EQUIVALENCE AND SIMILARITY IN PITCH AND THEIR INTERACTION WITH PCSET THEORY

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A good deal of Western music theory rests on the distinction between pitch and pitch-class. Nevertheless, the term pitch-class (pc) was introduced by Milton Babbitt relatively recently—about 45 years ago; and the first general exposition elaborating the implications of the pitch/pc distinction[1] was published only in 1980 by John Rahn.[2] In my book, *Composition with Pitch-Classes: a Theory of Compositional Design*,[3] I divide pitch function into three categories, each encapsulated by a different pitch space: *contour space*, *pitch space*, and *pitch-class space*.[4] Along the way, I develop the basis for a set theory for pitch sets in analogy to the now "standard" pcset theory of Allen Forte[5] and others.[6] Still, my book as a whole treats the relation of pitch-class to pitch rather than pitch alone. This is because, for me at least, the study of an atonal composition considers the relations between the underlying pitch-class materials and entities and their musical *realizations* as pitches in various categories of time.

While this approach is particularly valuable in the study of twelve-tone and other types of serial music, it is awkward—even wrong-headed—in some other sorts of music, most notably in a good deal of progressive European art music written since the 1950s by composers such as Ligeti, Messiaen, and Xenakis. While some of this literature seems resistant to any kind of pitch function, a good deal of it would seem to be based on pitch rather than pitch-class relations either on the faith of composer testimony or by little evidence of musical unity or coherence using pitch-class theory as a heuristic. We should therefore expect such music to be insensitive to octave equivalence so that the occurrence of an octave is as likely as any other interval. Jonathan Bernard takes such a view when he asserts that "octave equivalence must be ruled out" in the music of Edgard Varèse.[7] Yet a good deal of music, not only by Varèse, is marked by a characteristic lack of proximate octaves or by their special treatment. This suggests that pitch-class relations still affect this repertoire in some special way.

This paper explores the relations between pitch and pitch-class function in two ways. First, it defines three context-sensitive definitions of pitch equivalence, the context being pitch-class space. These equivalences are called PSC, PCINT, and FB. Second, the paper proposes that similarity relations can be used on a par with equivalence relations in pitch since similarity among pitch sets does not have the problems of pitch-class similarity we will discuss below.

I begin with a brief review of pitch set theory (example 1). We adopt the convention of labeling middle-C as the pitch 0 and pitches n semitones higher or lower than 0 are labeled by plus or minus n. The ordered interval between two pitches is computed by taking the first pitch number from the second. Ordered intervals are labeled by (signed) integers. Unordered intervals, called *interval-classes*, are distances between either simultaneous or conceptually unordered pitch pairs. We use unsigned (positive) integers to label interval-classes, or *ics* in pitch. The reader should remember that in this paper, ics are between pitches, not pcs. A pset is an unordered collection of pitches. The pset { -12 -5 2 9 } is the collection of open strings of the viola; the pset { -12 -4 -1 4 9 } is the "Farben" chord of Schoenberg's op. 16, no. 3. Ordered sets of pitches are called psegs. < -3 -2 1 0 -1 > is the pitch sequence of the opening phrase of Bartók's *Music for Strings Percussion and Celesta*. In correspondence to pc-space relations and as the examples show, curly brackets denote unordered sets and angle brackets denote ordered sets.[8]

The interval content of a pset is a roster of the interval-classes between each of its pairs of pitches. It is analogous to an interval-class vector in pc theory. We can derive the roster from an ic-matrix. The ic between the mth and nth pitch in a pset is in the mth row and nth column of the matrix. The roster tells us how many times each ic n occurs in the upper right triangle of the ic-matrix, which contains the interval content of the pset. The function SP(X), "the spacing of the pset X," is an ordered list of the adjacent but unordered pitch-intervals in the pset X from low to high. For instance, SP(<-4 -2 5 8 10>) = [2 7 3 2]. The left-most diagonal of the ic-triangle of X's ic-matrix contains the SP(X).

GLOSSARY

Pitch-space Definitions.

Pitches are shown on the staff or by integers, with pitch 0 = to middle-C.

Ordered interval: interval from pitch a to pitch b = b - a.

Unordered interval, distance, interval-class (ic): interval between pitch a and pitch b = | b - a |; the number of semitones between a and b.

psets: unordered set of pitches.

Examples:

open strings of viola: { -12 -5 2 9 }

Schoenberg's "Farben" chord: { -12 -4 -1 4 9 }

psegs (pitch segments): ordered set of pitches.

Example:

Bartok: opening phrase from

Music for Strings Percussion and Celesta, I.

< -3 -2 1 0 -1 >

Example 1

ic content of X = {–4 –2 5 8 10}.

ic roster:

ic	freq.
2	2
3	1
5	1
7	1
9	1
10	1
12	2
14	1

ic-matrix for X = {–4 –2 5 8 10}

ic	–4	–2	5	8	10
–4	0	**2**	**9**	**12**	**14**
–2	2	0	**7**	**10**	**12**
5	9	7	0	**3**	**5**
8	12	10	3	0	**2**
10	14	12	5	2	0

The spacing of X or SP(X) is [2 7 3 2] and is found in the diagonal with underlined entries.

N.B.: Bold-face entries form the "ic-triangle" from which the ic-roster is drawn.

Example 1 (continued)

Proximate realization of pcseg X' as pset X.

pcset X' = < 4 2 8 1 >

INT(X') = < A 6 5 >

pset X = { -8 2 8 13 }

SP(X) = [10 6 5]

INT(X') is realized as SP(X).

Example 2

Example 2 presents a pc-segment X' and its INT. The INT of a pc-segment is the list of its successive adjacent ordered-intervals.[9] The example shows that the INT of a pc-space segment X' is related to the spacing of a pset X if X is a *proximate realization* of X' in pitch, ordered low to high, such that the pitches of X are as close as possible. In the example, X' is < 4 2 8 1> and the INT(X') is < A 6 5 >. The proximate realization of X' in pitch-space is { -8 2 8 13 } and its spacing is [10 6 5] the same as INT(X').

Pitch-space transposition and inversion are defined in example 3. The example shows that: 1) the interval from pitch a to b, a's transposition by n, is n; 2) the I operator (inversion) inverts pitches around pitch 0, middle-C; and 3) two pitches related by T_nI add up to n. In addition, the axis of symmetry between two pitches related by T_nI is n/2. This axis is either a pitch or quarter-tone between two pitches.

A pitch set-class (PSC) is the collection of psets related by T_n.[10] A pitch set-class that includes the pset X will be denoted by the function PSC(X). A PSC is named by its members' spacing. In principle each PSC contains an infinity of psets, but human hearing and use limits the number of functional members. A few members of the PSC({-6 7 9}) are given in example 4.

In sum, pitch-space relations correspond to relations in pc-space but without doing the arithmetic mod-12. A major difference between pitch and pc is the number of entities in each space. In pc-space there are 12 pcs, 12 ordered intervals, 7 ics, and 4096 pcsets. In pitch-space there are 88 pitches, 176 ordered intervals, 88 ics, and 2^{88} psets. As for SCs, in pc-space there are 223; in pitch-space the number of PSC is astronomical and impossible to calculate without a computer programmed to execute standard combinatorial formulae. A comparison of the number of trichordal SCs with PSCs, however, gives some idea of the scale; there are 12 trichordal SC in pc-space and 2838 trichordal PSCs in pitch-space.[11]

a, b $\in$ pitches

Transposition (T_n): $b = T_n a = a + n$
(and $n = b - a$).

Inversion (I): $b = Ia = -a$.

Transposition and Inversion: $b = T_n Ia = n - a$
(and $n = a + b$)

Axis (center) of $T_n I = n/2$.

Example 3

PSC (pitch T_n-type set-class)

X = {−6 7 9} SP(X) = [13 2]

PSC(X) is named [13 2] and includes:

{−6 7 9} = X
{0 13 15} = T_6X
{−8 5 7} = T_{-2}X
etc.

Example 4

Thus, the sheer number of PSCs would seem to diminish their general usefulness in composition and analysis. For while it is not very difficult to memorize and internalize 12 trichord types, 2838 poses a significant musical challenge.

Fortunately, our interest in defining some context-sensitive equivalences and similarities in pitch-space—those that reflect pcset equivalence—can help solve the problem of PSC abundance. Some new definitions of pitch equivalence will collect psets into a more modest number of equivalence-classes. (Of course, each class will have many more members than any PSC.) The key to this project is to consider the set of pitch realizations of the members of a pitch-class SC. We denote all possible pitch realizations of the pc SC X by the function PR(X). Example 5a gives the 24 members of the 3-3[014] set-class. Example 5b provides a few of the members of PR(3-3). The example only shows psets, but the pitch realization of a SC includes ordered psets, with and without duplications.

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{ {014} {67A} {0B8} {652}
  {125} {78B} {109} {763}
  {236} {890} {21A} {874}
  {347} {9A1} {32B} {985}
  {458} {AB2} {430} {A96}
  {569} {B03} {541} {BA7} }
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Set-Class 3-3[014].

Example 5a

Some pset members of PR(3-3).

PR(3-3) is the set of all pitch realizations of the pc SC(3-3).

Example 5b

The number of pitch realizations is, of course, infinite, or in practice very high, but we can divide this collection into smaller and manageable subcollections using a pitch relation S. When S is an equivalence-relation, S partitions PR(X) into (non-overlapping) subcollections. And by definition, each subcollection is made out of pitch-realizations of the members of SC X in pc-space. Thus when we apply S to pitch-relations of *all* pc SCs, the partitions induced by S will not cut across pc-space SC boundaries. In this way, S is context-sensitive to SC membership in pc-space.

