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Hamilton Emmons • George Vairaktarakis

# Flow Shop Scheduling 

Theoretical Results, Algorithms, and Applications

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# $\Sigma \tau \eta \mu \nu \dot{\eta} \mu \eta \tau \omega \nu$ фí $\lambda \omega \nu \mu o v$ Гı́́ $\gamma \gamma$ ov $A \gamma \gamma \epsilon \lambda \lambda$ ótov $\lambda$ ov к $\alpha \iota \Delta \eta \mu \dot{\eta} \tau \rho \eta$ K $\omega \nu \sigma \tau \alpha \nu \tau \alpha ́ \kappa \eta$ G.V. 

For Lin, with love
H.E.

## Preface

The idea for this monograph was born during lunch conversations between the authors. By that time, both of us had a lot of exposure in scheduling research, and observed that a constant stream of invitations to review was filling our desks with papers on flow shop research. We couldn't help noticing that, even though several monographs and edited volumes have appeared on scheduling in general, most of these works survey the field by contributing a single chapter to every production system such as the flow shop. In contrast, flow shop systems appear to be the most studied in all of scheduling literature, offering unique contributions, occupying the research efforts of many researchers around the globe.

This monograph is all about flow shops. We have organized selected results into ten distinct flow shop systems and whenever possible, we have attempted to exploit the connections. We used our subjective personal discretion in selecting the results presented, and we apologize in advance to those (hopefully few) in the audience whom we leave disappointed. We also include a limited number of results that have not been published before. We have sought to make the material accessible to a broad readership, and have tried hard to simplify notation and reveal unifying concepts. In all, we know of no other text dedicated to flow shop research at the breadth attempted here.

It is our hope that, by organizing in one place a huge body of flow shop knowledge along distinct design features, we help scholars and practitioners to identify easily what is known in the literature on problems of interest. Moreover, we hope that results unique to flow shop research will provide the seed for research in other areas of scheduling and in optimization in general. Finally, our monograph may provide the impetus for methods and techniques that have been tried successfully in other areas of optimization to be applied to flow shop problems that are intractable as of now.

We are very grateful for the helpful suggestions of many friends and colleagues, notable among whom are Kenneth Baker, Xiuli Chao, Selçuk Karabati, Michael Pinedo, Ruben Ruiz, and Gideon Weiss. Also, many thanks to our Editor and friend, Frederick Hillier, for his patience and encouragement.

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## Acronyms

ACO Ant Colony Optimization: a type of metaheuristic
API Adjacent Pair Interchange: The reordering of jobs in a sequence, often used to show the optimality of a simple sequencing rule
AS/RS Automatic Storage/Retrieval Stations: a term in robotic manufacturing
CAD Computer Aided Design
CDF Cumulative Distribution Function: a way of expressing the probability distribution of a random variable
CDS Campbell, Dudek, and Smith: a well-known heuristic for $F m \mid$ perm $\mid C_{\max }$
CNC Computer Numerically Controlled: a type of automation for equipment
CPO Complete Priority Ordering: A property of some sequencing problem objectives which allows adjacent pairs of jobs to be ordered, thus giving simple polynomial solutions.
CPU Central Processor Unit: the functional heart of a computer
CT Cycle Time: the interval of repetition in cyclic production
CTV Completion Time Variance: a nonlinear objective function
EA Evolutionary Algorithm: a type of metaheuristic
EDD Earliest Due Date: the rule $\nearrow d_{j}$ for sequencing a set of jobs
EDDP Earliest Due Date over Processing time: the rule $\nearrow d_{j} / P_{j}$ for sequencing a set of jobs
EWDD Earliest Weighted Due Date: the rule $/ d_{j} / w_{j}$ for sequencing a set of jobs
FAM First Available Machine: a rule to assign jobs in a hybrid flow shop
FBFS First Buffer, First Served: A rule for sequencing jobs at a work station, in a reentrant system
FCFS First Come, First Served: a rule for sequencing jobs at a work station
GA Genetic Algorithm: a type of metaheuristic
GP Grabowski and Pempera: a TS implementation
ILS Iterative Local Search: a type of metaheuristic
JR Johnson's Rule: a simple algorithm for minimizing makespan in a two machine flow shop

LBFS First Buffer, First Served: A rule for sequencing jobs at a work station, in a reentrant system
LBM Last Busy Machine: a rule for job assignment in a hybrid flow shop
LPT Longest Processing Time: the rule $\searrow P_{j}$ for selecting the job to be processed next
LS Least Slack: the rule $\nearrow\left(d_{j}-C(\sigma j)\right)$ for selecting the job to be processed next
MA Moore's Algorithm: a procedure to minimize the number of tardy jobs on one machine
MMS Merged Machines Shop: an approximation to a hybrid flow shop in which parallel machines are replaced by one "superserver"
MPS Minimal Part Set: the smallest production run that includes the desired proportion of each product
MST Minimal Slack Time: a rule for selecting the job to be processed next
MJR Modified Johnson's Rule: Johnson's Rule applied to the two-machine flow shop with lags
MRM Minimax Regret Makespan: A criterion when task times are uncertain
NEH Nawaz, Enscore, and Ham: a heuristic for $F m \mid$ perm $\mid C_{\text {max }}$
NP Nondeterministic Polynomial: an NP complete [NP hard] problem belongs to $\mathcal{N P \mathcal { C }}[\mathcal{N} \mathcal{P H}]$
$\mathcal{N P C}$ Nondeterministic Polynomial Complete: a class of "hard" problems
$\mathcal{N} \mathcal{P H}$ Nondeterministic Polynomial Hard: a class of "hard" problems
$\mathcal{P} \quad$ Polynomial: the class of problems solvable in polynomial time
PCA Parallel Chain Algorithm: A simple procedure for solving the problem $F 2 \mid$ (perm), chains $\mid C_{\max }$, covered in Sect. 3.2.5
PCB Printed Circuit Board
PF $\quad$ Profile Fitting: a heuristic for $F m \mid$ block, perm $\mid C_{\text {max }}$
RA Rapid Access: a heuristic for $F m \mid$ perm $\mid C_{\text {max }}$
RACS Rapid Access with Close order Search: a heuristic for $F m \mid$ perm $\mid C_{\max }$
RAES Rapid Access with Extended Search: a heuristic for $F m \mid$ perm $\mid C_{\max }$
SA Simulated annealing: a type of metaheuristic
SEPT-LEPT Shortest Expected Processing Time-Longest Expected Processing Time: a rule for job sequencing in a stochastic flow shop
SPT Shortest Processing Time: the rule $\nearrow P_{j}$ for selecting the job to be processed next
TCF Time Complexity Function: a measure of the running time of an algorithm
TS Tabu Search: a type of metaheuristic
TSAB Tabu Search Algorithm with Back jump tracking: a TS implementation
TSP Traveling Salesman Problem: the problem of finding the shortest tour of a given set of locations on the plane
VLSI Very Large Scale Integration: a term used in PCB assembly

## Chapter 1 INTRODUCTION


#### Abstract

We start by presenting the basic structure and language of flow shops, and the conventions and notation to be adopted. The concepts of precedence, permutation schedules and when they are optimal, graphic schedule representations, dominance properties, heuristics and error bounds, cyclic shops and other fundamentals are introduced.


In a manufacturing environment, the items being produced, or jobs, usually can be decomposed into a number of tasks or operations each of which is the smallest piece of work it is convenient to consider. Each task may require a variety of raw materials and subassemblies, machinery and equipment, conveyors and operators, and has other characteristics, notably its required processing time. Thus, the job generally goes through several stages of work at a series of work stations, or machine groups. Each stage consists of one or more productive facilities (drill presses, ovens, human inspectors, etc.), collectively referred to as machines or most generally as processors.

There are many constraints on the times at which, or time intervals in which, any task may be performed. It may be delayed because the needed workers and/or equipment may be otherwise engaged, or raw materials may not yet have been delivered. Some tasks may have to await the completion of other tasks, usually when they are two parts of the same job; such precedence relations are discussed in the next section. Other constraints may further restrict the timing of tasks. A schedule is a specification of when each task of a given job set is to be processed, and with what personnel, equipment, etc. When speaking of schedules it will normally be understood that we restrict attention to feasible schedules: those that satisfy all constraints. The classical problem of job shop scheduling is to determine an optimal schedule: a feasible schedule that best achieves any one of several possible objectives. These objectives are almost always expressed as functions
of the task completion times, and our goal is then to minimize this function over all feasible schedules.

In full generality, this is often difficult to do, because the problem may be very large and complex, and there may be multiple and possibly ill-defined objectives. Solution procedures may be slow and laborious, and, when optimality is unattainable, are sometimes heuristic. A heuristic algorithm produces a satisficing schedule, which is the best one we can find and which gives satisfactory performance, but which is not guaranteed to be truly optimal.

A certain manufacturing layout has widespread use and great practical importance, and it is considerably more tractable. This is the flow shop, and it will be the subject of this monograph.

Definition 1.1. A flow shop is a processing system in which the task sequence of each job is fully specified (this is called chain precedence, illustrated in Fig. 1.1), and all jobs visit the work stations in the same order.

Most authors add the requirement that a job never revisits any stage. Thus the stations can be numbered $1,2, \ldots, m$, and every job visits them in numerical order. In a pure flow shop each job has $m$ tasks and visits all stages. More generally, jobs may have fewer than $m$ tasks and may skip over some stations. Still, it is assumed that a job never visits any stage $i^{\prime}<i$ after it has visited stage $i$. This will be our assumption throughout most of this monograph.

However, there is one other case that some authors include in the flow shop category, and that we shall discuss in Chapt. 9: the reentrant flow shop. Jobs in a reentrant shop may cycle back and be reprocessed at the same station, or sequence of stations, multiple times. As long as all jobs follow the same path, the system retains the defining characteristic of a flow shop. Nevertheless, outside the one chapter on reentrant flow shops, we will require that jobs move always forward, with no turning back.


Fig. 1.1 Chain precedence for a job in a flow shop

### 1.1 Assumptions and Conventions

There are many variations on this general flow shop structure. The most fundamental model is the simple flow shop, which has the following characteristics:

- Each of the $m$ stages or work stations can handle one operation at a time: it consists of a single processor which is always available. When an $m$-station flow shop has one machine at each station, it is called an $m$-machine flow shop.
- A given number, $n$, of independent jobs require processing, each made up of $m$ well-ordered tasks with known requirements: task $i$ of job $j$ requires processor $i$ for a processing time $p_{i j} \geq 0$. Thus, each task of a job requires a different machine (no job visits a work station more than once), and the machine sequence is the same for all jobs.
- All jobs are available simultaneously at the start (at time zero), and remain available without interruption until all work on them is finished.
- Each job can be in only one place (in process on one machine) at a time.
- No preemption is allowed: once started, a task must be processed to completion without interruption.
- When a job completes processing on one machine, it is immediately available to begin on the next. That is, there are no delays due to machine setups, or transfer lags caused by transportation between machines, etc.
- Intermediate storage (that is, room for work in process to queue up) between successive machines is unlimited.

A more broadly encompassing model, to which we shall make brief reference from time to time, is the general job shop, which is described exactly as the simple flow shop, with the one exception that the required machine sequence of different jobs may differ. A more important generalization, from our point of view, is the hybrid (or compound or flexible) flow shop, in which work stations may consist of several functionally identical processors in parallel.

We will first present results for the simple flow shop, and will relax each assumption in subsequent chapters.

### 1.2 Terminology

Whenever possible, the following notation will be adhered to throughout this monograph. We start with the given parameters that may be needed to define a particular scheduling problem.

- $\boldsymbol{n}$ : number of jobs to be scheduled.
- $\boldsymbol{m}$ : number of stages (or work stations or machine groups) in the shop. In a simple shop, each stage has one processor or machine. In a hybrid shop, a work station may have a number of machines, not necessarily identical in all respects, but all capable of procesing the tasks assigned to that station.
- $\boldsymbol{J}_{j}$ : job $j(j=1, \ldots, n)$. Jobs are indexed in arbitrary order unless otherwise specified.
- $\boldsymbol{M}_{\boldsymbol{i}}$ : machine $i$, i.e., the machine at stage $i(i=1, \ldots, m)$ in a simple flow shop, or sometimes any one of the identical processors at stage $i$ of a hybrid flow shop.
- $G_{i}$ : Stage $i$ (or work station $i$ or machine group $i$ ) in a hybrid flow shop.
- $\boldsymbol{k}_{\boldsymbol{i}}$ : the number of machines at $G_{i}$ in a hybrid flow shop.
- $\boldsymbol{M}_{\boldsymbol{h i}}$ : machine $h$ at $G_{i}\left(h=1, \ldots, k_{i} ; i=1, \ldots, m\right)$ in a hybrid flow shop.
- $\boldsymbol{T}_{i j}$ : task $i$ of $J_{j}$, i.e., the task of $J_{j}$ processed at $G_{i}$; also called the stage- $i$ task. Tasks are often referred to as operations in the job shop literature, and as activities in project scheduling parlance; we will generally use the term task. We may write simply $T_{i}$ when referring to an arbitrary task without concern for the job it is part of.
- $\boldsymbol{p}_{\boldsymbol{i j}}$ : processing time of $T_{i j}$. However, for small shops, we generally use the following simplified notation.
- $\boldsymbol{a}_{\boldsymbol{j}}, \boldsymbol{b}_{j}, \boldsymbol{c}_{\boldsymbol{j}}$ : processing times of $J_{j}$ at $G_{1}, G_{2}, G_{3}$, respectively, in shop with two or three stages. Sometimes $a_{j}\left[b_{j}, c_{j}\right]$ may be used to denote the task $T_{1 j}\left[T_{2 j}, T_{3 j}\right]$, as well as its processing time; or we may refer to a task processed at $G_{1}$ as an $a$-task, etc.
- $\overline{\boldsymbol{p}}_{\boldsymbol{j}}=<p_{1 j}, p_{2 j}, \ldots, p_{m j}>$ : processing times of $J_{j}$ on all machines.
- $\boldsymbol{p}_{\boldsymbol{i}}=\left(p_{i 1}, p_{i 2}, \ldots, p_{i n}\right)$ : processing times of all jobs at stage $i$.
- $\boldsymbol{r}_{\boldsymbol{j}}$ : ready time or release date of $J_{j}$ : the earliest that $J_{j}$ is available to be processed. In a simple shop, $r_{j}=0$.
- $\boldsymbol{d}_{j}$ : due date of $J_{j}$. A job completion time in excess of its due date results in a late job and generally incurs a penalty.
- $\boldsymbol{w}_{\boldsymbol{j}}$ : the weight of $J_{j}$, a measure of its relative importance. Often, the cost per unit time of lateness.

Other notation, for variables that are schedule-dependent:

- $\boldsymbol{S}$ : arbitrary schedule or sequence. For permutation schedules, we write $S=(i, j, \ldots, k)$, where each component is a job index.
- $S_{i}$ : schedule on $M_{i}$ in a simple flow shop, or at $G_{i}$ in a hybrid shop, when sequences differ across stages.
- $\boldsymbol{W}_{i j}(\boldsymbol{S})$ : time delay or waiting time between completing $J_{j}$ at stage $i-1$ and starting it at stage $i$.
- $C_{i j}(S)$ : completion time of $T_{i j}$.
- $C_{i}(\boldsymbol{S})$ :completion time of all the tasks scheduled on $M_{i}$. Thus, $C_{m}(S)$ is the makespan, often abbreviated $C(S)$, and sometimes written $C_{\max }(S)$.
- $\boldsymbol{L}_{j}(\boldsymbol{S})=\boldsymbol{C}_{\boldsymbol{m} j}(\boldsymbol{S})-\boldsymbol{d}_{\boldsymbol{j}}$ : lateness of $J_{j}$. Note that $L_{j}$ may be positive or negative (in which case it is early).
- $T_{j}(S)=\max \left(0, L_{j}(S)\right)$ : tardiness of $J_{j}$, the positive part of lateness. An early job has negative lateness but zero tardiness.
- $\boldsymbol{U}_{\boldsymbol{j}}(\boldsymbol{S})$ : tardiness indicator, where $U_{j}(S)=1$ if $J_{j}$ is late, $U_{j}(S)=0$ otherwise.

To simplify notation, the dependence on $S$ will be suppressed when the context makes it clear. Thus, we often write $C_{i j}(S)$ as $C_{i j}$, etc.

A script or caligraphic capital will be used to denote sets. Two examples that will be commonly used are:

- $\mathcal{J}=\left\{\boldsymbol{J}_{1}, \boldsymbol{J}_{2}, \ldots, \boldsymbol{J}_{n}\right\}$ : the set of all jobs. More often, as convenient, the elements of $\mathcal{J}$ may be the job indices: $\mathcal{J}=\{1,2, \ldots, n\}$.
- $\mathcal{M}=\left\{\boldsymbol{M}_{1}, \boldsymbol{M}_{2}, \ldots, \boldsymbol{M}_{\boldsymbol{m}}\right\}$ : the set of all processors. Alternatively, as above, $\mathcal{M}$ may contain just the machine indices: $\mathcal{M}=\{1,2, \ldots, m\}$.

Fig. 1.2 uses some of this notation to show the progress of a typical $J_{j}$ through an $m$-station shop using a Gantt chart: a graphical presentation of a work schedule in which each task is represented by a bar laid out along a horizontal time axis, the bar's location and length showing the time position and duration of task. Note that $J_{j}$ is completed before its due date $\left(C_{j}<d_{j}\right)$; it is early. We will generally think of earliness as negative lateness, as indicated in the figure.


Fig. 1.2
Gantt chart for $J_{j}$

### 1.2.1 Problem Classification and Notation

To specify problems precisely and economically, the following shorthand notation was introduced by Graham et al. (1979). The classification system consists of three fields separated by bars: $\alpha|\beta| \gamma$.

## Type and Size

In the $\alpha$-field is entered the type and size of the shop, starting in our case with $F$ for flow shop. For example:

- $\boldsymbol{F m}$ : the simple flow shop with $m$ machines.
- $\boldsymbol{F}\left(\boldsymbol{k}_{1}, \boldsymbol{k}_{\boldsymbol{2}}, \ldots, \boldsymbol{k}_{\boldsymbol{m}}\right)$ : the hybrid flow shop with $k_{i}$ identical machines in parallel at stage $i$.

We will mention other types of job shops from time to time, to wit:

- $\boldsymbol{G m}$ : the general job shop. Each job visits some or all of the $m$ onemachine stations with chain precedence, but the machine sequences of jobs differ.
- Om: the open shop with $m$ one-machine stations. The tasks of a job may be processed in any order: there is no precedence.
- $\boldsymbol{S m}$ : the super shop. Some jobs have machine sequences specified, as in $G m$, and others have no required order, as in Om .


## Special Features

In the second field are listed the special features of the shop in question that deviate from the simple flow shop. Some examples are:

- $\boldsymbol{r}_{j}$ : jobs arrive or become ready intermittently. That is, $r_{j} \geq 0$, as opposed to the default assumption that $r_{j}=0$.
- prec: precedence relationships exist between jobs.
- pmtn: preemption (i.e., interruption) of a task is permitted. The preempted task may be resumed at a later time without penalty.
- ( $\boldsymbol{p m t n}$ ): preemption may or may not be permitted. This signifies that, even if preemption is allowed, we do not use it because preemptive and nonpreemptive optima always have the same objective function value.
- perm: in the simple flow shop, this signifies that permutation schedules only are considered : those schedules in which the same job order is maintained at all machines. Such a shop is called a permutation flow shop. For hybrid shops, the concept of a permutation schedule may be extended as follows: given an ordering or job list, each job is scheduled sequentially in list order to the successive work stations at the earliest feasible time that a machine at that stage becomes available. In the literature, this restriction to permutations is sometimes imposed when the general problem is intractable, to facilitate finding a usable, even if suboptimal, solution. Also, in some shops, it may be a technological necessity or a managerial requirement that no job ever overtakes another.
- (perm): as with (pmtn), parentheses are used when a permutation schedule will always be chose even if it is not required, either because there always exists an optimal schedule that is permutation, or because all optimal schedules are permutation.

Other features will be introduced as needed.

## Objective Function

In the third or $\gamma$-field we enter the criterion to be minimized (maximization criteria rarely arise naturally in scheduling problems, and can easily be converted to minimizations by sign reversal). We will sometimes refer to the objective as a cost function to emphasize that we are minimizing. Scheduling objectives are functions of the completion times of the tasks. Indeed, throughout this monograph all objectives are functions of job completion times; that is, they depend only on the times that the last tasks of each job are completed. Properly, the criterion is a function of the schedule chosen, but we understand that it depends on the schedule only through the completion times that result. Some common objectives are encoded as follows:

- $C_{\max }$ : the maximal or latest completion time of any job. This criterion, commonly called the makespan, has been by far the most exhaustively studied.
- $\boldsymbol{\Sigma} \boldsymbol{C}_{\boldsymbol{j}}$ : total completion time of all jobs, $\sum_{j=1}^{n} C_{j}$, which is equivalent to the mean completion time (they differ only by a constant factor $n$ ).
- Reg: the class of all objective functions with the property of regularity. Regular objectives or measures are functions $R(\bar{C})$ of the job completion times, $\bar{C}=\left(C_{1}, C_{2}, \ldots, C_{m}\right)$ that increase only if at least one $C_{j}$ increases. Simply put, it is always desirable to complete a job earlier, if it can be done without making any other job later. More formally stated, given any two schedules producing completion times $\bar{C}$ and $\overline{C^{\prime}}$, regularity means that, if $C_{j}^{\prime} \leq C_{j}$ for all $j$, then $R\left(\overline{C^{\prime}}\right) \leq R(\bar{C})$. There are many regular objectives, including $C_{\max }$ and $\Sigma C_{j}$. It is worth defining this class of criteria because we can sometimes show that all such objectives share certain useful properties.
- Any: any function of job completion times. This includes all regular measures, plus for example objectives that involve earliness penalties.

Sometimes we are concerned with two objectives, both of which we would like to make as small as possible. While we cannot generally minimize both simultaneously, we want our objective to somehow incorporate both goals. Such objectives are called multicriteria objectives. There are three ways to formulate them:

- hierarchical objective, $(\boldsymbol{A} \mid \boldsymbol{B})$ : minimize $A$ subject to a constraint on $B$. For example, we might minimize total flow time while keeping the makespan under some prespecified upper bound, written $\left(\Sigma C_{j} \mid C_{\max } \leq D\right)$.
- composite objective, $(\boldsymbol{\alpha} \boldsymbol{A}+\boldsymbol{\beta} \boldsymbol{B})$ : minimize a linear combination of the two objectives, for prespecified relative weights $\alpha$ and $\beta$.
- bicriteria objective, $(\boldsymbol{A}, \boldsymbol{B})$ : instead of somehow combining the two objectives into one, we find the schedules that produce the "best" pairs of values, and present these options to the decision maker for final selection. To define "best", suppose arbitrary schedule $S_{i}$ has values $\left(a_{i}, b_{i}\right)$ for the two objectives $(A, B)$ that we wish to make small. If $a_{i} \leq a_{j}$ and $b_{i} \leq b_{j}$, with at least one inequality strong, we say that $S_{i}$ dominates $S_{j}$. Clearly, only non-dominated or efficient solutions are of interest, and it is the set of all these solutions that we wish to determine.

As a simple example of this notation, the classic problem of scheduling $n$ jobs in a simple $m$-machine flow shop so as to complete all work as soon as possible is encoded $F m \| C_{\text {max }}$. Note how, since the simple shop contains all default assumptions, there are no entries in the middle or $\beta$ field.

### 1.3 Precedence

In rare cases, it may be possible to do the tasks of a job in any order, but it is almost always necessary to complete certain tasks before others can be
started. This may be true even when the tasks are parts of different jobs. Such constraints on the order or sequence of tasks are referred to as precedence requirements. Sometimes a single task awaits the completion of several others, as when various parts must all be ready before they can be assembled. In other situations the opposite is true: the completion of one task releases several others. A large complex job may involve a great many tasks interrelated in a tangled web of precedence relationships.

We say $T_{a}$ has precedence over $T_{b}$, written $T_{a} \rightarrow T_{b}$, when $T_{a}$ must be completed before $T_{b}$ can be started. The precedence structure of a job can be graphically shown as an acyclic directed graph or network in which tasks are shown as small circles or nodes, and an arrow or arc connecting node $a$ to node $b$ denotes precedence of $T_{a}$ over $T_{b}$. A job consisting of eight tasks is shown in Fig. 1.3. The precedence interrelationships are quite complicated. For example, task $S 3$ cannot begin until the last of its predecessor tasks $P 1, P 2$, and $P 3$ is done, while $P 1$ also delays two other successor tasks $S 1$ and $S 2$.


Fig. 1.3
Example of a precedence diagram

There are several particular precedence relationships, often encountered in practice, whose special structures lend themselves to simple analysis.

- A Chain as defined above, is a set of tasks that are completely sequenced, resulting in a precedence network like that in Fig. 1.1. In flow shops, chains are usually referred to as flow shop constraints.
- A String is a chain of tasks that must additionally be done contiguously, i.e., consecutively without interruption. Thus, a string is a quantity of work that behaves like a single task.
- An In-Tree is a connected acyclic graph where every node has precisely one direct successor, except for the root node which has no successor. It may be called a terminally rooted tree or assembly tree. If we eliminate nodes $S 1, S 2$ and $X$ in Fig. 1.3, we are left with an in-tree.
- An Out-Tree or initially rooted tree or branching tree is an in-tree with all precedence relations reversed.

Two graphs (or two sets of tasks interrelated by precedence) are in parallel if every task in one set is unrelated to, or independent of, every task in the other.

We extend the concept of precedence from tasks to jobs. We say $J_{j}$ has precedence over $J_{k}$, written $J_{j} \rightarrow J_{k}$, if $T_{i j}$ has precedence over $T_{i k}$ for all $M_{i}$ on which both jobs require processing.

### 1.3.1 Precedence versus Dominance

We have already defined precedence: the requirement that one task be completed before another is started. The nature of the work imposes this constraint, which may not be violated. Thus, each job in a simple flow shop has tasks in chain precedence: $T_{i j}$ has precedence over $T_{i+1, j}$ for $i=1, \ldots, m-1$.

Another kind of ordering between two jobs (it could equally be defined between tasks, but this has not been found useful) we shall call ordering by dominance (or by preference or priority). The set of conditions under which we prefer to schedule one job earlier than another is called a dominance relation. Note that such relations are defined with respect to a criterion.

It will be useful to define two types of dominance: local and global.
Definition 1.2. $J_{j}$ locally dominates $J_{k}$, if it would be less costly (or at least no more costly) to schedule $J_{j}$ before $J_{k}$ when they are adjacent in a schedule. Put another way, local priority signifies that $J_{j}$ should not immediately follow $J_{k}$ (or equivalently, $J_{k}$ should not immediately precede $J_{j}$ ).

Definition 1.3. $J_{j}$ globally dominates $J_{k}$, if there always exists an optimal schedule with $J_{j}$ positioned earlier than $J_{k}$, if it is feasible to do so.

See Sect. 1.6 for a full discussion.
We shall use the following notation for these two types of ordering:

- $\boldsymbol{a} \rightarrow \boldsymbol{b}: T_{a}$ has precedence over $T_{b}: T_{a}$ must be completed before $T_{b}$ is started.
- $\boldsymbol{j} \rightarrow \boldsymbol{k}$ : $J_{j}$ has precedence over $J_{k}$ : each task of $J_{j}$ must be completed before the corresponding task of $J_{k}$ is started. In a pure flow shop, $j \rightarrow k$ means $T_{i j} \rightarrow T_{i k}, i=1, \ldots, m$.
- $\boldsymbol{j} \leadsto \boldsymbol{k}: J_{j}$ has local dominance over $J_{k}$.
- $\boldsymbol{j} \xrightarrow{\text { global }} \boldsymbol{k}: J_{j}$ has global dominance over $J_{k}$. We use the simpler notation for local preference because we shall encounter it more often.
- $\boldsymbol{j} \sim \boldsymbol{k}: j \sim k$ and $k \sim j$. That is, the jobs have equal priority.

This is a convenient place to mention one other piece of ordering notation that will be used from time to time.

