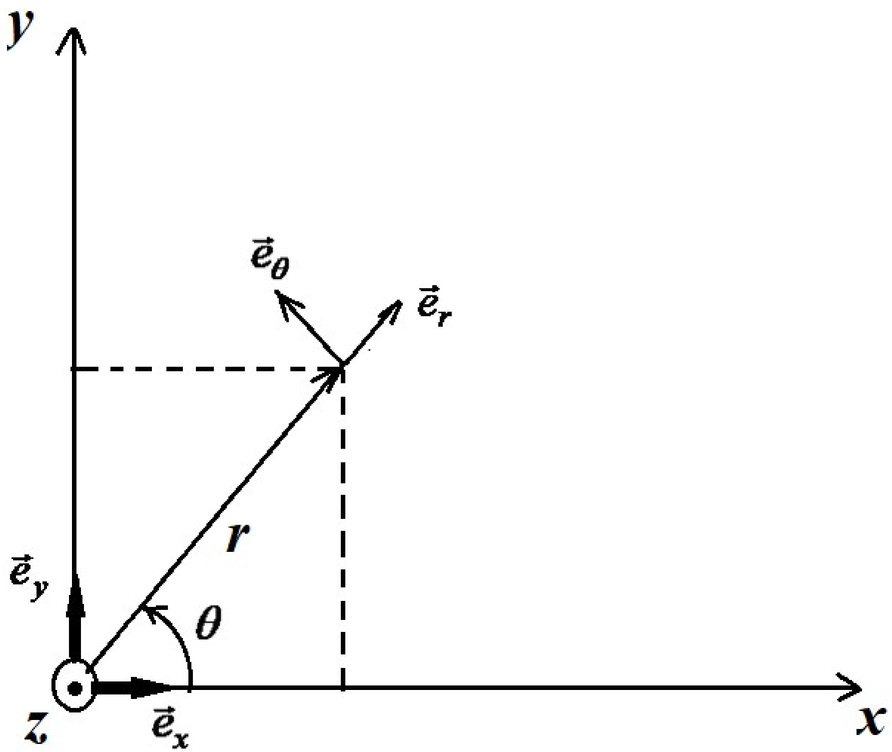
Coordenadas Cilíndricas

Aceleração, Divergente, Rotacional em coordenadas cilíndricas

1) Relação entre os sistemas de coordenadas cartesiano e cilíndrico



$$r\,\vec{e}_r = x\,\vec{e}_x + y\,\vec{e}_y \quad \Rightarrow \quad \vec{e}_r = \frac{x}{r}\vec{e}_x + \frac{y}{r}\vec{e}_y \quad \Rightarrow \quad \boxed{\vec{e}_r = \cos\theta\;\vec{e}_x + \mathrm{sen}\,\theta\,\vec{e}_y}$$

$$\vec{e}_\theta = \vec{e}_z \times \vec{e}_r \quad \Rightarrow \quad \boxed{\vec{e}_\theta = -\,\mathrm{sen}\theta\;\vec{e}_x + \cos\theta\,\vec{e}_y}$$

Das expressões dos versores $\vec{e}_r$ e $\vec{e}_\theta$:

$$\boxed{\frac{d\vec{e}_r}{d\theta} = -\operatorname{sen}\theta\ \vec{e}_x + \cos\theta\,\vec{e}_y = \vec{e}_\theta}$$

$$\boxed{\frac{d\vec{e}_\theta}{d\theta} = -\cos\theta\ \vec{e}_x - \operatorname{sen}\theta\,\vec{e}_y = -\vec{e}_r}$$

O sistema de coordenadas cilíndrico é um sistema de coordenadas não-inercial, e devido à dependência dos versores $\vec{e}_r$ e $\vec{e}_\theta$ com a coordenada θ temos o aparecimento de termos adicionais nas equações da aceleração, do divergente de um vetor e do rotacional de um vetor.