I propose three candidates for S, a pitch equivalence that partitions any PR(X): PSC, PCINT, and FB.

We have already defined PSC. Example 6 shows a few members of PSCs that partition SC(3-3). Thus, as in the example, the PSC [1 3] contains the following psets: {5 6 9}, {19 20 23}, and {-21 -20 -17}. Each PSC has a dual PSC in which the SP succession is backward. For example, the dual of PSC [4 5 11] is PSC [11 5 4]. The psets in a PSC are the p-space inversions of the pset members of its dual. Some PSCs are their own duals; PSC [23532] is self-dual.[12]

The large number of PSCs can be reduced significantly by setting a limit on a PSC's outside pitch interval, the interval from the lowest pitch

Members from two of the PSCs (pitch set-classes) that partition PR(3-3).

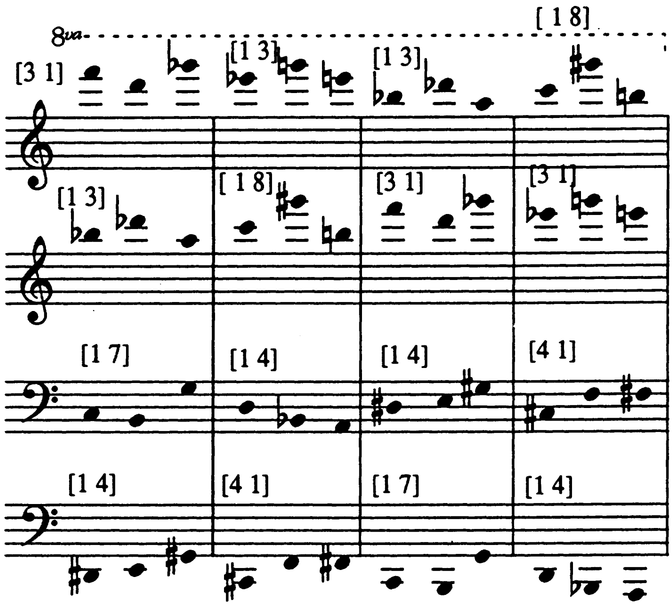
Example 6

PSCs containing members of PR(3-3) with outside intervals less than 12 semitones.

Example 7

to the highest in a PSC member. The outside interval is the sum of the intervals in the PSC's name, its members' spacing. If we set the outside interval limit of PSCs to 11 semitones, there are only six PSCs that contain members of PR(3-3) as shown in example 7. Such a limitation is particularly apt in serial pieces where array lynes are articulated in narrow registral spans.

Example 8 indicates how band-limited pitch areas help project pc lynes in Milton Babbitt's *Partitions* for piano. The example comes from mm. 32–35, which is based on a typical four-aggregate trichordal array. Each array lyne is articulated in a different 11-semitone span from some A♮ to G♯. The trichordal segments which generate the array are therefore mapped one-to-one from pc to pitch into their registral slots. This means that the unordered content of each trichord is mapped to one and only one PSC. Since the trichords in an array line are related under twelve-tone operations, the PSCs that result are those that partition one trichordal SC. On the top line, for example, each trichord's content is of SC3-3[014] and due to the constraints under discussion, only PSCs [3 1], [1 3], and [1 8] occur in the music. Note that two of the three are duals. Since the second lyne is a rotation of the first by six order-positions, the same PSCs occur in the pitch realization of that line. The bottom two lines are also related by rotation, but are based on members a different trichordal SC, 3-4[015]. The corresponding PSCs are therefore different, but once again



PSCs in Babbitt's *Partitions* for piano, mm. 32-35.

The top two lines contain pitch realizations of (pc) trichordal segment-class <5 2 6> whose content is of (pc) SC(3-3)[014]. Only three PSCs are found realizing the trichords, two of which are duals: [1 3], [3 1], and [1 8].

The bottom two lines contain pitch realizations of (pc) trichordal segment-class <0 B 7> whose content is of (pc) SC(3-4)[015]. Only three PSCs are found realizing the trichords, two of which are duals: [1 4], [4 1], and [1 7].

Example 8

(pc) segment-Class 1	<0 1 4> INT = <13 > → pitch PCINT-class [13]
(pc) segment-Class 2	<0 1 9> INT = <18 > → pitch PCINT-class [18]
(pc) segment-Class 3	<0 0 4 1> INT = <049 > → pitch PCINT-class [049]

Three pc segment-classes whose (unordered) contents are members of SC(3-3)[014] that map to three (pitch) PCINT-classes.

(NB: pc segment-classes contain pcsegs related (only) by T_n.)

Example 9a

Members of the PCINT-class [1 3] derived from segment-class 1 in example 9a.

Members of the PCINT-class [1 8] derived from segment-class 2 in example 9a.

Members of the PCINT-class [0 4 9] derived from segment-class 3 in example 9a.

Example 9b

there are three and two of them are duals. The reduction of the ordered pc trichords of the array to a handful of PSCs induced by the tight registral assignments helps project the pc structure of the array in a redundant and hence effective way. In addition, the pc ordering in the array with the PSC registrations projects a small set of contours that further unifies the passage.[13]

A second equivalence, called PCINT (for pitch-class INT equivalence) involves psets whose spacing intervals from low to high are identical, or expanded (or diminished but without order interchange) by any number of octaves. We name a PCINT-class by the spacing of its most compressed member(s), in proximate realization. Thus a few of the members of the PCINT-class [2,4,5] are psets {0,2,6,11}, {0,14,18,23}, {-5, -3, 13,18}. The members of a particular PCINT-class are pitch realizations of members of a pitch-class segment-class that contains the transpositions of a pcseg. The spacings of any member of a PCINT-class, when taken mod-12, will reduce to the same INT of some pcsegment.[14] The retrogrades, inversions, and retrograde-inversions of the same pcsegment form three other PCINT-classes. Since the pc-space sources of PCINT-classes are ordered pcsets, the reader may wonder if PR(X) is partitioned by a set of PCINT-classes. The assertion goes through when we remember that PR(X) contains all permutations of all members of SC(X). Since each permutation of the pcs of X generates a different INT from X (to within any pc invariances of X) and these INTs define each PCINT-class, so the PCINT-classes in question partition PR(X). Example 9a provides three different pc segment-classes whose members' unordered content are members of SC 3-3. Three segment-classes map to three PCINT-classes. Example 9b shows several members of these PCINT-classes. As with PSCs, The dual of a PCINT-class P is that class which takes the name of P in reverse. [4 7] is the dual [7 4]. The PCINT-class [111] is its own dual and contains the psets, {0123}, {-3, -2, 11, 12}, {-24, -11, 2, 27}, among others.

When the pcset X in PR(X) has invariance under pc transpositions beside T_0, the number of different PCINT-classes is diminished according to the following formula. D_X is the degree of symmetry of X.

$$\#\text{PCINT-classes} = (\#X)! * 2 / D_X$$

The third equivalence is well-known to music theorists; it is called FB, for figured bass.[15] The pc ordered-intervals between the non-bass pitches and the lowest, bass pitch of a pset are given as the integers 0 to B. These intervals are listed in ascending numeric order and identify an FB-class. Two psets with the same (chromatic) figured bass are FB-equivalent—in the same FB-class—which takes its "normal form" from the set of figures. Each member of an FB-class can be interpreted as a partially-

Members of the FB-class 3478.

Example 10a

orderéd pcset realized in pitch-space. One pc occurs before all the others in the set. The concept of "before" is interpreted in pitch as lower. Thus the pcs that follow the first pc can be realized in any register of pitch-space as long as they are higher than the first pc's realization. Example 10a shows samples of psets related by FB-equivalence. Note also that a FB or its class may have a zero or multiple instances of intervals.

How do the FB-classes partition PR(X)? I have just shown that FB-classes are pitch realizations of partially ordered pcsets. We can derive all the possible posets of the required type as follows. To construct an FB-class we take a pc of X and follow it by the other members of X to produce the partially ordered set. The rest of the partially-ordered sets are derived likewise from the (pc) inversion of X. These posets define the non-duplicating FB-classes without zeros or multiple intervals. In example 10b, we see the three rotations of X and I(X) which are members of SC 3-3. The rotations of the inversion of X are 034, 340, and 403. Taking for instance the last, 403, the example generates the partially ordered set <4, {03}> which means that the pc 4 comes before the pcs 0 and 3, which are unordered with respect to each other. The intervals between 4 and 0 and 4 and 3 are listed as subtractions on line three of the example, vis. "[0-4, 3-4]." The intervals are then computed; this yields the FB-class 8B. Every member of this class is a pitch trichord with two pitches 8 and 11 semitones (plus any number of octaves) above its lowest note. From this generative algorithm, the number of non-duplicating FB-classes is given by the following formula:

$$\#(\text{non-duplicating FB-classes}) = \#X * 2 / D_X$$

The formula indicates that, for example, there are 8 FB-classes (without duplication) derivable from SC(4-2) and there are 3 classes from SC(3-6) since each member of 3-6 has a degree of symmetry of 2. A hexachordal SC may generate anywhere from 10 to 1 FB-classes. The total number of non-duplicating FB-classes is easy to determine. Since a non-duplicating FB-class is an unordered pcset preceded by a different

$X = \{0\ 1\ 4\} \in$ 3-3[014]