- $\boldsymbol{x} \prec[\succ] \boldsymbol{y}: T_{x}$ precedes [follows] $T_{y}$ (or, in a permutation schedule, $J_{x}$ precedes [follows] $J_{y}$ ).
- $\boldsymbol{x} \prec \prec[\succ \succ] \boldsymbol{y}: T_{x}$ (or $J_{x}$ ) immediately precedes [follows] $T_{y}$ (or $J_{y}$ ).

Note that " $x$ precedes [follows] $y$ " does not imply required precedence ordering, nor that one task or job is preferred before the other, but only that in the schedule under discussion they happen to be so ordered.

### 1.4 The Permutation Flow Shop

In a simple flow shop, as jobs follow each other through a fixed sequence of processors, is it ever desirable for one job to overtake another? First consider a machine where several jobs have piled up waiting for service. Without loss of time, we can pick any of them to process next. Downstream considerations might well induce us to violate "first come, first served" discipline. Thus, we should not always expect the best schedule to be a permutation schedule, which maintains the same job order on all machines.

However, suppose that after $J_{j}$ completes on $M_{i}$ it is free to start immediately on $M_{i+1}$. If the following job, $J_{k}$, is to pass it, we must shunt aside $J_{j}$ to wait while $J_{k}$ is processed on $M_{i}$ and, overtaking $J_{j}$, on $M_{i+1}$. Thus, $M_{i+1}$ is left idle for a time when it could have been occupied; an occurrence known as inserted idle time. Surprisingly, even in this situation it may be desirable to interchange jobs. That is, not only is it not always best to use a permutation schedule, it may not even be optimal to use a nondelay schedule: one that keeps processors busy whenever possible. Instead, inserted idleness may be beneficial.

The simplest example of this involves two jobs in a four-machine flow shop, with the makespan objective: $F 4 \| C_{\text {max }}$. Suppose the processing times are $\langle 1,5,5,1\rangle$ and $\langle 5,1,1,5>$. In Fig. 1.4a and b we show the two possible permutation schedules, while in Fig. 1.4c the optimal nonpermutation schedule is shown. We use Gantt charts laid out by machine, with jobs distinguished by the color of shading. Note how the makespan (and the average completion time, too) is shortened when we switch the job order after two machines; the inserted idle time (one unit on $M_{3}$ ) is worth it.


Fig. 1.4 Schedules for an instance of $F 4 \| C_{\text {max }}$ with two jobs

Of course, if we could restrict our attention to permutation schedules only, our search for an optimal schedule would be greatly simplified, since there are just $n$ ! job orderings, but $(n!)^{m}$ ways the jobs can be independently ordered on each machine. There are situations where only permutation schedules are acceptable, either because it is technologically impossible for one job to pass another, or by management fiat, to keep things operationally simple. In addition, there are cases when it can be shown that a permutation schedule will always be optimal over all schedules. Two such situations will now be presented.

Theorem 1.1 (Johnson, 1954; Yeung et al., 2009) For Fm\|Any, there exists an optimal schedule with the same job order on the first two processors.

Proof: In any other schedule, there must be two jobs scheduled successively on $M_{1}$ which are reversed on $M_{2}$. We can interchange their tasks on $M_{1}$ leaving all other tasks unmoved, without violating any constraints and without affecting any job completion time.

Note that this result holds for any objective that is a function of the job completion times (as are almost all criteria considered in this monograph). Until very recently, it was thought to be true only for regular criteria, as defined in Sect. 1.2.1. Most reasonable objectives are regular, but not all. For example, penalties are not regular when jobs have due dates and earliness, as well as lateness, penalties accrue. Since 1954, all papers and texts referencing this result have included the regularity requirement until Yeung et al. (2009) noted that it was not needed.

Theorem 1.2 (Johnson, 1954) For $F m \| C_{\max }$, there exists an optimal schedule with the same job order on the last two processors.

The proof is very similar to that of Theorem 1.1, and involves interchanging two adjacent jobs on the last machine, $M_{m}$. Further details are omitted. Note that Theorem 1.1 applies to a large class of criteria, but Theorem 1.2 is true only for the makespan objective because the job swap on $M_{m}$, while leaving the makespan unaffected, does change two of the completion times.

These two results tell us that we need only consider permutation schedules in two-machine flow shops, and in three-machine flow shops with makespan objective.

### 1.5 Graphic Representation of Flow Shop Schedules

We have seen two kinds of diagrams that help us visualize aspects of task scheduling. In Figs. 1.1 and 1.3, precedence relationships are shown graphically. In a flow shop, we have the simple precedence structure of $n$ parallel chains: $n$ separate independent jobs each with chain precedence, as in Fig. 1.1. The other kind of diagram is the Gantt chart, as illustrated in Figs. 1.2 and 1.4. This shows the exact timing of tasks, given the precedence
constraints and assuming a particular schedule has been chosen. It will be useful to extend the precedence diagram to a precedence network, which also includes the ordering imposed by a chosen schedule. For example, consider the Gantt charts in Fig. 1.4. The initial precedence diagram for these two jobs is given in Fig. 1.5a, where the chains are vertical so that each row represents a machine. Figure 1.5b shows the precedence network for the schedule in Fig. 1.4c, with additional arrows showing the job order on each processor. Two extra arrows show the start and end of the schedule.

Note that each path from start to finish through a precedence network represents a sequence of tasks that must be done consecutively. Thus, adding up the processing times along such a path gives a lower bound on the makespan for this schedule. Since the time to complete all work is limited only by these enforced sequences, it follows:

Theorem 1.3 The longest path, or critical path, through a precedence network equals the makespan of the corresponding schedule.

We shall find this insight very useful.

(a)

(b)

Fig. 1.5 (a) Precedence diagram, and (b) Precedence network for jobs scheduled in Figure 1.4c

### 1.6 Dominance Properties

The simplest scheduling problems involve finding the optimal ordering of a set of objects (in our case, jobs). We call them sequencing problems. Seeking the optimal permutation schedule in a flow shop is such a problem, because the schedule is fully determined once the job order is fixed (although admittedly a few more calculations will be needed to specify the start times of each task on each processor).

Sequencing problems can be difficult to solve. By "easy" or "difficult" we shall mean polynomially solvable or NP-complete, as defined in the Theory of

Algorithmic Complexity which is reviewed in Appendix A. "Difficult" problems usually require some sort of implicit search procedure, such as dynamic programming or branch-and-bound. To shorten the search for a "best" schedule, it is sometimes possible to find characteristics that optimal schedules possess, allowing us to limit our search to only sequences having those characteristics. We call these desirable characteristics of schedules dominance properties or elimination criteria, since the subset of schedules having them constitutes a dominant set.

We usually prove the optimality of schedules with some characteristic by starting with an arbitrary schedule that lacks it, and demonstrating that the schedule is improved (or at least cannot become worse) by introducing it. Most often, the desirable characteristic involves a partial job ordering, such as that one job can always be scheduled before another, or a certain job should (or should not) be scheduled last, etc. Thus, to show that $J_{k}$ need never be considered before $J_{j}$, we might start with an arbitrary schedule in which $J_{k}$ precedes $J_{j}$ and show that the schedule is improved by interchanging the jobs. We introduced this idea of "preference" or "priority" orderings between two jobs earlier. Dominance instead relates two sets of schedules. Thus, if $J_{i}$ has priority over $J_{j}$, then schedules in which $J_{i}$ precedes $J_{j}$ constitute a dominant subset.

As mentioned earlier, dominance or priority orderings between two jobs may be local or global, a local ordering resulting when we find we can only prove that one job should precede another when the two jobs in question are adjacent in the schedule. For example, for $F 2 \mid($ perm $) \mid \Sigma_{j} T_{j}$, if $a_{j} \leq a_{k}$, $b_{j}=b_{k}$, and $d_{j}-b_{j} \leq d_{k}-b_{k}$, then $J_{j}$ has local priority over $J_{k}$; that is, any schedule with $J_{k}$ immediately before $J_{j}$ can be improved (or at least made no worse) by interchanging the jobs.

Technically, not all optimal schedules need have the property; alternative optima may exist. For instance, in the example above, to be strictly correct the conclusion should be, "there exists an optimal sequence in which $J_{j}$ precedes $J_{k}$." Still, this is enough to permit us to truncate the search tree, and we will generally omit the language "there exists an optimal solution ...". As introduced in Sect. 2, when we show that $J_{j}$ locally [globally] precedes $J_{k}$ in some optimal schedule, we write $j \leadsto k[j \stackrel{\text { global }}{\sim} k]$ and we say " $J_{j}$ dominates $J_{k}$ locally [globally]" or " $j$ has local [global] priority over $k$ ". The set of conditions that lead to this job ordering are called priority rules or dominance relations.

Of course, we cannot expect that every pair of jobs can be ordered by priority. For instance, in the $F 2 \mid($ perm $) \mid \Sigma_{j} T_{j}$ example mentioned above, the three conditions will clearly not hold for most pairs of jobs, and only partial ordering can be determined. Still, the results of such quickly-applied tests can significantly shorten the search time, especially if several dominance relations can be found. In case every pair of jobs can be ordered by priority (the ordering is complete), the full sequence will thereby be established without any need to search, as discussed in the next section.

Sometimes we look for dominance relationships between unscheduled jobs in the presence of an already established partial schedule $\sigma$. Given an initial $\sigma$, we generally ask which jobs should be considered, or can be eliminated from consideration, for the position immediately following $\sigma$. We defer further discussion of this topic until Chap. 4.

One final remark is motivated by the possibility of applying several dominance relations simultaneously. Since each property is existential ("there exists an optimal schedule ..."), can they be accumulated? Logically, they cannot: we might have one optimal schedule with property A, another with property B, but none having both properties. However, it turns out that the kind of properties we will be concerned with, though existential, are cumulative, as can be seen by following the logic of the proofs.

### 1.6.1 Sequencing using Dominance

"Easy" problems, for which efficient or polynomial algorithms exist, are often solved to optimality using priority ordering, without any searching. We can do this if we can find a priority rule ordering pairs of jobs that is complete: every pair of jobs is ordered by it. In such cases, we need to establish the ordering criterion only for adjacent pairs of jobs (local dominance, which is easier to prove), to fully determine the sequence. We now formalize this idea.

Definition 1.4. For any sequencing problem with objective function $M(S)$ assigning a real value to any sequence $S$ of the $n$ jobs, $M(S)$ has the Complete Priority Ordering (CPO) property if there exists a local priority relation $j \leadsto k$, defined on all pairs of jobs $J_{j}$ and $J_{k}$, which is:
(a) Transitive: $(i \sim j$ and $j \leadsto k) \Rightarrow i \leadsto k$;
(b) Complete: $j \leadsto k$ or $k \leadsto j$ or both;
(c) Priority-Indicating: $j \sim k \Rightarrow M(u, j, k, v) \leq M(u, k, j, v)$,
$\forall S=(u, j, k, v)$, where $u$ and $v$ are arbitrary subsequences.
Keep in mind that this relation depends solely on the characteristics of the two jobs, and is independent of the number and properties of the other jobs, and how they are ordered in the schedule.

If $j \leadsto k$ and $k \leadsto j$, we say that $j$ and $k$ have equal priority, written $j \sim k$. It follows:

$$
j \sim k \Rightarrow M(u, j, k, v)=M(u, k, j, v) \forall u, v
$$

The following is an immediate consequence:
Theorem 1.4 If $M(S)$ satisfies the $C P O$ property, then any permutation which orders the jobs from highest priority to lowest minimizes $M(S)$ (and the reverse order maximizes $M(S)$ ).

Very often, when the CPO property is present, job priority can be established even more simply using a priority index, $\boldsymbol{I}(j)$, a number which is
calculated for each $J_{j}$ depending only on the given attributes of that job. If we can find an index such that, give two adjacent jobs, the one with smaller index should always come first, then we can establish priority order simply by comparing index values:

$$
i \leadsto j \Longleftrightarrow I(i) \leq I(j)
$$

We can then conclude:
Theorem 1.5 If there exists a function $I(j)$ assigning a real value to each $J_{j}$, such that

$$
I(j) \leq I(k) \Longrightarrow M(u, j, k, v) \leq M(u, k, j, v), \quad \forall S=(u, j, k, v)
$$

then any permutation that orders the jobs from lowest index value to highest minimizes $M(S)$ (and the reverse order maximizes $M(S)$ ).

Proof: Considering the minimization objective, start with any sequence and find the job with smallest $I(j)$. If it is not at the start of the sequence, interchange it with the job to its left and repeat until it is in first place. Each interchange can only decrease $M(S)$. Similarly, move the job with second smallest index into second place, and repeat for all jobs in turn.

A priority list is a preference ordering of jobs based on a priority index. To specify whether the jobs are to be in nondecreasing or nonincreasing order of $I(j)$, we shall use the following shorthand notation. Suppose the list of jobs is to be sequenced in nondecreasing order of $I(j)$. As we move forward in the schedule $(\rightarrow)$, the value of $I(j)$ increases $(\uparrow)$. Combining these two directions, we shall represent such a schedule as
$\nearrow I(j)=$ the sequence in which jobs are in nondecreasing order of $I(j)$.
Similarly, for the sequence in nonincreasing order of $I(j)$, we write $\searrow I(j)$.
Priority indices are clearly useful, if we can find them. This is usually done, not surprisingly, by considering the change in objective function value caused by interchanging the positions of an arbitrary pair of adjacent jobs: $M(u, j, k, v)-M(u, k, j, v)$. If the difference in values depends only on the parameters of the two jobs interchanged (regardless of which jobs are chosen, where they are in the schedule, and how all the other jobs are arranged around them), and furthermore if this difference can be written as a function of one of the job's attributes minus that same function of the other job, then that function can be used as a priority index to sequence all the jobs. That is, if $M(u, j, k, v)-M(u, k, j, v)$, once all common terms have been cancelled, reduces to $I(j)-I(k)$ for some function $I($.$) , then I(i) \leq I(j)$ implies $M(u, i, j, v) \leq M(u, j, i, v)$ so $I(j)$ is a priority index.

### 1.7 Heuristics and Worst Case Analysis

When combinatorial problems are "difficult" (see Appendix for the precise meaning of this word), we can only solve to optimality by some form of exhaustive enumeration. There are techniques such as branch-and-bound and dynamic programming (we will not discuss them here, referring the reader to any Operations Research textbook) that with some ingenuity can curtail the tedious process of considering every possible solution in our search for the best one; these are called implicit enumeration techniques. Still, finding an optimum remains laborious, and at some point, as the problem size grows, it gets out of hand.

In such cases, we are often willing to settle for a good, but not necessarily optimal, solution. Any procedure or algorithm that generates a solution to a problem without any guarantee of optimality is called a heuristic. An effective heuristic produces a good answer in a short time. But what does "good" mean? Generally we measure the quality of a solution by its deviation from the optimum. Thus, in a minimization problem, we might agree that an objective function value $2 \%$ more than the best possible is "good", while $20 \%$ is not. But this raises another question. If we cannot find the optimum, how can we tell how close the heuristic solution is to it? Researchers proposing new heuristics often report extensive computational results, using simulation or implicit enumeration to find the optimum for a set of randomly generated problem instances. They can then solve the same instances using their heuristic, and make the comparison. Given a large and representative sample of instances (how to choose representative instances is another can of worms, not to be discussed here; see Hall and Posner (2001)), they can then report the average performance of the heuristic. The difficulty often is that only small to moderate sized instances can be solved to optimality, even using a great deal of computer time, so we must extrapolate the performance of the heuristic on smaller problems to its performance on larger ones; a risky business.

Alternatively, one may develop a lower bound for the optimal value by means of optimization techniques. Then, the average per cent relative gap of the heuristic solution from the lower bound can be computed on randomly generated problem instances. The lower bound must be easy to compute even for large instances, and is hopefully not too far from optimum. If we can show that our heuristic gives answers close to the lower bound, they must be very close to optimum.

Another way to assess the quality of a heuristic is through worst case analysis. Briefly, the worst case error bound or worst case performance ratio, $\rho$, for a given heuristic H is the multiple of the optimal makespan such that the heuristic always gives a makespan at least that good. Technically, if for any instance, $C_{\max }^{\star}$ is the minimal makespan and $C_{\max }^{H}$ is the makespan produced by the heuristic, and if, for every instance, $C_{\max }^{H} \leq \rho C_{\max }^{\star}$ for some positive constant $\rho$, then $\rho$ is a worst case error bound for H , which
may be written $\rho_{H}$. For example, if $\rho_{H}=1.2$, then heuristic H is guaranteed to find a solution to any instance with makespan value no more than $20 \%$ more than the optimum. If $\rho_{H}$ is as small as possible for H (that is, if there is an instance for which $C_{m a x}^{H}=\rho_{H} C_{\text {max }}^{\star}$ ), then the heuristic is said to be tight. We remark that worst case performance is very conservative; average performance of the heuristic is likely to be much better.

### 1.7.1 Cyclic Flow Shops

So far we have discussed the problem of scheduling a given set of $n$ distinct jobs, to be processed just once. Call this the static problem; most of this monograph will be concerned with it. However, from time to time we shall look at a different kind of flow shop scheduling that is often encountered in just-in-time and many other environments, where there are a finite number $q$ of job types or products to be manufactured in quantity. All jobs of one type have the same processing times. Let $r_{j}, j=1, \ldots, q$ be the number of units for a type-j job over a planning horizon. The goal is to maintain a nearly constant flow of parts through the shop. To achieve this, the production target is subdivided into identical subsets, as small as possible given the required product mix ratio. This Minimal Part Set or MPS therefore contains $r_{j} / d$ items of type $j$, where $d$ is the greatest common divisor of the integers $r_{1}, \ldots, r_{q}$. If we let $n$ be the total number of jobs in an MPS, then $n=$ $\sum_{j=1}^{q} r_{j} / d$. Despite the motivation for this model, we will henceforth, as is customary, ignore the fact that many of the jobs are identical.

In a cyclic flow shop, we wish to produce the $n$ jobs in an MPS repeatedly and indefinitely, so as to achieve some objective in steady state production. The time required to complete an MPS is called the production cycle, and it is this cycle time that is usually the objective, which we denote $C T$, to be minimized.

We should mention one caveat with respect to this objective. Clearly, given the requirement that we produce one MPS repeatedly, minimizing the cycle time is equivalent to maximizing the output rate or throughput, which is presumably the underlying goal. Suppose, however, we decide to produce two MPS's in each production run. The minimal cycle time to produce two sets will generally be less than double the time for one set (it cannot be more), thus increasing the throughput. Larger multiples of the MPS would give greater gains in throughput. The downside is the possible loss of the smooth flow of output, each product in small quantities proportional to need. Instead, we might stay too long with one product, while stocking out for other products.

## Cycle Time Compared to Makespan

A substantial part of this monograph will deal with minimizing makespan in the static flow shop under various conditions and constraints. Clearly, minimizing cycle time in the cyclic shop is closely related. To begin with, we could approximate the cycle time by the makespan for a single MPS
in isolation. Iterating this static schedule yields a feasible, but usually not optimal, solution to the cyclic problem: an upper bound.

The relationship between the two objectives (makespan in the static case, cycle time in the cyclic setting) can be easily seen. For a permutation schedule over $m$ machines, let

$$
S_{k j}\left[C_{k j}\right]=\text { start time [completion time] on } M_{k} \text { of the job in position } j .
$$

Then, of course, the makespan is $C_{m n}-S_{11}$. In a cyclic schedule, the cycle time, $Z$, turns out to be the maximal machine occupancy of the MPS: $Z=$ $\max _{k=1, \ldots, m}\left\{C_{k n}-S_{k 1}\right\}$. This can be easily seen with reference to Figure 1.6, which, for an instance with $m=n=4$, gives the Gantt chart for an arbitrary schedule. The first iteration shows static production of the $n$ jobs, while the rest gives the cyclic continuation when the jobs make up an MPS. Note how, for cyclic production, the block of work that constitutes an MPS repeats at intervals that are determined by the longest occupied machine, $M_{3}$.


Fig. 1.6 Comparison of makespan and cycle time

This is not to say that the cycle time shown in the figure is minimal, even for the given permutation of jobs. In fact, if you look at the first production run for a moment, you can see how tasks on each machine except $M_{1}$ can be moved to the right, to make the machine loading more compact. Actually, on the last machine, they can always be made contiguous without affecting the makespan, as they already are on $M_{1}$. We will discuss in Sect. 4.19.1 how to find the smallest cycle time for a given job sequence.

Given a mathematical program for one problem, it is now easy to formulate a similar one for the other. The makespan objective is to minimize $C_{m n}-S_{11}$; for cycle time, we minimize $Z$, subject to $Z \geq C_{k n}-S_{k 1}, k=1, \ldots, m$. All other constraints, whose function it is to interrelate the start and completion times of the various tasks, will be identical.

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## Chapter 2 THE TWO-MACHINE FLOW SHOP


#### Abstract

The two-machine flow shop has attracted significant attention, especially when it comes to extensions of the basic makespan model to include release times, setups, or other complications. As expected, we start our coverage with a full discussion of Johnson's Rule for makespan minimization. Then, we present extensions that incorporate setup/teardown times in automated manufacturing cells. When arbitrary positive release times are assumed for jobs, the problem of minimizing makespan in the two-machine flow shop is shown to be NPcomplete, and hence we discuss solution procedures, optimal and heuristic. Even when a single server is used to perform setups, the problem is shown strongly NP-complete, though special cases accept simple solutions. Results on optimal lot streaming of a product are also presented. Precedence constraints are postponed to a later chapter. Subsequently, we survey results on objectives like $\Sigma C_{j}, L_{\max }, T_{\max }, \Sigma T_{j}, \Sigma U_{j}$, as well as corresponding multicriteria. Various manifestations of these models come with setups, scarce resources, common deadlines, etc. Whenever possible, we provide dominance properties, lower bounds, branch-andbound approaches, heuristics and computational results.


The simple flow shop with two stations, or two-machine flow shop, is the most elementary of all multistage processing facilities. The following examples demonstrate how fundamental this system is.

### 2.1 Examples

Surprisingly, few examples of two-machine flow shops are presented in the literature, most likely because in terms of applications, researchers focus on
larger flow shops. The following examples are obvious applications with just two stations.

- Al-Anzi and Allahverdi (2001) showed that the internet connectivity problem in 3 -tiered client-server databases is a 2 -machine flow shop with $\Sigma f_{j}$ objective. In this, end users seek connectivity to local or remote databases through the Web using two separate servers: an application server and a database server.
- See Nagar et al. (1995) for an application in the steel industry where each job undergoes wire-drawing first, followed by annealing.
- Allahverdi and Al-Anzi (2002) showed that the scheduling of multimedia data objects for WWW applications reduces to a 2-machine flow shop with $L_{\text {max }}$ objective.
- Modern manufacturing systems may involve automated storage and retrieval stations, robotic loading and unloading of machines, transportation robots, etc. Jobs are transported on pallets by a circulating conveyor, and require special fixtures and tooling at each machine. Cells of this type with two Computer Numerically Controlled (CNC) machines served by robots are modeled by Levner et al. (1995) and Kogan and Levner (1998). They can be efficiently analyzed as two-machine flow shops with transfer lags (see Sect. 3.2.6).

We organize the rest of our discussion around the objective function to be minimized. The most general single objectives to have been considered (multiple objectives have received little attention) are $\sum_{j=1}^{n} f_{j}\left(C_{j}\right)$ (written $\Sigma f_{j}$ for short) and $\max _{j=1}^{n} f_{j}\left(C_{j}\right)$ (abbreviated $f_{\max }$ ), where $f_{j}(t)$ is an arbitrary nondecreasing function giving the cost of completing $J_{j}$ at time $t$. The simplest special cases, when $f_{j}(t)=t$, are the total completion time and the maximal completion time or makespan. We deal first with the makespan, which has been far more studied than any other objective.

## 2.2 $\operatorname{F2} 2 \mid$ (perm), (pmtn) $\mid C_{\text {max }}$ : Johnson's Rule

First we note that, for any measure, preemption is never helpful on $M_{1}$ (in any schedule, a task on $M_{1}$ that has been broken up can be consolidated at its latest completion time without delaying any job, hence without necessitating any change on $M_{2}$ ), and for the makespan objective we never need preemption on $M_{2}$ (parts of a preempted task can be collected at its earliest start time), so we will consider the nonpreemptive case only, understanding that the solution remains optimal even if preemption is allowed. A problem that may be preemptive or not is encoded ( $p m t n$ ) in the three-field notation.

The basic result, to minimize the makespan in the nonpreemptive twomachine flow shop, is one of the most famous in all of scheduling theory. It is known as Johnson's Rule, and provides a simple optimal scheduling procedure running in time $O(n \log n)$. To simplify notation, we use $<a_{j}, b_{j}>$ for $\left\langle p_{j 1}, p_{j 2}\right\rangle$. Recall that, by Theorem 1.1, we can confine our attention
to permutation schedules, whether or not such a schedule is required for technological or other reasons (this idea is conveyed by (perm) in the problem name). Thus, a schedule can be specified simply by a job ordering. Since job numbering is arbitrary, a typical schedule can be written $S=\left(J_{1}, J_{2}, \ldots, J_{n}\right)$, which we will abbreviate as $S=(1,2, \ldots, n)$. The precedence network for this schedule is shown in Fig. 2.1, where we label the tasks of $J_{j}$ by their processing times rather than $T_{j k}$. We easily see that there are $n$ routes through


Fig. 2.1 Precedence network for schedule $(1,2, \ldots, n)$ in a two-machine flow shop
the network, with route $j$ traversing nodes $a_{1}, \ldots, a_{j}, b_{j}, \ldots, b_{n}$. Thus, since the makespan for this schedule is the longest route or critical path, it can be written

$$
\begin{equation*}
C_{\max }(S)=\max _{j=1, \ldots, n} R_{j}, \tag{2.1}
\end{equation*}
$$

where $R_{j}$ is the length of route $j$ :

$$
R_{j}=\sum_{i=1}^{j} a_{i}+\sum_{i=j}^{n} b_{i} .
$$

Theorem 2.1 (Johnson's Relation: Johnson, 1954)
For $F 2 \mid$ (perm), $($ pmtn $) \mid C_{\text {max }}$, provided $a_{j} \neq b_{j} \forall J_{j}$, a Complete Priority Ordering exists, with

$$
j \leadsto k \Longleftrightarrow \min \left(a_{j}, b_{k}\right) \leq \min \left(a_{k}, b_{j}\right) .
$$

Proof: By Definition 1.4, the CPO property must be transitive, complete, and priority-indicating. It is to establish transitivity that we need the extra assumption: $a_{j} \neq b_{j} \forall J_{j}$, as will be seen. We will discuss this further after completing the proof. Taking each requirement in turn:
(a) Transitive: We must show that

$$
\left.\begin{array}{l}
\min \left(a_{i}, b_{j}\right) \leq \min \left(a_{j}, b_{i}\right) \\
\min \left(a_{j}, b_{k}\right) \leq \min \left(a_{k}, b_{j}\right)
\end{array}\right\} \Rightarrow \min \left(a_{i}, b_{k}\right) \leq \min \left(a_{k}, b_{i}\right)
$$

Consider the four processing times that appear on the left of the two hypotheses: $a_{i}, a_{j}, b_{j}$, and $b_{k}$. If $a_{i}=\min \left(a_{i}, a_{j}, b_{j}, b_{k}\right)$, then $a_{i}$ is less than all four of the numbers on the right of the hypotheses: $a_{i} \leq \min \left(a_{j}, a_{k}, b_{i}, b_{j}\right)$, and the conclusion follows. The same arguement applies if $b_{k}=\min \left(a_{i}, a_{j}, b_{j}, b_{k}\right)$. If $a_{j}=\min \left(a_{i}, a_{j}, b_{j}, b_{k}\right)$, then:

$$
a_{j} \leq \min \left(a_{i}, b_{j}\right) \leq\left(\min \left(a_{j}, b_{i}\right),\right.
$$

which would be a contradiction unless both weak inequalities hold as equalities. Since $a_{j} \neq b_{j}, a_{j}<b_{j}$, and so:

$$
a_{j}=a_{i}=\min \left(a_{j}, b_{i}\right)
$$

Thus, $a_{j} \leq b_{i}$. Also, from the second hypothesis, $a_{j} \leq a_{k}$. Combining these results, $a_{j}=a_{i} \leq \min \left(a_{k}, b_{i}\right)$, which gives our conclusion. A similar argument takes care of the last case: $b_{j}=\min \left(a_{i}, a_{j}, b_{j}, b_{k}\right)$.
(b) Complete: Any two numbers are ordered.
(c) Priority-indicating: Consider the change in objective function value if we interchange $J_{j}$ and $J_{k}$ in any $S=(u, j, k, v)$, to produce $S^{\prime}=(u, k, j, v)$, where $u$ and $v$ are partial schedules containing all other jobs. For this problem, $M(u, j, k, v)=C_{\max }(S)$, given in (2.1). After cancelling like terms in $C_{\max }(S)$ and $C_{\max }\left(S^{\prime}\right)$ (note in Fig. 2.1 how the lengths of most routes are unchanged by the job interchange), we get:

$$
\begin{aligned}
& C_{\max }(S)-C_{\max }\left(S^{\prime}\right)= \\
& \quad \max \left(a_{j}+b_{j}+b_{k}, a_{j}+a_{k}+b_{k}\right)-\max \left(a_{k}+b_{k}+b_{j}, a_{k}+a_{j}+b_{j}\right)
\end{aligned}
$$

Using the identity for any real $x, y, z$ :

$$
\max (x+y, x+z)=x+\max (y, z)=x+y+z-\min (y, z),
$$

we have:

$$
\begin{aligned}
& C_{\max }(S)-C_{\max }\left(S^{\prime}\right)= \\
& \quad\left(a_{j}+b_{k}\right)+b_{j}+a_{k}-\min \left(b_{j}, a_{k}\right)-\left[\left(a_{k}+b_{j}\right)+b_{k}+a_{j}-\min \left(b_{k}, a_{j}\right)\right]
\end{aligned}
$$

from which, after cancelling like terms, the result follows.
The constraint $a_{j} \neq b_{j}$ is needed for the following reason. Transitivity implies $i \sim j$ and $j \sim k \Rightarrow i \sim k$, but this is not always true if $a_{j}=b_{j}$. Consider jobs $J_{i}, J_{j}$ and $J_{k}$, with processing times $\left.<2,3>,<1,1\right\rangle$ and $<4,5>$, respectively. Johnson's Relation gives $i \sim j$, but both orderings $i \leadsto j$ and $j \leadsto i$ depend on the properties of $J_{j}: a_{j}=b_{j}=1$. The values of $a_{i}$ and $b_{i}$ are irrelevant (as long as they are both greater than 1). The same thing is true when we compare $J_{j}$ and $J_{k}$. Thus, knowing that both $J_{i}$ and $J_{k}$ have equal priority with $J_{j}$ tells us nothing about the ordering of $J_{i}$ and $J_{k}$. In fact, in our example, $i \sim j$ and $j \sim k$ but $i \nsim k$. In one sense this is not contradictory, but it does violate the strict definition of transitivity.

