy) =$$

$$= \oint_C (u\,dx + v\,dy) + i\oint_C (u\,dy - v\,dx) =$$

$$= \oint_c d\Phi + i\oint_c d\Psi = \Gamma + iQ$$

Summary of Relevant Potential flows

Flow	$F(z)$	$W(z)$
Uniform	Az	$A \in \mathbb{C}$
Corner of Angle $\alpha = \pi/n$ $(n>1)$	Az^n, $A \in \mathbb{R}$	Anz^{n-1}
Source at $z=0$, Strength $Q = 2\pi A$, $A \in \mathbb{R}$	$A\log(z)$	$\frac{A}{z}$
Vortex at $z=0$, Circulation $\Gamma = -2\pi b$, Positive $\Rightarrow$ counter-clockwise	$ib \log z$	$\frac{ib}{z}$
Doublet at $z=0$, Axis in the x direction, strength $\mu = 2\pi A$, $A \in \mathbb{R}$	$-\frac{A}{z}$	$\frac{A}{z^2}$

around Closed Curves (C)

Circulation and Source strength:

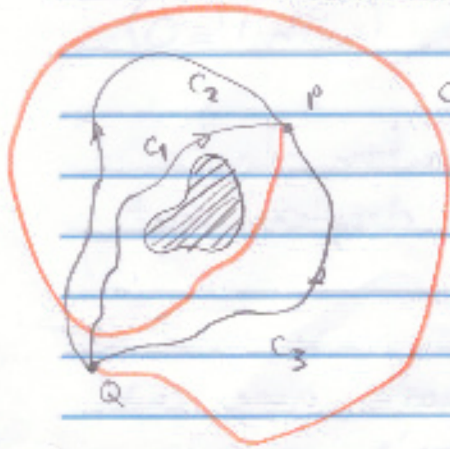
$Q \Rightarrow$ volumetric flow rate

$$\Gamma \equiv \oint_C \vec{q} \cdot d\vec{s} = \oint_C (u\,dx + v\,dy) = \oint_C d\Phi$$

$$Q = \oint_C (u\,dy - v\,dx) = \oint_C d\Psi$$

$$\Gamma + iQ = \oint_C W(z)\,dz = F(z_2) - F(z_1)$$ where z_1 and z_2 are coincident points on C

A short additional note on the 2-D flows with circulation



$$\int_{c_1 Q}^{P} d\phi - \int_{c_2 Q}^{P} d\phi = 0$$ because they form a closed contour that does not encircles the body. As a result of that, we get:

$$[\Phi(P) - \Phi(Q)]_{c_1} = [\Phi(P) - \Phi(Q)]_{c_2}$$ independent of the path

However, for the closed contours that encircle the body, we have: (for Γ may be nonzero)

$$\int_{c_3 Q}^{P} d\phi - \int_{c_1 Q}^{P} d\phi = \Gamma \Rightarrow \begin{cases} [\Phi(P) - \phi(Q)]_{c_3} - [\Phi(P) - \Phi(Q)]_{c_1} = \Gamma \\ [\Phi(P) - \phi(Q)]_{c_4} - [\Phi(P) - \Phi(Q)]_{c_3} = \Gamma \end{cases} \oplus$$

$$[\Phi(P) - \Phi(Q)]_{c_4} - [\Phi(P) - \Phi(Q)]_{c_1} = 2\Gamma$$

Then, on taking point (Q) as a reference point, we can write:

$$\left[\Phi(P)_{c_4} - \Phi(P)_{c_1}\right] = 2\Gamma + \left[\Phi(Q)_{c_4} - \Phi(Q)_{c_1}\right]$$

and, since Q is the reference point, we make $\Phi(Q)_{c_4} = \Phi(Q)_{c_1}$, which leads to

$\Phi(P)_{c_4} - \Phi(P)_{c_1} = 2\Gamma$ where the contour $C_1 \cup C_4$ would encircle the body twice!

Well, anyway, the potential here is multivalued....

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Creeping flow past a Cylinder (with $\Gamma = 0$)
(Van Dyke pages 8-11)

$Re \ll 1$ there is no b.L. separation, but the b.L. is too thick and its displacement cannot be ignored.

On the other hand, for most any case where Re is a bit higher, there's separation which undermines the validity of potential flow results.

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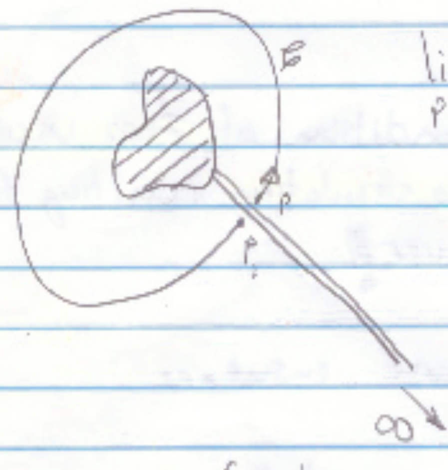
the presence of circulation in the flow solves the D'Alembert Paradox, since it brings forces into the steady flow. However it does so at the cost of introducing the Multivalued Potentials

Another point that is worth making is to say that, eventhough there are **no** forces acting upon the body in steady potential flow in the absence of circulation, there are moments that act upon the body under those conditions — despite the fact we have not sutudied them yet!.

But we will...