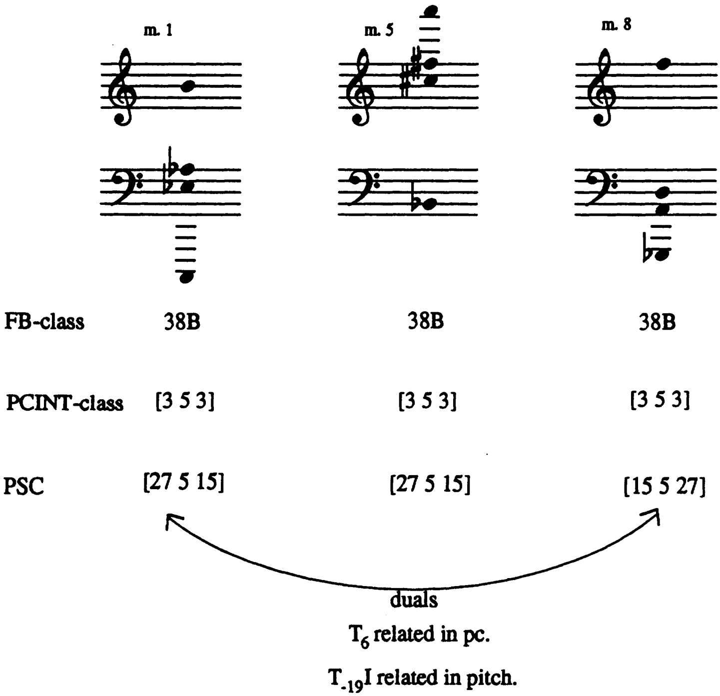
014	rotations:	014	140	401
	posets:	< 0,{14} >	< 1,{04} >	< 4,{01} >
	FB intervals:	[1–0, 4–0]	[4–1, 0–1]	[0–4, 1–4]
	FB-class:	14	3B	89
034	rotations	034	340	403
	posets	< 0,{34} >	< 3,{04} >	< 4,{03} >
	FB intervals	[3–0, 4–0]	[4–3, 0–3]	[0–4, 3–4]
	FB-class	34	19	8B

(posets = partially-ordered sets)

FB-class Duals:

FB-class	k =	FB	n →(k–n)		0→k	Dual FB
14	4	14	→	30	→	34
3B	B	3B	→	80	→	8B
89	9	89	→	10	→	19

Example 10b



Referential pitch equivalences in Babbitt's *Partitions.*

Example 11

pc and the size of the pcset can vary from 0 to 11 elements, there are 2^{11} or 2048 FB-classes. Incidentally and surprisingly, such enumerative facts about chromatic figured bass were worked out and published in 1884 by the French music theorist Anatole Loquin.[16]

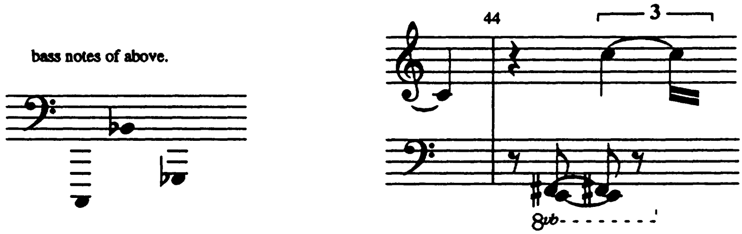
As the bottom of example 10b shows, we can arrange the FB-classes in pairs of duals, such that for each interval n in a FB-class, its dual has the interval k-n where k is the largest interval in the FB-name; in addition, we replace the 0 by k. For instance, to produce the dual of the FB-class 349 ; k =9, so 9-3 = 6, 6-4 = 5, and 9-9=0 which is replaced by 9. The dual is 569.[17] The reason the definition of dual is a little more complicated than with other equivalences is that we want two psets related by p-space inversion to be respectively included in dual FB-classes.

A brief musical illustration, also from Babbitt's *Partitions,* will help point out how these pitch equivalences can be used in analysis. The point of the discussion is not that pitch equivalence is an analytic substitute for pc-equivalence, but that pitch-equivalence enables the deeper, underlying pc relations to be discovered and heard.

The first section of *Partitions,* which serves as a kind of introduction, presenting the materials of the entire piece, opens dramatically with the pitches on the left of example 11. The pc SC is 4-17[0347], a telling subset of the work's generating hexachord, an ordering of the B all-combinatorial source-set. At the very end of the section in m. 8, the music also articulates a member of 4-17, a T_6 transposition of pcs of the opening chord. In the middle of the opening passage, in m. 5, there is one other articulation of 4-17. This is a T_A pc transposition of the first chord. When we write down the pitches of the three chords, as in example 11, we see that each is a member of the same PCINT-class [3 5 3]. This obliges the FB-classes of the three chords to be the same, since, as we will see, PCINT-classes partition FB-classes. Two PSCs are represented by the three psets; the first two are identical, the last is the dual of the first two. This pitch symmetry gracefully frames this opening section and suggests many other relations as well. Among these is a $T_{-19}I$ relation between the first and last psets of the example. This symmetric pitch relation prompts the listener to suspect that a T_5I pc relation underlies the pitch symmetry. This conjecture is well rewarded as the hexachords of the array are so related. And together with the T_6 relation already mentioned, T_5I forms a four-group of pc relations that generates the entire four-aggregate array of the opening section. The transformations are T_0, T_6, T_5I, and T_BI, which together with retrograde and rotation define the symmetries upon which the array and much of the rest of the composition is based.[18]

But we haven't addressed the role of the middle chord of example 11. Aside from having its own pc symmetry under T_5I, this pset provides an origin for a special event crossing bar 44 shown in example 12. There the pc 0 is played twice an octave apart signalling an array boundary. The low major second joining the two C♮s produces a memorable, if odd, moment. Measure 44 perplexed me for many years before I recognized its origin in the opening section. Of course, the reader has the benefit of example 11; thus it doesn't take much work to understand that the music at m. 44 plays out the T_0, T_A, T_6 relation of the chords as the pcset {046}. And hearing the three chords as connected by their shared figured bass suggests that the pseg < C, B♭, G♭ >is derived from the chords' bass tones; this underlines the connection to m. 44 that much more since the pseg under T_6 becomes the notes of the later passage. So it is Babbitt's *realization* of the pcs of the opening array of *Partitions* as psets related by pitch equivalence relations that "explains" the event at m. 44. Conversely, these pitch equivalences allow the listener to appreciate the pitch-class symmetries in the array.

I now turn to relations among the three equivalences that partition



{ 0 A 6 } $\xrightarrow[T_6]{\hspace{3cm}}$ { 0 4 6 }

Relation of opening to m.44 of *Partitions.*

Example 12

PR(X). Example 13 summarizes the definitions of PSC, PCINT and FB. While I trust my reasons for selecting these pset relations are not counterintuitive, there is a property that gives them special status; they form a hierarchy of relations under set-inclusion. As example 13 shows, for PR(X), each member of SC(X) exclusively includes certain FB-classes, each of these FB-classes exclusively includes certain PCINT-classes, each of these PCINT-classes exclusively includes certain PSCs, which in turn include certain psets. Or, working up from left to right, psets partition the PSCs, which partition the PCINT-classes, which partition the FB-classes, which partition PR(X).

The entire system of equivalences forms a huge tree-structure of depth 4. Example 14 indicates some of the tree's branches for the SC (3-3). The diagram also indicates how the duals in the three pitch-space equivalences interact. Note that two psets related by (p-space) T_nI are found respectively in dual PSC, PCINT, and FB-classes. But a PCINT-class and its dual need not be respectively included in a FB-class and its dual. For instance, PCINT [4 9] is included in FB-class 14, but PCINT [9 4], the dual of [4 9], is not included in FB-class 34, the dual of 14, but in 19.

Another issue in the relations between duals defined in each pitch-equivalence is revealed when we look at the members of duals of PSCs. The pset {0 1 4} is the pitch-space inversion of {0 -1 -4}; these two sets are dually related under PSC. Yet the spacing of their dual PSC and PCINT-classes are related by retrograde. [1 3] is the retrograde of [3 1]. Since spacing is analogous to the INT of a pcseg, the two spacings can taken as INTs of two pcsegs related by retrograde-inversion. This may seem counterintuitive, but if we are ordering a pcseg Y under a proximate realization in pitch-space, the RI (not the I) of Y similarly interpreted in

Hierarchy of pitch and pc equivalence classes:

psets $\subset$ PSC $\subset$ PCINT-classes $\subset$ FB-classes $\subset$ PR(X)

Summary of pitch and pc equivalence classes:

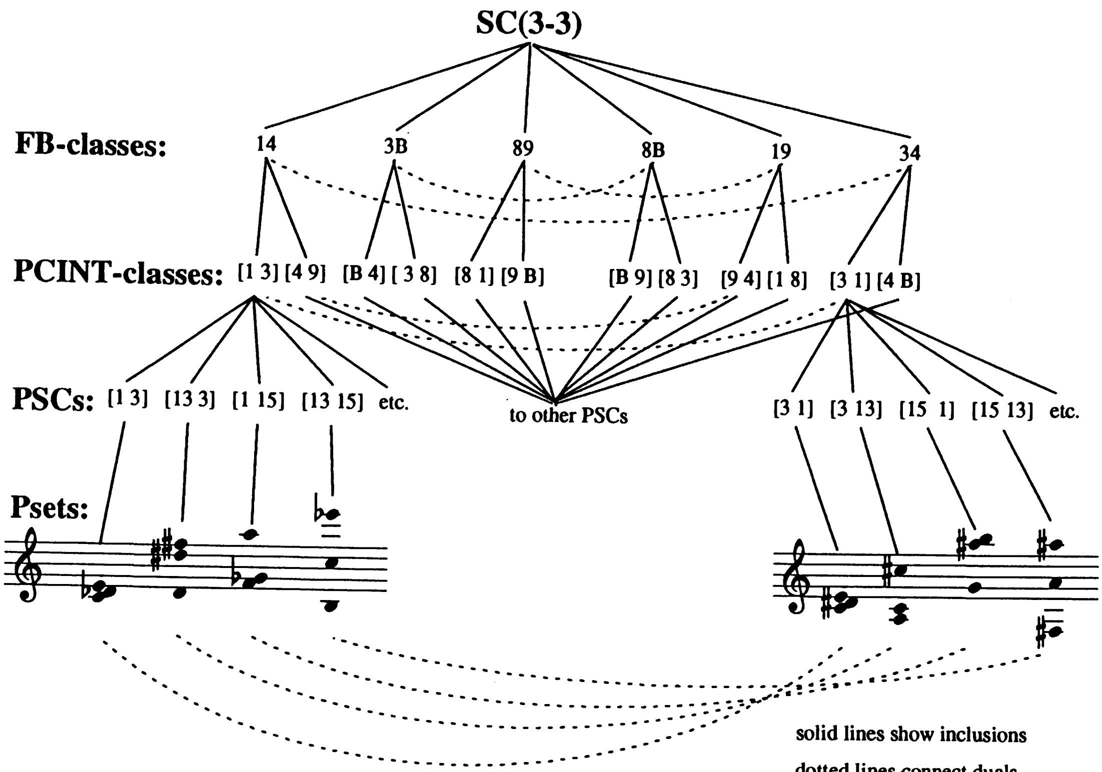
PR(X)	=	all pitch realizations of members of SC(X).
FB-classes	=	partially ordered members of SC(X) realized in pitch-space to within pitch transposition.
PCINT-classes	=	members of SC(X) partitioned into classes of pcsegs with identical INTS realized in p-space from low to high.
PSC	=	members of SC(X) realized in p-space as psets with identical spacings.
psets	=	unordered sets of pitches; members of PSCs.

Example 13

pitch produces an ordering of intervals that is the reverse of Y's realization. So inversional duality among pcsegs maps to inversional duality among psets under retrograde.

While these pitch equivalences can model music in pitch-space without postulating any underlying pitch-class relations yet do not contradict potential pitch-class SC correlations, if any, the large number of equivalence-classes remains a major issue. Even though there are pieces that will be modeled adequately by a small number of classes, many others have psets from literally hundreds of classes. Defining similarity relations between psets is one recourse to such situations, and it is of some merit since it does not have the problems associated with pitch-class similarity.

There are two basic difficulties with applying pitch-class set-class similarity in analysis: 1) different pcsets can articulate the same pitch-class SC; and 2) each pcset has a multitude of realizations in pitch and time. If we base a similarity measure on a feature shared by all members of a set-class, such as interval-class content or abstract subset content, then we address the first difficulty. However, as example 15a indicates, the second difficulty remains a severe stumbling block to the meaningful use of pitch-class similarity in most analytic contexts. Each pset in the example is a realization of the same eight pitch-classes, an octatonic


SC(3-3)
FB-classes:
14
3B
89
8B
19
34
PCINT-classes:
[1 3] [4 9]
[B 4] [3 8]
[8 1] [9 B]
[B 9] [8 3]
[9 4] [1 8]
[3 1] [4 B]
PSCs:
[1 3]
[13 3]
[1 15]
[13 15]
etc.
to other PSCs
[3 1]
[3 13]
[15 1]
[15 13]
etc.
Psets:
solid lines show inclusions
dotted lines connect duals

Example 14

Four psets that realize SC(8-28) (the octatonic scale).

Example 15a

SC(6-23)[023568] SC(3-11)[037] SC(5-35)[02479] SC(8-9)[01236789]

Four psets, each very similar to its corresponding pset in Example 15a. These psets are not realizations of SC(8-28).

Example 15b

scale. Yet the four sets are highly differentiated by spacing and interval-class adjacency. Moreover, example 15b has four other psets that pair up with the psets in 15a to produce similarity pairs—yet these psets are realizations of different SCs, none of which is octatonic. The example demonstrates that similarity in pitch (and time) is just as likely to cut across as confirm SC equivalence in pitch-class space. Consequently, paths of pcset similarity have to be carefully realized in pitch according to criteria of pitch similarity; and more importantly, pitch similarity may operate independently of pitch-class equivalence. Because psets do not need to be realized through some other musical space or dimension—that is, they can be directly perceived as chords or arpeggiated chords—this difficulty is answered.[19]

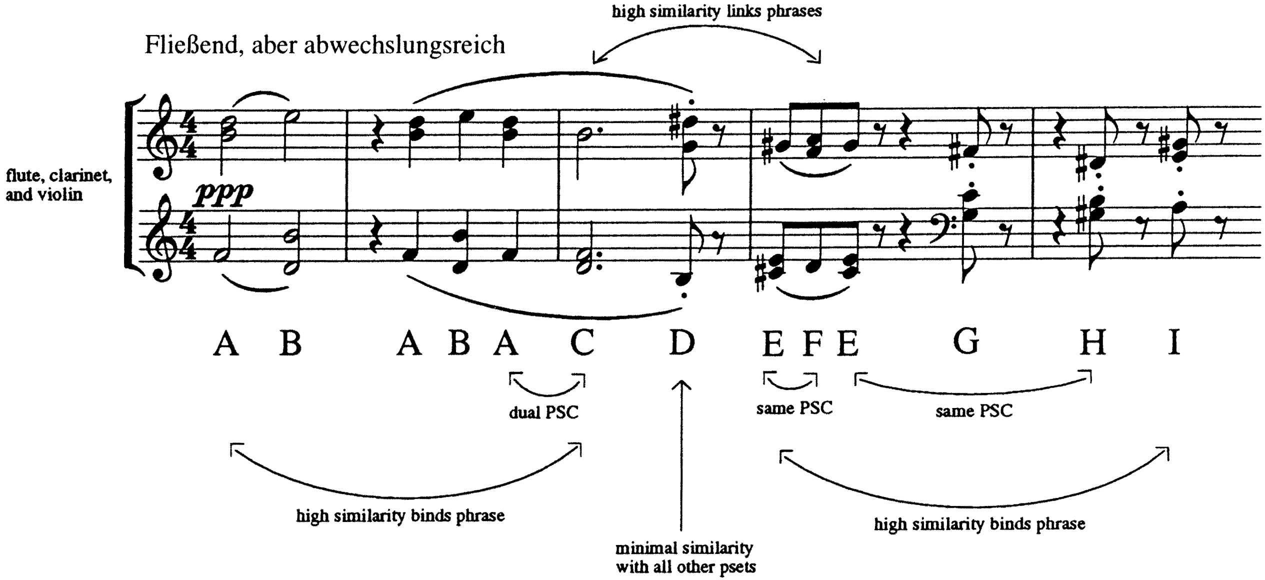
While a number of similarity measures among the three types of pitch equivalences can be constructed, I will discuss and evaluate what is perhaps the most direct of all of them. This measure is called PM (for pitch-measure) and compares any two psets. It registers two things: the number of pitches shared by two psets; and number of (pitch) ics of the same size shared by the two. The measure therefore returns two numbers, the first for pitch intersection, the second for ic intersection. We write,

$$p,i = PM(X,Y)$$

where X and Y are two psets and p and i are the cardinalities of pitch and ic intersection. For the psets {0 4 6} = X and {-1 2 4} = Y, the number of pitches shared is 1 and the number of ics shared is 1 so the PM(X,Y) is 1,1.

To illustrate how PM can aid pitch analysis, I include a few analytic remarks on the opening of "Eine blasse Wäscherin" from Arnold Schoenberg's *Pierrot Lunaire*.[20] Example 16a presents the first five measures of the piece, an introduction immediately preceding the entrance of the *sprechstimme*. The 13 chords are labeled A to I with repeated chords given the same letter. In example 16b, each chord is listed as a pset with its interval roster. The chords are then compared according to the PM measure in the similarity matrix at the bottom of the example. The entry in the cell in row R and column C of the matrix gives the PM similarity between the psets associated with row R and column C. Bold face entries highlight the higher degrees of similarity. The entries in the pitch-similarity matrix are used to support the findings in example16a. As the example shows, the passage exhibits a retrograde symmetry according to the PM measure. This comes from a parallelism between the first three measures and the next two. Each group of measures each exhibits a high similarity between its beginning and ending pset, possesses adjacent PSC connections, and is linked to the other by high similarity symmetrically placed around the chord labeled D. D makes a good pivot since it is almost inert, having only the weakest connections with the other chords.