2) Aceleração em coordenadas cilíndricas

A expressão de aceleração é:

$$\vec{a} = \frac{\partial \vec{\upsilon}}{\partial t} + \left(\vec{\upsilon}.\nabla\right)\vec{\upsilon}$$

Num sistema cilíndrico de coordenadas, o vetor da velocidade é dado por:

$$\vec{\upsilon} = \upsilon_r\,\vec{e}_r + \upsilon_\theta\,\vec{e}_\theta + \upsilon_z\,\vec{e}_z$$

E o vetor nabla (operador gradiente):

$$\nabla = \vec{e}_r\,\frac{\partial}{\partial r} + \vec{e}_\theta\,\frac{\partial}{r\,\partial\theta} + \vec{e}_z\,\frac{\partial}{\partial z}$$

A expressão da aceleração resulta:

$$\vec{a} = \frac{\partial}{\partial t}\left(\upsilon_r\,\vec{e}_r + \upsilon_\theta\,\vec{e}_\theta + \upsilon_z\vec{e}_z\right) + \left(\upsilon_r\,\frac{\partial}{\partial r} + \upsilon_\theta\,\frac{\partial}{r\,\partial\theta} + \upsilon_z\,\frac{\partial}{\partial z}\right)\left(\upsilon_r\,\vec{e}_r + \upsilon_\theta\,\vec{e}_\theta + \upsilon_z\vec{e}_z\right)$$

Ou seja:

$$\vec{a} = \frac{\partial\left(\upsilon_r\,\vec{e}_r\right)}{\partial t} + \frac{\partial\left(\upsilon_\theta\,\vec{e}_\theta\right)}{\partial t} + \frac{\partial\left(\upsilon_z\vec{e}_z\right)}{\partial t} + \upsilon_r\,\frac{\partial\left(\upsilon_r\,\vec{e}_r\right)}{\partial r} + \upsilon_\theta\,\frac{\partial\left(\upsilon_r\,\vec{e}_r\right)}{r\,\partial\theta} + \upsilon_z\,\frac{\partial\left(\upsilon_r\,\vec{e}_r\right)}{\partial z}$$

$$+\upsilon_r\,\frac{\partial\left(\upsilon_\theta\,\vec{e}_\theta\right)}{\partial r} + \upsilon_\theta\,\frac{\partial\left(\upsilon_\theta\,\vec{e}_\theta\right)}{r\,\partial\theta} + \upsilon_z\,\frac{\partial\left(\upsilon_\theta\,\vec{e}_\theta\right)}{\partial z} + \upsilon_r\,\frac{\partial\left(\upsilon_z\vec{e}_z\right)}{\partial r} + \upsilon_\theta\,\frac{\partial\left(\upsilon_z\vec{e}_z\right)}{r\,\partial\theta} + \upsilon_z\,\frac{\partial\left(\upsilon_z\vec{e}_z\right)}{\partial z}$$

Continuando:

$$\vec{a}=\frac{\partial\upsilon_r}{\partial t}\vec{e}_r+\frac{\partial\upsilon_\theta}{\partial t}\vec{e}_\theta+\frac{\partial\upsilon_z}{\partial t}\vec{e}_z+\upsilon_r\frac{\partial\upsilon_r}{\partial r}\vec{e}_r+\upsilon_\theta\frac{\partial\upsilon_r}{r\,\partial\theta}\vec{e}_r+\frac{\upsilon_\theta\upsilon_r}{r}\underbrace{\frac{\partial\vec{e}_r}{\partial\theta}}_{\vec{e}_\theta}+\upsilon_z\frac{\partial\upsilon_r}{\partial z}\vec{e}_r$$

$$+\upsilon_r\frac{\partial\upsilon_\theta}{\partial r}\vec{e}_\theta+\upsilon_\theta\frac{\partial\upsilon_\theta}{r\,\partial\theta}\vec{e}_\theta+\frac{{\upsilon_\theta}^2}{r}\underbrace{\frac{\partial\vec{e}_\theta}{\partial\theta}}_{-\vec{e}_r}+\upsilon_z\frac{\partial\upsilon_\theta}{\partial z}\vec{e}_\theta$$

$$+\upsilon_r\frac{\partial\upsilon_z}{\partial r}\vec{e}_z+\upsilon_\theta\frac{\partial\upsilon_z}{r\,\partial\theta}\vec{e}_z+\upsilon_z\frac{\partial\upsilon_z}{\partial z}\vec{e}_z$$