However, as the above proof shows, this problem will never arise as long as $a_{j} \neq b_{j}, \forall J_{j}$. This is not a seriously constraining requirement. Whenever we find a job with equal processing times, we need only perturb one of them slightly; say, by adding $\epsilon$ to $a_{j}$. Assuming $\epsilon$ is much smaller than any processing time, makespans are changed by at most $n \epsilon$, so the optimal sequence remains optimal. We can thus solve the perturbed problem, and then restore the original processing times.

In fact, even this will not be necessary. The importance of Johnson's Relation lies in the simple and famous algorithm that comes from it, and that algorithm, which we now state, automatically takes care of this ambiguity.

Theorem 2.2 (Johnson, 1954) For $F 2 \mid($ perm $),($ pmtn $) \mid C_{\max }$, the following algorithm gives an optimal sequence:

## Johnson's Rule (JR) - Version 1

1. Find the smallest of all processing times: $t=\min _{j}\left[\min \left(a_{j}, b_{j}\right)\right]$.
2. If $t=a_{k}$ for some $k$, schedule $J_{k}$ first; If $t=b_{k}$ for some $k$, schedule $J_{k}$ last; break ties arbitrarily.
3. Remove $J_{k}$ and repeat.

Proof: Suppose in some schedule $J_{k}$ (as defined in the algorithm) is not first [last] except for ties. Using Johnson's relation, we can interchange it successively with each job that precedes [follows] it.

If there is more than one smallest task in Step 2, we can trace all tie-breaking options and each will give us an optimal schedule. Additionally, there are sometimes other optima that JR does not generate. For example, if one job, say $J_{1}$, has a very large task time on $M_{1}$, so that $a_{1}>\sum_{j=2}^{n} b_{j}$, and at the same time $b_{1}=\min _{j} b_{j}$, then all schedules with $J_{1}$ last are optimal, with $C_{\text {max }}^{\star}=\sum_{j=1}^{n} a_{j}+b_{1}$.

We will denote the schedule (or any of the schedules) obtained using this rule $J R(\underline{a}, \underline{b})$. where $\underline{a}=\left(a_{1}, \ldots, a_{n}\right)$ and $\underline{b}=\left(b_{1}, \ldots, b_{n}\right)$. Thus, if $S^{\star}$ denotes an optimal schedule, we can say:

For $F 2 \| C_{\max }, \quad S^{\star}=J R(\underline{a}, \underline{b})$, or simply $\quad S^{\star}=J R$.
Another way to state the algorithm (as should be clear without proof):

## Johnson's Rule (JR) - Version 2

Let $\mathcal{F}=\left\{J_{j}: a_{j}<b_{j}\right\}$, and $\mathcal{L}=\left\{J_{j}: a_{j} \geq b_{j}\right\}$. Then

$$
J R(\underline{a}, \underline{b})=\left(\mathcal{F}: \nearrow a_{j}, \quad \mathcal{L}: \searrow b_{j}\right)
$$

That is, start with the jobs in $\mathcal{F}$ (for first) in increasing order of $a_{j}$, and after them schedule set $\mathcal{L}$ (for last) in decreasing order of $b_{j}$. Stated this way, it is easy to see that this schedule has the following useful property: if one or more jobs are added to or removed from the set to be processed, after the sequence is determined, the optimal order of the other jobs is unaffected.

Actually, the jobs with $a_{j}=b_{j}$ could be put in either $\mathcal{F}$ or $\mathcal{L}$; we put them all in $\mathcal{L}$ arbitrarily. Each partition of such jobs between $\mathcal{F}$ and $\mathcal{L}$ gives an optimal schedule.

Still another way to express Johnson's Rule is:

## Johnson's Rule (JR) - Version 3

$$
J R(\underline{a}, \underline{b})=\nearrow I(j), \quad \text { where } I(j)=\frac{\operatorname{sign}\left(a_{j}-b_{j}\right)}{\min \left(a_{j}, b_{j}\right)}= \begin{cases}-1 / a_{j}, & a_{j}<b_{j} \\ 1 / b_{j}, & a_{j} \geq b_{j}\end{cases}
$$

Note how the jobs in $\mathcal{F}$ are given negative indices, assuring that they precede jobs in $\mathcal{L}$. Reciprocals are taken to reverse the ordering of jobs, as needed. Clearly, now that we have reduced Johnson's Rule to a listing by priority index, transitivity is assured; we need not worry whether $a_{j}=b_{j}$ for some $j$.

One more property of this simple algorithm may be noted: Johnson's Rule remains optimal if the two machines are not simultaneously available. To see this, let $R_{i}$ be the ready time of $M_{i}$. Then, having $R_{2}>R_{1}$ is like adding a dummy job $J_{d}$ with $a_{d}=0$ and $b_{d}=R_{2}-R_{1}$. The case $R_{2}<R_{1}$ is trivial.

A final remark is in order. We have assumed that every job visits both machines: $a_{j}>0, b_{j}>0$ for all $j$. If there are additional jobs that only require processing on one machine, they can be simply handled by assigning them zero processing time on the other machine, and letting Johnson's rule position them. If $a_{j}=0$, they will be scheduled first, and if $b_{j}=0$, last. This leads to the intuitive solution: schedule the "both-machines" jobs by JR, appending the " $M_{1}$-only" jobs at the end of $M_{1}$ and putting the " $M_{2}$-only" jobs at the start of $M_{2}$, both in any order.

### 2.2.1 $\boldsymbol{F} 2 \mid$ (perm),$r_{j} \mid C_{\text {max }}$ and $F 2 \mid($ perm $) \mid L_{\text {max }}$

Still with the makespan objective, suppose jobs are not all available simultaneously, but have different release dates, $r_{j}$. We start with the observation that this is equivalent to the problem where jobs all have the same release time ( $r_{j}=0$ for all $j$ ) but have different due dates, with the objective of minimizing maximal lateness. To understand this equivalence, it is convenient to consider the decision versions of the two problems. Thus, for the first problem, namely $F 2 \mid($ perm $), r_{j} \mid C_{\max }$, we ask: for any $D \geq 0$, does there exist a schedule with $C_{\max } \leq D$ ? A three-job instance is shown in Fig. 2.2a, where $\underline{r}=(1,2,6), \underline{a}=(3,7,4)$, and $\underline{b}=(7,2,5)$. The schedule given is optimal, and clearly if we choose $D=21$ as illustrated (or any $D \geq 19$ ), the answer is "yes".


Fig. 2.2 Corresponding schedules to show the equivalence of
(a) $F 2 \mid$ (perm),$r_{j} \mid C_{\text {max }}$ and (b) $F 2 \mid($ perm $) \mid L_{\text {max }}$

To show the equivalence of this problem to $F 2 \mid($ perm $) \mid L_{\text {max }}$, we must state how an instance of one corresponds to an instance of the other. To do this, we simply reverse time, interchange the roles of $M_{1}$ and $M_{2}$, define $d_{j}=D-r_{j}$ for each $J_{j}$, and ask the question: does there exist a schedule with $L_{\max } \leq 0$ ? Applying this to our numerical instance, we get the schedule in Fig. 2.2b. Note how $L_{\max }=L_{3}=-2$, and approaches zero as $D$ diminishes to $C_{\max }$, so that each problem has the answer "yes" if and only if the other does.

In what follows we work exclusively with the makespan objective.
Theorem 2.3 (Lenstra et al., 1977)
$F 2 \mid$ (perm),$r_{j} \mid C_{\max }$ is ordinary NP-complete.
Proof Outline: The reduction is from the NP-complete problem (a special case of the Knapsack problem):

## SUBSET SUM

INSTANCE: An integer $V$, and $k$ positive integers $v_{i}: i \in \mathcal{T}=\{1,2, \ldots, k\}$. QUESTION: Is there a subset $\mathcal{S} \subset \mathcal{T}$ such that $\Sigma_{i \in \mathcal{S}} v_{i}=V$ ?
to the decision version of our problem:
F2 $\mid$ (perm),$r_{j} \mid C_{\text {max }} \leq B$ ?
INSTANCE: An integer $B$, and $n$ jobs $J_{j}: j \in \mathcal{N}=\{1,2 \ldots, n\}$, each with parameters $\left\langle a_{j}, b_{j} ; r_{j}\right\rangle$, to be scheduled in a two-machine flow shop, where $a_{j}\left[b_{j}\right]$ is the processing time on $M_{1}\left[M_{2}\right]$, and $r_{j}$ is the ready time, of $J_{j}$.
QUESTION: Does there exist a schedule with $C_{\max } \leq B$ ?
For any instance of Subset Sum, define an instance of $F 2 \mid($ perm $), r_{j} \mid C_{\max }$ as follows:

- $B:=W+1$, where $W=\Sigma_{i} v_{i}$.
- $n:=k+1$
- $\left\langle a_{j}, b_{j} ; r_{j}\right\rangle:=\left\langle v_{j}, 0 ; 0\right\rangle, \quad j \in \mathcal{T}$
- $<a_{n}, b_{n} ; r_{n}>:=<1, W-V ; V>$

To see why the two problems will always have the same answer, we need only note that we can only achieve a makespan of $B$ (the answer "yes") if $J_{n}$ is scheduled immediately upon arrival, and if the other jobs are schedulable on $M_{1}$ in the two intervals defined by the positioning of $J_{n}$.

Since $F 2 \mid($ perm $), r_{j} \mid C_{\max }$ is NP-complete, only implicit search or approximation schemes have been proposed. Tadei et al. (1998) present a number of lower bounds that are used at each node of a branch-and-bound algorithm. Even though the authors do not state it explicitly, their bounds are valid only when either $b_{j}>0 \forall J_{j}$ or $a_{j}>0 \forall J_{j}$. Then, consider partial schedule $\sigma=(1,2, \ldots, r)$ with makespan $C(\sigma)$ and let $\mathcal{U}$ be the set of unscheduled jobs. With the above assumption, all tasks in $\mathcal{U}$ will contribute at least $b_{j}$ to the makespan and hence

$$
L B_{1}=C(\sigma)+\sum_{j \in \mathcal{U}} b_{j} .
$$

Alternatively, consider the Johnson schedule of jobs in $\mathcal{U}$ and add it to the earliest time, say $\bar{R}$, that these jobs can start on $M_{1}$. Clearly, $\bar{R} \geq \min _{j \in \mathcal{U}} r_{j}$. Moreover, $\bar{R}$ should be greater than the completion time of jobs in $\sigma$ on $M_{1}$, i.e., $\bar{R} \geq C_{1}=\max _{j \in \sigma}\left\{r_{j}+\sum_{i=j}^{r} a_{i}\right\}$. In conclusion,
$L B_{2}=\max \left\{C_{1}, \min _{j \in \mathcal{U}} r_{j}\right\}+C_{\max }(J R(\mathcal{U}))$.
Hall (1994) and Kovalyov and Werner (1997) propose polynomial approximation schemes: optimization algorithms that, for any $\epsilon>0$, produce a schedule with makespan at most $(1+\epsilon) C_{\max }^{\star}$, with running time polynomial in $n$ but exponential (and very great) in $1 / \epsilon$. Hall's algorithm runs in time $O(f(\epsilon) \cdot n \log n)$, where the complex function $f(\epsilon)$, given in the paper, is a huge multiplier: for $\epsilon=1 / 2$, as Hall notes, $f(\epsilon)$ is already $5^{60} \cdot 61^{5} \approx 10^{51}$. Kovalyov and Werner propose a very different algorithm whose time performance is $O\left(g(\epsilon) \cdot(n+1)^{6 / \epsilon+2}\right)$. Here, the coefficient $g(\epsilon)$, again specified by the authors, is much smaller (for $\epsilon=1 / 2, g(\epsilon) \approx 10^{9}$ ) but the running time grows more rapidly in $n$. Thus, it performs better for small $n$, but Hall's algorithm will overtake it for large enough $n$. Such algorithms, with their enormous run times, have theoretical but little practical interest.

Heuristics for this problem have been proposed by Potts (1985), each evaluated according to its error bound, $\rho$ (see Sect. 1.7 for definition of error bounds). Potts proposes the following approaches.

Heuristic A Use arbitrary sequence.
Heuristic R Use $/ r_{j}$.
Heuristic J Use Johnson's Rule (JR).

## Heuristic RJ

0 . Initially, set $\mathcal{J}:=\{$ all jobs $\}, R:=\min _{j \in \mathcal{J}} r_{j}$, and $k:=0$.

1. Let $\mathcal{F}=\left\{J_{j}: j \in \mathcal{J}, r_{j} \leq R, a_{j}<b_{j}\right\}$ and $\mathcal{L}=\left\{J_{j}: j \in \mathcal{J}, r_{j} \leq R, a_{j} \geq b_{j}\right\}$. If $\mathcal{F} \neq \phi$, find any $J_{i} \in \mathcal{F}$ with smallest $a_{i}$. If $\mathcal{F}=\phi$, find any $J_{i} \in \mathcal{L}$ with largest $b_{i}$.
2. Set $k:=k+1, R:=R+a_{i}, \mathcal{J}:=\mathcal{J}-\{i\}$, and schedule $J_{i}$ in position $k$.
3. If $\mathcal{J}=\phi$, stop. Else, set $R:=\max \left\{R, \min _{j \in \mathcal{J}} r_{j}\right\}$ and go to Step 1 .

Theorem 2.4 (Potts 1985) The worst case bounds for algorithms A, R, J, and RJ are: $\rho_{A}=3, \rho_{R}=2, \rho_{J}=2$, and $\rho_{R J}=2$, and these bounds are tight.

He also proposes an elaboration of heuristic RJ (details omitted) that has error bound of $5 / 3$.

### 2.2.2 $\mathrm{F} 2 \mid$ (perm), prec| $\mathrm{C}_{\text {max }}$

When precedence constraints between jobs are present in two-machine flow shops, it will be convenient to use the theory of "transfer lags" to find the schedule minimizing the makespan. This topic will be discussed in Chap.
3. Thus, although the present problem does not directly involve lags, our discussion of it will be postponed to Sect. 3.2.5.

### 2.2.3 F2|perm, $s_{i j}, t_{i j} \mid C_{\text {max }}$

Sometimes, a machine must be prepared in some way before it can process a certain job. It may need retooling, adjusting, etc. The time needed for such preparation is called setup time. Similarly, after completing a job, we may need an additional teardown time to clean up, put away tools and materials, etc. We say that the setup (or teardown) of $J_{j}$ is sequence-independent if it depends only on $J_{j}$, not on the job that precedes (or follows) it. We call it separable or anticipatory if the job does not have to be physically present: the setup for a job on $M_{i+1}$ (or teardown on $M_{i-1}$ ) can proceed at the same time that it is being processed on $M_{i}$.

The two-machine flow shop with sequence-independent and separable setups and teardowns is another case that can be handled using the theory of transfer lags. Suppose the processing of $J_{j}$ on $M_{i}$ is immediately preceded by a setup of length $s_{i j}$ and followed by a teardown lasting $t_{i j}$, both of which can proceed on $M_{i}$ in the absence of $J_{j}$. This can be solved by a simple extension of JR, as discussed in Sect. 4.18, where it is shown that, for $F 2 \mid$ perm, $s_{i j}, t_{i j} \mid C_{\max }$,

$$
S^{\star}=J R\left(\underline{a}+\underline{s}_{1}-\underline{s}_{2}, \underline{b}+\underline{t}_{2}-\underline{t}_{1}\right) .
$$

### 2.2.4 Manufacturing Cells

Again, we mention a complex two-machine makespan minimization problem efficiently solved by the techniques of Chap. 3. Modern manufacturing systems may include two Computer Numerically Controlled (CNC) machines with robotic loading and unloading, automated storage and retrieval stations, etc. Jobs are transported on pallets by a circulating conveyor, and require special fixtures and tooling at each machine. For complete coverage, see Sect. 3.2.6.

### 2.2.5 $F 2\left|s_{i j}\right| C_{\max }$ with a Single Server

A problem of practical relevance arises when separable setups are required for each task on each machine, and a single server (a worker, perhaps, or a robot) must move among the machines, setting them up before the corresponding task can be processed. Thus, setups cannot overlap each other, and we must simultaneously schedule machines and server. In shorthand notation, we indicate a shop with a single server by adding $S$ in the first of the three fields. The problem we now consider is $F 2, S\left|s_{i j}\right| C_{\text {max }}$.

The Complexity of $F 2, S\left|s_{i j}\right| C_{\text {max }}$
Theorem 2.5 (Glass et al., 2000) $F 2, S\left|s_{i j}\right| C_{\max }$ is strongly NP-complete.
Proof Outline: We reduce the following strongly NP-complete problem:

## 3-PARTITION

INSTANCE: An integer $V$, and $3 k$ positive integers $v_{i}: i \in \mathcal{T}=\{1,2, \ldots, 3 k\}$, with $\Sigma_{i \in \mathcal{T}} v_{i}=k V$ and $V / 4<v_{i}<V / 2$, for all $i \in \mathcal{T}$.
QUESTION: Can $\mathcal{T}$ be partitioned into $k$ disjoint sets $\mathcal{S}_{1}, \mathcal{S}_{2}, \ldots, \mathcal{S}_{k}$ with $\left|\mathcal{S}_{j}\right|=3$ and $\Sigma_{i \in \mathcal{S}_{j}} v_{i}=V$ for $j=1,2, \ldots, k$ ?
to the decision version of our problem:
$\boldsymbol{F} 2, S\left|s_{i j}\right| C_{\text {max }} \leq \boldsymbol{B}$ ?
INSTANCE: An integer $B$, and $n$ jobs $J_{j}, j=1,2, \ldots, n$, each with parameters $\left\langle s_{i j}, p_{i j}\right\rangle$, to be scheduled in a two-machine flow shop, where $s_{i j}$ is the separable setup time and $p_{i j}$ is the processing time of $J_{j}$ on $M_{i}, i=1,2$, and a single server who can only do one setup at a time.
QUESTION: Does there exist a schedule with $C_{\max } \leq B$ ?
For any instance of 3-Partition, an instance of $F 2, S\left|s_{i j}, p_{i j}\right| C_{\max }$ is defined as follows:

- $B:=4 k V$
- $n:=6 k+1$
- $\left\langle s_{1 j}, p_{1 j} ; s_{2 j}, p_{2 j}\right\rangle:=\left\langle v_{j}, V ; 0,0\right\rangle, j \in \mathcal{T}=\{1,2, \ldots, 3 k\}$
- $<s_{1 j}, p_{1 j} ; s_{2 j}, p_{2 j}>:=<0,0 ; V, V>, j \in \mathcal{U}=\{3 k+1, \ldots, 5 k+2\}$
- $<s_{1 j}, p_{1 j} ; s_{2 j}, p_{2 j}>:=<0,0 ; 0, V>, j \in \mathcal{V}=\{5 k+3, \ldots, 6 k+1\}$

Observe that each job requires processing on just one machine; it has zero processing (hence setup) time on the other. The jobs in $\mathcal{T}$ are processed only on $M_{1}$ (their zero-length tasks can be placed at the end of the schedule); jobs in $\mathcal{U}$ and $\mathcal{V}$ only on $M_{2}$, with null tasks on $M_{1}$ at the start.

Now, suppose an instance of 3-Partition has the answer "yes". The schedule in Fig. 2.3, where each job has been identified only by the set it belongs to and setups are cross-hatched, is therefore feasible and clearly optimal. For the opposite implication, note that, to avoid overlapping setups, each setup on $M_{2}$ must be precisely synchronized with the processing of a job on $M_{1}$. This partitions the time on $M_{1}$ so that sets of three jobs occupy time $4 V$, implying that their setup times must total $V$.


Fig. 2.3 Optimal schedule for NP-complete instance of $F 2, S\left|s_{i j}\right| C_{\text {max }}$

Brucker et al. (2005) strengthened this result by showing that $F 2, S\left|s_{i j}\right| C_{\max }$ is strongly NP-complete even when all setup times are equal $\left(s_{i j}=s\right)$. They also showed that another special case, $F 2, S\left|s_{i j}, p_{i j}=p\right| C_{\max }$, is NPcomplete.
$F 2, S \mid($ perm $),(p m t n), s_{i j}=s, p_{i j}=p \mid \Sigma f_{j}, \max f_{j}$
Brucker et al. (2005) found polynomial solutions for the special cases where the objective functions are very general, but all tasks have identical processing characteristics $\left(s_{i j}=s, p_{i j}=p\right)$, differing only in their cost contributions. The cost of each job is an arbitrary nondecreasing function of its completion time, with the overall objective being to minimize either the total or the maximum of these costs. There are two cases to consider (see Fig. 2.4).

1. $s \geq \boldsymbol{p}$ The server is now always busy, back and forth between the machines, as in Fig. 2.4(a). It is easy to see that preemption is never, and permutation scheduling is always, desirable. The completion time for the job in position $k$ is $C_{k}=2 k s+p$, independent of the schedule. We thus have to assign each job to a position to minimize our objective. For the Total Cost case, this is the well-known Assignment Problem, solved in time $O\left(n^{3}\right)$ by the Hungarian Method (Kuhn, 1955). To minimize the Maximal Cost of such an assignment, we can adapt Lawler's Rule (Lawler, 1973), an $O\left(n^{2}\right)$ algorithm for single machine sequencing: schedule last the job that is least costly there. More formally:

Lawler's Rule for $F 2, S \mid$ (perm), $($ pmtn $), s_{i j}=s, p_{i j}=p, s \geq p \mid \max f_{j}$
0 . Set $\mathcal{R}:=\{1,2, \ldots, n\}$ and $k:=n$.

1. Assign $J_{j}$ to position $k$, where $f_{j}(2 k s+p)=\min _{i \in \mathcal{R}}\left\{f_{i}(2 k s+p)\right\}$.
2. Set $\mathcal{R}:=\mathcal{R}-\{j\}$ and $k:=k-1$. If $k=0$, stop. Otherwise, go to Step 1 .
3. $s \leq \boldsymbol{p}$ Now, an optimal nonpreemptive permutation schedule clearly exists, as in Fig. 2.4(b). Again, the $k^{t h}$ completion time, $C_{k}=k s+(k+1) p$, is sequence-independent, and the same two algorithms can be adapted.


Fig. 2.4 Schedules for $F 2, S \mid($ perm $),($ pmtn $), s_{i j}=s, p_{i j}=p \mid \Sigma f_{j}$, $\max f_{j}$ when

$$
\text { (a) } s \geq p \text {, and (b) } s \leq p
$$

### 2.2.6 Lot Streaming

So far the assumption has been that each job is typically different from any other, even though this does not preclude many jobs having the same char-
acteristics. Suppose now we are producing many identical copies (identical in their processing requirements; they may differ in other respects) of a product. We will consider each copy to be a job. We may prefer to process them in batches due for example to limited material handling equipment, difficulty in tracking individual jobs, etc. Often, the initial quantity of like jobs, called a job lot, is a single customer's order, and is processed in one or a small number of sublots. The jobs of a sublot advance from machine to machine as a group: the last job of a sublot must complete on $M_{1}$ before the first can start on $M_{2}$. This is called lot streaming; in the three-field description we call it lots. Clearly, the makespan is minimized when jobs are processed separately. Therefore, we would like to know the makespan loss resulting from grouping the job lot into sublots.

To define the problem precisely, we make the following additional assumptions:

- Consistent sublots The sublots remain fixed in size as they move from machine to machine. Actually, although varying sublot sizes over many machines can be advantageous, it is never helpful when $m=2$, as Theorem 2.1 below shows.
- No machine idling Each machine must process each lot without interruption. This again is no restriction for $m=2$, since on $M_{1}\left[M_{2}\right]$ there is no loss in assuming that all jobs are processed consecutively starting [ending] at time 0 [ $C_{\max }$ ], but for $m>2$ idle time on $M_{k}, 1<k<m$, could shorten the makespan.
- One type of job We shall confine ourselves to discussing this case; there does not seem to be much research on the multiproduct case.

Potts and Baker (1989) considered problem $F 2 \mid$ (perm), lots $\mid C_{\max }$ when $n$ copies of a single product must be produced, in $v$ sublots. The $n$ jobs all have processing times $\langle a, b\rangle$. We let $x_{i k}$ denote the number of jobs in sublot $i$ on $M_{k}$ (we'll call this sublot $(i, k)$, or simply $(i, k)$ ). They first establish the optimality of consistent sublots for $m=2$. We give a simpler proof.

Theorem 2.1. For $F 2 \mid$ (perm), lots $\mid C_{\max }$, if $x_{i k}$ is the lot size of sublot $i$ on $M_{k}$, then there exists an optimal schedule in which $x_{i 1}=x_{i 2}, i=1, \ldots, v$.

Proof: Suppose we are given an optimal schedule that lacks consistency. For the given partition into sublots on $M_{1},\left\{x_{i 1}: i=1, \ldots, v\right\}$, we will argue that the same partition on $M_{2}$ is also optimal. Consider the smallest index $r, 1 \leq r \leq v$, such that $x_{r 1} \neq x_{r 2}$.

- $x_{r 1}<x_{r 2}$. The extra items in $(r, 2)$ can be moved to the following sublot, delaying them if necessary to make the new $(r+1,2)$ a contiguous set. Clearly, this is feasible and does not affect the makespan.
- $x_{r 1}>x_{r 2}$. The last $x_{r 1}-x_{r 2}$ items of $(r, 1)$ have been detached from $(r, 2)$ and make up the first part of $(r+1,2)$. Since they were part of $(r, 1)$, they are available to be reattached to ( $r, 2$ ), shifting them earlier if necessary. Again, this cannot increase the makespan.

We have now made sublot $r$ consistent. We can proceed iteratively through the schedule, making successive sublots consistent as needed, in the same way, until $x_{i 1}=x_{i 2}$ for all $i$.

