Derivation of the Beltrami Identity in Variational Calculus

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1 Introduction

Let the problem of finding the minimum for the functional: funcional:

$$F = \int_{x_i}^{x_f} L\left(y, y', x\right)\,\mathrm{d}x \tag{1}$$

From the Euler-Lagrange equation, functions $y(x)$ that extremize (1) are also solution of:

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = 0 \tag{2}$$

In particular, if $\partial L/\partial y = 0$, then there is a constant K such that:

$$\frac{\partial L}{\partial y'} = K \tag{3}$$

This is a very useful result, as it trivially transforms the differential equation in (2), potentially a 2nd order one, into the first order equation in (3).

Naturally, this result is only available whenever $\partial L/\partial y = 0$. One may ask if there is an equivalent result when $\partial L/\partial x = 0$.

2 Inverting the variable dependency

The functional in (1) is an integral over x and the curve that extremizes it is written as $y = y(x)$, with y as dependent variable and x as independent variable. The property (3) is valid when the dependent variable does not appear in the functional.

On the other hand, there's nothing particular about that description. The curve very well be written as $x = x(y)$, with y as independent variable and x as dependent variable. In this case,

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{{}^{\mathrm{d}x}\!/\!{}_{\mathrm{d}y}} = \frac{1}{x'} \tag{4}$$

and

$$\int_{x_i}^{x_f} L\left(y, y', x\right)\,\mathrm{d}x = \int_{y_i}^{y_f} L\left(y, {}^{1}\!/\!{}_{x'}, x\right) x'\,\mathrm{d}y \tag{5}$$

From the Euler-Lagrange equation, the function $x(y)$ that extremizes (5) is solution of:

$$\frac{\partial}{\partial x}\left(L\left(y, {}^{1}\!/_{x'}, x\right) x'\right) - \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial}{\partial x'}\left(L\left(y, {}^{1}\!/_{x'}, x\right) x'\right)\right) = 0 \tag{6}$$

3 Beltrami Identity

Whenever $L = L\left(y, {}^{1}\!/_{x'}\right)$ and $\partial L/\partial x = 0$, there is a constant K such that:

$$\frac{\partial}{\partial x'}\left(L\left(y, {}^{1}\!/_{x'}\right) x'\right) = K \tag{7}$$

Also,

$$\frac{\partial}{\partial x'}\left(L\left(y, {}^{1}\!/_{x'}\right) x'\right) = L + x' \frac{\partial}{\partial x'} L\left(y, {}^{1}\!/_{x'}\right) \tag{8}$$

And,

$$\frac{\partial}{\partial x'} L\left(y, {}^{1}\!/_{x'}\right) = \frac{\partial}{\partial\left(1/x'\right)} L\left(y, {}^{1}\!/_{x'}\right) \frac{\partial(1/x')}{\partial x'} = -\frac{1}{x'^2} \frac{\partial}{\partial\left(1/x'\right)} L\left(y, {}^{1}\!/_{x'}\right) \tag{9}$$

Replacing (9) in (8),

$$\frac{\partial}{\partial x'}\left(L\left(y, {}^{1}\!/_{x'}\right) x'\right) = L - \frac{1}{x'} \frac{\partial}{\partial\left(1/x'\right)} L\left(y, {}^{1}\!/_{x'}\right) \tag{10}$$

Finally, replacing $1/x' = y'$ in (10),

$$L - y' \frac{\partial}{\partial y'} L = K \tag{11}$$

Equation (11) is known as the *Beltrami Identity*, and it applies whenever the functional does not explcitly depends of the independent variable, that is, $\partial L/\partial x = 0$.