Resultam as componentes:

$$\boxed{a_r = \frac{\partial \upsilon_r}{\partial t} + \upsilon_r \frac{\partial \upsilon_r}{\partial r} + \upsilon_\theta \frac{\partial \upsilon_r}{r\,\partial \theta} + \upsilon_z \frac{\partial \upsilon_r}{\partial z} - \frac{\upsilon_\theta{}^2}{r}}$$

$$\boxed{a_\theta = \frac{\partial \upsilon_\theta}{\partial t} + \upsilon_r \frac{\partial \upsilon_\theta}{\partial r} + \upsilon_\theta \frac{\partial \upsilon_\theta}{r\,\partial \theta} + \upsilon_z \frac{\partial \upsilon_\theta}{\partial z} + \frac{\upsilon_r\,\upsilon_\theta}{r}}$$

$$\boxed{a_z = \frac{\partial \upsilon_z}{\partial t} + \upsilon_r \frac{\partial \upsilon_z}{\partial r} + \upsilon_\theta \frac{\partial \upsilon_z}{r\,\partial \theta} + \upsilon_z \frac{\partial \upsilon_z}{\partial z}}$$

3) Divergente de um vetor

O divergente da velocidade é dado por:

$$\nabla.\vec{\upsilon}=\left(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\theta\frac{\partial}{r\partial\theta}+\vec{e}_z\frac{\partial}{\partial z}\right).\left(\upsilon_r\vec{e}_r+\upsilon_\theta\vec{e}_\theta+\upsilon_z\vec{e}_z\right)$$

Isso resulta:

$$\begin{aligned}\nabla.\vec{\upsilon}=&\ \vec{e}_r.\frac{\partial(\upsilon_r\vec{e}_r)}{\partial r}+\vec{e}_\theta.\frac{\partial(\upsilon_r\vec{e}_r)}{r\partial\theta}+\vec{e}_z.\frac{\partial(\upsilon_r\vec{e}_r)}{\partial z}\\&+\vec{e}_r.\frac{\partial(\upsilon_\theta\vec{e}_\theta)}{\partial r}+\vec{e}_\theta.\frac{\partial(\upsilon_\theta\vec{e}_\theta)}{r\partial\theta}+\vec{e}_z.\frac{\partial(\upsilon_\theta\vec{e}_\theta)}{\partial z}\\&+\vec{e}_r.\frac{\partial(\upsilon_z\vec{e}_z)}{\partial r}+\vec{e}_\theta.\frac{\partial(\upsilon_z\vec{e}_z)}{r\partial\theta}+\vec{e}_z.\frac{\partial(\upsilon_z\vec{e}_z)}{\partial z}\end{aligned}$$

Continuando:

$$\nabla.\vec{\upsilon} = \vec{e}_r . \frac{\partial \upsilon_r}{\partial r}\vec{e}_r + \vec{e}_\theta . \frac{\partial \upsilon_r}{r\partial \theta}\vec{e}_r + \vec{e}_\theta . \frac{\upsilon_r}{r}\underbrace{\frac{\partial \vec{e}_r}{\partial \theta}}_{\vec{e}_\theta} + \vec{e}_z . \frac{\partial \upsilon_r}{\partial z}\vec{e}_r$$