Actually, by a time reversal argument, we could keep the partition on $M_{2}$ unchanged and adjust the sublots on $M_{1}$ to conform. Thus, this result holds, not just for the makespan criterion, but for all criteria, even those that are not regular, since every job completion time on the last machine is unaltered. However, given that all jobs are identical, without job-dependent characteristics such as weights or due dates, no other objective seems meaningful.

Now, with consistent sublots, we can denote the $i^{\text {th }}$ sublot simply $x_{i}$. We seek the vector $\left(x_{1}, x_{2}, \ldots, x_{v}\right)$ to minimize makespan for given $v$. Although of course $x_{i}$ must be an integer, we first relax this constraint for simplicity. The fractional sublot sizes can then be rounded to give a good solution, or with a little more care, the optimum.

It is first shown that in the optimal schedule every sublot (except the first) is critical: processing on $M_{1}$ concludes exactly at the time $M_{2}$ becomes free from the previous sublot, so that there is no idle time on $M_{2}$ after the initial wait for the first sublot (see Fig. 2.5). This implies that $a x_{i+1}=b x_{i}$ which, together with $\Sigma_{i=1}^{v} x_{i}=n$, quickly gives:

$$
\begin{equation*}
x_{i}=n q^{i-1}(1-q) /\left(1-q^{v}\right), \quad \text { where } q=b / a \tag{2.2}
\end{equation*}
$$

and the resulting makespan is


Fig. 2.5 Optimal schedule for sample instance of $F 2 \mid$ (perm), $\operatorname{lots} \mid C_{\text {max }}$

For example, suppose a lot of 250 jobs is to be run in $v=5$ sublots on 2 machines, each job requiring times $\langle a, b\rangle=<3,4\rangle$. Then (2.2) gives $\left(x_{1}, \ldots, x_{5}\right)=(25.93,34.57,46.09,61.46,81.95)$, with a makespan of 1077.8, as shown in Fig. 2.5. If every job is scheduled separately $(v=250), C_{\max }=$ $3+4(250)=1003$, while if all jobs are batched in one big lot, $C_{\max }=$ $250(3+4)=1750$. Note how a small number of sublots gives us most of the benefit of independent scheduling.

Since each of the $v$ sublots must actually contain an integer number of jobs, we really want to solve the discrete version of $F 2 \mid($ perm $), l o t s \mid C_{\max }$. The continuous solution gives us a makespan, $C^{\star}$, that is a tight lower bound on the optimal discrete makespan, so it is reasonable to search for the latter starting with the rounded up value of $C^{\star}$. For each trial value $C$ of the
makespan, we check its feasibility and, if no set of $v$ sublots can achieve it, we increment $C$ and try again.

To check the feasibility of $C$, let $X_{i}=\sum_{j=1}^{i} x_{j}$ be the total number of jobs in the first $i$ sublots. Then, given sublots $\left(x_{1}, \ldots, x_{v}\right), C$ must be large enough to satisfy, for $i=1, \ldots, v$ :

$$
C \geq a \sum_{j=1}^{i} x_{i}+b \sum_{j=i}^{v} x_{i}=a X_{i}+b\left(n-X_{i-1}\right), \quad \text { with } X_{0}=0
$$

and equality for at least one value of $i$. For fixed $C$ this gives the recursion

$$
\begin{equation*}
X_{i} \leq\left[C-b\left(n-X_{i-1}\right)\right] / a \tag{2.3}
\end{equation*}
$$

Observe that $X_{i}$ is increasing in $X_{i-1}$ and hence it is beneficial to select the largest possible integral $X_{i}$ value that satisfies (2.3), except of course that we also require $X_{i} \leq n$, with $X_{v}=n$. If $X_{v}<n$, then the trial value $C$ must have been infeasible: too small to accommodate all $n$ jobs. We then must increment the value of $C$ and try again, stopping when at last (2.3) permits $X_{v} \geq n$. Trietsch (1989) showed that the above procedure can be implemented by a polynomial time algorithm that searches over possible values for $C$.

For the earlier example, with $n=250, v=5$ and $\langle a, b\rangle=\langle 3,4\rangle$, we got $C^{\star}=1077.8$. An initial trial value of 1078 gives

$$
\begin{aligned}
& X_{1} \leq \min \{(1078-4 \cdot 250) / 3,250\}=26, \text { so let } X_{1}=26, \\
& X_{2} \leq \min \{(1078-4 \cdot 224) / 3,250\}=60.7, \text { so let } X_{2}=60, \\
& X_{3} \leq \min \{(1078-4 \cdot 190) / 3,250\}=106, \text { so let } X_{3}=106, \\
& X_{4} \leq \min \{(1078-4 \cdot 144) / 3,250\}=167.3, \text { so let } X_{4}=167, \\
& X_{5} \leq \min \{(1078-4 \cdot 83) / 3,250\}=248.7, \text { so } X_{5}<250 .
\end{aligned}
$$

Thus, the trial makespan was too small. Using an increment of 1 , we try $C=$ 1079, getting $\left(X_{1}, X_{2}, X_{3}, X_{4}, X_{5}\right)=(26,61,107,169,250)$ which is clearly optimal.

### 2.3 The Choice of Objective

For the two-machine flow shop, the makespan is the first and most studied objective function. Now that we have introduced it, we pause to consider what other objectives might be useful. Another popular choice is the sum of the completion times of all jobs (or equivalently the average completion time), denoted $\Sigma C_{j}$. This objective is also referred to in the literature as flow time. Assuming that each job is delivered to a customer upon its completion $C_{j}$, the sum of completion times reflects the total manufacturing waiting (or total flow) time experienced by all customers. Hence, flow time is a service objective. Assuming further that all material costs are incurred at time $t=0$, the sum of completion times is a measure of capital utilization or work-inprocess costs which is a popular metric for plant performance.

Gupta and Dudek (1971) argue that a general cost objective is better approximated by $\Sigma C_{j}$ than by $C_{\max }$; a criterion that somehow combines multiple objectives is better still. One way to incorporate two objectives in a hierarchy is to optimize the secondary objective function, $O_{2}$, subject to a constraint on the primary criterion $O_{1}$. The constraint may be a bound (e.g., minimize $\Sigma C_{j}$ while keeping the makespan at ten weeks or less), or we may require $O_{1}$ to be kept at its optimal value if the remaining feasible solution set is not too small. Another way, the composite criterion, includes both functions in the objective, usually as a weighted average.

## $2.4 \mathrm{~F} 2 \mid($ perm $) \mid \Sigma C_{j}$

Minimizing total completion times in the two-machine flow shop, though an apparently simple problem, is strongly NP-complete (Garey et al., 1976) even though there always exists an optimal permutation schedule (Conway et al., 1967). A few special cases, which we present next, have simple polynomial solutions. Otherwise, the bulk of the literature on problem $F 2 \mid($ perm $) \mid \Sigma C_{j}$ is devoted to the development of branch-and-bound algorithms supported by lower bounding schemes. The rest of this section reviews these results.

### 2.4.1 Polynomially Solvable Cases

The following special cases have simple efficient solutions.

- When $a_{i} \leq a_{j} \Rightarrow b_{i} \leq b_{j}$ (Panwalkar and Khan, 1976), we say that the jobs are ordered by processing times (one job is shortest on both machines, etc.). Now, Johnson's Rule is the same as SPT, and is clearly optimal.
- When $a_{j} \geq b_{j}, \quad S^{\star}=\nearrow a_{j}$.
- When $b_{j}=b$ (Van de Velde, 1990), $S^{\star}=\nearrow a_{j}$.
- When $a_{j} \leq b_{j}$, a much more complicated algorithm is given in Hoogeveen and Kawaguchi (1999), with running time $O\left(n^{2} \log n\right)$.


### 2.4.2 Lower Bounds

Lower bounds, hopefully tight and easy to find, are useful to evaluate heuristics when the optimum is incalculable, and in a branch-and-bound algorithm to bound the completions of an intermediate solution at a node. Where possible, we will assume the second case: a partial schedule $\sigma$ is already determined, occupying each $M_{k}$ up to time $C_{k}(\sigma)$ and leaving a set $\mathcal{U}$ of unscheduled jobs. If a simple lower bound is wanted, just set $\sigma=\phi, C_{k}(\sigma)=0$ and $\mathcal{U}=\mathcal{N}$.

Since $C_{1}(\sigma)<C_{2}(\sigma)$, we really only care about the "stagger": $\Delta=C_{2}(\sigma)-$ $C_{1}(\sigma)$. The extra delay $C_{1}(\sigma)$ must be added to the flow times of all remaining jobs, and does not affect their scheduling; it will be ignored hereafter, as will the already incurred cost of the jobs in $\sigma$. Also, for notational simplicity, we
will redefine $n$ as the number of jobs remaining to schedule: $|\mathcal{U}|=n$. Thus, the only effect of a predetermined partial schedule is that $M_{2}$ is unavailable until time $\Delta$.

A single-machine lower bound $L B_{1}$ is presented in Ahmadi and Bagchi (1990) who observe that, since no job can start on $M_{2}$ until it finishes on $M_{1}$, one may consider problem $1\left|r_{j}\right| \Sigma C_{j}$ on $M_{2}$ where $r_{j}=\max \left(a_{j}, \Delta\right)$ for $j=1,2, \ldots, n$. This problem is NP-complete, but the preemptive version $1\left|r_{j}, p m t n\right| \Sigma C_{j}$ is solvable in $O(n \log n)$ time by the Shortest Remaining Processing Time (or SRPT) rule (see Schrage, 1968). Therefore, $L B_{1}$ is produced in $O(n \log n)$ time.

Two related bounds are presented in Ignall and Schrage (1965). For the first, reindex the unscheduled jobs in SPT order of $b_{j}: b_{1} \leq b_{2} \leq \ldots \leq b_{n}$. Then, if $J_{[j]}$ is the job in position $j$, clearly

$$
C_{[j]} \geq \max \left\{\Delta, \min _{j} a_{j}\right\}+\sum_{i=1}^{j} b_{i}, \quad j=1, \ldots, n .
$$

Thus:

$$
L B_{2}=n \max \left\{\Delta, \min _{j} a_{j}\right\}+\sum_{j=1}^{n}(n-j+1) b_{j}, \text { where } b_{1} \leq b_{2} \leq \ldots \leq b_{n} .
$$

By symmetry, the SPT order of $a$-tasks yields lower bound

$$
L B_{3}=\sum_{j=1}^{n}(n-j+1) a_{j}+\sum_{j=1}^{n} b_{j}, \text { where } a_{1} \leq a_{2} \leq \ldots \leq a_{n} .
$$

A lot of work has appeared in the literature on lower bounds based on Lagrangean relaxation schemes. The first such scheme was developed by Van de Velde (1990) and Hoogeveen and Van de Velde (1995) who worked on a formulation that uses "positional processing times". Using [ $j$ ] to index the job in position $j$ of the permutation schedule, they considered the following formulation, where the minimization is over all such schedules:
$\mathbf{P} \quad$ minimize $\sum_{j=1}^{n} C_{[j]}$
subject to $\quad C_{[j]} \geq C_{1[j]}+b_{[j]}, \quad j \in \mathcal{J}$ $C_{[j]} \geq C_{[j-1]}+b_{[j]}, \quad j \in \mathcal{J}$
$C_{1[j]}=\Sigma_{k=1}^{j} a_{[k]}, \quad j \in \mathcal{J}$
where $C_{1[j]}\left[C_{[j]}\right]$ is the completion time of $J_{[j]}$ on $M_{1}\left[M_{2}\right]$, with $C_{[0]}=\Delta$. Let $\underline{\lambda}=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ be nonnegative Langrangean multipliers associated with inequalities (2.5). Then, inequalities

$$
\lambda_{j}\left[C_{1[j]}-C_{[j]}+b_{[j]}\right] \leq 0, \quad j=1, \ldots, n
$$

are summed up and added to the objective function in (2.4) to obtain

$$
\begin{align*}
& \operatorname{minimize}  \tag{2.8}\\
& \text { subject to }(\underline{\lambda})=\sum_{j=1}^{n}\left[\lambda_{j} C_{1[j]}+\left(1-\lambda_{j}\right) C_{[j]}+\lambda_{j} b_{[j]}\right] \\
& \hline(2.7)
\end{align*}
$$

where we minimize over all permutations for fixed $\underline{\lambda}$. We require that $\lambda_{j} \leq 1$ for $j=1, \ldots, n$ so as to avoid unecessarily small values for $L(\underline{\lambda})$.

We first note that, without the constraint (2.5) to link the machines, (2.6) can be written as an equality, yielding $C_{[j]}=\Delta+\Sigma_{k=1}^{j} b_{[k]}$. Substituting this simplified (2.6) and (2.7) into the objective function, we now are left with the unconstrained minimization of

$$
\begin{equation*}
L(\underline{\lambda})=\sum_{j=1}^{n}\left[\lambda_{j} \cdot \sum_{k=1}^{j} a_{[k]}+\left(1-\lambda_{j}\right)\left(\Delta+\sum_{k=1}^{j} b_{[k]}\right)+\lambda_{j} b_{[j]}\right] \tag{2.9}
\end{equation*}
$$

Now consider the restricted version of the above relaxation that we obtain by setting all $\lambda_{j}$ equal to a given constant: $\lambda_{j}=c$ for $j=1, \ldots, n$. Note that the last term in (2.9) and the term involving $\Delta$ become constants and hence do not affect the optimization. Without them, the objective function becomes

$$
L(c)=\sum_{j=1}^{n}\left[\sum_{k=1}^{j}\left(c a_{[k]}+(1-c) b_{[k]}\right)\right]
$$

which is the total flowtime on a single machine with job processing times $p_{j}=c a_{j}+(1-c) b_{j}$. The well-known optimum for $1 \| \Sigma C_{j}$ being SPT, the restricted relaxed problem is solved in $O(n \log n)$ time. Binary search may then be employed to solve the Lagrangean dual problem: $\max _{0 \leq c \leq 1} L(c)$.

Let $L\left(c^{\star}\right)$ be the resulting lower bound. This bound was first developed in Van de Velde (1990) and was later improved by Della Croce et al. (2002) who showed that a sufficient condition for a permutation $\pi$ of the $n$ jobs to solve the problem minimize $L(\underline{\lambda})$ subject to $(2.6),(2.7)$ is:

$$
\begin{equation*}
\lambda_{i} a_{j}+\left(1-\lambda_{i}\right) b_{j} \geq \lambda_{j} a_{i}+\left(1-\lambda_{j}\right) b_{i} \text { when } J_{i} \text { precedes } J_{j} \text { in } \pi . \tag{2.10}
\end{equation*}
$$

Hoogeveen and Van de Velde (1995) showed that value $L\left(c^{\star}\right)$ is generally inferior to the one obtained if we strengthen inequalities (2.5) as follows. First, introduce slack variables $W_{j}$ to get:

$$
\begin{equation*}
C_{[j]}=C_{1[j]}+W_{[j]}+b_{[j]}, \quad j=1, \ldots, n \tag{2.11}
\end{equation*}
$$

Here, $W_{[j]} \geq 0$ is the time $J_{[j]}$ must wait between machines; that is, between completing $a_{[j]}$ and starting $b_{[j]}$. Note that

$$
\begin{equation*}
W_{[j]}=\max \left(0, W_{[j-1]}+b_{[j-1]}-a_{[j]}\right), \quad j=1, \ldots, n, \tag{2.12}
\end{equation*}
$$

as can be verified recursively, starting with $W_{[1]}=\max \left(0, \Delta-a_{[1]}\right)$. It follows that $W_{[j]} \geq \max \left(0, b_{[j-1]}-a_{[j]}\right)$, with $b_{[0]} \equiv \Delta$, so substitution for $W_{[j]}$ in (2.11) gives

$$
\begin{equation*}
C_{[j]} \geq C_{1[j]}+b_{[j]}+\max \left(0, b_{[j-1]}-a_{[j]}\right), \quad j=1, \ldots, n \tag{2.13}
\end{equation*}
$$

which are more binding constraints than (2.5). If we replace (2.5) with (2.13) in $\mathbf{P}$, then the same relaxation yields the problem
minimize $L^{\prime}(\underline{\lambda})=L(\underline{\lambda})+\sum_{j=1}^{n}\left[\lambda_{j} \max \left(0, b_{[j-1]}-a_{[j]}\right)\right]$
subject to (2.6), (2.7).
When $\lambda_{j}=c$ for all $j$, we get $L^{\prime}(c)=L(c)+c \sum_{j=1}^{n} \max \left(0, b_{[j-1]}-a_{[j]}\right)$. The first term is the problem given in (2.8). The second term is equivalent to the 2-machine no-wait flow shop problem of minimizing makespan, as will be discussed in Chap. 6, where it will be shown to be solvable in $O(n \log n)$ time. Thus, this is the time required to compute $L^{\prime}(c)$ for given $c \in[0,1]$. Bisection search over $c \in[0,1]$ provides an approximation for $\max _{0 \leq c \leq 1} L^{\prime}(c)$, and hence a lower bound for the original problem. Let $L^{\prime}\left(c^{\star \star}\right)$ be the resulting value. By construction of the 2 relaxed formulations it is expected that $L^{\prime}\left(c^{\star \star}\right)$ is a better lower bound than $L\left(c^{\star}\right)$.

In a related article, Hoogeveen et al. (2006) further characterized relaxations for model $\mathbf{P}$. They studied relaxations based on slack variables $I_{j}$ for inequalities (2.6), as in

$$
\begin{equation*}
C_{[j]}=C_{[j-1]}+b_{[j]}+I_{[j]}, \quad j \in \mathcal{J} . \tag{2.15}
\end{equation*}
$$

Here, $I_{[j]} \geq 0$ reflects the idle time on $M_{2}$ after finishing the $(j-1)^{s t}$ task, with $I_{[1]}=\max \left(0, a_{[1]}-\Delta\right)$. Note that when $I_{[j]}>0, W_{[j]}=0$, and vice versa. Working recursively using $C_{[1]}=\max \left(a_{[1]}, \Delta\right)+b_{[1]}$, we see that

$$
\begin{equation*}
C_{[j]}=\max \left(a_{[1]}, \Delta\right)+\sum_{i=1}^{j} b_{[i]}+\sum_{i=2}^{j} I_{[i]}, \quad j \in \mathcal{J} . \tag{2.16}
\end{equation*}
$$

Using (2.16) in (2.4) - (2.7), the following relaxation is obtained:
$\mathbf{L P}_{1} \quad$ minimize $n \cdot \max \left(a_{[1]}, \Delta\right)+\sum_{j=1}^{n}\left[\sum_{i=1}^{j} b_{[i]}+\sum_{i=2}^{j} I_{[i]}\right]$

$$
\text { subject to } \quad \sum_{i=1}^{j-1} b_{[i]}+\sum_{i=2}^{j} I_{[i]}-\sum_{i=2}^{j} a_{[i]} \geq 0, \quad j=2, \ldots, n,
$$

$$
I_{[1]}=\max \left(0, a_{[1]}-\Delta\right), \quad I_{[j]} \geq 0, \quad j=2, \ldots, n .
$$

Working similarly, one can replace (2.5) by (2.13) to obtain another relaxation $\mathbf{L} \mathbf{P}_{\mathbf{2}}$ (details omitted). Hoogeveen et al. (2006) showed that $\mathbf{L} \mathbf{P}_{\mathbf{1}}$ and $\mathbf{L} \mathbf{P}_{\mathbf{2}}$ are equivalent formulations with $O\left(n^{2}\right)$ variables and $O(n)$ constraints and yield the same objective function value. They, too, investigated Lagrangean relaxations obtained by relaxing inequalities (2.5) so as to obtain the formulation minimize $L(\underline{\lambda})$ subject to $(2.6),(2.7), \underline{\lambda} \geq 0$. Similarly, they relax inequalities (2.6) using multipliers $\mu_{1}, \ldots, \mu_{n}$ and obtained the formulation minimize $L^{\prime}(\underline{\mu})$ subject to (2.5), (2.7), $\underline{\mu} \geq 0$. The authors proved
 maximize $\underline{\mu} \geq 0$ 数 $(\underline{\mu})$ is equivalent to $\mathbf{L} \mathbf{P}_{\mathbf{2}}$. They also performed experiments showing that, on average, the deviation of the lower bound produced by $\mathbf{L} \mathbf{P}_{\mathbf{1}}$ from the optimal is under $1 \%$ for instances with 35 or 40 jobs.

A very different but equally creative formulation for $F 2 \| \Sigma C_{j}$ is developed in Akkan and Karabati (2004) for jobs with integer processing times. As before, $W_{k} \geq 0$ will denote the waiting time or delay of $J_{k}$ between $M_{1}$ and
$M_{2}$. We first rewrite (2.11):

$$
\begin{equation*}
C_{[j]}=\sum_{i=1}^{j} a_{[i]}+W_{[j]}+b_{[j]}, \quad j=1, \ldots, n \tag{2.17}
\end{equation*}
$$

Adding over all positions, and regrouping terms, our objective becomes to minimize:

$$
\sum_{j=1}^{n} C_{[j]}=\sum_{j=1}^{n}\left[(n-j+1) a_{[j]}+W_{[j]}+b_{[j]}\right]
$$

We can now assign the cost

$$
\begin{equation*}
C_{[j]}^{\prime}=(n-j+1) a_{[j]}+W_{[j]}+b_{[j]} \tag{2.18}
\end{equation*}
$$

to the job in position $j$. This cost depends only on the parameters of $J_{[j]}$, and on the time that job must wait for $M_{2}$.

In fact, we can gain further insight by introducing the variable $\Delta_{j}$, the time from the finish of $J_{j}$ on $M_{1}$ until $J_{j}$ finishes on $M_{2}$ :

$$
\Delta_{[j]}=C_{[j]}-C_{1[j]}=W_{[j]}+b_{[j]}, \quad j=1, \ldots, n
$$

$\Delta_{[j]}$ is the difference in time, or stagger, between the occupancies of the two machines as seen by the next job to be scheduled, $J_{[j+1]}$. It determines whether $J_{[j+1]}$ will have to wait (that is, $W_{[j+1]}>0$, which results if $a_{[j+1]}<$ $\Delta_{[j]}$ ) or not. Using (2.12):

$$
\begin{equation*}
\Delta_{[j]}=\max \left\{0, \Delta_{[j-1]}-a_{[j]}\right\}+b_{[j]}, \quad j=1, \ldots, n \tag{2.19}
\end{equation*}
$$

with $\Delta_{[0]}=\Delta$ (assuming a partial schedule $\sigma$ already exists producing an initial stagger $\Delta$; otherwise $\Delta_{[0]}=0$ ).

The problem can now be formulated as a transshipment-type branching network (for an introduction to transshipment problems, see for example Winston (2003)) in which schedules are built, left to right, one job at a time. Nodes represent partial schedules, with emanating arcs for each job that can be scheduled next. All nodes with exactly $j$ jobs scheduled make up stage $j$. When branching from a node at stage $j$ where the last job scheduled was $J_{[j]}$, we need only know the stagger $\Delta_{[j]}$. If the next job is to be $J_{h}$ (i.e., $[j+1]=h$ ), then the new stagger is, from (2.19), $\Delta_{h}=\max \left\{0, \Delta_{[j]}-a_{h}\right\}+b_{h}$, and the cost contribution is, from (2.18), $C_{h}^{\prime}=(n-j+1) a_{h}+\Delta_{h}$.

Note that, at any stage, a given stagger value may be attained by many jobs, and the number of different staggers is usually not very large. Therefore, to keep the number of nodes from growing to $n$ ! (as when we generate all job sequences), the authors suggest that at each stage there be one node for every distinct stagger value. This, as we have just seen, is the only datum needed to extend the network to the next stage. On the down side, this means that we cannot know what sequence of $j$ jobs a level- $j$ node represents: several partial schedules may converge on this node. Thus, each node must have $n$ branches,
and therefore will be generated with the same job(s) appearing more than once.

Example 2.1: The following small example illustrates the network (see Akkan and Karabati, 2004). It assumes no initial partial schedule, hence $\Delta=\Delta_{0}=0$. Let $n=3$ with $\left\langle a_{j}, b_{j}\right\rangle=\langle 10,7\rangle,\langle 7,3\rangle$ and $\langle 1,3\rangle$, for $j=1,2$, and 3 . Since any job in position 1 has $\Delta_{[1]}=b_{[1]}, \Delta_{[1]}$ will take as many values as the distinct $b$-tasks, i.e., 3 or 7 , as in Figure 2.6. Observe that


Fig. 2.6
The transshipment network
two arcs end in node 1 because there are two jobs with $b_{j}=3$. According to (2.18), the cost of the arcs from node 0 to node 1 are $3 a_{2}+b_{2}=24$ and $3 a_{3}+b_{3}=6$ for $J_{2}$ and $J_{3}$ respectively. The job sequence $(1,3,2)$ has cost $3 a_{1}+7+2 a_{3}+9+a_{2}+5=60$. Moreover, $C_{1}=17, C_{2}=23, C_{3}=20$ and $\Sigma_{j} C_{j}=60$.

Evidently, a mincost unit flow from the source node to the sink that passes through arcs that correspond to each job exactly once, provides an optimal solution. By construction, the number of nodes in the network is of order $O\left(\max _{j} b_{j} \cdot n^{2}\right)$, hence the number of $\operatorname{arcs}$ is $O\left(\max _{j} b_{j} \cdot n^{3}\right)$ because at most $n$ arcs emanate from each node. Relaxing this integrality constraint yields a tight linear programming formulation.

### 2.4.3 Optimal Algorithms and Heuristics

Nearly every article in the literature where a lower bound is developed, contains an associated branch-and-bound algorithm. Using the relaxation of the transshipment network-based lower bound described earlier, at the root node of their branch-and-bound algorithm, Akkan and Karabati (2004) were able
to solve problems with as many as 60 (45) jobs when processing times are drawn uniformly from $[1,10]([1,100])$. However, the size of the network renders this lower bound too expensive for repeated use within a branch-andbound framework.

Hoogeveen and Van de Velde (1995) experimented with problems having up to 30 jobs and processing times drawn uniformly from [1, 10]. For each combination of parameters they randomly generated 40 problems and found that in all combinations of parameters the median value of the relative per cent error of $L^{\prime}(\bar{c})$ from the optimal flowtime value is slightly better than the corresponding per cent error for $L\left(c^{\star}\right)$.

Della Croce et al. (1996) experimented with various branch-and-bound implementations based on combinations of the lower bounds $\mathrm{LB}_{1}, \mathrm{LB}_{2}, \mathrm{LB}_{3}$, $L\left(c^{\star}\right)$ and $L^{\prime}(\bar{c})$ and were able to consistently solve problems with up to $n=25$ jobs. Also, they developed a heuristic which was found to deviate from the best amongst the 5 lower bounds by no more than $2.62 \%$ on average, on randomly generated problems of up to $n=100$ jobs. By exploiting the sufficient condition (2.10), Della Croce et al. (2002) later improved upon the latter branch-and-bound algorithm and were able to solve problems with up to $45(30)$ jobs for processing times drawn uniformly from $[1,10]([1,100])$.

The branch-and-bound algorithm of Della Croce et al. (1996) runs faster when using the heuristic solution obtained in T'kindt et al. (2002) at the start node which results to about $25 \%$ reduction in the search tree. The algorithm presented in T'kindt et al. (2002) is referred to as SACO and is a metaheuristic based on ant colony optimization (ACO). Metaheuristics are algorithms that make extensive use of neighborhood search and are equipped with mechanisms that allow the search to deviate from local optima, temporarily accepting inferior solutions so as to direct the search to other (hopefully more promising) areas of the search space. Metaheuristic algorithms are mimetic in nature. ACO is a type of metaheuristic that emulates the manner in which ants locate food sources - optimal solutions in our case. Specifically, a pheromone matrix $\left\{\tau_{i j}\right\}$ records the probability of scheduling $J_{j}$ at position $i$ in a good schedule for the $\Sigma C_{j}$ criterion. Hence, value $\tau_{i j}$ represents the frequency by which a pheromone trail is used by ants as a means of sharing information across the colony. $J_{i}$ is scheduled on position $j$ with probability $p_{i j}$ if it is found in the same position in previously identified "good" schedules, or $1-p_{i j}$ to create new promising schedules. Probability $p_{i j}$ is usually optimized experimentally by trial and error.