The natural solution to this conundrum is to draw a branch cut through the domain.



$$\lim_{P \to P_1} \int_{P}^{P_1} d\Phi = \lim_{P \to P_1} \left[\Phi(P) - \Phi(P_1) \right] = \Gamma$$

and the value of Φ jumps by the amount of Γ everytime the path crosses the branch cut once.

the sign of the jump obviously depends on the direction in which the branch cut is crossed.

On the magnus effect.

On combining a uniform flow with a doublet with its axis opposing the stream it leads to the potential flow past a cylinder: $(\mu = 2\pi A)$

$$F(z) = U_\infty z + \frac{A}{z} = U_\infty (x + iy) + \frac{A}{r} e^{-i\theta}$$

$$= \left(U_\infty x + \frac{A}{r} \cos\theta \right) + i \left(U_\infty y - \frac{A}{r} \sin\theta \right) =$$

$$= \Phi + i \Psi$$

$$\Phi = U_\infty x + \frac{A}{r} \cos\theta \quad ; \quad \Psi = U_\infty y - \frac{A}{r} \sin\theta$$

$\Psi = 0$ on the circle: $\Psi = U_\infty r \sin\theta - \frac{A}{r} \sin\theta = 0$

$$U_\infty r^2 = A \Rightarrow \boxed{r = + \sqrt{\frac{A}{U_\infty}} = + \sqrt{\frac{\mu}{2\pi U_\infty}}} \quad (*)$$

1) Comment on the "Reflection Principle of Aerodynamics" as means of achieving a symmetry condition or a slip-wall.

2) If we simply impose the condition of zero viscosity, that is, inviscid, there would be no circulation resulting from the cylinder rotation whatsoever!

Hence, no lift force would ever ensue...

Therefore, we can say that circulation is a "trick" to capture the physics of the "indirect" effect the viscosity has on the pressure field, whithout having to deal with its direct effect on the boundary layer, vorticity and viscous stresses.

$$W = \frac{dF}{dz} = U_\infty - \frac{\mu}{2\pi z^2} + \frac{i\Gamma a}{2\pi z} = U_\infty + \frac{i\Gamma a}{2\pi z} - \frac{\mu}{2\pi z^2}$$

$$W = U_\infty - \frac{\mu e^{-i2\theta}}{2\pi r^2} + \frac{i\Gamma a\, e^{-i\theta}}{2\pi r}$$

$$U = \overline{W} = U_\infty - \frac{\mu e^{i2\theta}}{2\pi r^2} - \frac{i\Gamma a\, e^{i\theta}}{2\pi r}$$

$$\Gamma_T + i Q_T = \oint_C W(z)\, dz = F(z_2) - F(z_1) = \frac{i\Gamma}{2\pi}\left[\log\left(\frac{z_2}{a}\right) - \log\left(\frac{z_1}{a}\right)\right] =$$

$z_1 = R e^{i\theta}$, $z_2 = R e^{i(\theta+2\pi)}$

$$= \frac{i\Gamma}{2\pi}\log\left(\frac{z_2}{z_1}\right) = \frac{i\Gamma}{2\pi}\left\{\ln\left(\frac{R}{R}\right) + i\left(\theta + 2\pi - \theta \pm 2k\pi\right)\right\}$$

$\boxed{k=0}$

$z_1 = X + iY = z_2$

$$\frac{1}{z_1} = \frac{1}{z_2}$$

$$= \frac{i\Gamma}{2\pi}\left\{i2\pi(1 \pm \cancel{k})\right\} = -\Gamma \Rightarrow \begin{cases} \Gamma_T = -\Gamma \\ iQ = i0 \end{cases}$$

For any closed body $\underline{Q = 0}$!

Circulation is added to the picture by superimposing a point vortex with clockwise circulation:

$$F(z) = U_\infty z + \frac{A}{z} - ib\log\left(\frac{z}{a}\right) = U_\infty z + \frac{\mu}{2\pi z} + i\frac{\Gamma}{2\pi}\log\left(\frac{z}{a}\right)$$

where the circulation is taken to be clockwise, in order for the resulting lift force to be directed upwards. The parameter $(a \in \mathbb{R})$, on the other hand, is but an algebraic convenience, as we'll see next.

$$F(z) = U_\infty(x+iy) + \frac{\mu}{2\pi r}(\cos\theta - i\sin\theta) + \frac{i\Gamma}{2\pi}\left[\ln\left(\frac{r}{a}\right) + i(\theta \pm 2k\pi)\right]$$

$$F(z) = \left[U_\infty x + \frac{\mu}{2\pi r}\cos\theta - \frac{\Gamma}{2\pi}(\theta \pm 2k\pi)\right] + i\left[U_\infty y - \frac{\mu}{2\pi r}\sin\theta + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right)\right]$$

$$F(z) = \Phi + i\Psi$$

And the stream-function vanishes at: $r = a \Rightarrow \Psi = 0$

$$\Psi = U_\infty a\sin\theta - \frac{\mu}{2\pi a}\sin\theta + \cancel{\frac{\Gamma}{2\pi}\ln\left(\frac{a}{a}\right)}^{\to 0} = 0$$

$$U_\infty a^2 = \frac{\mu}{2\pi}$$

$$\boxed{a = +\sqrt{\frac{\mu}{2\pi U_\infty}}}$$

Which is the same result as before.

The lift can be computed by the Kutta-Joukowski theorem for the 2-D case:

$$\vec{F} = \rho\vec{U} \times \int_{b_1}^{b_2} \Gamma(b)\,db \Rightarrow \boxed{\vec{L} = \rho U_\infty \Gamma}$$