PRO 5971 - Statistical Process Monitoring

Shewhart control chart: X-bar chart

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Outline

Introduction

- Proposed by W. Shewhart at 30's
- Two main types: Attribute and Variable control chart
- Variable Control chart: quality characteristic expressed in terms of a numerical measurement
- Attribute Control Chart: quality characteristics cannot be conveniently represented numerically

Main variable control charts

- Two main type variables charts:
- For monitoring process mean: $\overline{X}$ chart
- For monitoring the process variability: R, S and S^2 charts

Variable Control chart: Monitoring process mean - $\overline{X}$ Chart

- Assumption: Quality characteristic X normally distributed with mean μ and σ, both available
- X_1, $X_2, \ldots$, X_n, is a sample of size n
- Monitored statistic to draw in the control chart: the average sample $\overline{X}$
  - Estimator of μ
  - Normally distributed with mean μ and $\sigma/\sqrt{n}$
- In-control process mean: μ_0
- Fixed error of type I equal α such that $P(\overline{X} < \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = P(\overline{X} > \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = \alpha/2$
- The control limits: $LCL = \mu_0 - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$; $UCL = \mu_0 + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$
- $z_{1-\alpha/2}$ is the quantile of a standard normal distribution such that $P(Z < z_{1-\alpha/2}) = \alpha/2$

$\overline{X}$ Control Chart: Performance - the power

- Process out-of-control: Process mean shifts to $\mu_1 = \mu_0 + k\sigma$
- Error of type II:
  $\beta = P(LCL < \overline{X} < UCL | \mu_1 = \mu_0 + k\sigma)$
- $$\beta = \Phi\left[\frac{UCL - (\mu_0 + k\sigma)}{\sigma/\sqrt{n}}\right] - \Phi\left[\frac{LCL - (\mu_0 + k\sigma)}{\sigma/\sqrt{n}}\right]$$
  $$\beta = \Phi\left[z_{1-\alpha/2} - k\sqrt{n}\,\right] - \Phi\left[-z_{1-\alpha/2} - k\sqrt{n}\,\right]$$
- 1-β = Probability that shift will be detected=Power
  $1 - \left[\Phi\left(z_{1-\alpha/2} - k\sqrt{n}\,\right) - \Phi\left(-z_{1-\alpha/2} - k\sqrt{n}\right)\right]$

Exercises

1. For sample of sizes $n = 3, 5, 9$ obtain the control limits for bilateral shifts for $\bar{X}$ chart considering $\alpha = 0.05, 0.0027$ and at a stable situation the quality characteristic follows a normal distribution with $\mu_0 = 10$ and $\sigma = 2$
2. Obtain the power of the $\bar{X}$ when the mean shifts to $\mu_1 = \mu_0 + \delta\sigma$, $\delta = 0.25, 0.5, 1, 1.25, 1.5, 2, 3$ for the item 1
3. Discuss the results of items 1 and 2
4. Redo the items 1 and 2 considering only one-sided shifts and discuss the results.

Measuring the performance of a control chart by the RUN LENGTH

The parameters of a control chart: sample size n, the control limits (UCL and LCL)

Action: whenever the statistic is plotted out of the control limits, a search for special causes starts.

Performance measure: the number of samples until a signal. In Figure 1, 4 samples for the first signal; 6 samples for second signal; 6 samples for third signal, etc

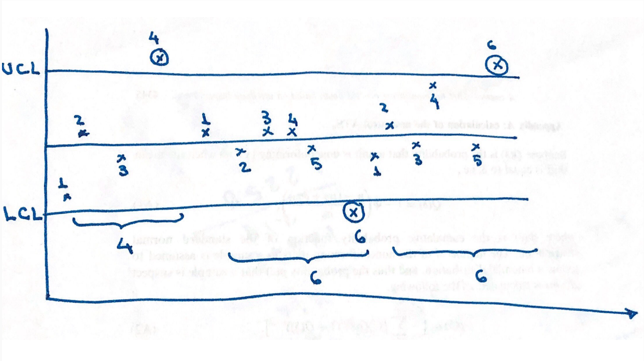

Figure 1: # of samples until a signal

Run length

Y$=$ # of samples until a signal - is a random variable known as RUN LENGTH

If the monitored statistics are **independent** $\rightarrow$ Y follows a Geometric distribution with parameter p, the probability to signal

Its probability function is

$$P(Y = y) = p(1-p)^{y-1}, y = 1, 2, \ldots$$

For example: $\begin{cases} P(Y=1) = p, & \text{the probability to signal at the first sample} \\ P(Y=2) = p(1-p), & \text{the probability to signal at the second sample} \\ \ldots P(Y=5) = p(1-p)^4, & \text{the probability to signal at the fifth sample} \end{cases}$

$E(Y) = \dfrac{1}{p}$ is the average number of samples until a signal.

It is known as AVERAGE RUN LENGTH (ARL) and has been the most used a performance metric in SPC.

Other measures: median run length (MRL); standard deviation run length(SDRL).

In-control average run length: ARL_0

$$\text{When the process is stable (in control or under } H_0) \begin{cases} \alpha - \text{probability to signal} \\ 1-\alpha, \ \text{probability to not signal} \\ ARL_0 = \dfrac{1}{\text{probability to signal}} = \dfrac{1}{\alpha} \end{cases}$$

$$ARL_0 \begin{cases} \text{ is the average number of sample until a signal when the process is in-control} \\ \text{It is desirable to have larger values for } ARL_0 \\ \text{Usually values like 370 or 500 are used as } ARL_0 \text{ in process monitoring} \end{cases}$$

Out-of-control average run length: ARL_1

$$\text{When the process is out of control or under } H_1) \begin{cases} 1-\beta - \text{probability to signal} \\ \beta, \text{ probability to not signal} \\ ARL_1 = \dfrac{1}{\text{probability to signal}} = \dfrac{1}{1-\beta} \end{cases}$$

$$ARL_1 \begin{cases} \text{ is the average number of sample until a signal when the process is out-of-control} \\ \text{It is desirable to have } ARL_1 \approx 1 \end{cases}$$

Plan $\overline{X}$ control charts (assuming σ known and remains stable) to have $ARL_0 = 200, 370, 500$ with $n = 3, 5, 9$. Obtain ARL_1 when μ_0 shifts to $\mu_1 = \mu_0 \pm \delta\sigma$ for $\delta = 0.25, 0.5, 1, 1.5, 2, 2.5$.

References