Next, after the entry of the voice halfway through m. 5, the instruments play a run of seven eighth-note trichords as shown in example 17a. The chords, all different, are labeled from A to G and are compared in a new similarity matrix in example 17b. Even without the matrix it is easy to see/hear that chords D and E are related by T_4 in pitch, and chord A and F are PSC duals. As before, there is an augmented-chord labeled G at the end of the passage with little connection to the psets right before it. In fact, its similarity rating is 0,0 with all other chords except B and C. From a pitch-class point of view, G has the same content as chord D in



Retrograde symmetry among psets at the beginning of Schoenberg's "Eine blasse Wäscherin

Example 16a

Psets in example 16a.

pset	interval-class roster
A = {5 11 14}	[3 6 9]
B = {2 11 16}	[5 9 14]
C = {2 5 11}	[3 6 9]
D = {-1 7 15}	[8 8 16]
E = {1 4 8}	[3 4 7]
F = {2 5 9}	[3 4 7]
G = {-5 0 6}	[5 6 11]
H = {-4 -1 3}	[3 4 7]
I = {-3 4 8}	[4 7 11]

Similarity Matrix for psets A through H.

The entry in the cell in row R and column C of the matrix gives the PM similarity between the psets associated with row R and column C. Bold face entries highlight the higher degrees of similarity.

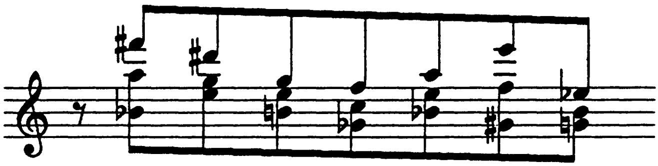
	A	B	C	D	E	F	G	H	I
A	—	1,1	**2,3**	0,0	0,1	1,1	0,1	0,1	0,0
B		—	**2,1**	0,0	0,0	1,0	0,1	0,0	0,0
C			—	0,0	0,1	**2,1**	0,1	0,1	0,0
D				—	0,0	0,0	0,0	1,0	0,0
E					—	**0,3**	0,0	**0,3**	**2,2**
F						—	0,0	**0,3**	**0,2**
G							—	0,0	0,1
H								—	**0,2**
I									—

(Note that the lower-left triangle of the matrix is left unfilled since these entries are redundant.)

Example 16b

the previous passage, but we need not rely on pc relations to connect the two chords. They are at the end of phrases, unrelated to their neighbors, and share two pitches. Besides, all spacings of augmented chords are of the same FB-class (and PCINT-class).

But we gain finer relations among the seven chords from consulting the similarity matrix. Example 17c writes the chord labels connected by curves. The curves show moderately high PM values from the matrix. Numbers above each chord-label indicate the sum of all PM ratings involving that chord. The example's network of chords indicates the separation of G from the rest save one and that chord E is the nexus of the seven, having more similarity with its fellows than the others. We can



Example 17a

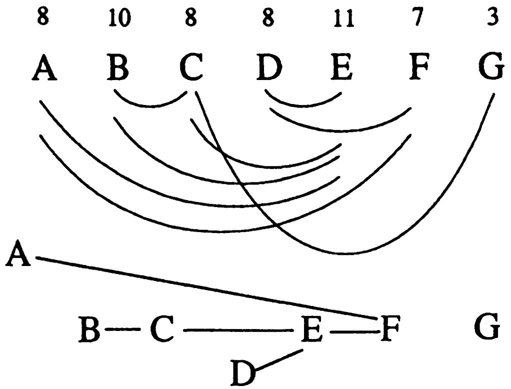
Psets in example 17a.

pset	interval-class roster
A = {10 21 30}	[9 11 20]
B = {16 19 27}	[3 8 11]
C = {11 16 19}	[3 5 8]
D = {6 12 17}	[5 6 11]
E = {10 16 21}	[5 6 11]
F = {8 17 28}	[9 11 20]
G = {7 11 15}	[4 4 8]

Similarity Matrix for psets A through H.

	A	B	C	D	E	F	G
A	—	0,1	0,0	0,1	2,1	0,3	0,0
B		—	2,2	0,1	1,1	0,1	0,1
C			—	0,1	1,1	0,0	1,1
D				—	0,3	1,1	0,0
E					—	0,1	0,0
F						—	0,0
G							—

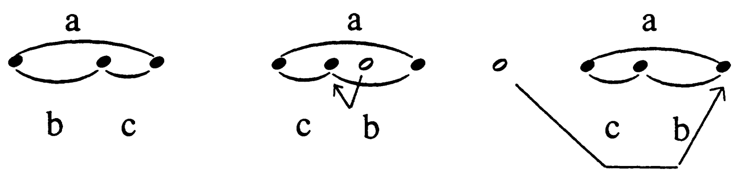
Example 17b



Example 17c

rewrite the diagram as a partially ordered set as shown at the bottom of the example. Now we see that the first chord, A, connects most directly but non-adjacently to E and especially F, its dual. Chords B, C, E, and F form a chain of similarity, with D acting as a local embellishment of E. Note how E is part of three continuities, from B, from C and from D. These associations make E stand out of the flow which makes sense since E (and D) are duals to chord G of the previous example, a chord with little affinity with its surroundings. A strategy seems to be emerging from our observations. Psets that are most deviant from their local environment stand out and are closely tied via PM similarity to each other across longer musical spans.

While the PM measure does help illuminate the passage, I do not mean to imply that ordinary SC analysis would not point to some of the same or similar conclusions. My point is that SC analysis using pc similarity relations like Forte's R_1, R_2, and/or R_p, Morris's SIM, or Rahn's TMEMB underdetermines the analysis.[21] Consider that the degrees of similarity determined by such measures would remain exactly the same if the chords of the Schoenberg were each randomly transposed or inverted and/or if each pitch of the piece were randomly transposed by any number of octaves. Rather, it is the exact pitch of each chord of the unaltered Schoenberg that accounts for our sense of pitch similarity therein. As my discussion of examples 15a and b implies, pc similarity, if it is to elicit any sense of heard "similarity," is essentially dependent on realizations that promote pitch similarity. Thus, in the absence of pitch-class SC relations, we need not appeal to pc relations as explanatory, but remain within the domain of pitch, using PM or other kinds of pitch similarity to explicate a passage.



Example 18

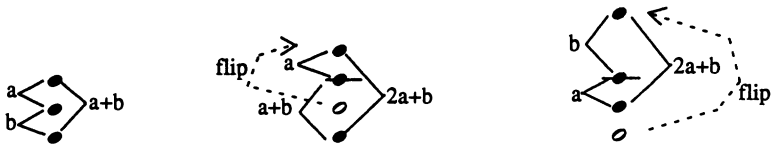
The transparency of the PM similarity index is a strength; it is easy to calculate and its uses are many. But behind the simplicity there is another, quite remarkable feature that involves what is often called *maximal similarity* in the literature on pc similarity measures. Two sets are maximally similar if they are as similar as possible but not identical. Now two psets from the same or dual PSC will contain the same ics, but the number of pitches in common will vary from from all to none.[22] In the case of trichords, maximal similarity is therefore all (three) ics in common and two pitches in common.

A more general question can be posed: when the two psets differ by only one pitch what is the maximal number of ics that can be preserved? The diagram in example 18 shows the trichordal case; it indicates how the change of one pitch can preserve all ics in a three-pitch pset. Either the middle note between the two others is changed so the ics b and c interchange while the ic a remains constant, or the first note is flipped around the last two it so that the loss of ic b from the first pitch to the second is regained by the final position of the changed pitch. When there are more than 3 pitches, then the intervals between the moving pitch and the other pitches change but not in the subset of unaltered pitches. Thus, the number of ics that change in a pset S when one pitch is altered is the number of pitches in S minus 1.[23] We write:

#(Changed ics) = #S - 1.

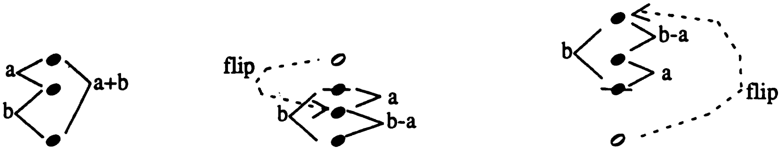
Furthermore, when pset X is changed to pset X' by changing one pitch the PM measure between X and X' is C,D, where C = #X - 1 and D $\geq$ #(ics in X) - C.

As mentioned above, the PSC of a trichordal pset can be preserved or changed to its dual with maximal similarity under the PM measure. It is more interesting to consider a special kind of maximal similarity among trichords: a change of one pitch and a change of one ic. This similarity relates trichords of two *different* PSCs provided the ic roster of either of the two psets has no ic duplication. This kind of maximal similarity will be called FOLDSIM. We have seen an example of FOLDSIM in exam-



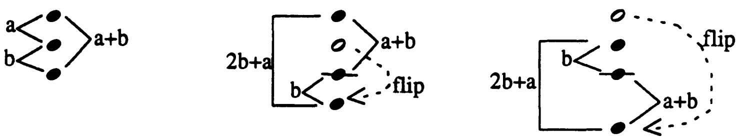
(Bernard's unfolding.)

Example 19a

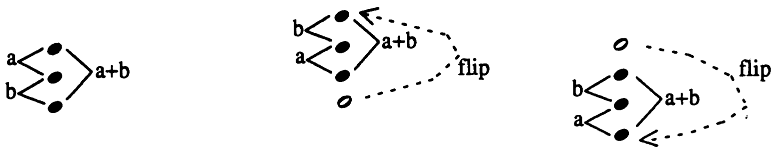


(Bernard's infolding.)

Example 19b



Example 19c



Example 19d

In the above, dotted arrows show a flip from ⊘ over pivot note –●– to ● ; solid lines show intervals.

ple 16a. There trichordal psets E and I share two out of three pitches and two out of three intervals.