Using an improved pheromone updating rule, Lin et al. (2008) developed an alternative heuristic referred to as $A C O$ and tested it on problems $F 2 \mid($ perm $) \mid \Sigma C_{j}$ and $F 2 \mid($ perm $) \mid\left(\alpha \Sigma C_{j}+\beta C_{\max }\right)$. For the former problem, the authors experimented with problems with up to $n=500$ jobs, running 50 iterations using 10 ants (i.e., simultaneous trials). It is found that, on average, ACO yields slightly better solutions than SACO. Rajendran and Ziegler (2004) proposed two other ant colony optimization metaheuristics referred to as $M$-MMAS and $P A C O$. The former makes use of the idea of a max-min ant
system; the latter incorporates relative distances between 2 scheduling positions for the same job. Algorithms M-MMAS and PACO outperform ACO and SACO in terms of solution quality but require more CPU time.

A worst case analysis has been conducted for the simple heuristic: use the permutation schedule with jobs sequenced in nondecreasing order of their total processing time, $a_{j}+b_{j}$.

Theorem 2.6 (Hoogeveen and Kawaguchi, 1999)
For $F 2 \mid($ perm $) \mid \Sigma C_{j}$, let
(a) $F_{a+b}=$ the total flow time of schedule $\nearrow\left(a_{j}+b_{j}\right)$.
(b) $F^{\star}=$ the total flow time of the optimal schedule.

Then

$$
F_{a+b} / F^{\star} \leq 2 y /(x+y), \quad \text { where } x=\min _{j}\left\{a_{j}, b_{j}\right\} \text { and } y=\max _{j}\left\{a_{j}, b_{j}\right\},
$$

and this bound is tight.

### 2.4.4 $\operatorname{F2} \mid($ perm $) \mid \Sigma C_{j}$ with Setups

Consider the mean flow time objective when there are separable and sequenceindependent setup times on $M_{2}$ (there may also be setups on $M_{1}$, but with no preceding stage there is never the need to consider setup and processing separately, so their times can be combined). The proof of Theorem 1.1 requires only minor modification to accommodate setup times, so we can continue to confine ourselves to permutation schedules. Let $s_{j}$ be the time required to set up $M_{2}$ for $J_{j}$, and denote the problem as $F 2 \mid($ perm $), s_{j} \mid \Sigma C_{j}$.

Allahverdi (2000), besides proposing and comparing three heuristics, proposes a branch-and-bound algorithm. As usual with sequencing problems, schedules are built by adding jobs, one at a time, to a partial schedule. We may build forwards in time, adding jobs at the end of an initial partial schedule, as we assume here, or we may start with the final job and grow the schedule backwards. We shall adopt the notation:

- $\sigma$ : variously, a node or the initial partial sequence at that node,
- $\mathcal{S}$ : the set of scheduled jobs that make up $\sigma$, with cardinality $s$.
- $\mathcal{U}$ : the set of jobs that remain unscheduled, with cardinality $u=n-s$.

The author develops dominance relations and lower bounds to be used in the search algorithm. We now summarize his results.

## Dominance Properties

The following results state conditions under which jobs can be partially ordered, thus eliminating some job sequences from the search. They are given by Allaverdi (2000) unless otherwise referenced. The proofs are by job interchange; details omitted.

Theorem 2.7 (Bagga and Khurana, 1986) For $F 2 \mid($ perm $), s_{j} \mid \Sigma C_{j}$,

$$
\left.\begin{array}{l}
a_{j} \leq a_{k}, \quad b_{j} \leq b_{k} \\
\min \left\{a_{j}-s_{j}, b_{k}\right\} \leq \min \left\{a_{k}-s_{k}, b_{j}\right\} \\
s_{j}+b_{j} \leq s_{k}+b_{k}
\end{array}\right\} \Rightarrow j \stackrel{\text { global }}{\sim} k .
$$

Theorem 2.8 For $F 2 \mid$ (perm), $s_{j} \mid \Sigma C_{j}$,

$$
\left.\begin{array}{l}
b_{k} \leq b_{j} \\
a_{j}-s_{j} \leq a_{k}-s_{k} \\
s_{j}+b_{j} \leq s_{k}+b_{k}
\end{array}\right\} \Rightarrow j \stackrel{\text { global }}{\sim} k .
$$

Theorem 2.9 For $F 2 \mid($ perm $), s_{j} \mid \Sigma C_{j}$,

$$
\left.\begin{array}{l}
a_{j} \leq s_{j}+b_{j} \\
a_{j}-s_{j} \leq a_{k}-s_{k} \\
s_{j}+b_{j} \leq s_{k}+b_{k}
\end{array}\right\} \Rightarrow j \leadsto k
$$

## Lower Bounds

To bound total completion time for all schedules beginning with partial schedule $\sigma$, Bagga and Khurana (1986) propose lower bounds separately for each job. First, for jobs in $\mathcal{S}$, the exact completion time can be used (we note that the authors do not do so for the first lower bound, instead using the same lower bound for these jobs as for the unscheduled ones, and thus producing a somewhat weaker overall bound). It is probably easiest to compute these completion times recursively. Thus, for $j \leq s$ :

$$
\begin{equation*}
C_{[j]}=\max \left\{\sum_{i=1}^{j} a_{[i]}, C_{[j-1]}+s_{[j]}\right\}+b_{[j]}, \text { with } C_{[0]}=0, \tag{2.20}
\end{equation*}
$$

where $[j]$ is the index of the job in position $j$.

- First bound, $L B_{1}$

Consider the job in position $j>s$. Its completion time must be at least the time to process the first $j$ jobs on $M_{1}$, plus $b_{[j]}$. The first $s$ jobs have already been scheduled; their time on $M_{1}$ is, of course, $\Sigma_{i=1}^{s} a_{[i]}=\Sigma_{i \in \mathcal{S}} a_{i}$. Since we do not know which jobs occupy the next $j-s$ positions, we use the smallest of the remaining $a$-tasks to give a lower bound. Thus, for $j>s$ :

$$
\begin{equation*}
C_{[j]} \geq \sum_{i \in \mathcal{S}} a_{i}+\sum_{i=1}^{j-s} a^{[i]}+b_{[j]}, \tag{2.21}
\end{equation*}
$$

where $a^{[i]}$ is the $i^{t h}$ smallest $a$-task in $\mathcal{U}$. Note that we do not know which job contributes $b_{[j]}$, but this will not matter since we add them all up to bound the total completion time, getting:

$$
\sum_{j=1}^{n} C_{[j]} \geq L B_{1} \equiv \sum_{j=1}^{s} C_{[j]}+u \sum_{i \in \mathcal{S}} a_{i}+\sum_{i=1}^{u}(u-i+1) a^{[i]}+\sum_{j \in \mathcal{U}} b_{j}
$$

- Second bound, $L B_{2}$

Applying a similar logic to $M_{2}$, we have for $j>s$ :

$$
\begin{equation*}
C_{[j]} \geq C_{[s]}+\sum_{i=1}^{j-s}\left(s^{[i]}+b^{[i]}\right) \tag{2.22}
\end{equation*}
$$

where $s^{[i]}+b^{[i]}$ is the $i^{\text {th }}$ smallest value of $s_{j}+b_{j}$ among jobs in $\mathcal{U}$. Thus:

$$
\sum_{j=1}^{n} C_{[j]} \geq L B_{2} \equiv \sum_{j=1}^{s} C_{[j]}+u C_{[s]}+\sum_{i=1}^{u}(u-i+1)\left(s^{[i]}+b^{[i]}\right)
$$

We can now use for our lower bound $L B=\max \left\{L B_{1}, L B_{2}\right\}$.

## Heuristics and Computational Results

Three heuristics are proposed for $F 2 \mid($ perm $), s_{j} \mid \Sigma C_{j}$. Heuristic $\mathrm{H}_{1}$ simply schedules the jobs sequentially, starting at time zero, each time choosing from the unscheduled set the job that completes earliest. Heuristics $\mathrm{H}_{2}\left[\mathrm{H}_{3}\right]$ are similar, scheduling at each iteration the ordered pair [triple] of jobs whose total completion time is minimal.

Clearly, $\mathrm{H}_{2}$ will require more ( $\mathrm{H}_{3}$, much more) time at each step, but even for the largest instances tested (ten 35 -job replicates), where $\mathrm{H}_{3}$ took about 20 times as long as $\mathrm{H}_{1}$, the CPU time was negligible. For such simple greedy algorithms, all three heuristics performed remarkably well, with the more complex giving better results. Thus, $\mathrm{H}_{3}$ gave results with average error (deviation of the heuristic value from the optimum, divided by the optimum) an order of magnitude smaller than those of $\mathrm{H}_{1}$. For the 35 -job instances, the average error for $\mathrm{H}_{1}, \mathrm{H}_{2}$ and $\mathrm{H}_{3}$ was $0.04,0.008$ and 0.003 , respectively.

Using the above lower bounds and dominance relations, the branch-andbound optimization required about 30 minutes to sequence 35 jobs, using C-programming on a Sun Sparc 10. Rather slow, but still probably useful. The relative usefulness of the three dominance relations was reported in terms of their average frequency, i.e. the average number of times they ordered a pair of jobs by preference, in any one problem instance. For Theorems 2.7, 2.8 , and 2.9 , for each 35 -job instance, the average frequencies were 110,30 , and 140 , respectively. Theorem 2.7 was the most useful, giving global ordering information fairly often. Theorem 2.9 also provided quite a few priority orderings, but of the less useful local variety. Theorem 2.8, while global, was not often applicable.

## $2.5 \operatorname{F2}|(\operatorname{perm})|\left(\Sigma C_{j} \mid \min C_{\max }\right)$

This problem seems to have been first considered in Rajendran (1992) where a branch-and-bound algorithm is presented that uses lower bounds $L B_{2}, L B_{3}$ of Ignall and Schrage (1965). Problems of size up to $n=10$ jobs are solved. For larger problems, the author presents 2 heuristics that revise the Johnson schedule by interchanging adjacent jobs as long as the makespan doesn't increase and $\Sigma C_{j}$ decreases. The order in which these interchanges are made is of the essence, and they are based on a couple of preference relationships. The best of the 2 heuristics has average deviation from $L B_{2}, L B_{3}$ ranging from $5 \%$ to $14 \%$ for problems up to $n=24$ jobs.

Using the heuristics in Rajendran (1992) as benchmark, Neppalli et al. (1996) developed a genetic algorithm that, with appropriate choice of pa-
rameters, performs better by an average of $3 \%$ for problems with $n=10$, and $10 \%$ for problems with $n=80$ jobs. The improved performance of the genetic algorithm over the heuristics in Rajendran (1992) exhibits a linear increasing trend as $n$ increases.

Gupta et al. (2001) and T'Kindt et al. (2003) presented heuristics that served as benchmark in T'Kindt et al. (2002) who developed the metaheuristic SACO mentioned earlier for $F 2 \mid($ perm $) \mid \Sigma C_{j}$. To ensure that the search is over schedules that attain minimal makespan, let $\sigma(1), \sigma(2), \ldots, \sigma(i-1)$ be a partial schedule for $F 2|(\operatorname{perm})| \Sigma C_{j}$. Then, $\sigma(i)=j$ is chosen with probability $p_{0}$ to intensify amongst the "existing trails", or $1-p_{0}$ to diversify. If $\sigma(1), \sigma(2), \ldots, \sigma(i-1), j$ can be extended to an optimal schedule with respect to the makespan objective, then the choice $\sigma(i)=j$ is admissible and value $\tau_{i j}$ is updated accordingly. T'Kindt et al. (2002) find that the best trade-off between CPU time and solution quality is obtained when SACO uses 20 ants and 100 iterations. They experiment with problems with up to $n=200$ jobs and find that SACO yields solutions that slightly improve upon the performance of the heuristics presented in Gupta et al. (2001) and T'Kindt et al. (2003). In problems with less than $n=25$ jobs, it is shown that SACO is nearly optimal.

### 2.6 F2|(perm) $\mid\left(\alpha \Sigma C_{j}+\beta C_{\text {max }}\right)$

Nagar et al. (1995) found that, when $b_{j} \geq a_{j}$ for all jobs, the greedy algorithm that schedules next a job that minimizes the marginal increase in the objective function value, is near optimal. For this special case, the authors develop a branch-and-bound algorithm that consistently solves problems with $n=200$ jobs within a few seconds. However, when the processing times are randomly generated, results on problems with up to $n=14$ jobs are reported. Their branch-and-bound algorithm builds schedules from the start. Let $V$ be the value of the partial schedule at a node, and $\mathcal{U}=\{$ unscheduled jobs $\}$. Then, disregarding the $a$-tasks and indexing the $b$-tasks in SPT order, the quantity

$$
V+\alpha \sum_{j \in \mathcal{U}}(n-j+1) b_{j}+\beta \sum_{j \in \mathcal{U}} b_{j}
$$

gives a lower bound for the node.
Lin et al. (2008) tested the metaheuristics ACO, SACO, M-MMAS and PACO on $F 2 \mid($ perm $) \mid\left(\alpha \Sigma C_{j}+\beta C_{\max }\right)$ and found that PACO is dominant in terms of solution quality while on average, ACO requires the least time.

### 2.7 F2|(perm) $\mid f_{\text {max }}$

Recall that the objective $f_{\max }$ seeks to minimize the largest job cost, where each $J_{j}$ incurs a cost $f_{j}(t)$ when completed at time $t$. By Theorem 1 of Chap. 1, we can limit our search to permutation schedules. To begin with,
consider the problem without precedence constraints. It is shown in Lenstra et al. (1977) that even $F 2 \mid($ perm $) \mid L_{\text {max }}$ (the special case with $\left.f_{j}(t)=t-d_{j}\right)$ is NP-complete, so no efficient algorithm is to be expected for the general objective. Townsend (1977) gives a branch-and-bound algorithm, building schedules backwards. Since Johnson's Rule minimizes the completion time, hence the cost, of the last job scheduled, it is a reasonable starting point. We then consider modifying it using the single-machine result of Lawler (1973): the sequence that minimizes the maximal cost of any job puts last the job that is least costly there. This may lead us to place a different job in the final position. We leave the rest in JR-sequence, thus minimizing the cost of the last job. We now repeat this, for the penultimate job. We could continue this way through all the positions, except that, when we place a job in any position, we generally change the completion times of all the following jobs which have already been scheduled. This may force us to reconsider earlier decisions, hence the need to search. Consult the paper for further algorithmic details; no computational results are given.

Finally, as usual when solving a scheduling problem by implicit search, adding precedence constraints limits the number of feasible schedules, and so only makes the problem easier to solve.

### 2.7.1 $\mathrm{F} 2 \mid($ perm $), s_{j} \mid L_{\text {max }}$ or $T_{\text {max }}$

As noted above, minimizing $L_{\max }$ (or $T_{\max }$, which is equivalent) in a twomachine flow shop is already NP-complete. Allahverdi and Al-Anzi (2002) propose a branch-and-bound algorithm. Each $J_{j}$ now has a separable and sequence-independent setup time $s_{j}$ on $M_{2}$ (setups on $M_{1}$ may be included in task times without loss of generality), and a due date $d_{j}$, as well as processing times $\left.<a_{j}, b_{j}\right\rangle$. The presentation is similar to the one used to minimize $\Sigma C_{j}$ by Allahverdi (2000), as discussed in Sect. 2.4.4. Please review that material, as our approach and notation will be the same.

## Dominance Properties

The following dominance orderings are due to Allahverdi and Al- Anzi (2002) unless otherwise credited.

Theorem 2.10 (Dileepan and Sen, 1991) For F2 $\mid($ perm $), s_{j} \mid L_{\max }$,

$$
\left.\begin{array}{l}
d_{j} \leq d_{k}, \\
\min \left\{a_{j}-s_{j}, b_{k}\right\} \leq \min \left\{a_{k}-s_{k}, b_{j}\right\}
\end{array}\right\} \Rightarrow j \leadsto k .
$$

Theorem 2.11 For $F 2 \mid$ (perm), $s_{j} \mid L_{\text {max }}$,
$\left.\begin{array}{l}b_{k} \leq b_{j}, d_{j} \leq d_{k} \\ a_{j}-s_{j} \leq a_{k}-s_{k} \\ s_{j}+b_{j}-d_{j} \leq s_{k}+b_{k}-d_{k}\end{array}\right\} \Rightarrow j \stackrel{\text { global }}{\sim} k$.

Theorem 2.12 For $F 2 \mid$ (perm), $s_{j} \mid L_{\max }$,

$$
\left.\begin{array}{l}
s_{k}+b_{k} \leq d_{k}-d_{j} \\
b_{j} \geq a_{k}-s_{k} \geq \min \left\{b_{k}, a_{j}-s_{j}\right\}
\end{array}\right\} \Rightarrow j \leadsto k
$$

Theorem 2.13 For $F 2 \mid$ (perm), $s_{j} \mid L_{\max }$, $\left.\begin{array}{l}a_{k} \geq s_{k}+b_{k} \geq d_{k}-d_{j} \geq 0 \\ b_{j} \geq \min \left\{a_{k}-s_{k}, b_{k}\right\}\end{array}\right\} \Rightarrow j \leadsto k$.

## Lower Bound

At an arbitrary node of the search tree, to bound maximal lateness for all completions of the $s$-job initial partial schedule $\sigma$, we first bound $C_{[j]}$, where [ $j$ ] indexes the job in position $j$. As before, for jobs already scheduled ( $j \leq s$ ), $C_{[j]}$ is fixed and can be computed recursively using (2.20). The two bounds for $C_{[j]}, j>s$ are also the same as those given in (2.21) and (2.22), except that in (2.21) we do not know which job contributes $b_{[j]}$ so must replace it with $b_{\text {min }}=\min _{j \in \mathcal{U}} b_{j}$. Using the greater of these as the overall bound on completion time, we get as a lower bound, $L B_{[j]}$, on the lateness $L_{[j]}$ of the job in position $j$, for $j>s$ :

$$
L B_{[j]}=\max \left\{\sum_{i \in \mathcal{S}} a_{i}+\sum_{i=1}^{j-s} a^{[i]}+b_{\min }, C_{[s]}+\sum_{i=1}^{j-s}\left(s^{[i]}+b^{[i]}\right)\right\}-d_{\max },
$$

where $d_{\max }=\max _{j \in \mathcal{U}} d_{j}$. Since the lateness of an already-scheduled job is known, we get finally:

$$
\begin{equation*}
L_{\max } \geq \max \left\{\max _{j \in \mathcal{S}} L_{j}, \max _{j=s+1}^{n} L B_{[j]}\right\} \tag{2.23}
\end{equation*}
$$

## Heuristics

Dileepan and Sen (1991) propose two heuristics: (1) Heuristic $\mathrm{H}_{\text {EDD }}$, in which the EDD schedule is improved by neighborhood search using adjacent pair job interchanges; and (2) $\mathrm{H}_{\mathrm{JR}}$, which uses the $J R(\underline{a-s}, \underline{b})$ schedule. Starting with each of these schedules, Allahverdi and Al-Anzi (2002) give two algorithms for further improvement. We outline one of them as an example. In it, a sequence of initial partial schedules $\sigma_{i}$ is developed, adding one job at a time, so that $\sigma_{i}$ contains $i$ jobs, $i=1, \ldots, n$.

## Insertion Algorithm

1. Let $\left(\pi_{1}, \pi_{2}, \ldots, \pi_{n}\right)$ be the starting schedule, from either $\mathrm{H}_{\text {EDD }}$ or $\mathrm{H}_{\mathrm{JR}}$.
2. Set $\sigma_{1}:=\left(\pi_{1}\right)$, and $i:=2$.
3. Create $i$ candidate sequences for $\sigma_{i}$ by inserting $\pi_{i}$ into each of the $i$ possible positions in $\sigma_{i-1}$.
4. For each candidate, compute the lower bound on $L_{\max }$ as given by (2.23). Let $\sigma_{i}$ be the one with the smallest bound, hence hopefully smallest $L_{\text {max }}$.
5. Increment $i$. If $i=n+1$, then stop, with final schedule $S=\sigma_{n}$; else go to Step 3.

This algorithm is applied to $\mathrm{H}_{E D D}$ and $\mathrm{H}_{J R}$, and followed by a final adjacent-pair-interchange procedure, to give two heuristic procedures; call them EDDINSERT and JR-INSERT. Another improvement process, called the Merge Algorithm (details omitted) gives two more, EDD-MERGE and JR-MERGE. All have time complexity $O\left(n^{3}\right)$.

## Computational Results

Limited results are given for the branch-and-bound algorithm, with and without the enhancement of dominance properties. The largest instances solved to optimality have twelve jobs, so presumably running times become prohibitive at that point. It is clear that preference orderings are very useful. For nine-job instances, the time required to solve the problem with the help of dominance properties is one tenth the time needed without them; and results for smaller instances show that this superiority is growing rapidly with problem size. Running times are not given directly, but we assume them to be proportional to the number of nodes searched, which is reported. Regrettably, no comparison is made between dominance properties.

Heuristic performance is reported for several problem sizes; we will note results only for the largest instances solved to optimality (12 jobs) and the largest for which heuristic solutions are found ( 75 jobs). When the optimal solution is available, performance is measured by average error (deviation of the heuristic value from the optimum, as a fraction of the optimum), and by the percentage of instances solved to optimality. For the large problems where no optimal solution can be found, we replace the optimum with the best value found by any of the heuristics. The results are:

|  | 12 jobs |  | 75 jobs |  |
| :---: | :---: | :---: | :---: | :---: |
| Heuristic | Avg Err | $\%$ Opt | Avg Err | $\%$ Best |
| EDD-INSERT | 0.24 | 81.2 | 0.03 | 77.6 |
| EDD-MERGE | 0.14 | 85.9 | 0.02 | 86.5 |
| JR-INSERT | 0.14 | 88.2 | 0.01 | 87.8 |
| JR-MERGE | 0.10 | 89.7 | 0.01 | 92.7 |

$H_{J R}$ gives a clearly better initial schedule than $H_{\text {EDD }}$, and JR-MERGE gives a somewhat better final result than JR-INSERT. It should be remarked that, though MERGE has the same time complexity as INSERT, $O\left(n^{3}\right)$, it is considerably more complicated and so will take longer to run. Also, keep in mind that the results for the 75 -job cases are deceptively impressive, since comparison is to the best solution found, not the true optimum.

## 2.8 $\boldsymbol{F} 2 \mid($ perm $) \mid \Sigma T_{j}$

We now consider the case in which each $J_{j}$ has a due date $d_{j}$, and we wish to minimize total tardiness. Koulamas (1994) presents a survey of the literature on this objective. As always, permutation schedules constitute a dominant
set. Despite this, the problem is still among the hardest problems: even the special case with all due dates set to zero becomes simply $F 2 \mid($ perm $) \mid \Sigma C_{j}$, earlier shown to be strongly NP-complete.

### 2.8.1 Dominance properties

Sen et al. (1989) found the following priority relation for adjacent jobs:
Theorem 2.14 For $F 2 \mid($ perm $) \mid \Sigma T_{j}$ :

$$
j \leadsto k \Longleftrightarrow\left\{\begin{array}{l}
d_{j} \leq d_{k} \\
b_{j}-d_{j} \leq b_{k}-d_{k} \\
a_{j} \leq \min \left\{b_{j}, a_{k}\right\}
\end{array}\right.
$$

Kim (1993) proposes the following dominance test:
Theorem 2.15 For $F 2 \mid($ perm $) \mid \Sigma T_{j}$, to schedule jobs at the end of the sequence:

1. Find the longest makespan any nondelay schedule can have. This is given by the reverse of JR; the proof parallels the proof that JR minimizes makespan.
2. If any $d_{j}$ is greater than this, schedule $J_{j}$ last. $J_{j}$ will never be tardy.
3. Remove $J_{j}$; recompute makespan and repeat as often as necessary.

Kim found this to give useful priority information more often than the previous Theorem.

Pan and Fan (1997) give other dominance criteria; for example:
Theorem 2.16 For $F 2 \mid($ perm $) \mid \Sigma T_{j}$, for any two jobs $J_{j}$ and $J_{k}$,
(a) if $a_{j} \leq a_{k}, b_{j} \leq b_{k}, d_{j}-b_{j} \leq d_{k}-b_{k}$, then do not put $J_{j}$ last;
(b) if $a_{j} \leq a_{k}, b_{j}=b_{k}, d_{j}-b_{j} \leq d_{k}-b_{k}$, then $j \stackrel{\text { global }}{\sim} k$;
(c) if $a_{j} \leq \min \left\{b_{j}, a_{k}\right\}, b_{j} \leq b_{k}, d_{j} \leq d_{k}$, then $j \leadsto k$.

### 2.8.2 Lower Bounds

Lower bounds have a variety of uses, such as to test heuristic solutions when optima cannot be found, and to bound the completions of partial schedules at a node of a branch-and-bound algorithm.

## A first bound

Kim (1993) gives a branch-and-bound algorithm that builds schedules one job at a time, starting with the last job, so that each node in the search tree represents an ordered subset, $\sigma$, of jobs that will end the schedule. Kim's principal contribution is to derive a lower bound on the objective function at any node. If a lower bound is sought for the entire problem, with no partial schedule, the formula of Theorem 2.17 below (with $u=n$ ) gives it directly.

Suppose the partial schedule $\sigma$ at an arbitrary node includes $s$ jobs, with the set $\mathcal{U}$ of $u=n-s$ jobs remaining to be sequenced at the start of the schedule. We wish to find a lower bound on the total tardiness of the schedule. Two separate total tardiness bounds are found, first for the jobs in $\sigma$, and second for the unscheduled jobs in $\mathcal{U}$. Their sum will then give the overall bound. The two bounds are found as follows.

Since the jobs in $\sigma$ are well ordered, their tardiness can be precisely calculated if we know the times the two machines become available. Of course, since $M_{1}$ is never idle, that machine is occupied with the unscheduled jobs for $\Sigma_{j \in \mathcal{U}} a_{j}$, at which time it becomes available. The availability of $M_{2}$ is unknown, depending on the undetermined first part of the schedule; instead, we can use the makespan given by applying Johnson's Rule to the unscheduled set, as a lower bound.

Now, to find a bound on the tardiness of the $u$ unscheduled jobs, note that, if $c_{j}$ is a lower bound on the completion time, $C_{[j]}$, of the job in position $j$ (call that job $J_{[j]}$ ), then

$$
c_{j}=\max \left\{a_{(1)}+B_{(j)}, A_{(j)}+b_{(1)}\right\}
$$

where $a_{(j)}\left[b_{(j)}\right]$ is the $j^{\text {th }}$ smallest task time on $M_{1}\left[M_{2}\right]$, and $A_{(j)}\left[B_{(j)}\right]$ is the sum of the $j$ smallest task times on $M_{1}\left[M_{2}\right]$. This is because the completion of $j$ jobs requires at least that one task on one machine and $j$ tasks on the other be processed sequentially.

Next note the following, proven by a pairwise interchange argument: given two sets of $u$ positive numbers, $\left\{x_{j}\right\}_{j=1}^{u}$ and $\left\{y_{j}\right\}_{j=1}^{u}$, the sum of the positive parts of pairwise differences, $\sum_{j=1}^{u}\left(x_{j}-y_{j}\right)^{+}$is minimized if the two sets are reindexed so that $x_{1} \leq x_{2} \leq \ldots \leq x_{u}$ and $y_{1} \leq y_{2} \leq \ldots \leq y_{u}$. Applying this result, we can bound the tardiness sum of any schedule by matching each $C_{[j]}$ with the $j^{\text {th }}$ smallest due date, which we denote $d_{(j)}$. Thus, we have

Theorem 2.17 (Kim, 1993) For $F 2 \mid($ perm $) \mid \Sigma T_{j}$ with u jobs, if $d_{(j)}$ is the $j^{\text {th }}$ smallest due date, then:

$$
\sum_{j=1}^{u} T_{j}=\sum_{j=1}^{u}\left(C_{[j]}-d_{[j]}\right)^{+} \geq \sum_{j=1}^{u}\left(\max \left\{a_{(1)}+B_{(j)}, A_{(j)}+b_{(1)}\right\}-d_{(j)}\right)^{+}
$$

Finally, the separate bounds on the scheduled and unscheduled jobs can be added to give a lower bound on the total tardiness at the branch-and-bound node corresponding to the partial schedule $P S$. Pan and Fan (1994) give a slightly improved version of this bound.