$$+\vec{e}_r . \frac{\partial \upsilon_\theta}{\partial r}\vec{e}_\theta + \vec{e}_\theta . \frac{\partial \upsilon_\theta}{r\partial \theta}\vec{e}_\theta + \vec{e}_\theta . \frac{\upsilon_\theta}{r}\underbrace{\frac{\partial \vec{e}_\theta}{\partial \theta}}_{-\vec{e}_r} + \vec{e}_z . \frac{\partial \upsilon_\theta}{\partial z}\vec{e}_\theta$$

$$+\vec{e}_r . \frac{\partial \upsilon_z}{\partial r}\vec{e}_z + \vec{e}_\theta . \frac{\partial \upsilon_z}{r\partial \theta}\vec{e}_z + \vec{e}_z . \frac{\partial \upsilon_z}{\partial z}\vec{e}_z$$

Finalmente:

$$\nabla.\vec{\upsilon} = \frac{\partial \upsilon_r}{\partial r} + \frac{\upsilon_r}{r} + \frac{\partial \upsilon_\theta}{r\partial \theta} + \frac{\partial \upsilon_z}{\partial z}$$

Em geral, os dois primeiros termos são combinados usando a regra da cadeia:

$$\nabla . \vec{\upsilon} = \frac{1}{r}\frac{\partial\left(r\,\upsilon_r\right)}{\partial r} + \frac{\partial \upsilon_\theta}{r\,\partial\theta} + \frac{\partial \upsilon_z}{\partial z}$$

Assim, para um escoamento incompressível:

$$\boxed{\nabla . \vec{\upsilon} = \frac{1}{r}\frac{\partial\left(r\,\upsilon_r\right)}{\partial r} + \frac{\partial \upsilon_\theta}{r\,\partial\theta} + \frac{\partial \upsilon_z}{\partial z} = 0}$$

4) Rotacional de um vetor

O rotacional do vetor da velocidade é:

$$\nabla \times \vec{\upsilon} = \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{\partial}{r \partial \theta} + \vec{e}_z \frac{\partial}{\partial z} \right) \times \left(\upsilon_r \vec{e}_r + \upsilon_\theta \vec{e}_\theta + \upsilon_z \vec{e}_z \right)$$

E podemos escrever:

$$\begin{aligned}
\nabla \times \vec{\upsilon} = {} & \vec{e}_r \times \frac{\partial(\upsilon_r \vec{e}_r)}{\partial r} + \vec{e}_\theta \times \frac{\partial(\upsilon_r \vec{e}_r)}{r \partial \theta} + \vec{e}_z \times \frac{\partial(\upsilon_r \vec{e}_r)}{\partial z} \\
& + \vec{e}_r \times \frac{\partial(\upsilon_\theta \vec{e}_\theta)}{\partial r} + \vec{e}_\theta \times \frac{\partial(\upsilon_\theta \vec{e}_\theta)}{r \partial \theta} + \vec{e}_z \times \frac{\partial(\upsilon_\theta \vec{e}_\theta)}{\partial z} \\
& + \vec{e}_r \times \frac{\partial(\upsilon_z \vec{e}_z)}{\partial r} + \vec{e}_\theta \times \frac{\partial(\upsilon_z \vec{e}_z)}{r \partial \theta} + \vec{e}_z \times \frac{\partial(\upsilon_z \vec{e}_z)}{\partial z}
\end{aligned}$$

Continuando:

$$\nabla \times \vec{\upsilon} = \vec{e}_r \times \frac{\partial \upsilon_r}{\partial r}\vec{e}_r + \vec{e}_\theta \times \frac{\partial \upsilon_r}{r\,\partial \theta}\vec{e}_r + \vec{e}_\theta \times \frac{\upsilon_r}{r}\underbrace{\frac{\partial \vec{e}_r}{\partial \theta}}_{\vec{e}_\theta} + \vec{e}_z \times \frac{\partial \upsilon_r}{\partial z}\vec{e}_r$$