I will now show that trichordal FOLDSIM occurs as a byproduct of two pitch transformations called *unfolding* and *folding in* invented by Jonathan Bernard for analyzing the music of Varèse.[24] I shall refer to all of these transformations as *foldings*.[25] Consider the three trichords shown schematically in example 19a. On the left is a trichord whose three pitches are represented by black note-heads with the ics a, b, and a+b. The center trichord shows the result of transforming the left trichord by

Pset spacing [b a] and pset ic roster {a b a+b} become in

Ex. 19a [a+b a] or [a a+b] and {a a+b 2a+b},

Ex. 19b [b-a a] or [a a-b] and {a a-b b},

Ex. 19c [b a+b] or [a+b b] and {a a+b 2b+a},

Ex. 19d [a b] and {a b a+b}.

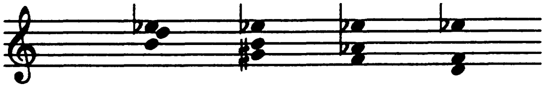
Example 19e

symmetrically flipping its middle pitch around its top pitch. This is indicated by a white note representing the old position of the changed pitch and a line through the pivot note—in this case, the top one. Since the flip is symmetric, ic a is preserved, as is the ic a+b (between the unchanged pitches). A new ic however is formed between the flipped note and the bottom pitch; it is 2a+b. Bernard calls this kind of flip an *unfolding*. The right trichord shows what happens when the flipped note is originally on the bottom, keeping the top one as the pivot. This flip transformation is not described as such by Bernard. In any case, the result is the dual of the center trichord.

Examples 19b and c continue the flipping around the middle and bottom pitches respectively. The center trichord of 19b is an example of Bernard's other transformation, *folding in*. Some other foldings not described by Bernard are given in example 19d. Here the pivot is not a note but two pitches, or the ic between them. The result is pset duality.

Example 19e produces a summary of the six basic foldings, plus the two others in 19d. The example shows that a given trichord can be changed into three pairs of dual trichords or its own dual. Four pairs of duals are involved. Because two out of three pitches and ics are preserved in each case, the flipping transformations produce either FOLDSIM or maximal similarity between the unflipped trichord and any of the others.

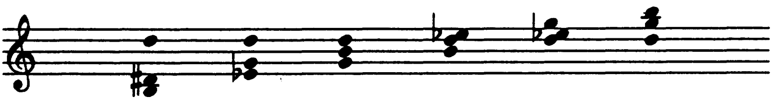
We can illuminate an important aspect of folding if we repetitively perform one of the six basic types given above. In such cases, one pitch remains invariant throughout while the other two change one at a time and alternately. In example 20a the pset {11 14 15} is unfolded downward. The process involves the bottom two pitches exchanging the roles of flip and pivot. Since the first pset has 3 as its lower ic, the set of spacings retains an ic 3. The other ic in the spacing expands by 3 as does the outside ic, not noted in the spacing. Note that the moving pitches form cycles of one pitch interval. This observation couples folding, FOLDSIM, and pitch transposition to the compositional theories of George Perle.[26]



spacings: [3 1] [3 4] [3 7] [3 10]

Repeated unfolding of pset {11 14 15}.

Example 20a



spacings: [4 11] [4 7] [4 3] [3 1] [1 4] [5 4]

Repeated infolding (then unfolding) of pset {-1 3 14}.

Example 20b

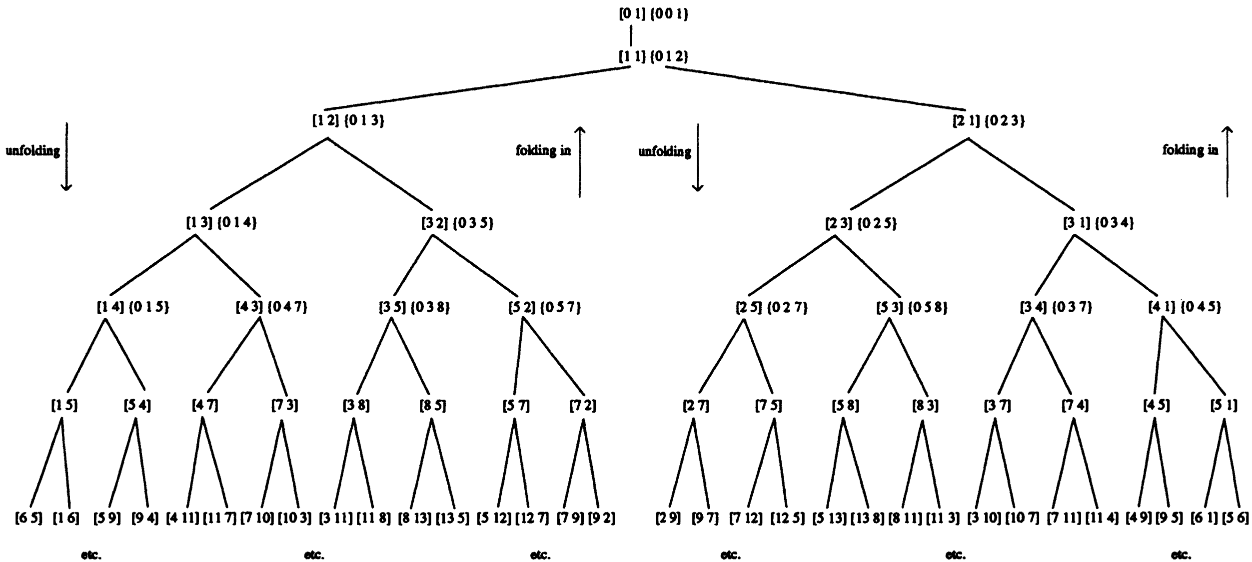
Example 20b displays folding in—or at least up to a point. The bottom pitches exchange the flip and pivot roles until they reach the invariant pitch. Then the process changes so that the lowest note flips not only around the middle but over the top note. This happens once more, after which the process is a matter of unfolding upward. In this example the ic 4 is preserved in the spacings that are produced by in- or unfoldings. In the crossing of the invariant pitch, the 4 is an outside interval. The other two intervals in each pset's ic roster (either in the spacing or the outside interval) are contracted or expanded by 4.

The next step in our exploration of trichordal FOLDSIM is to connect all PSCs whose psets are related by in- and unfolding to form a FOLDSIM network.[27] The most basic and perhaps most important of the FOLDSIM networks is shown in example 21. As before, we represent a PSC by its spacing. At the top of the network we start out with the PSC [01], containing the pset {0 0 1}. This pset and all other members of its PSC can be folded into only psets of two PSC: another member of its own PSC or members of the PSC [11]. This shown by the line connecting [01] to [11]; the self-mappings under folding (producing maximal similarity) are suppressed. The psets of PSC [11] can be folded into psets of four PSCs: itself, back to [01], to [12], or [21]—this shown by connecting lines on the network. Psets of [12] can be folded into four different PSCs; [11], [12], [13], and [32]. On the other side of the network, we have the pset [21] connected to [11], [23], [31], and itself. As the reader can see, the entire network is left/right symmetric under pitch inversion. Since folding can connect pset duals, we should regard the FOLDSIM network as having connections between its dual psets. The reader can imagine the network as folded up around a middle axis of symmetry so that each pset coincides with its dual. With the psets [12] and [21] and all others below we have the norm for trichords given above in example 19e; that is, folding connects a pset with psets of four pairs of dual PSCs, one of which its own PSC or its dual; the top two psets of the network have less folding mappings because they are invariant under T_nI.

As can be readily seen, the network forms a tree structure. This is of some significance since nodes in a tree structure are related to each other by one and only one path.[28] The distance between two PSCs is the number of PSCs on the path between the two plus one. Duals are related by a distance of 1. The network can also be interpreted as relating psets. This means that one can get from a specific pset to another by following a unique path on the network by in- or unfolding pitches; this automatically preserve two pitches and two ics from one pset to the next. Then the members of the connected PSCs may not be related by FOLDSIM or any other degree of PM similarity dependent on pitch intersection. The distance between a pset of [57] and a pset of [54] with FOLDSIM is found by tracing the unique path between them. The PSCs on the path are [57], [52], [32], [12], [13], [14], [54]; there are 5 PSCs between the beginning and end so the distance is 6.[29]

Since trichordal FOLDSIM networks are trees, they induce a hierarchy on their nodes. Moving down the tree is done by unfolding, moving up the tree involves infolding. To get from one node to another, one either unfolds, infolds, or infolds then unfolds. In the last move, there is a node at which the folding changes from in- to un-. This is called the *controlling pset* (or *controlling PSC*) and has the smallest ics of all the other psets on the path. Another relation between nodes of the tree is asserted by Bernard's *constellation*, which in our theoretic context is a node N and the set of nodes that surround N on the tree. Thus the PSC [2 5] and the PSCs that surround it, [2 3], [7 5], and [2 7] form a constellation "around" [2 5]. Bernard seems to treat his constellations as if they were akin to equivalence-classes, but since they are actually similarity sets, constellations are not mutually exclusive. Bernard therefore advances a number of criteria for limiting the PSCs found in a piece to the members of a few constellations. This is done empirically, presuming that the analyst will be able to show that only a small number of PSCs dominate a work and these in turn represent the pcsets in their respective constellations.[30]

Two additional aspects of FOLDSIM networks are worthy of comment. First, as mentioned above, the network in example 21 is not the only tree of trichords. In general, any number of trees can be generated from top node containing the PSC [0 n], where n is any non-negative integer.[31] The various trees partition all the trichordal PSCs into non-over-



A FOLDSIM Network

(To save space, spacings are only given for PSCs after the fifth level from the top of the network.)