## A Tighter Bound

The two-machine specialization of a bound given by Chung et al. (2006) for $m$ machines (see Theorem 4.17) is worth mentioning. Assume that $\sigma$ is an initial partial schedule occupying $M_{k}$ for a time $C_{k}(\sigma)(k=1,2)$ and $\mathcal{U}$ is the set of $u$ unscheduled jobs that must follow it. For simplicity, renumber the jobs in $\mathcal{U}$, and the positions they will occupy, starting from 1 , omitting the jobs already scheduled. Let $s_{k i}$ be a lower bound on the start time of
the task in position $i$ of $M_{k}$, obtained by ordering the tasks on each machine independently in SPT order. For $i=1, \ldots, u$ :

$$
\begin{aligned}
& s_{1 i}=C_{1}(\sigma)+A_{(i-1)}, \text { with } s_{11}=C_{1}(\sigma) \\
& s_{2 i}=\max \left\{s_{21}+B_{(i-1)}, C_{1}(\sigma)+A_{(i)}\right\}, \text { with } s_{21}=\max \left\{C_{2}(\sigma), C_{1}(\sigma)+a_{(1)}\right\}
\end{aligned}
$$

Now suppose some job $J_{j} \in \mathcal{U}$ is scheduled in position $i$. Given $s_{1 i}$, a lower bound on that job's tardiness is $\left(s_{1 i}+a_{j}+b_{j}-d_{j}\right)^{+}$. Defining $\delta_{j}=$ $d_{j}-a_{j}-b_{j}$, we now ask: what assignment of jobs to positions minimizes the total tardiness, $\Sigma_{i}\left(s_{1 i}-\delta_{j}\right)^{+}$, thus assuring that we have a lower bound? As we saw above, we need $\delta_{j}=\delta_{(i)}$, the $i^{\text {th }}$ smallest value of $\delta_{j}$. This gives for our first bound:

$$
L B_{1}=\Sigma_{i=1}^{u}\left(s_{1 i}-\delta_{(i)}\right)^{+}, \text {where } \delta_{i}=d_{i}-a_{i}-b_{i} .
$$

Analogously, starting with $s_{2 i}$, we get a second bound:

$$
L B_{2}=\Sigma_{i=1}^{u}\left(s_{2 i}-\delta_{(i)}^{\prime}\right)^{+}, \text {where } \delta_{i}^{\prime}=d_{i}-b_{i}
$$

and the larger of the two is our bound.

### 2.8.3 Branch-and-Bound Algorithms

All three of the articles cited above present branch-and-bound algorithms based on their results. Successive papers naturally report improved performance, though each successive algorithm is less than an order of magnitude faster than the previous. In the first paper, Sen et al. (1989) could handle instances with job numbers in the low teens, while the latest results from Pan and Fan (1997) could push it only to the high teens.

### 2.8.4 Heuristics

Sen et al. (1989) present a heuristic (the best of several they considered), suggested by the three conditions given in Theorem 2.14:

## The SPT-PI Algorithm

1. SPT: Find schedule $S=\nearrow a_{j}$, breaking ties first by $\nearrow\left(b_{j}-d_{j}\right)$, then by $\nearrow d_{j}$, then arbitrarily.
2. PI (pairwise improvement) : find a local optimum by interchanging adjacent pairs whenever this reduces the value of the objective.

Computational experiments show the heuristic works well, often giving the optimal solution, at least for small problems and for some parameter ranges.

### 2.8.5 F2|(perm), prep| $\boldsymbol{\Sigma} T_{j}$

A special case of the two-machine flow shop, often referred to as preprocessing and hence denoted here by prep, occurs when the major work of each job is performed on one machine, preceded by a relatively brief preparatory processing on another machine. We also assume that larger jobs require more preprocessing. Thus:

$$
\text { prep implies }\left\{\begin{array}{ll}
a_{j} \leq b_{j}, & \text { for all } j \\
b_{i} \geq b_{j}
\end{array} \Rightarrow a_{i} \geq a_{j}, \quad \text { for all } i, j\right.
$$

Here, the SPT schedule is unambiguous: $S P T=\nearrow a_{j}=\nearrow b_{j}$. Koulamas (1996) discusses the total tardiness objective in a preprocessing environment. Although permutation schedules constitute a dominant set, the problem is still NP-complete: even with $a_{j}=0$, it reduces to $1 \| \Sigma T_{j}$, which is well known to be NP-complete. We have a simple solution in the following special case:

Theorem 2.18 For $F 2 \mid$ (perm), prep $\mid \Sigma T_{j}, S P T$ is optimal if all jobs are tardy in SPT.

Otherwise, dominance conditions are developed, and used in a forward branch-and-bound algorithm, BB . An $O\left(n^{2} \log n\right)$ heuristic is also proposed, which is tested against BB and is found to give solutions deviating about $3 \%$ from optimal for problems with 35 jobs.

## 2.9 $\mathrm{F} 2 \mid($ perm $), d_{j}=d \mid \Sigma w_{j} U_{j}$

There has been no published research on the problem of minimizing the weighted number of late jobs in the two-machine flow shop (recall that $U_{j}$ is a lateness indicator variable: $U_{j}=1$ if $J_{j}$ is late; 0 otherwise) except in the special case where all jobs have a common due date, which has been considered by Józefowska et al. (1994). They also give results for the open and the general job shop on two machines, but we shall stick to flow shops.

### 2.9.1 The Complexity of $F 2 \mid($ perm $), d_{j}=d \mid \Sigma U_{j}$

The authors first prove that even the simpler problem with unweighted objective is NP-complete.

Theorem 2.19 $F 2\left|d_{j}=d\right| \Sigma U_{j}$ is ordinary $N P$-complete
Proof Outline: The reduction is from the NP-complete problem:

## 2-PARTITION

INSTANCE: An integer $V$, and $2 k$ positive integers
$v_{i}: i \in \mathcal{T}=\{1,2, \ldots, 2 k\}$ such that $\Sigma_{i \in \mathcal{T}} v_{i}=2 V$.
QUESTION: Can $\mathcal{T}$ be partitioned into two disjoint sets $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ with $\left|\mathcal{S}_{1}\right|=\left|\mathcal{S}_{2}\right|=k$ and $\Sigma_{i \in \mathcal{S}_{1}} v_{i}=\Sigma_{i \in \mathcal{S}_{2}} v_{i}=V ?$
to the decision version of our problem:
$\boldsymbol{F} 2\left|d_{j}=d\right| \Sigma U_{j} \leq B ?$
INSTANCE: A real number $B>0$, and $n$ jobs with common due date $d>0$ and processing times $\left\langle a_{j}, b_{j}\right\rangle, j \in \mathcal{J}$ to be scheduled in a two-machine flow shop.
QUESTION: Does there exist a schedule with $\Sigma_{j} U_{j} \leq B$ ?
2-Partition is known to remain NP-complete under the assumptions, which we shall make, that $k \geq 2, V$ is divisible by $k$ (so $V \geq k$ ), and $v_{i} \leq V$.

Given an instance of 2-Partition, construct an instance of $F 2\left|d_{j}=d\right| \Sigma U_{j}$ as follows:

- $n:=6 k$;
- $\left\langle a_{j}, b_{j}\right\rangle:=<3 V+v_{j}, 3 V+2 V / k-v_{j}>, j \in \mathcal{T}=\{1,2, \ldots, 2 k\}$;
- $\left.\left\langle a_{j}, b_{j}\right\rangle:=<1, m\right\rangle, \quad j \in \mathcal{U}=\{2 k+1, \ldots, 4 k\}$;
- $\left\langle a_{j}, b_{j}\right\rangle:=<m, 1>, \quad j \in \mathcal{V}=\{4 k+1, \ldots, 6 k\}$;
- $B:=k$;
- $d:=1+2 k m+3 k V+V+2 k$;
where $m=\min \left\{\min _{i} a_{i}, \min _{i} b_{i}\right\}$.
We claim that, if one of the two instances has the answer "yes", then the other must, too. First, assume the partitioning subsets $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ exist. Consider the schedule $S=\left(\mathcal{U}, \mathcal{S}_{1}, \mathcal{V}, \mathcal{S}_{2}\right)$, with jobs arbitrarily sequenced within each of the four sets. This schedule is illustrated in Fig. 2.7.


Fig. 2.7 Optimal schedule when a partition of $\mathcal{T}$ exists

To understand the figure, first note that, since $\Sigma_{i \in \mathcal{S}_{1}} v_{i}=V$ by hypothesis, $\Sigma_{j \in \mathcal{S}_{1}} a_{j}=\Sigma_{j \in \mathcal{S}_{1}} b_{j}=3 k V+V$. It follows that the total processing time of jobs in $\mathcal{U}, \mathcal{V}$ and $\mathcal{S}_{1}$ is $d-1$ on each machine, so that all (and only) these jobs are on time. It can also be shown that $t_{1} \leq t_{2}$ (see Fig. 2.7), so $S$ is feasible (the operations of a job do not overlap) and has $B=k$ tardy jobs.

To show the reverse implication, suppose the scheduling problem has a solution with at most $B=k$ tardy jobs. Assume exactly $k$ are tardy (we can always make more jobs tardy if necessary), and all of them are in $\mathcal{T}$ (by definition of $m$, all jobs in $\mathcal{T}$ take longer on both machines than any job in $\mathcal{U}$ or $\mathcal{V}$, so interchanging any tardy job not in $\mathcal{T}$ with an early job in $\mathcal{T}$ can
only improve the schedule). This implies that exactly $k$ jobs in $\mathcal{T}$ are early (call this set of jobs $\mathcal{S}_{1}$ ), as well as all jobs in $\mathcal{U}$ and $\mathcal{V}$.

It remains to show that $\mathcal{S}_{1}$ defines a partition; that is, that $\Sigma_{i \in \mathcal{S}_{1}} v_{i}=V$. But if $\Sigma_{i \in \mathcal{S}_{1}} v_{i}>V$, then the total time of all early tasks on $M_{1}$ exceeds $d-1$ and the last job is tardy; while if $\Sigma_{i \in \mathcal{S}_{1}} v_{i}<V$, then the total on $M_{2}$ exceeds $d-1$, and since jobs cannot start until time 1 , the last of them must be tardy. In each case, we reach a contradiction, so indeed $\Sigma_{i \in \mathcal{S}_{1}} v_{i}=V$.

### 2.9.2 $F 2 \mid($ perm $),($ pmtn $), d_{j}=d \mid \Sigma w_{j} U_{j}$

Observe that an optimal schedule always exists in which the early jobs come first, ordered by Johnson's Rule (JR minimizes makespan, so makes all jobs early when possible). This makes it clear that preemption can never be helpful, nor can reordering the jobs between stages; they are disallowed hereafter. Assuming jobs are indexed by JR, Józefowska et al. (1994) give a dynamic program that builds the schedule backwards, considering jobs successively to be discarded (put into the late set). Defining

$$
\begin{aligned}
f_{k}(u, v)= & \text { the minimal weighted number of late jobs for the jobs } \\
& \text { in } \mathcal{J}_{k}=\left\{J_{k}, \ldots, J_{n}\right\}, \text { given that the first job in } \mathcal{J}_{k} \\
& \text { starts at time } u \text { on } M_{1} \text { and no earlier than } v \text { on } M_{2},
\end{aligned}
$$

The recurrence is

$$
f_{k}(u, v)=\left\{\begin{array}{l}
f_{k+1}(u, v)+w_{k}, \quad \text { if } \max \left\{u+a_{k}, v\right\}+b_{k}>d \\
\min \left\{f_{k+1}(u, v)+w_{k}, f_{k+1}\left(u+a_{k}, \max \left\{u+a_{k}, v\right\}+b_{k}\right)\right\} \\
\quad \text { otherwise },
\end{array}\right.
$$

with initial conditions $f_{n+1}(u, v)=0$ for $0 \leq u, v \leq d$; where the first case is when $J_{k}$ alone cannot finish by $d$ given the start times $u$ and $v$. We evaluate $f_{k}(u, v)$ for decreasing values of $u, v(0 \leq u \leq v \leq d)$ and $k \in\{1, \ldots, n\}$ until we reach the cost of an optimal schedule, $f_{1}(0,0)$. Since each step takes a fixed time, the program runs in time $O\left(n d^{2}\right)$.

### 2.10 F2 $\mid($ perm $), p_{i j}=u_{i j}-v_{i j} r_{i j} \mid\left(\Sigma_{i j} r_{i j} \mid C_{\max } \leq D\right)$

Janiak (1989) considers the problem of completing all jobs by a given deadline, $D>0$, using minimal resources. The processing time $p_{i j}$ of task $T_{i j}$ has an upper bound or maximal value $u_{i j}$ which can be shortened by the expenditure of a resource. Thus, if $r_{i j}$ is the amount of resource allotted to $T_{i j}$, and if $u_{i j}$ and $v_{i j}$ are given positive constants, then

$$
p_{i j}=u_{i j}-v_{i j} r_{i j}, \quad i=1,2, \quad j=1,2, \ldots, n, \text { with } u_{i j}>0, v_{i j}>0
$$

Let $w_{i j}$ be the largest amount of the resource that can be used to expedite $T_{i j}$. Then

$$
0 \leq r_{i j} \leq w_{i j} \leq u_{i j} / v_{i j}
$$

where the last constraint assures that processing times are not negative.
We seek a list schedule $S$ (since the criterion is a regular measure, only permutation schedules need be considered) and a feasible resource allocation $R=\left\{r_{i j}, i=1,2, j=1,2, \ldots, n\right\}$ that minimize total resource consumption $\Sigma_{i} \Sigma_{j} r_{i j}$ while achieving a makespan no greater than $D$. Using the notation $A \mid B$ to denote the heirarchical objective of minimizing $A$ given $B$, we write this $\Sigma_{i j} r_{i j} \mid C_{\max } \leq D$.

The complexity of this problem is stated without proof.
Theorem 2.20 (Janiak 1989)
$F 2\left|p_{i j}=u_{i j}-v_{i j} r_{i j}\right|\left(\Sigma_{i j} r_{i j} \mid C_{\max } \leq D\right)$ is NP-complete, even for $v_{i 1}=v, v_{i 2}=0$. The reduction is from the KnApsACK problem.

Consider the extreme cases, in which
(1) no resources are allocated, producing the min-cost solution which gives the longest makespan, $C^{\text {max }}$;
(2) the maximal amount is allocated to each task: the max-cost solution giving the shortest possible makespan, $C^{\text {min }}$.

In each case, we can use Johnson's Rule (JR) to compute the makespan.
Clearly, the problem is infeasible whenever $D<C^{\text {min }}$, and is trivial if the deadline is already achieved at no cost $\left(D \geq C^{\max }\right)$ so we assume $C^{\min } \leq$ $D<C^{\max }$. Note also:

- For any fixed $S$, the problem reduces to an LP. Furthermore, it is shown to be solvable by an $O\left(n^{3}\right)$ algorithm; call it Algorithm A.
- For any fixed $R$, we can determine using JR whether there exists a feasible solution.
Finally, the following heuristic for $F 2\left|p_{i j}=u_{i j}-v_{i j} r_{i j}\right|\left(\Sigma_{i j} r_{i j} \mid C_{\max } \leq D\right)$ is given:

Heuristic $\mathbf{H}_{\mathbf{J}}$ (Janiak, 1989)

1. Find the list schedule $S_{J}=J R\left(\underline{u}_{1}, \underline{u}_{2}\right)$, where $\underline{u}_{i}=\left(u_{i 1}, u_{i 2}, \ldots u_{i n}\right)$.

That is, use Johnson's Rule with no resource allocation.
2. Find a resource allocation $R_{J}$ for schedule $S_{J}$, using Algorithm A.

A number of alternatives to Step 1 and an elaboration of Step 2 were proposed, but computational tests showed the alternatives to be inferior, and the extension to Step 2 gave no significant improvement. The heuristic gave total resource usage $50 \%$ to $60 \%$ higher than a calculated lower bound on the optimum.

### 2.11 Bicriteria Objectives

When we wish to give simultaneous consideration to two measures, there are several approaches. We have already presented problems with a hierarchical
objective (e.g., $\Sigma C_{j} \mid \min C_{\max }$, discussed in Sect. 2.5), or a compound objective (we considered $\alpha \Sigma C_{j}+\beta C_{\max }$ in Sect. 2.6). The third way to deal with two measures is the bicriteria approach, in which the two objectives are not combined into one function, but are left as a pair of measures of a schedule's merit. Generally, it is no longer possible to find a schedule that is "best" on both counts. Instead, the most we can do is to eliminate all schedules that are clearly not candidates for selection: the dominated schedules.

Definition 2.1. Schedule $S$ dominates schedule $S^{\prime}$ with respect to objectives $O_{1}$ and $O_{2}$ if $O_{1}(S) \leq O_{1}\left(S^{\prime}\right)$ and $O_{2}(S) \leq O_{2}\left(S^{\prime}\right)$, with at least one strict inequality. An undominated schedule is called efficient.

Our goal, then, is to present the decision maker with the entire set of efficient (that is, undominated) solutions from which to choose. We denote by $\left(O_{1}, O_{2}\right)$ the objective of finding all efficient solutions for the pair of measures $O_{1}$ and $O_{2}$, and we call $\left(O_{1}, O_{2}\right)$ a bicriteria objective (note that all of our other objectives, including hierarchical and compound, have single objective functions by which each schedule is measured). Since each separate objective function satisfies Theorem 1.1, it remains sufficient to consider only permutation schedules as candidates for the efficient set.

### 2.11.1 Branch-and-Bound for a Bicriteria Objective

Bicriteria problems are generally difficult, and published results often report branch-and-bound algorithms. We briefly summarize here the special features of this approach when applied to finding the efficient set for two objective functions. As usual with sequencing problems, schedules are built job by job either forwards from the start or backwards from the end. We will here assume backwards generation of the schedule. Thus, each node represents a terminating partial schedule, with a descendant node for each of the remaining jobs that could be added at the start of it. Recall the notation:

- $\sigma$ : a node, and also the partial sequence at that node.
- $\mathcal{S}$ : the set of jobs making up the partial sequence $\sigma$.
- $\mathcal{U}$ : the set of jobs that remain unscheduled at node $\sigma$.

Normally, at a node $\sigma$ we compute a lower bound $L B_{\sigma}$ on the value of the objective function for all completions of this partial schedule (that is, for all descendants of this node), and compare it with an upper bound, $U B$, the value of the best complete schedule yet generated. If $L B_{\sigma} \geq U B$, the node can be discarded or fathomed. Now, with bicriteria, we require two lower bounds, $L B_{\sigma}\left(O_{1}\right)$ and $L B_{\sigma}\left(O_{2}\right)$. Instead of a single schedule setting the upper bound, we must define:

- $\mathcal{E}$ : the set of candidate efficient schedules.

Initially empty, $\mathcal{E}$ is updated at each node. We may generate a complete schedule $S_{\sigma}$ at node $\sigma$, perhaps by appending $\mathcal{U}$ in JR order to the front of
$\sigma$ if $C_{\max }$ is one of the criteria, or in EDD order if a measure of tardiness is of interest. We then add $S_{\sigma}$ to $\mathcal{E}$ if no schedule in $\mathcal{E}$ dominates it, and then eliminate from $\mathcal{E}$ any schedule that $S_{\sigma}$ dominates. The node $\sigma$ can then be fathomed if, for any $S^{\prime} \in \mathcal{E}, O_{1}\left(S^{\prime}\right) \leq L B_{\sigma}\left(O_{1}\right)$ and $O_{2}\left(S^{\prime}\right) \leq L B_{\sigma}\left(O_{2}\right)$. If $\mathcal{E}$ was changed, other nodes may also be fathomable.

If $\sigma$ is not fathomed, descendant nodes are created, one for each job in $\mathcal{U}$ except those ruled out by dominance properties: conditions under which a certain job need not be considered to immediately precede $\sigma$. We then select the next node to analyze, perhaps choosing the one with smallest $S_{\sigma}$, or smallest lower bound on one of the criteria, continuing as usual until all nodes are fathomed or fully expanded.

### 2.11.2 $\operatorname{F2} \mid($ perm $) \mid\left(\Sigma C_{j}, C_{\text {max }}\right)$

An objective closely related to $\alpha \Sigma C_{j}+\beta C_{\max }$ is the bicriteria objective $\left(\Sigma C_{j}, C_{\max }\right)$. For $F 2 \mid($ perm $) \mid\left(\Sigma C_{j}, C_{\max }\right)$, Sayin and Karabati (1999) developed a branch-and-bound algorithm that solves a series of hierarchical optimization problems where the makespan objective is required to be no greater than an upper bound. The authors show that in some cases, the time required to solve $F 2 \mid($ perm $) \mid\left(\Sigma C_{j}, C_{\max }\right)$ is not greater than the time needed for $F 2 \mid$ perm $\mid\left(\Sigma C_{j} \mid \min C_{\max }\right)$. In their computational experiment they are able to solve problems with up to $n=28$ jobs.

### 2.11.3 Bicriteria Involving $C_{\max }$ and a Measure of Tardiness

Often, when considering multiple criteria in scheduling a set of jobs, one important consideration is meeting customers' due dates, as measured by some function of job tardiness. Another is usually to maintain a high level of resource utilization, achieved by keeping the makespan small. Daniels and Chambers (1990) consider the dual objectives of $C_{\max }$ and $T_{\max }$. Building on their work, Liao et al. (1997) deal with the bicriteria ( $C_{\max }, \Sigma T_{j}$ ) and $\left(C_{\max }, \Sigma U_{j}\right)$, where $U_{j}$ is a zero-one tardiness indicator, so that $\Sigma U_{j}$ denotes the number of tardy jobs.

Both papers develop similar branch-and-bound algorithms, and we discuss them together. Note: in this section, schedules are built backwards.

## Dominance Properties

For each bicriteria objective, conditions are given under which jobs can be partially ordered, thereby enabling the pruning of some branches of the search tree. When the conclusion of a Theorem is some property A, we mean "we can limit our search for efficient schedules to those with property A".

- $\left(C_{\text {max }}, T_{\max }\right)$

Theorem 2.21 For $F 2 \mid($ perm $) \mid\left(C_{\max }, T_{\max }\right)$,

$$
\left.\begin{array}{l}
b_{k} \leq \min \left\{a_{k}, b_{j}\right\} \\
d_{j} \leq d_{k}
\end{array}\right\} \Rightarrow j \stackrel{\text { global }}{\sim} k .
$$

Theorem 2.22 For $F 2 \mid($ perm $) \mid\left(C_{\text {max }}, T_{\text {max }}\right)$,

$$
\left.\begin{array}{l}
a_{j} \leq \min \left\{a_{k}, b_{j}\right\} \\
d_{j} \leq d_{k}
\end{array}\right\} \Rightarrow j \leadsto k
$$

Note how these elimination criteria combine Johnson's Rule, which minimizes $C_{\max }$, and Earliest Due Date $\left(E D D=\nearrow d_{j}\right)$ sequencing, a reasonable heuristic for minimizing $T_{\text {max }}$.

- $\left(C_{\max }, \Sigma T_{j}\right)$

The following condition was first given for $F 2 \mid($ perm $) \mid \Sigma T_{j}$ by Sen et al. (1989), but it works for $C_{\max }$, too, since the first hypothesis implies $\min \left(a_{j}, b_{k}\right) \leq \min \left(a_{k}, b_{j}\right)$.

Theorem 2.23 For $F 2 \mid($ perm $) \mid\left(C_{\text {max }}, \Sigma T_{j}\right)$,

$$
\left.\begin{array}{l}
a_{j} \leq \min \left\{a_{k}, b_{j}\right\} \\
d_{j} \leq d_{k} \\
b_{j}-d_{j} \leq b_{k}-b_{k}
\end{array}\right\} \Rightarrow j \leadsto k
$$

- $\left(C_{\max }, \Sigma U_{j}\right)$

Theorem 2.24 For $F 2 \mid($ perm $) \mid\left(C_{\max }, \Sigma U_{j}\right)$, suppose that $J_{j}$ is the first job in $\sigma$. Then
$\left.\begin{array}{l}\text { 1. } J_{k} \text { is tardy when last in } \mathcal{U} \\ \text { 2. } \min \left\{a_{j}, b_{k}\right\} \leq \min \left\{a_{k}, b_{k}\right\}\end{array}\right\} \Rightarrow j \leadsto k$.
The first hypothesis means: $J_{k} \in \mathcal{U}$ is tardy when immediately before $J_{j}$. This can be tested by scheduling the rest of the jobs in $\mathcal{U}$ using Johnson's Rule, in front of $J_{k}$. This completes them as early as possible, so if it makes $J_{k}$ tardy, any other arrangement will, too. Finally, for the last two Theorems, recall that, when building the schedule backwards, a useful interpretation of $j \leadsto k$ is " $J_{k}$ should not be scheduled immediately before $J_{j}$."

## Lower Bounds

The following bounds are given for the four objectives under discussion.

- $C_{\text {max }}$

At any node $\sigma$, since the earliest completion of the initial partial schedule results from sequencing $\mathcal{U}$ by JR , a lower bound on the makespan is:

$$
L B_{\sigma}\left(C_{\max }\right)=C_{\max }\left(S_{\sigma}\right), \text { where } S_{\sigma}=(J R(\mathcal{U}), \sigma)
$$

- $T_{\text {max }}$

For $T_{\max }$, since $S_{\sigma}$ puts jobs in $\sigma$ as early as possible, one lower bound is clearly

$$
L B 1_{\sigma}\left(T_{\max }\right)=\max _{j \in \sigma} T_{j}\left(S_{\sigma}\right)
$$

For another bound, this time based on $\mathcal{U}$, suppose we schedule only $M_{2}$, ignoring $M_{1}$. Now, $T_{\max }$ is minimized by scheduling $M_{2}$ in $E D D$ order. Adding a minimal delay at the start of the schedule (nothing can start on $M_{2}$ until at least $a_{\text {min }}=\min _{j \in \mathcal{U}} a_{j}$ ), and renumbering the jobs in $\nearrow d_{j}$ order, we have

$$
L B 2_{\sigma}\left(T_{\max }\right)=\max _{k \in \mathcal{U}}\left(a_{\min }+\sum_{j=1}^{k} b_{j}-d_{k}\right)
$$

A third lower bound comes from relaxing the $M_{2}$ constraints, and scheduling $\mathcal{U}$ on $M_{1}$ by $E D D$. However, each job $J_{j}$, when completed on $M_{1}$, must at least incur the additional delay of $b_{j}$. We adjust for this by redefining the due date to be $d_{j}-b_{j}$. Reindexing in $\nearrow\left(d_{j}-b_{j}\right)$ order, this results in

$$
L B 3_{\sigma}\left(T_{\max }\right)=\max _{k \in \mathcal{U}}\left(\sum_{j=1}^{k} a_{j}+b_{k}-d_{k}\right)
$$

We can then evaluate the three bounds and select the best, $L B_{\sigma}\left(T_{\max }\right)=$ $\max _{i=1}^{3} L B i_{\sigma}\left(T_{\max }\right)$.

## - $\Sigma T_{j}$

As we have repeatedly noted, $S_{\sigma}$ starts $\sigma$ as early as possible, so it makes the total tardiness of the jobs in $\sigma$ as small as possible:

$$
L B 1_{\sigma}\left(\sum T_{j}\right)=\sum_{j \in \sigma} T_{j}\left(S_{\sigma}\right)
$$

For the overall bound, we can add to this a lower bound on the total tardiness of the jobs in $\mathcal{U}$. We start by finding lower bounds on the completion times of the jobs in each position. Suppose there are $u$ jobs in $\mathcal{U}$. Recall that, as in (2.1), for an arbitrary schedule $S=(1,2, \ldots, u)$, $C_{\max }=\max _{1 \leq k \leq u}\left\{\Sigma_{j=1}^{k} a_{j}+\Sigma_{j=k}^{u} b_{j}\right\}$. For a lower bound on this quantity, perform a monotone reindexing of the task times, separately on each machine, defining $a_{(i)}\left[b_{(i)}\right]$ as the $i^{\text {th }}$ smallest $a$-task [ $b$-task] in $\mathcal{U}$. Clearly, $\Sigma_{j=1}^{k} a_{j}$ is minimized if $a_{j}$ is replaced by $a_{(j)}$, and so, if $C_{[i]}$ denotes a lower bound on the completion time of the job in position $i$ :

$$
C_{[i]}= \begin{cases}\min _{j \in u}\left\{a_{j}+b_{j}\right\}, & i=1 \\ \max _{k=1, \ldots, i}\left\{\sum_{j=1}^{k} a_{(j)}+\sum_{j=1}^{i-k+1} b_{(j)}\right\}, & i=2, \ldots, u-1 \\ C_{\max }(J R(\mathcal{U})), & i=u\end{cases}
$$

where, in the first and last positions ( $i=1$ and $u$ ), stronger bounds (which are self evident) have been substituted. The formula for the intermediate positions is stronger than the one in Liao et al. (1997).