$$+\vec{e}_r \times \frac{\partial \upsilon_\theta}{\partial r}\vec{e}_\theta + \vec{e}_\theta \times \frac{\partial \upsilon_\theta}{r\,\partial \theta}\vec{e}_\theta + \vec{e}_\theta \times \frac{\upsilon_\theta}{r}\underbrace{\frac{\partial \vec{e}_\theta}{\partial \theta}}_{-\vec{e}_r} + \vec{e}_z \times \frac{\partial \upsilon_\theta}{\partial z}\vec{e}_\theta$$

$$+\vec{e}_r \times \frac{\partial \upsilon_z}{\partial r}\vec{e}_z + \vec{e}_\theta \times \frac{\partial \upsilon_z}{r\,\partial \theta}\vec{e}_z + \vec{e}_z \times \frac{\partial \upsilon_z}{\partial z}\vec{e}_z$$

Isso resulta:

$$\nabla \times \vec{\upsilon} = -\frac{\partial \upsilon_r}{r\,\partial \theta}\vec{e}_z + \frac{\partial \upsilon_r}{\partial z}\vec{e}_\theta + \frac{\partial \upsilon_\theta}{\partial r}\vec{e}_z + \frac{\upsilon_\theta}{r}\vec{e}_z - \frac{\partial \upsilon_\theta}{\partial z}\vec{e}_r - \frac{\partial \upsilon_z}{\partial r}\vec{e}_\theta + \frac{\partial \upsilon_z}{r\,\partial \theta}\vec{e}_r$$

Reorganizando os termos:

$$\nabla \times \vec{\upsilon} = \left(\frac{\partial \upsilon_z}{r\,\partial \theta} - \frac{\partial \upsilon_\theta}{\partial z} \right)\vec{e}_r + \left(\frac{\partial \upsilon_r}{\partial z} - \frac{\partial \upsilon_z}{\partial r} \right)\vec{e}_\theta + \left(\frac{\partial \upsilon_\theta}{\partial r} - \frac{\partial \upsilon_r}{r\,\partial \theta} + \frac{\upsilon_\theta}{r} \right)\vec{e}_z$$

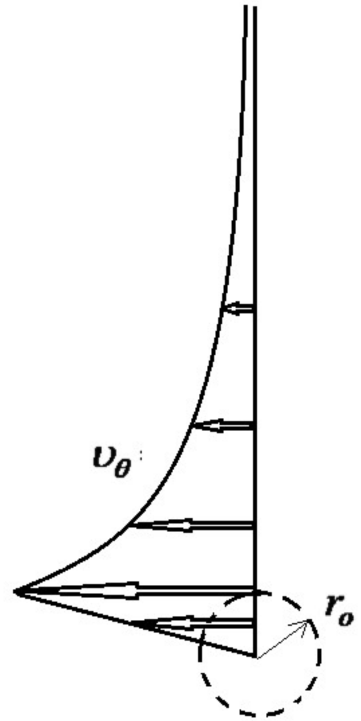
O rotacional da velocidade é chamado de vorticidade $\vec{\omega}$. Na forma de determinante:

$$\vec{\omega} = \nabla \times \vec{\upsilon} = \begin{vmatrix} \vec{e}_r & \vec{e}_\theta & \vec{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{r\partial \theta} & \dfrac{\partial}{\partial z} \\ \upsilon_r & \upsilon_\theta & \upsilon_z \end{vmatrix} + \frac{\upsilon_\theta}{r}\vec{e}_z$$

Exercício: O vórtice de Rankine é um modelo usado para representar furacões ou tornados. Ele é dado pelo campo de velocidades:

$$\upsilon_r \cong 0 \qquad \upsilon_z \cong 0 \qquad \upsilon_\theta = \begin{cases} \Omega r \;\; \text{para} \;\; r \leq r_o \\ \dfrac{\Omega r_o{}^2}{r} \;\; \text{para} \;\; r > r_o \end{cases}$$

Calcule as acelerações e verifique se o escoamento é incompressível e irrotacional.



