PRO 5971 - Statistical Process Monitoring

About the control limits

Linda Lee Ho

March 15, 2023

Department of Production Engineering
University of São Paulo

Outline

Asymptotic control limits

- Let X, be a random variable with PDF $f(X)$ and CDF as $F(X)$ with $E(X) = \mu$ and $Var(X) = \sigma^2$, both $< \infty$
- Let us consider a sample random of size n **independent identical distributed** (iid) random variables: $X_1, X_2, \ldots, X_n$.
- By Central Limit Theorem, the **asymptotic distribution** (that is, when $n \to \infty$) of $\overline{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$ follows a normal distribution $\sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$
- For the $\overline{X}$ control chart, the control limits are of the form $\mu_0 \pm z_{1-\alpha/2} \dfrac{\sigma}{\sqrt{n}}$
- These control limits are known as **asymptotic** one
- $z_{1-\alpha/2} = 3$, the control limits are known as **3 sigma control limits**
- They are symmetrical and easier to deal with them

About the control limits

- If $X \sim N(\mu, \sigma^2)$ then the **exact distribution** of $\overline{X}$ is also normal $\sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$
- So if the qualty characteristic X follows a normal distribution, the asymptotic and the exact distribution of $\overline{X}$ are **identical**.
- However when the quality characteristic X does not follow a normal distribution, the **exact distribution** of $\overline{X}$ **may be far from the asymptotic distribution of** $\overline{X}$
- Control limits determined by using the exact distribution of $\overline{X}$ is known as **probability control limits**.
- If $X \sim N(\mu, \sigma^2)$ then the probability and asymptotic control limits of $\overline{X}$ are **identical**.

Probability control limits

- Assumption: The quality characteristic X does not follow a normal distribution
- Let $F_{\overline{X}}(.)$ be the CDF of $\overline{X}$; $q_{\alpha/2}$ and $q_{1-\alpha/2}$ are respectively the $\alpha/2$ and $1-\alpha/2$ quantiles of $F_{\overline{X}}$ such that:

$$F_{\overline{X}}(q_{\alpha/2}) = P(\overline{X} < q_{\alpha/2}) = \alpha/2$$

  and

$$F_{\overline{X}}(q_{1-\alpha/2}) = P(\overline{X} < q_{1-\alpha/2}) = 1-\alpha/2$$

- Consequently $P(q_{\alpha/2} < \overline{X} < q_{1-\alpha/2}) = 1-\alpha$
- The quantiles $q_{\alpha/2}$ and $q_{1-\alpha/2}$ are **the probability control limits** considering as error of type I equal to α
- However sometimes the exact distribution of $\overline{X}$ is unknown and the probability control limits are determined by simulation.

Some exact distributions

- If X is r.v. following a Bernoulli distribution with parameter p, then $\sum_{i=1}^{n} X_i \sim$ Binomial (n;p)
- If X is r.v. following a Poisson distribution with parameter λ, then $\sum_{i=1}^{n} X_i \sim$ Poisson ($n\lambda$)
- If X is r.v. following an exponential distribution with parameter λ, then $\sum_{i=1}^{n} X_i \sim$ Gamma (n;λ)
- etc

- X, some quality characteristic with PDF $f_X(.)$ and CDF $F_X(.)$
- Let $X_1, X_2, \ldots, X_n$ a random sample of X; $\tilde{\Theta} = G(X_1, X_2, \ldots, X_n)$ a statistic used to monitor some parameter of interest with $E(\tilde{\Theta})$ and $Var(\tilde{\Theta})$ both known/available.
- The **asymptotic control limits** are: $E(\tilde{\Theta}) \pm z_{1-\alpha/2}\sqrt{Var(\tilde{\Theta})}$
- The quantiles $q_{\alpha/2}$ and $q_{1-\alpha/2}$ are **the probability control limits** with as error of type I equal to α such that

  $$F_{\tilde{\Theta}}(q_{\alpha/2}) = P(\tilde{\Theta} < q_{\alpha/2}) = \alpha/2$$

  and

  $$F_{\tilde{\Theta}}(q_{1-\alpha/2}) = P(\tilde{\Theta} < q_{1-\alpha/2}) = 1 - \alpha/2$$

  $F_{\tilde{\Theta}}(.)$ is the CDF of $\tilde{\Theta}$
- Consequently $P(q_{\alpha/2} < \tilde{\Theta} < q_{1-\alpha/2}) = 1 - \alpha$