Reindexing the due dates so that $d_{(1)} \leq d_{(2)} \leq \ldots \leq d_{(u)}$, it is straightforward to show that a lower bound on the total tardiness of the jobs in $\mathcal{U}$ is

$$
L B 2_{\sigma}\left(\sum T_{j}\right)=\sum_{j=1}^{u} \max \left\{0, C_{[j]}-d_{(j)}\right\}
$$

Finally, $L B_{\sigma}\left(\sum T_{j}\right)=L B 1_{\sigma}\left(\sum T_{j}\right)+L B 2_{\sigma}\left(\sum T_{j}\right)$.

- $\Sigma U_{j}$

Using a similar approach, since $S_{\sigma}$ starts $\sigma$ as early as possible, it gives the fewest tardy jobs in $\sigma$, so:

$$
L B 1_{\sigma}\left(\sum U_{j}\right)=\sum_{j \in \sigma} U_{j}\left(S_{\sigma}\right)
$$

To this can be added a bound on the number tardy in $\mathcal{U}$. Two such bounds are given, each based on reducing the problem to single machine sequencing. The following well-known procedure for minimizing the number of tardy jobs on one machine, where $J_{j}$ has parameters $\left(p_{j}, d_{j}\right), j=1, \ldots, n$, starts with the jobs in $E D D=\nearrow d_{j}$ order (guaranteed to complete all jobs on time if any schedule can), and removes one job at a time until no tardy jobs remain.


1. Let the first tardy job in $\nearrow d_{j}$ be in position $f$.
2. Let $J_{g}$ be the longest job in the first $f$ positions of $\nearrow d_{j}$. Remove $J_{g}$.
3. Repeat Steps 1 and 2 until there are no more tardy jobs in $\nearrow d_{j}$.
4. $S^{\star}$ is the remaining $\nearrow d_{j}$ followed by the removed jobs in any order.

As above, we can relax the $M_{1}\left[M_{2}\right]$ constraints and schedule the jobs solely on $M_{2}\left[M_{1}\right]$, making the same adjustments, but now using MA rather than EDD. This produces the two lower bounds based on $\mathcal{U}$ :

$$
L B 2_{\sigma}\left(\sum U_{j}\right)=\sum_{j \in \mathcal{U}} U_{j}\left(M A\left(\underline{b}, \underline{d}^{\prime}\right)\right), \text { where } d_{j}^{\prime}=d_{j}-a_{\min }
$$

and

$$
L B 3_{\sigma}\left(\sum U_{j}\right)=\sum_{j \in \mathcal{U}} U_{j}\left(M A\left(\underline{a}, \underline{d}^{\prime \prime}\right)\right), \text { where } d_{j}^{\prime \prime}=d_{j}-b_{j} .
$$

Finally, the overall bound for $\Sigma U_{j}$ at node $\sigma, L B=L B 1+\max \{L B 2, L B 3\}$.

## Heuristics

The above tree search becomes unwieldy for instances with more than about 30 jobs. While Liao et al. (1997) give no alternative approach for ( $C_{\max }, \Sigma T_{j}$ ) and $\left(C_{\max }, \Sigma U_{j}\right)$, Daniels and Chambers (1990) propose the following heuristic for $\left(C_{\max }, T_{\max }\right)$. We seek to solve: minimize $C_{\max }$ subject to $T_{\max } \leq B$, as $B$ is varied over an appropriate range of values. For each $B$, we make $C_{\max }$ as small as possible by using Johnson's Rule constrained by the tardiness bound. The schedule is built backwards, job by job. At any point, if $\mathcal{U}$ is the set of remaining unscheduled jobs, a job $J_{k} \in \mathcal{U}$ is eligible to be last if $\Sigma_{j \in \mathcal{U}} a_{j}+b_{k}-d_{k} \leq B$. The eligible job latest in $J R$ is placed last, and the process repeated until $\mathcal{U}$ is empty. Of course, eligibility does not guarantee that $T_{k} \leq B$, so some backtracking may be necessary: if a job's tardiness is excessive, move it one position earlier, etc.

We start by finding $J R$ unconstrained, and letting the resulting maximal job tardiness be our initial value for $B$. By decrementing $B$ iteratively over a suitable range, all efficient schedules can be generated.

## Computational Experiments

The optimizing algorithm for each of the three bicriteria was encoded and tested quite extensively. The program could solve problem instances of up to about thirty jobs. The most significant findings were:

- Problems are harder to solve when a larger proportion of jobs is tardy. This may be because, when jobs are mostly on time, good schedules depart little from $J R$.
- The number of efficient schedules is very small. In Liao et al., this number was reported as the average of over 20 randomly generated instances, for about 30 categories. For $\left(C_{\max }, \Sigma T_{j}\right)$, there tended to be an average of two or three efficient schedules for small (under 20 job) instances, dropping below two for larger instances ( 25 to 30 jobs). For ( $C_{\max }, \Sigma U_{j}$ ), the average was rarely more than two throughout the range of instance sizes ( 5 to 30 jobs). Daniels and Chambers (1990) give similar results for ( $C_{\max }, T_{\max }$ ). For the 675 instances generated, the number of efficient schedules averaged about two, invariant over problem size and other parameter variations. They also report that "a maximum of ten efficient solutions was encountered, while in over $50 \%$ of the problems only one schedule was efficient". This seems to be a surprising and unexplained result.

The $\left(C_{\max }, T_{\max }\right)$ heuristic performed quite well. The solution found was identical to the optimum in over $70 \%$ of the problems, and overall more than $80 \%$ of the efficient solutions were identified. The authors propose a reasonable measure of error for those heuristic solutions that were not in fact efficient, and using it they report an average error of $.29 \%$ with a maximum of $10 \%$.

### 2.12 Related Problems

We conclude by presenting a few simple extensions of some of the results presented above, for manufacturing configurations similar to two-machine flow shops.

### 2.12.1 $G 2|(p m t n)| C_{\max }$

In the general job shop (the $m$-machine general shop is denoted $G m$ ), the machine sequence is specified for each job, but different jobs may follow different paths. With $m=2$ and assuming each machine is visited at most once (when jobs can visit machines more than once, we have a reentrant job shop which is dealt with in another chapter), the following solution due to Jackson
(1956) is a straightforward extension of $J R$, still with run time $O(n \log n)$, which we present without proof. It can also be shown that we cannot improve this solution using preemption (the argument used for the flow shop extends readily to this case).

Clearly, all jobs are in one of the following two sets:

$$
\mathcal{J}_{a b}=\left\{\text { jobs requiring first } M_{a}, \text { then } M_{b}\right\}, \quad(a, b)=(1,2) \text { or }(2,1)
$$

In case there are jobs requiring only one machine, they can be assigned a zero -time task on the other machine, given an arbitrary machine sequence, and added to the appropriate set. Then the optimal schedule on each machine is

$$
S_{1}^{\star}=\left(\mathcal{J}_{12}(J R), \mathcal{J}_{21}\right) ; \quad S_{2}^{\star}=\left(\mathcal{J}_{21}(J R), \mathcal{J}_{12}\right)
$$

That is, schedule $\mathcal{J}_{12}$ first on $M_{1}$, ordered by Johnson's Rule. Similarly, on $M_{2}$ start with $\mathcal{J}_{21}$ using JR. Thereafter, any nondelay sequence can be followed on both machines.

### 2.12.2 O2|(pmtn) $\mid C_{\text {max }}$

Tasks in an open shop have no precedence ordering. The two-machine nonpreemptive open shop allows us so much freedom to schedule jobs that most schedules are optimal. If, whenever a machine becomes idle, we simply throw at it any available job not yet processed by it, we may expect to produce a schedule with no idle time on either machine most of the time, giving a makespan $C_{\max }=\max \left\{\Sigma_{j} a_{j}, \Sigma_{j} b_{j}\right\}$, an obvious lower bound. Only occasionally will this procedure be suboptimal, as when, at the time the last job should start on $M_{1}$, that same job is being processed on $M_{2}$. This may introduce unnecessary delay on $M_{1}$, and thereby lengthen the makespan. Thus, it requires some care to specify a procedure that will always guarantee optimality.

Pinedo (2008) proposes the following selection procedure:
Longest Alternate Processing Time Rule: whenever a machine becomes idle, select for it the available job not yet processed by it with the longest processing time on the other machine. A job already processed on the other machine is assigned zero time there, hence lowest priority.

The algorithm resulting from iterating this rule has complexity $O(n \log n)$, and we call the schedule it generates, $L A P T$. Then:

Theorem 2.25 (Pinedo 2008) For $O 2 \| C_{\max }$, an optimal schedule is $S^{\star}=L A P T$, and the resulting makespan is

$$
C_{\max }^{\star}=\max \left(\max _{j}\left(a_{j}+b_{j}\right), \sum_{j} a_{j}, \sum_{j} b_{j}\right)
$$

The proof results from a careful enumeration of cases, and will be omitted. An algorithm developed in Gonzalez and Sahni (1976), although significantly more complicated, solves $O 2 \| C_{\max }$ in $O(n)$ time. Note that the smallest
conceivable makespan is always achieved, so there can be no benefit from allowing preemption.

### 2.12.3 $S 2 \| C_{\text {max }}$ and $S 2|p m t n| C_{\max }$

The super shop was defined by Strusevich (1991) as a combination of general job shop and open shop. In the two-machine case, assuming all jobs visit both stations, we have jobs in three sets: $\mathcal{J}_{12}$ and $\mathcal{J}_{21}$, which contain the jobs that must visit the machines in subscript order, and $\mathcal{J}_{0}$, jobs whose tasks can be done in either order. Let $n_{12}, n_{21}$, and $n_{0}$ be the cardinalities of these sets, so that $n=n_{12}+n_{21}+n_{0}$. Setting $n^{\prime}=\max \left\{n_{12}, n_{21}\right\}$, the author presents algorithms with time complexity $O\left(n+n^{\prime} \log n^{\prime}\right)$ for the preemptive and nonpreemptive cases, which may have different solutions. He breaks the proof down into thirteen cases, and presents the optimal solution for each. Though polynomial, the algorithms are too complex to present here.

### 2.13 Closing Remarks

Johnson's Rule and its extensions are the most important ideas behind two-machine flow shop scheduling, and from them we can gain general insights that are helpful in scheduling variations of the two-machine flow shop with additional constraints. The main idea is to maximize the utilization of facilities by scheduling jobs so as to:
(a) get all the machines to work as quickly as possible, so start with eligible jobs that pass quickly through the first few machines;
(b) keep the machines working as steadily as possible, so schedule early those eligible jobs that tend to have increasing times on successive machines, to give the later machines a backlog;
(c) let all the machines complete their work as nearly synchronously as possible, so schedule late those eligible jobs that tend to have decreasing times, so that once the early machines are done with them the remaining work is soon over.

Such common-sense ideas may offer reasonable approaches for future research on two-machine models not yet investigated.

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## Chapter 3 <br> TRANSFER LAGS IN THE FLOW SHOP


#### Abstract

We define the types of transfer lags, positive and negative, between stages of a flow shop, and discuss applications (transport times, nonbottleneck machines, manufacturing cells, batch transfers, setup and teardown times, master-slave systems, etc.). We show that the twomachine makespan problem is NP-hard, but that a variety of models can be efficiently solved using the Modified Johnson's Rule and its extensions when our search is restricted to permutation schedules. We extend the analysis to the two-stage hybrid shop, and then to the $m$ machine shop, for which bounds and heuristics are given. Further results involving transfer lags are given in Chap. 4.


### 3.1 Preliminaries

Up to now, we have assumed that each successive operation of a job could begin processing as soon as (but no sooner than) the upstream operation was completed, subject only to machine availability. Now consider the possibility that an additional time delay must elapse between completing a job at one stage and starting it at the next. We call this extra delay a transfer lag, time lag, or simply lag. We emphasize: at least this much time must pass even if the required processor is ready and waiting. Let us define
$\ell_{i j}=$ the minimal additional delay or lag required, between completing $J_{j}$ at stage $i$ and starting it at stage $(i+1)$.

It follows that, when we do not mention lags, we are assuming $\ell_{i j}=0$. In a two-machine environment where each job has only one lag, we write $\ell_{1 j}$ as simply $\ell_{j}$. Note that throughout this section $M_{i}$ will denote either the unique processor at stage $i$ in a simple flow shop or any one of the identical processors in a hybrid shop.

### 3.1.1 Applications

Transfer lags are almost always present between stages of processing, but often they are short enough to be negligible. The most commonly encountered lag is probably the time to transport a job between processors. Another kind of lag arises when we must allow each job to undergo a self-contained process, such as cooling, paint drying, the settling of impurities out of a liquid, etc., before proceeding to the next stage of processing. Here are some more elaborate examples of lags.

- The time for processing on nonbottleneck processors. Suppose, for example, there are $m$ stations or machine groups through which each job must flow. Single processors at the first and last stations are the constraining resources. The intermediate stations are nonbottleneck, meaning that any number of tasks can be simultaneously processed. Perhaps there are many parallel machines at a station, or perhaps one machine, like a large furnace for heat treatment, can accommodate many jobs at once. Each job $j$ requires a certain amount of processing time $p_{i j}$ at each station $G_{i}$. Such a system can be viewed as a two-machine flow shop with lags, where $a_{j}=p_{1 j}, b_{j}=p_{m j}$, and $\ell_{j}=\sum_{i=2}^{m-1} p_{i j}$. It is, in fact, useful to observe that positive transfer lags are always equivalent to processing times at an intermediate nonbottleneck station (i.e., one with effectively unlimited processors).
- As detailed in Sect. 3.2.6, two-stage manufacturing cells, involving automated storage and retrieval, robotic machine loading/unloading, computerized conveyors, etc. can be efficiently modeled as flow shops with lags.
- Industrial applications of $F 2\left|\ell_{j}\right| C_{\max }$ include the case of consolidators that receive orders to procure quantities of various items whose manufacture is farmed out to subcontractors. The consolidator provides front-end support through its Supply Department and completes the transaction with its Delivery Department. The Supply Department assembles the raw materials (from the inventory), loads the trucks which deliver this material to the manufacturer, and performs an inspection before the consignment leaves. All of these make up the $a$-task of the consolidator. The actual manufacture of the orders makes up the transfer lag. After this, the $b$-task involves loading the finished goods onto the trucks of the Delivery Department and taking them to their destination.
- Another two-machine application is found in parallel computer scheduling. A common parallel programming paradigm employs the fork and join concept. In this, a single computational job is broken up into a thread of fork operations (or tasks) which involve collection and transmission of data needed by a remote processor. Each remote processor performs the necessary work to receive, and analyze the data. Upon execution of a task, the corresponding remote processor returns the results to a single postprocessing computational thread that synchronizes the completed tasks. Evidently, fork and join operations can be executed by two $n$ Cube hy-
percube processors, one responsible for preprocessing tasks, and the other for postprocessing tasks. They make up a flow shop, with the spawned threads functioning as lags. A particular application of this is the VLSI CAD problem that involves visualizations of objects in a 2-dimensional region. The region may be divided into $n$ parts by a stage -1 processor; each part is sent to a slave computer, and the results are returned to a postprocessing site.

In some situations, negative lags may arise. That is, it may be possible to start a job at a following stage before it is finished at the preceding stage. Such overlapping processing is sometimes called lap scheduling or lap phasing. Here are some ways this can happen.

- Setup times (except for those that are sequence-dependent, which are not in question here) are not considered explicitly when they are attached (the term used in Baker (2005)), meaning that they are tied to the jobs so that setup of $J_{j}$ on $M_{i}$ cannot begin until $J_{j}$ is present. In this case, setups can be simply added to processing times. However, when setups are separable, so that $M_{i+1}$ can be set up for $J_{j}$ while that job is still being processed on $M_{i}$, then the setup time can be treated as a negative lag.
- The job may be to construct or renovate a large machine or structure, with the processors being workers and equipment. For example, while exterior painting must follow installation of siding, windows, etc. on a new house, painting can start on one wall while construction continues on the others.
- Negative lags often arise when the operations to be performed on each processor are actually compound tasks, and each subtask can be passed on to the next processor when ready. For example, the job may be to manufacture many identical items (e.g., iron bars or plastic moldings), and may be subdivided into several transfer batches or sublots for scheduling flexibility (see Fig. 3.1a, further discussed below).


### 3.1.2 Types of Lags

So far, we have defined a lag as the minimal time from completing a job at one stage to starting it at the next. This is the most natural, intuitive meaning of the term, and we will continue to use it. However, other types of "lags" can be defined.

Example 3.1: $\quad$ Suppose a job is divided into $b$ equal batches, as in the final example above, and let $h_{1}$ and $h_{2}$ be the batch times (in hours) on $M_{1}$ and $M_{2}$, respectively (or on any two processors at successive stages). Now, although total processing time on $M_{1}$ is $b h_{1}$, the job can start on $M_{2}$ as early as $h_{1}$ hours after it starts on $M_{1}$ (see Fig. 3.1a for the case $h_{1}<h_{2}$ ). However, if $h_{1}>h_{2}$ (see Fig. 3.1b), $M_{2}$ will be idle between batches, which is wasted time. To economize, all batches but the last should be consolidated
on $M_{2}$, i.e., delayed until they are contiguous (see Fig. 3.1c). Now the timing constraint becomes: completion time on $M_{2}$ must be at least $h_{2}$ hours after completion time on $M_{1}$.

We can now recognize four types of lags, one or more of which could be a constraint on any job:

- A start-start lag, $\boldsymbol{S} \boldsymbol{S}_{\boldsymbol{i j}}$, often called simply a start lag, is the minimal additional delay required between starting $J_{j}$ on $M_{i}$ and starting it on $M_{i+1}$. In Example 3.1, with $h_{1}<h_{2}$, we had $S S_{1 j}=h_{1}$.
- A finish-finish lag, $\boldsymbol{F} \boldsymbol{F}_{\boldsymbol{i j}}$, or stop lag for short, is the minimal additional delay required between completing $J_{j}$ on $M_{i}$ and completing it on $M_{i+1}$. In Example 3.1, with $h_{1}>h_{2}$, we had $F F_{1 j}=h_{2}$.
- A start-finish lag, $\boldsymbol{S} \boldsymbol{F}_{\boldsymbol{i j}}$, is the minimal additional delay required between starting $J_{j}$ on $M_{i}$ and completing it on $M_{i+1}$.
- A finish-start lag, $\boldsymbol{F} \boldsymbol{S}_{\boldsymbol{i j}}$, is the minimal additional delay required between completing $J_{j}$ on $M_{i}$ and starting it on $M_{i+1}$. This is the same as our original lag $\ell_{i j}$.


Fig. 3.1a A job with $b$ transfer batches, with times $h_{1}=4, h_{2}=7$


Fig. 3.1b A job with $b$ transfer batches, with times $h_{1}=7, h_{2}=4$


Fig. 3.1c Same job with batches on $M_{2}$ consolidated to the right

Denoting the start time [finish time] of $J_{j}$ on $M_{i}$ as $S_{i j}\left[C_{i j}\right]$, the four constraints can be mathematically expressed:
$S_{i+1, j}-S_{i j} \geq S S_{i j}$
$C_{i+1, j}-C_{i j} \geq F F_{i j}$
$C_{i+1, j}-S_{i j} \geq S F_{i j}$
$S_{i+1, j}-C_{i j} \geq F S_{i j}$

Fortunately, it will not be necessary to work with all these lag types, because each one can be reduced to any other. Referring to Fig. 3.2 (which shows the two tasks scheduled as close together as possible), we can see how the other three types are equivalent to the basic (finish-start) lag:

$$
\begin{aligned}
& S S_{i j}=\ell_{i j}+p_{i j} \\
& F F_{i j}=\ell_{i j}+p_{i+1, j} \\
& S F_{i j}=\ell_{i j}+p_{i j}+p_{i+1, j} \\
& F S_{i j}=\ell_{i j}
\end{aligned}
$$

Thus, whenever a start-start lag is specified, we can replace it with a lag $\ell_{i j}=S S_{i j}-p_{i j}$; and similarly for the others.

Finally, we may need to satisfy more than one of these constraints. For instance, in Example 3.1, we may not know the relative magnitudes of $h_{1}$ and $h_{2}$ in advance. If so, we can simply require that both the start-start and the finish-finish constraints be observed. Since both specify earliest times at which the second operation can be processed, we take the maximum to enforce both. We conclude that the general expression for the lag of $J_{j}$ between $M_{i}$ and $M_{i+1}$ is

$$
\ell_{i j}=\max \left\{S S_{i j}-p_{i j}, F F_{i j}-p_{i+1, j}, S F_{i j}-p_{i j}-p_{i+1, j}, F S_{i j}\right\}
$$

where we only take the maximum over those lag types that are operative (others may be given the value $-\infty$ ). If no lags are defined, $\ell_{i j}=0$. In summary, then, when we introduce a lag $\ell_{i j}$, we are imposing the constraint: $S_{i+1, j} \geq C_{i j}+\ell_{i j}$.


Fig. 3.2
The basic finish-start lag, $\ell_{i j}$

Whenever we allow negative lags, we need to constrain how negative they can be, as mentioned in Johnson (1959). If, for instance, the second operation of a job could start earlier than the first $\left(\ell_{1 j}<-p_{1 j}\right)$, the very meaning of the term "flow shop" becomes ambiguous: the first job scheduled, for example, could start processing on $M_{2}$ before $M_{1}$. Similarly, if the magnitude of a negative lag exceeded the processing time at the following stage, say $\ell_{1 j}<-p_{2 j}$, the job might finish on $M_{2}$ before finishing on $M_{1}$. If this were true of the last job scheduled, the makespan could be determined on some machine other than the last. Of course, even more negative lags could move a succeeding task completely ahead of its predecessor. The following constraint alleviates these difficulties:

$$
\ell_{i j} \geq \max \left\{-p_{i j},-p_{i+1, j}\right\}=-\min \left\{p_{i j}, p_{i+1, j}\right\}
$$

and we shall assume it unless otherwise stated.

### 3.1.3 Notation

For problem classification, in the usual three-field notation, we will signal the presence of lags by introducing $\ell_{i j}$ (or $\ell_{j}$ if there are only two stages) in the second field. If the nature of the situation rules out negative lags, we write $\ell_{i j} \geq 0$. When listing the parameter values for an arbitrary job $J_{j}$ across all stages, we will write

$$
<p_{1 j}, p_{2 j}, \ldots, p_{m j} ; \ell_{1 j}, \ell_{2 j}, \ldots, \ell_{m-1, j}>
$$

When there are only two or three stages, we simplify the notation as usual by writing $p_{1 j}:=a_{j}, p_{2 j}:=b_{j}, p_{3 j}:=c_{j}$.

We review some other notation introduced in earlier chapters. The processing times for all jobs on an arbitrary machine $M_{i}$ will be given by the vector $\underline{p}_{i}=\left(p_{i 1}, p_{i 2}, \ldots, p_{i n}\right)$. A job sequence (or permutation schedule) that specifies the order $J_{u}, J_{v}, \ldots, J_{w}$ will be written $S=(u, v, \ldots, w)$. If $J_{j}$ has precedence over $J_{k}$, we write $j \rightarrow k$, while if $J_{j}$ has local priority over $J_{k}\left(J_{j}\right.$ is preferred before $J_{k}$ when they are adjacent), we use $j \leadsto k$.

### 3.2 The Two-Machine Flow Shop with Lags

Here, we limit ourselves to the case in which the flow shop has just two stages, and each stage has only one machine. As we have seen, when there are no lags, a permutation schedule is always optimal in a two-machine flow shop for any regular measure. With lags, this is no longer true. Consider, for example, the instance of $F 2\left|\ell_{j}\right| C_{\max }$ with two jobs whose parameters $<a_{j}, b_{j} ; \ell_{j}>$ are $<1,1 ; 6>$ and $<2,2 ; 2>$ for $j=1$ and 2, respectively. In Fig. 3.3a and b, the permutation schedules $(1,2)$ and $(2,1)$ are shown, each with $C_{\max }=10$. Clearly, Fig. 3.3c gives the optimal schedule, with $C_{\max }=8$.


Fig. 3.3 Why permutation schedules are not always optimal

Of course, there are frequently situations in practice where only permutation schedules are feasible. This may be due to production line or other physical constraints which make it impossible or at least impractical to rearrange the jobs once their sequence has been established. It may be because the information system at the first stage uses a queue to maintain a record
of job status. On the other hand, second stage processing may be simplified if a stack is used to keep track of work in process. This leads to a "last in, first out" strategy, in which the order of jobs on $M_{2}$ is the exact reverse of their order on $M_{1}$. Finally, in some settings, no constraints are operative and we seek the minimal makespan over all schedules.

These three versions of the two-machine makespan problem are denoted:

- $F 2 \mid$ perm, $\ell_{j} \mid C_{\max }$, when the job order cannot change from $M_{1}$ to $M_{2}$;
- $F 2 \mid$ rurs,$\ell_{j} \mid C_{\max }$, when the job order on $M_{2}$ is the reverse of that on $M_{1}$;
- $F 2\left|\ell_{j}\right| C_{\max }$, when the orders of jobs on each machine are unrelated;
and we will discuss them in turn.


### 3.2.1 $\mathbf{F} 2 \mid$ perm, $\ell_{j} \mid C_{\text {max }}$

It was Mitten (1959a,b) who first observed that the $O(n \log n)$ algorithm of Johnson (1954) could be extended to handle SS and FF lags (though, as pointed out in Johnson (1959), the same idea was used for a different application in Johnson (1954)). In fact, the result holds for general lags $\ell_{j}$ :

Theorem 3.1 Modified Johnson's Rule (MJR) (Mitten, 1959b)
For $F 2 \mid$ perm, $\ell_{j} \mid C_{\max }$, the optimal schedule is $S^{\star}=J R(\underline{a}+\underline{\ell}, \underline{b}+\underline{\ell})$.
That is, apply Johnson's Rule (JR) to the $n$ jobs as though they had processing times $a_{j}+\ell_{j}$ and $b_{j}+\ell_{j}$ on $M_{1}$ and $M_{2}$, respectively. This extension of JR to incorporate lags is generally referred to as the modified Johnson's Rule, and will be abbreviated MJR (which could also stand for the Mitten-Johnson Rule).

To see this, note that the digraph representing the task sequence of a typical schedule $S=(1,2, \ldots, n)$ looks like Fig. 3.4a. For future reference, we give in Fig. 3.4b the equivalent portrayal of a three-machine flow shop without lags. Recall how the arrows represent enforced sequence, and why the lags are therefore not interconnected by arrows, since lags are equivalent to a stage with many processors and no waiting. If instead we have a middle station with just one machine, tasks must wait their turn for processing there, hence the additional arrows in Fig. 3.4b. Although finding the longest path through the latter digraph is difficult (see Sect. 4.7), Theorem 3.1 tells us that the longest path through the graph of Fig. 3.4a is easily found.

Proof of Theorem 3.1: For the schedule $S=(1,2, \ldots, n)$, Fig. 3.4a shows that the makespan (i.e., the longest path) can be expressed:

$$
\begin{aligned}
C_{\max }(S) & =\max _{j=1, \ldots, n}\left[\sum_{i=1}^{j} a_{i}+\ell_{j}+\sum_{i=j}^{n} b_{i}\right] \\
& =\max _{j=1, \ldots, n}\left[\sum_{i=1}^{j}\left(a_{i}+\ell_{i}\right)+\sum_{i=j}^{n}\left(b_{i}+\ell_{i}\right)-\sum_{i=1}^{n} \ell_{i}\right]
\end{aligned}
$$

Since the final sum is sequence-independent, the result is established.