Example 21

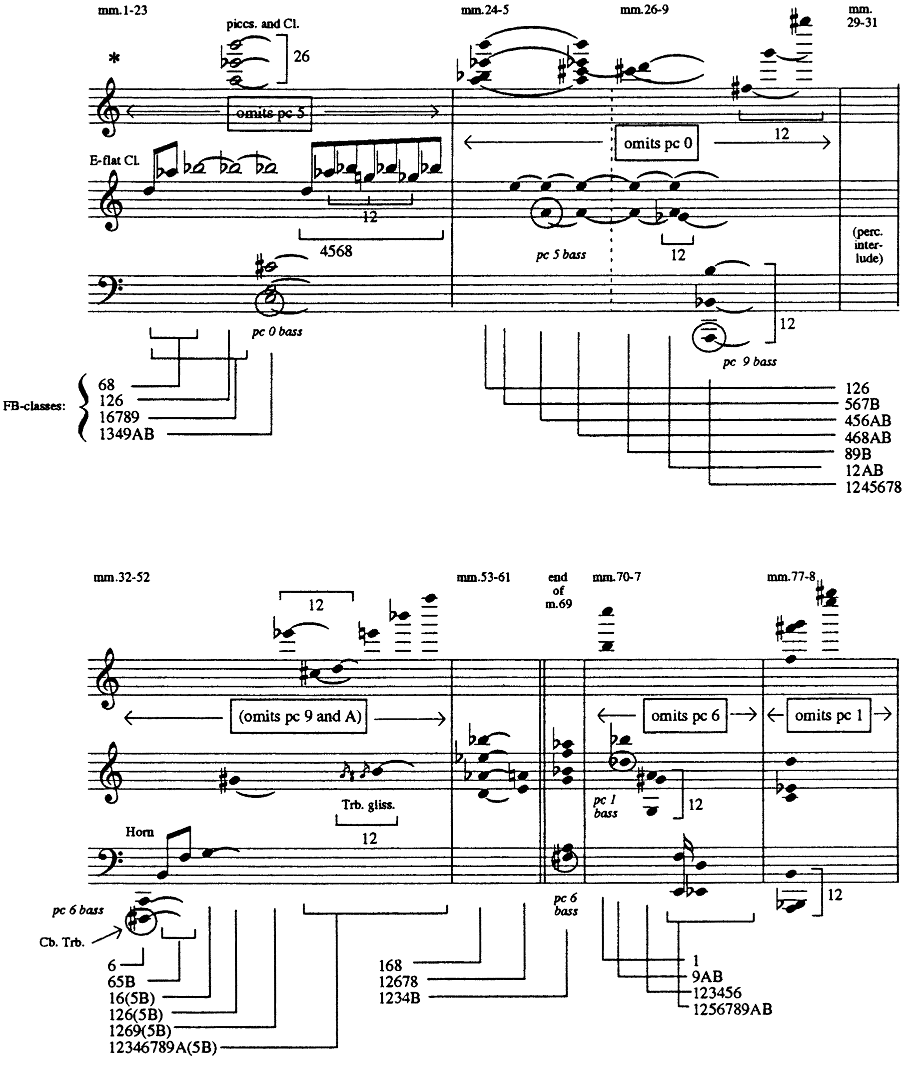
lapping collections internally related by folding and FOLDSIM. For example, the (augmented-chord) PSC [4 4] is a member of the tree with [0 4] as its top node. Thus, it is not related to members of the tree in example 21 by folding or FOLDSIM.[32] Only the top two nodes of a FOLDSIM network contain psets with T_nI invariance.

Second, when we generalize foldings to tetrachords and larger psets, the elegance of the trichordal networks is complicated. Thus with large psets, similarity and folding diverge; maximal similarity is better discussed directly than as the result of pitch flips.

Returning to the more general subject of pitch similarity, let us look at the opening of Varèse's *Intégrales*, with an eye on FB-classes, not PSC relations. Since FB-classes relate to T_n-type pitch-class SC-classes, and are notated via pc intervals, octave equivalence lurks in the background. Therefore the following discussion cannot conform to Bernard's general assertion that pitch-class relations do not function in Varèse.[33] Furthermore, FB-classes put emphasis on the bass pitches of psets. Nevertheless, we will show that pc-space and bass bias are reasonable contexts for at least one piece by Varèse.

Example 22 provides the pitches for the first 78 measures of *Intégrales* except for a brief patch of music from m. 61 to 69. As the pitch turnover is slow, the example represents the passage with some security even though the instrumentation and percussion are not notated. As in other pieces of Varèse, aggregate completion—pitch-class saturation—tends to define sections. In this case, the aggregate is "all-but-one-or-two," a ten- or eleven-pc chord. The missing pitch-class(es) for a given section are found in the bass of a neighboring chord or passage. For instance, the missing pc 5 of the first 23 bars occurs in the lowest note of the next bars, an F♮ in the trumpets. This passage from measures 24 to 29 produces a local climax but omits the pc 0, the lowest pitch in the first section. The pcs 9 and A are omitted from next large part of the work which begins at m. 32. The missing pcs are found as the lowest pitches in mm. 26–29. The pcs can also be found in the trombone glissando but are repressed as they slide up to the trombone's high B. The low F♯ in the contrabass trombone is also found later at the very end of the next section in m. 69. These instances of pc 6 provide the missing pc for mm. 70-7. A local lowest note at mm. 72–3 is pc 1, which is the missing pc in the next gesture in mm. 77–8.

Since aggregate completion is an integrating feature of *Intégrales* and that the pcs that confirm the completion are in the bass, we are motivated to use a theoretic tool that emphasizes the lowest tone.[34] I have therefore examined the psets in example 22 according to their FB-class membership. As for occurrences of the same FB-class, we find FB-class 12 most prominent, especially in mm. 24–9 and mm. 70-8. The opening three



FB relations and Aggregate Completion in mm.1-78 of Varese's *Intégrales.*

* Percussion (all indefinite pitch) omitted

Example 22

notes of the piece (played by the E♭ clarinet) form the FB-class 68, immediately followed by a 126 in the clarinets and piccolos. The same FB-classes are found in m. 32 where the horn takes the T_{-27} of the E♭ clarinet's notes of the beginning. Note that the FB-class 126 formed by the low F♯ and C, plus the horn G♮ and the higher G♯ is not analogous to the same FB-class at the opening; the collection of instrumental forces and their overlap is different. Another aspect of FB-classes occurs when notes collect over a sustained bass note as in mm. 32–52, or when successively lower notes are added to a high chord—this occurring in mm. 24–9. In the former place, the FB-classes formed by the passage are related by inclusion, merely adding more figures to the previous ones over the low F♯. Such passages promote FB-class similarity. In the descending passages however, each new bass forms a completely new FB-class. There can be a drama to such progressions as in measure 26, found in the top system after the dotted bar. The three FB-classes are 89B, 12AB, and 1245678. The move from the first preserves only the interval B, while the move from the second to the third shares only two intervals, 1 and 2, while changing two and adding three intervals more. The harmony at mm. 53–61, preceded and followed by fanfare-like writing in the trumpets sounds like a flashback to the opening. Why? Because the 68 and 126 of the opening measures match the FB-classes here, 168 and 12678, by inclusion. Moreover, the pitch-classes of the opening, {2 8 A} are also shared. But the opening pset {14 20 22} is neither literally nor abstractly included in the pset { 2 8 15 22} of measure 53, and the pitch-class connection between these chords cannot be registered in a world defined without pitch-classes, so only the FB-class can connect the two chords in pitch-space.

So despite Bernard's admirable work, I believe pitch-class connections do function in Varèse, regardless of appeals to the composer's writings or to other aspects of his music.

There is much more to say about the relation of pitch equivalence and similarity to pitch-class relations.[35] But the discussion of example 22 should portray my view of the influence of pitch-class on pitch relations well enough. Some music may not be modeled well by pc-relations, but this does not mean that in our use of pitch and contour we should ignore pc function altogether. For it is important that theory does not limit the potential richness inherent in the audition of its composers, performers or listeners—especially with those composers who have stretched and challenged our ears and minds.