Solução:

$$a_r = \frac{\partial \upsilon_r}{\partial t} + \upsilon_r \frac{\partial \upsilon_r}{\partial r} + \upsilon_\theta \frac{\partial \upsilon_r}{r \partial \theta} + \upsilon_z \frac{\partial \upsilon_r}{\partial z} - \frac{\upsilon_\theta{}^2}{r}$$

$$\boxed{a_r = -\frac{\upsilon_\theta{}^2}{r}}$$

Logo:

$$\boxed{a_r = -\Omega^2 r \quad \text{para } r \leq r_o} \quad \text{e} \quad \boxed{a_r = -\frac{\Omega^2 r_o{}^4}{r^3} \quad \text{para } r > r_o}$$

$$\boxed{a_\theta = \frac{\partial \upsilon_\theta}{\partial t} + \upsilon_r \frac{\partial \upsilon_\theta}{\partial r} + \upsilon_\theta \frac{\partial \upsilon_\theta}{r\,\partial \theta} + \upsilon_z \frac{\partial \upsilon_\theta}{\partial z} + \frac{\upsilon_r \upsilon_\theta}{r} = 0}$$

$$\boxed{a_z = \frac{\partial \upsilon_z}{\partial t} + \upsilon_r \frac{\partial \upsilon_z}{\partial r} + \upsilon_\theta \frac{\partial \upsilon_z}{r\,\partial \theta} + \upsilon_z \frac{\partial \upsilon_z}{\partial z} = 0}$$

$$\boxed{\nabla . \vec{\upsilon} = \frac{1}{r} \frac{\partial (r \upsilon_r)}{\partial r} + \frac{\partial \upsilon_\theta}{r\,\partial \theta} + \frac{\partial \upsilon_z}{\partial z} = 0}$$ para qualquer r.

Assim, o escoamento é incompressível.

Para $r \leq r_o$:

$$\boxed{\vec{\omega} = \nabla \times \vec{\upsilon} = \left(\underbrace{\frac{\partial \upsilon_z}{r\,\partial \theta}}_{0} - \underbrace{\frac{\partial \upsilon_\theta}{\partial z}}_{0} \right)\vec{e}_r + \left(\underbrace{\frac{\partial \upsilon_r}{\partial z}}_{0} - \underbrace{\frac{\partial \upsilon_z}{\partial r}}_{0} \right)\vec{e}_\theta + \left(\underbrace{\frac{\partial \upsilon_\theta}{\partial r}}_{\Omega} - \underbrace{\frac{\partial \upsilon_r}{r\,\partial \theta}}_{0} + \underbrace{\frac{\upsilon_\theta}{r}}_{\Omega} \right)\vec{e}_z = 2\Omega\,\vec{e}_z}$$

Assim, para $r \leq r_o$ o escoamento é <u>rotacional</u>.

Para $r > r_o$:

$$\boxed{\vec{\omega} = \nabla \times \vec{\upsilon} = \left(\underbrace{\frac{\partial \upsilon_z}{r\,\partial \theta}}_{0} - \underbrace{\frac{\partial \upsilon_\theta}{\partial z}}_{0} \right)\vec{e}_r + \left(\underbrace{\frac{\partial \upsilon_r}{\partial z}}_{0} - \underbrace{\frac{\partial \upsilon_z}{\partial r}}_{0} \right)\vec{e}_\theta + \left(\underbrace{\frac{\partial \upsilon_\theta}{\partial r}}_{-\frac{\Omega r_o^2}{r^2}} - \underbrace{\frac{\partial \upsilon_r}{r\,\partial \theta}}_{0} + \underbrace{\frac{\upsilon_\theta}{r}}_{\frac{\Omega r_o^2}{r^2}} \right)\vec{e}_z = 0}$$

Assim, para $r > r_o$ o escoamento é ir<u>rotacional</u>.

Bibliografia:

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