NOTES

1. Of course, composers have also identified the difference between pitch and pc, as in Boulez's distinction between absolute and relative pitch or John Cage's statement "I was free to hear that a high sound is different from a low sound even when both are called by the same letter." See *Boulez on Music Today* (Faber: Boston, 1960), 35, and Cage, *Silence* (Wesleyan University Press: Wesleyan, Ct, 1958), 116.
2. See Milton Babbitt, "Twelve-tone Invariants as Compositional Determinants," *Musical Quarterly* 46 (1960), John Rahn, *Basic Atonal Theory* (New York: Longman, 1980).
3. Robert Morris, *Composition with Pitch-Classes: a Theory of Compositional Design* (New Haven: Yale University Press, 1987).
4. The generality of Morris's definitions of p- and pc-space will not be needed here. For our purposes, p-space is the familiar set of pitches separated by semitones, from low to high; the set of integers models p-space. Pc-space is the set of pitch-classes; the set of residue-classes mod 12 models pc-space so that pitches related by any number of octaves belong to the same class.
5. See Allen Forte, *The Structure of Atonal Music* (New Haven: Yale University Press, 1973).
6. Morris's account of pitch relations has all the theoretic apparatus of pcset theory (including adaptions of Alphonce's invariance matrices) save complement-relations. (See Bo Alphonce, "The Invariance Matrix" (Ph.D. diss., Yale University, 1974).) It is also straightforward, having none of pitch-class theory's (interesting) complications such as the Z-relation or "embedded (abstract) complementation."
7. Jonathan Bernard, *The Music of Edgard Varèse* (New Haven: Yale University Press, 1987), 43.
8. In a different theoretic context we would continue to use integers to stand for pitches. For our present purposes however, it is just as easy and convenient to use music notation for pitches since the numerical notation is isomorphic to the notation of pitches on the staff to within enharmonic equivalence. This is not the case with pitch-classes, which must realized as particular pitches, a one-to-many, and therefore indeterminate, association.
9. See Morris, *Composition*, xviii.
10. I should point out that Morris's definition of "pitch set-classes" on page 55 of *Composition* includes T_nI as well as T_n as a criterion for set-class membership.
11. The comparison is not quite exact as T_nI is an equivalence operation in pc SCs, but not in pitch PSCs. If we define pc SCs under T_n alone, there are 19 trichordal SCs.
12. Dual PSCs taken in union form Morris "pitch set-class," or a T_n/T_nI PSC. See note 10.
13. Dora Hanninen has studied the effect of articulating rows in narrow registral spans in her paper, "Contour as a Medium for Musical Association in Milton Babbitt's *Tableaux* (1973) for Piano," delivered at the April 1994 meeting of the *New England Conference of Music Theorists* at Connecticut College, New London, and the November 1994 National Conference of the *Society for Music Theory* at Tallahassee.
14. Pcsegments are ordered sets, but we need not always interpret their ordering in time. In the case of PCINT-classes, a pcsegment's ordering interpreted in pitch

(register) generates the class. In fact, any pitch segment P can be considered as defined by two pcsegments; one determines the pitches of P from low to high, the other determines P's pitch's temporal succession.

15. Morris has discussed FB-classes in his article, "Some Recommendations for Atonal Music Pedagogy in General; Recognizing and Hearing Set-Classes in Particular," *Journal of Music Theory Pedagogy*, 8 (1994). Chromatic figured bass is also elaborated in Alan Chapman "Some Intervallic Aspects of Pitch-Class Set Relations," *Journal of Music Theory* 25/2 (1980). His AB is our FB (and his VP is our SP).
16. For a survey of Loquin's contribution to figured bass theory see "Anatole Loquin: Algèbra de l'Harmonie (1884)," an unpublished paper by Penelope Peters.
17. Note as stipulated above, the intervals of all FB-classes, including duals, are written in ascending order.
18. I should mention that the symmetries of the array and the 12-semitone disposition of pitch registers that articulate the array lines almost automatically produce the features pointed out here. One only has to articulate a certain four-pc begin-set from the beginning of the first aggregate, and the corresponding end-set from the last aggregate in some coherent, aurally cogent manner, to achieve these pitch and pc relations.
19. After these considerations, the reader might wonder if my reasons for applying similarity relations only in pitch might be used also to argue that grouping pcsets into (pc) SCs has comparable difficulties. While it is true that different spacings and temporal adjacencies can make two members of the same pc SC "sound" quite different, SCs have two basic properties that prevent the arguments from going through. First, the abstract subset content of each member of a SC is identical while pcsets from different SCs have different subset content. Pc similarity measures based on degrees of abstract subset intersection do not as a rule differentiate classes of SCs but only connect them in a network. Second, the members of a SC are related via the familiar mappings in the canonical group, usually T_n and T_nI. With pcsets from different SCs related by pc similarity relations, the mappings from one to another are various and arbitrary. If one can find a family of mappings that consistently relates pcsets from different SCs, one can posit the mappings as members of a canonical group and the similarity turns into equivalence.
20. John Roeder discusses the same passage in "A Geometric Representation of Pitch-Class Series," *Perspectives of New Music*, 25/1 & 2 (1987). He compares the chords of the Schoenberg using his "two-dimensional ordered interval space," and arrives at some of the same relations present herein. Nevertheless, his methodology only concerns ordered pcsets which can, but need not be, interpreted in p-space (our PCINT-classes).
21. See Forte *The Structure of Atonal Music*, Morris, "A Similarity Index for Pitch-Class Sets," *Perspectives of New Music* 18/2 (1980), and Rahn, "Relating Sets" *Perspectives of New Music* 18/2 (1980).
22. This works for members of the same PSC under transposition. A sum roster has to be generated for determining intersection under T_nI between (transposed) duals.
23. Of course, one or more of the new ics introduced by the changed pitch may be the same as one or more ics lost in the process. Thus, the number of *different* ics after the pitch change may be less than #S - 1.
24. See Bernard, *The Music of Edgard Varèse*, 74. I should also note that my presen-

tation of Bernard's transformations differs from his and takes a different theoretic and analytic tack. For instance, Bernard defines equivalence of psets under both pitch transposition and inversion; our dual PSCs are thus collapsed into one Bernard pitch SC. Consequently, he considers some of all of the possible unfoldings and infoldings as redundant—specifically those that produce duals of others.

25. David Lewin has also formalized Bernard's un- and infoldings as "serial transformations" called FLIPSTART and FLIPEND in *Generalized Musical Intervals and Transformations* (New Haven: Yale University Press, 1987), 189. Perhaps the earliest invocation of folding is found in Benjamin Boretz, "Sketch of a Musical System (Meta Variations Part II)," *Perspectives of New Music* 8/2 (1970). See Df. 2.13 on p.101 and its analytic use in Boretz, "Meta Variations, Part IV: Analytical Fallout (II)," *Perspectives of New Music* 11/2 (1973), 182.
26. See George Perle, Twelve-Tone Tonality (Berkeley: University of California Press, 1977).
27. Lewin makes the same move to develop chains (actually cycles) of overlapped trichords related by repeated applications of FLIPEND and FLIPSTART in alternation in his example 8-11a and b in *Generalized Intervals.* His cycles can also be generated from the network in example 21.
28. For the same reason, Tree structures are automatically mathematical "metrics" since the distance from node a to b or from node b to c is equal or less than the distance from node a to c—distance being the number of arcs traversed.
29. Consider the sequence of psets that can represent the path [57], [52], [32], [12], [13], [14], [45]. For instance, starting with pset {5 10 17} representing the PSC [5 7] can form different chains of psets having FOLDSIM: one chain is{5 10 17} {10 15 17} {12 15 17} {14 15 17} {14 15 18} {14 15 19} {14 19 23}; another is {5 10 17} {5 10 12} {5 8 10} {5 6 8} {4 5 8} {3 4 8} {3 8 12}. Such realization issues are discussed in my paper, "Compositional Spaces and Other Territories," *Perspectives of New Music* 33 (1995). The network in example 21 would be called an abstract compositional space and the much more complex graph of all the possible realization paths (including the two just given) is a literal space.
30. But despite our ability to describe Bernard's work on Varèse with the present conceptual framework, it has a very different feel in the original since Bernard relates his "basic forms" (p.74) with operations of folding and shows these relations as geometric configurations and transformations on a pitch/time grid.
31. Except where n is 0, and there is no tree, only the PSC [0 0].
32. According to Bernard, "A Theory of Pitch and Register for the Music of Edgard Varèse" (Ph.D. diss., Yale University, 1977), 114, the opening of Varèse's *Intégrales* is based on two "basic forms," our PSCs [4 9] and [2 6] and their duals. These two pairs of PSCs are not on the same tree and are therefore not related by folding or FOLDSIM. [4 9] and [2 6], with their constellations or networks, therefore represent highly differentiated, even conflicting sounds at the opening of the work. The two networks represented by [4 9] and [2 6] are generated by [0 0 1] (or ic 1) and [0 0 2] (ic 2). The conflict between the chromatic versus whole-tone worlds comes up in other pieces by Varèse. For instance, compare the end of *Density 21.5* and the rest of that composition.
33. Despite categorical statements about ruling out octave and inversional [read: complementary interval] equivalence on page 43, Bernard does discuss "octave doubling" and says on page 102 "an interval enlarged or shrunk by an octave (or com-

pound octave) has, *for some purposes only* [Bernard's emphasis], a meaning equivalent to its unaltered form" (*The Music of Edgard Varèse*). These octaves may be explained if we consider FB-classes to inform analysis. At least in traditional figured-bass theory, an interval number specified by the figured bass may result in the realization of more than one representation of a pitch-class in different octaves, which does not change the figured bass's function. In our theory however, an octave or unison doubling would be represented by a 0 in the FB name.

34. In the following I do not define similarity among FB-classes. Nevertheless, the following definition underlies the discussion. Two FB-classes are similar to the degree they have the same numbers (figures) in their names. Maximal FB similarity occurs when two FB-classes differ by only one number, either by change or omission: for instance, 678 is maximally similar to 67 and 679 and 4678.
35. For instance, I have not discussed PCINT similarity. John Roeder proposes a useful (non-Euclidian) measure in "A Geometric Representation," 383–84. "[I]t measures the total amount of discrepancy between [two INTs'] correspondingly situated intervals." Other equivalences can easily be derived from FB; consider FS (figured soprano) and FSB (combining FB and FS, wherein pitches n + 12q apart form the outside ics of equivalent psets). William Benjamin's "harmonic bip" and its derivatives form other equivalence functions. See Benjamin's review of Allen Forte's *The Structure of Atonal Music* in *Perspectives of New Music* 13/1 (1974), 183.