Fig. 3.4 Directed graph representation of schedule ( $1, \ldots, n$ )
for (a) $F 2 \mid$ perm, $\ell_{j} \mid C_{\max }$, and (b) $F 3 \mid$ perm $\mid C_{\text {max }}$

Johnson's relation now becomes
Theorem 3.2 F2|perm, $\ell_{j} \mid C_{\max }$ has the API property, with
$j \sim k \Longleftrightarrow \min \left(a_{j}+\ell_{j}, b_{k}+\ell_{k}\right)<\min \left(a_{k}+\ell_{k}, b_{j}+\ell_{j}\right)$
or, in terms of a priority index,
$j \leadsto k \Longleftrightarrow I(j)<I(k), \quad$ where $I(j)=\left\{\begin{array}{cl}-1 /\left(a_{j}+\ell_{j}\right), & a_{j} \leq b_{j} \\ 1 /\left(b_{j}+\ell_{j}\right), & a_{j}>b_{j}\end{array}\right.$
All the algorithms for $F 2 \| C_{\max }$ can be modified accordingly. For example:
Theorem 3.3 (Modified Johnson's Rule - Version 2)
For $F 2 \mid$ perm, $\ell_{j} \mid C_{\max }$, let
$\mathcal{F}=\left\{j: a_{j}<b_{j}\right\}$, and $\mathcal{L}=\left\{j: a_{j} \geq b_{j}\right\}$.
Then $S^{\star}=\left(\mathcal{F}: \nearrow\left(a_{j}+\ell_{j}\right), \quad \mathcal{L}: \searrow\left(b_{j}+\ell_{j}\right)\right)$
Example 3.2: In a three-stage flow shop, there are only single machines at stations $G_{1}$ and $G_{3}$, but enough processors at the intermediate station that no job ever has to wait here. At each stage, there is a setup time $s_{i j}$ as well as a processing time $p_{i j}(i=1,2,3 ; j=1, \ldots, n)$. All setups are separable: they can be done in advance of the job's arrival from the previous stage. We seek a permutation schedule of the following 5 jobs to complete all the work as soon as possible.

| $j$ | $s_{1 j}$ | $p_{1 j}$ | $s_{2 j}$ | $p_{2 j}$ | $s_{3 j}$ | $p_{3 j}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 3 | 3 | 2 | 8 | 4 | 8 |
| 2 | 5 | 8 | 2 | 5 | 8 | 3 |
| 3 | 3 | 7 | 2 | 9 | 3 | 4 |
| 4 | 3 | 4 | 2 | 9 | 4 | 6 |
| 5 | 7 | 5 | 2 | 3 | 7 | 6 |

First, observe that $G_{1}$ and $G_{3}$ can be treated as a two-machine flow shop, with $G_{2}$ providing a positive (finish-start) lag of $p_{2 j}$ for job $j$. Since each job effectively has its own dedicated processor at $G_{2}$, the setups there can be done in advance and so may be ignored for scheduling purposes. The third stage
setup contributes a negative component to the finish-start lag, as $M_{3}$ can be prepared while the job is still in process elsewhere. In Fig. 3.5, a typical job is shown, scheduled as early as possible.


Fig. 3.5 Earliest start times on three machines for a job with separable setups

We conclude that we effectively have an instance of $F 2 \mid$ perm, $\ell_{j} \mid C_{\max }$, with $a_{j}=s_{1 j}+p_{1 j}, \ell_{j}=p_{2 j}-s_{3 j}$, and $b_{j}=s_{3 j}+p_{3 j}$. In the following table, these parameters are calculated, and we find the optimal schedule, $S^{\star}=J R(\underline{a}+\underline{\ell}, \underline{b}+\underline{\ell})$ using Theorem 3.3. With $\mathcal{F}=\{1,4,5\}$ and $\mathcal{L}=\{2,3\}$, we compute the appropriate values of $a_{j}+\ell_{j}$ or $b_{j}+\ell_{j}$ and get $S^{\star}=(5,1,4,3,2)$.

| $j$ | $a_{j}$ | $\ell_{j}$ | $b_{j}$ | $a_{j}+\ell_{j}$ | $b_{j}+\ell_{j}$ |
| :---: | ---: | ---: | ---: | :---: | :---: |
| 1 | 6 | 4 | 12 | 10 |  |
| 2 | 13 | -3 | 11 |  | 8 |
| 3 | 10 | 6 | 7 |  | 13 |
| 4 | 7 | 5 | 10 | 12 |  |
| 5 | 12 | -4 | 13 | 8 |  |

The subject of separable setups will be discussed in greater generality in Sect. 4.18; for the two-machine case, see Theorem 4.1.

### 3.2.2 When are Permutation Schedules Optimal?

As we know, in the absence of lags (i.e., when $\ell_{i j}=0$ for all $i, j$ ), Johnson's Rule (Johnson, 1954) solves the two-stage flow shop in time $O(n \log n)$, giving a permutation schedule that is optimal over all schedules. Mitten's extension, above, finds the optimal permutation schedule when lags are present, still in $O(n \log n)$, but we have seen that nonpermutation schedules may be superior. However, we can state conditions under which a permutation schedule is always optimal. Both Lemma 3.1 and Theorem 3.4 (somewhat reinterpreted) come from Johnson (1959).

Lemma 3.1 For $F 2\left|\ell_{j}\right| C_{\text {max }}$, once the job order on $M_{1}$ is fixed, it is optimal to schedule jobs on $M_{2}$ in order of availability.

That is, the schedule on $M_{2}$ is $S_{2}^{\star}=\nearrow\left(C_{j 1}+\ell_{j}\right)$. The proof, a simple Adjacent Pair Interchange (API) argument, is omitted. As a consequence, we can say:

Theorem 3.4 (Johnson, 1959) For $F 2\left|\ell_{j}\right| C_{\max }$, if

$$
\ell_{i} \leq \min _{j}\left\{\ell_{j}+\max \left(a_{j}, b_{j}\right)\right\}, \text { for all jobs } i,
$$

then a permutation schedule is optimal over all schedules.
Proof: We need to show that, for the theorem to hold, it is sufficient that either $\ell_{i} \leq \ell_{j}+a_{j}$ or $\ell_{i} \leq \ell_{j}+b_{j}$ for each pair of jobs. First, corresponding to any instance of $F 2\left|\ell_{j}\right| C_{\max }$, consider the backward instance in which the roles of $a_{j}$ and $b_{j}$ are interchanged; or equivalently, in which the jobs must visit $M_{2}$ before $M_{1}$. By symmetry, every schedule of the original instance can be paired with a schedule of the backward instance in which the tasks are done in reverse order on each machine, both schedules having the same makespan. Thus, the backward instance is really the same instance with time reversed, and has the same minimal makespan.

Now consider any feasible schedule that is not permutation. There must be a pair of jobs, $J_{i}$ and $J_{j}$, which are consecutive on $M_{1}$ with $J_{i}$ before $J_{j}$, but in the reverse order (and not necessarily consecutive) on $M_{2}$.

- If $\ell_{i} \leq \ell_{j}+a_{j}$, then on $M_{2}$ move $J_{i}$ to the position immediately before $J_{j}$, allowing $J_{j}$ and perhaps other tasks on $M_{2}$ to shift later.
The new schedule is feasible, since $J_{i}$ is schedulable on $M_{2}$ at least as early as $J_{j}$ was, and all other tasks are at the same time or later. The makespan can only be shorter.
- If $\ell_{i} \leq \ell_{j}+b_{j}$, consider the backward instance. Now $b_{j}$ plays the role of $a_{j}$, and we interchange $J_{i}$ and $J_{j}$ on $M_{1}$ (the tasks being contiguous in this case). By symmetry, we can draw all the same conclusions.

Clearly, we can repeat the above operation as often as necessary, reordering pairs of jobs until the two sequences agree.

### 3.2.3 $F 2 \mid$ rves,$\ell_{j} \mid C_{\max }$

When we impose the requirement that the order of processing of jobs on $M_{2}$ must be precisely the reverse of the order on $M_{1}$, the problem is greatly simplified. Without lags, of course, the first job done on $M_{2}$ must be the last on $M_{1}$, so there can be no overlap, and we quickly see that any sequence is optimal, giving $C_{\max }=\sum_{i=1}^{n}\left(a_{j}+b_{j}\right)$.

With nonnegative lags, it remains true that all tasks on $M_{1}$ must be completed prior to initiating any task on $M_{2}$. Note how this model could reflect a situation in which all tasks are actually processed on the same machine. It has been studied as a master-slave model (see Sahni, 1993), in which both tasks of each job are done on a single master processor, while the lag
represents an intermediate task for which a large number of slave processors are available. Sahni (1993) presented the following $O(n \log n)$ algorithm under the assumption that $\ell_{j} \geq 0$, but it carries over without difficulty to the case of unconstrained lags.

Theorem 3.5 (Sahni, 1993) For $F 2\left|r v r s, \ell_{j}\right| C_{\max }$,

$$
S^{\star}=\left(S_{1}^{\star}, S_{2}^{\star}\right)=\left(\searrow \ell_{j}, \nearrow \ell_{j}\right),
$$

where $S_{i}^{\star}=$ the optimal schedule on $M_{i}$. In case some jobs have equal lags, ties may be broken arbitrarily provided $S_{2}^{\star}$ is the reverse of $S_{1}^{\star}$.
Proof: Consider any feasible schedule, $S$, other than $S^{\star}$. There must be two adjacent jobs on $M_{1}, J_{j}$ and $J_{k}$, with $J_{j}$ before $J_{k}$ but $\ell_{j}<\ell_{k}$. By the reverse requirement, $J_{k}$ must immediately precede $J_{j}$ on $M_{2}$. Now consider schedule $S^{\prime}$ which is identical to $S$ except that $J_{j}$ and $J_{k}$ are interchanged on both machines. Because $\ell_{j}<\ell_{k}, J_{j}$ can start at least as early on $M_{2}$ as $J_{k}$ did in $S$; earlier, if possible. Thus the makespan of $S^{\prime}$ can be no longer than that of $S$, and may be shorter. This process of job interchange can, of course, be repeated as often as needed to generate $S^{\star}$.

### 3.2.4 $F 2\left|\ell_{j}\right| C_{\text {max }}$

Lenstra (unpublished) is credited with showing that the version of our problem with unrestricted sequencing is strongly NP-complete (see, e.g., Lawler et al. (1993)). We present below a proof due to Dell'Amico (1996), who also shows that, even if preemptions are allowed, $F 2\left|p m t n, \ell_{j}\right| C_{\max }$ remains strongly NP-complete. He also gives some approximate approaches to the problem without preemption. The rest of this section is based on Dell'Amico (1996), where much more detail and other algorithms and results are to be found.

## The Complexity of $\boldsymbol{F} 2\left|\ell_{\boldsymbol{j}}\right| C_{\text {max }}$

Theorem 3.6 $F 2\left|\ell_{j}\right| C_{\max }$ is strongly $N P$-complete.
Proof Outline (Dell'Amico, 1996): We reduce the following strongly NPcomplete problem:

## 3-PARTITION

INSTANCE: An integer $V$, and $3 k$ positive integers $v_{i}: i \in \mathcal{T}=\{1,2, \ldots, 3 k\}$, with $\Sigma_{i \in \mathcal{T}} v_{i}=k V$ and $V / 4<v_{i}<V / 2$, for all $i \in \mathcal{T}$.
QUESTION: Can $\mathcal{T}$ be partitioned into $k$ disjoint sets $\mathcal{S}_{1}, \mathcal{S}_{2}, \ldots, \mathcal{S}_{k}$ with $\left|\mathcal{S}_{j}\right|=3$ and $\Sigma_{i \in \mathcal{S}_{j}} v_{i}=V$ for $j=1,2, \ldots, k$ ?
to the decision version of our problem:
$F \mathbf{2}\left|\ell_{j}\right| C_{\max } \leq \boldsymbol{B}$ ?
INSTANCE: An integer $B$, and $n$ jobs $J_{j}: j \in \mathcal{N}=\{1,2, \ldots, n\}$, each with
parameters $\left\langle a_{j}, b_{j} ; \ell_{j}\right\rangle$, to be scheduled in a two-machine flow shop, where $a_{j}\left[b_{j}\right]$ is the processing time of $J_{j}$ on $M_{1}\left[M_{2}\right]$, and $\ell_{j}$ is the lag between $M_{1}$ and $M_{2}$.
QUESTION: Does there exist a schedule with $C_{\max } \leq B$ ?
For any instance of 3-Partition, define an instance of $F 2\left|\ell_{j}\right| C_{\max }$ as follows:

- $B:=2 V(k+1)$
- $n:=4 k+2$
- $\left.\left\langle a_{j}, b_{j} ; \ell_{j}\right\rangle:=<v_{j}, v_{j}, V-v_{j}\right\rangle, \quad j \in \mathcal{T}$
- $\left\langle a_{j}, b_{j} ; \ell_{j}\right\rangle:=<V, V, 2 V>, \quad j=3 k+1, \ldots, 4 k$
- $\left.<a_{j}, b_{j} ; \ell_{j}\right\rangle:=<0,2 V, 0>, \quad j=4 k+1$
- $\left.<a_{j}, b_{j} ; \ell_{j}\right\rangle:=<2 V, 0,0>, \quad j=4 k+2$

We claim that, if one of the two instances has the answer "yes", then so does the other. First, note that $B=\sum_{j=1}^{n} a_{j}=\sum_{j=1}^{n} b_{j}$, which is an obvious lower bound on the makespan. Any schedule achieving this makespan would have no idle time on either machine and would certainly be optimal. This can occur only when the "filler" jobs, namely $\left\{J_{j}: j=3 k+1, \ldots, 4 k+2\right\}$, are scheduled precisely as shown in Fig. 3.6. This leaves the other jobs, $\left\{J_{j}: j \in \mathcal{T}\right\}$ to be fitted precisely into the remaining gaps, which is possible only if we can partition them into $k$ subsets whose processing times on each machine add to $V$.


Fig. 3.6 Optimal schedule for NP-complete instance of $F 2\left|\ell_{j}\right| C_{\max } \leq B$ ? .

## Lower Bounds on $C_{\text {max }}$

The following lower bounds, denoted $L B$, are easily obtained. They help us assess how good a heuristic solution is, and contribute to the design of heuristic algorithms. We mention in passing the simplest bounds: $\Sigma_{j} a_{j}, \Sigma_{j} b_{j}$, and $\max _{j}\left\{a_{j}+\ell_{j}+b_{j}\right\}$, which collectively have a worst case performance ratio of 0.5 . That is, the maximum of them is guaranteed to be at least half the optimal value. In computational experiments, they were greatly outperformed by the following bounds, which are based on relaxed versions of the basic problem, though these too have performance ratios of 0.5 .

- Define Problem $R$ as the relaxation of $F 2\left|\ell_{j}\right| C_{\max }$ in which we allow task overlap on $M_{1}$ (i.e., $M_{1}$ behaves like $n$ parallel identical machines). Thus, both stage 1 and the lag are nonbottleneck, and together they effectively provide release times for $M_{2}$. The problem of scheduling $M_{2}$ is essentially the same one covered by Lemma 3.1, and the same solution results, namely:
$\boldsymbol{R}: 1\left|r_{j}\right| C_{\max }$, with $r_{j}=a_{j}+\ell_{j}, p_{j}=b_{j}$, has $S^{\star}=\nearrow r_{j}$.
Now, $L B_{R}=C_{\max }\left(/ r_{j}\right)$.
- Similarly, if we make $M_{2}$ nonbottleneck, we get a single machine problem, call it Problem $Q$, where each $J_{j}$ has a delivery time $q_{j}=\ell_{j}+b_{j}$ and we wish to schedule the jobs on $M_{1}$ to minimize the latest time of delivery: $\max _{j}\left\{C_{j 1}+q_{j}\right\}$, which we abbreviate $(C+q)_{\max }$. As with Problem $R$, a simple API argument on $M_{1}$ shows that
$Q: 1\left|q_{j}\right|(C+q)_{\max }$, with $q_{j}=\ell_{j}+b_{j}, p_{j}=a_{j}$, has $S^{\star}=\searrow q_{j}$, and we have $L B_{Q}=C_{\max }\left(\searrow q_{j}\right)$.
- Another way to relax $F 2\left|\ell_{j}\right| C_{\max }$ to get an easily solved problem is to shorten the lags until a permutation schedule is optimal, according to Theorem 3.4. Thus, for Problem $T$, suppose we define new lags

$$
\ell_{j}^{\prime}=\min \left\{\ell_{j}, \min _{i}\left\{\ell_{i}+\max \left(a_{i}, b_{i}\right)\right\}\right\} .
$$

Then $\ell_{j}^{\prime} \leq \min _{i}\left\{\ell_{i}+\max \left(a_{i}, b_{i}\right)\right\}=\min _{i}\left\{\ell_{i}^{\prime}+\max \left(a_{i}, b_{i}\right)\right\}$, and so
$\boldsymbol{T}: F 2\left|\ell_{j}^{\prime}\right| C_{\max }$ has $S^{\star}=J R\left(\underline{a}+\underline{\ell}^{\prime}, \underline{b}+\underline{\ell}^{\prime}\right)$.
Since shortening the lags can only shorten the minimal makespan, we get $L B_{T}=C_{\max }\left(J R\left(\underline{a}+\underline{\ell}^{\prime}, \underline{b}+\underline{\ell}^{\prime}\right)\right)$.

## Heuristic Algorithms

Several heuristics (i.e., approximation algorithms) result from the above lower bounds. Keep in mind that a heuristic gives a feasible solution and so the makespan it generates is an upper bound, $B$, on the optimum.

- Algorithm $\mathrm{A}_{\mathrm{Q}}$ starts by solving Problem $Q$ to determine the sequence on $M_{1}$. Then, using Lemma 3.1, it schedules $M_{2}$ in order of availability. This is a feasible solution of the original problem, and the resulting makespan gives a bound $B_{Q}=C_{\max }^{\star}\left(A_{Q}\right)$.
- A similar algorithm, $\mathrm{A}_{\mathrm{R}}$, can be based on Problem $R$ with time reversed, producing another bound, $B_{R}=C_{\max }^{\star}\left(A_{R}\right)$.
- For a heuristic, $\mathrm{A}_{\mathrm{T}}$, based on adjusting the sizes of the lags, we increase the shortened lags, $\ell_{j}^{\prime}$, of Problem $T$ until they are all bigger than the original values $\ell_{j}$ (so that the makespan will give an upper bound), but in such a way that a permutation schedule will still be optimal. The inequality required for Theorem 3.4,

$$
\ell_{j}^{\prime} \leq \min _{i}\left\{\ell_{i}^{\prime}+\max \left(a_{i}, b_{i}\right)\right\},
$$

will be maintained if we add the same constant, $K$, to all lags. To make the bound as tight as possible, we set $K$ at the smallest value that will produce new lags, $\ell_{j}^{\prime \prime}=\ell_{j}^{\prime}+K$ satisfying $\ell_{j}^{\prime \prime} \geq \ell_{j}$ for all $j$. Thus, choose $K=$ $\max _{j}\left\{\ell_{j}-\ell_{j}^{\prime}\right\}$. Now, the solution of $F 2\left|\ell_{j}^{\prime \prime}\right| C_{\max }$ is $S^{\star}=J R\left(\underline{a}+\underline{\ell}^{\prime \prime}, \underline{b}+\underline{\ell}^{\prime \prime}\right)$, and this schedule, with the original $\ell_{j}$ replacing $\ell_{j}^{\prime \prime}$, produces bound, $B_{T}$.

Each of the three heuristics has a worst case performance ratio of 2 (i.e., the values obtained are never more than twice the true optimal makespan). However, computational results for a variety of randomly generated instances with up to 1000 jobs show much better average performance, which rapidly improves as problem sizes grow. The upper bound $B_{m}=\min \left\{B_{Q}, B_{R}, B_{T}\right\}$ generated by using the best of the three schemes was tested against the optimal makespan.

It was found that $B_{m}$ performs worst when lags tended to be larger than processing times. When lags averaged 5 times processing times, and for small (up to 200 jobs) problem instances, $B_{m}$ rarely equalled the optimum and averaged $6 \%$ over it. For instances with 400 to 1000 jobs, the optimum was found about half the time, and the error averaged less than $1 \%$.

When processing times averaged the same size as the lags or larger, the heuristic performed superbly, finding the optimum in $88 \%$ of small instances generated, and virtually all large ones.

### 3.2.5 $\mathbf{F 2 |} \mid$ (perm), prec| $C_{\text {max }}$

This problem does not seem to belong in the chapter on lags. However, as we shall see, certain special cases can be easily solved by grouping jobs without lags into "composite jobs" with lags. First, it is clear that, for the basic two-machine flow shop, even with precedence constraints added, there always exists a permutation schedule that minimizes any measure, even if such schedules are not mandated by technological or other requirements. A precedence constraint between two jobs, by definition, already forces the specified job order on both machines. For jobs not constrained by precedence, Theorem 1.1 applies. Thus, we will only consider permutation schedules in this section.

Now, recall that a chain of jobs must be executed in a given order, though not necessarily consecutively. By contrast, a group or contiguous set of jobs must be processed without interruption, but may be done in any order. A string of jobs has both properties: on every machine, all the work must be done continuously and in the same prespecified sequence. We assume now that the jobs to be scheduled in a two-machine flow shop form parallel chains or parallel groups or parallel strings: they can be partitioned into independent subsets, each of which is of one of these three types. We start with the most constrained case, which is easiest to handle, in which a set of parallel strings is to be scheduled to minimize makespan.

## F2|(perm), strings $\mid C_{\text {max }}$

Consider the early-start schedule of any string $\boldsymbol{S}$ in isolation, as illustrated in Fig. 3.7a for a five-job string. Its makespan, $C$, is easily computed. Suppose $\boldsymbol{S}$ consists of $u$ jobs having processing times $a_{i}$ and $b_{i}(i=1, \ldots, u)$ on $M_{1}$ and $M_{2}$, respectively. Since the idle time between certain tasks on $M_{2}$ is wasted, we can right-shift or consolidate tasks as shown in Fig. 3.7b without affecting
the makespan, thereby freeing up time on $M_{2}$ that an earlier job may utilize. Once each string has been consolidated, which involves a small amount of calculation, it becomes indistinguishable from a single job (we shall call it a composite job) with processing times $A=\sum_{i=1}^{u} a_{i}$ and $B=\Sigma_{i=1}^{u} b_{i}$, and with a negative lag $\ell=-(A+B-C)$, as shown in the figure. We assume hereafter that all strings are consolidated into composite jobs before scheduling, and denote the composite job for string $j$ by $\boldsymbol{S}_{\boldsymbol{j}}$.

If there are $w$ strings to be scheduled, with composite job $\boldsymbol{S}_{\boldsymbol{j}}(j=1, \ldots, w)$ having parameters $\left\langle A_{j}, B_{j} ; \ell_{j}\right\rangle$, then of course the optimal permutation schedule is given by MJR. We have seen that, in a two-machine flow shop with lags, a permutation schedule may not be the overall optimum. We know this is not a problem here, since it was not, in the original formulation. If further reassurance is needed, we can call upon Theorem 3.4. The lags for a composite job are defined in such a way that $\ell_{j} \leq 0$ and $\ell_{j}+\max \left(A_{j}, B_{j}\right)>0$ for all $j$, so the theorem holds and a permutation schedule is optimal over all schedules. To summarize:

Theorem 3.7 For $F 2 \mid($ perm $)$, strings $\mid C_{\max }$, compute for each composite job $\boldsymbol{S}_{j}(j=1, \ldots, w)$ its total processing times $A_{j}$ and $B_{j}$ on $M_{1}$ and $M_{2}$, its completion time in isolation $C_{j}$, and $\ell_{j}=-\left(A_{j}+B_{j}-C_{j}\right)$. Then:

$$
S^{\star}=J R(\underline{A}+\underline{\ell}, \underline{B}+\underline{\ell})=J R(\underline{C}-\underline{B}, \underline{C}-\underline{A})
$$



Fig. 3.7a Schedule with earliest start times on two machines


Fig. 3.7b Schedule with work consolidated to the right on $M_{2}$

## F2|(perm), chains $\mid C_{\text {max }}$

To schedule parallel chains, we shall first combine sequential jobs in chains into strings. That is, we give conditions under which two jobs ordered by precedence may be scheduled contiguously in an optimal schedule, and thus made into a two-job string. Indeed, our results will enable us to tie two sequential strings together into one long string. Of course, each new string is immediately consolidated into a composite job. Since one job is already a
rudimentary string, we shall use the term (composite) job to encompass both single jobs and composite jobs. Recall that precedence of $J_{j}$ over $J_{k}$ is denoted $j \rightarrow k$ while preference of $J_{j}$ over $J_{k}$ (that is, local preference, meaning, in the case of the two-machine flow shop with makespan objective, that JR would schedule $J_{j}$ before $J_{k}$ when the two jobs are adjacent) is indicated by $j \leadsto k$. Generalizing these ideas to composite jobs, $\boldsymbol{S}_{\boldsymbol{j}}$ has precedence over $\boldsymbol{S}_{\boldsymbol{k}}$, denoted $\boldsymbol{j} \rightarrow \boldsymbol{k}$ whenever there exists $J_{u}$ in $\boldsymbol{S}_{\boldsymbol{j}}$ and $J_{v}$ in $\boldsymbol{S}_{\boldsymbol{k}}$ with $u \rightarrow v$; and $\boldsymbol{S}_{\boldsymbol{j}}$ is preferred over $\boldsymbol{S}_{\boldsymbol{k}}$, written $\boldsymbol{j} \leadsto \boldsymbol{k}$ whenever MJR puts $\boldsymbol{S}_{\boldsymbol{j}}$ before $S_{k}$.

Theorem 3.8 For $F 2 \mid($ perm $)$, chains $\mid C_{\max }$, suppose $\boldsymbol{S}_{\boldsymbol{j}}$ and $\boldsymbol{S}_{\boldsymbol{k}}$ are consecutive (composite) jobs in a chain, with $\boldsymbol{j} \rightarrow \boldsymbol{k}$ but $\boldsymbol{k} \sim \boldsymbol{j}$. Then there is an optimal schedule in which $\boldsymbol{S}_{\boldsymbol{k}}$ immediately follows $\boldsymbol{S}_{\boldsymbol{j}}$, forming a single string ( $\boldsymbol{S}_{\boldsymbol{j}}, \boldsymbol{S}_{\boldsymbol{k}}$ ).
Proof: Let $S=\left(s_{1}, \boldsymbol{S}_{\boldsymbol{j}}, s_{2}, \boldsymbol{S}_{\boldsymbol{k}}, s_{3}\right)$ be any feasible schedule, where $s_{r}(r=$ $1,2,3$ ) are sequences of (composite) jobs which together include all other jobs, and $s_{2}$ is not empty. Note that, by the assumption of chain precedence, there is no required precedence between $\boldsymbol{S}_{\boldsymbol{j}}$ and $s_{2}$, or between $s_{2}$ and $\boldsymbol{S}_{\boldsymbol{k}}$. We write the makespan of a schedule $S$ as $C(S)$. We shall show that another schedule always exists with $\boldsymbol{S}_{\boldsymbol{j}}$ and $\boldsymbol{S}_{\boldsymbol{k}}$ adjacent which is at least as good (has makespan at least as small). Consider $s_{2}$ as a single (composite) job, $\boldsymbol{S}_{\mathbf{2}}$. If $\boldsymbol{k} \leadsto \mathbf{2}$, then the schedule with $\boldsymbol{S}_{\boldsymbol{k}}$ moved ahead of $\boldsymbol{S}_{\boldsymbol{2}}$ is preferred; that is, $C\left(s_{1}, \boldsymbol{S}_{\boldsymbol{j}}, \boldsymbol{S}_{\boldsymbol{k}}, s_{2}, s_{3}\right) \leq C\left(s_{1}, \boldsymbol{S}_{\boldsymbol{j}}, s_{2}, \boldsymbol{S}_{\boldsymbol{k}}, s_{3}\right)$. On the other hand, if $\mathbf{2} \leadsto \boldsymbol{k}$, then by the transitivity of the preference relation, $\mathbf{2} \leadsto \boldsymbol{j}$ so we prefer $\boldsymbol{S}_{\boldsymbol{j}}$ after $s_{2}$. One way or the other, we have united $\boldsymbol{S}_{\boldsymbol{j}}$ and $\boldsymbol{S}_{\boldsymbol{k}}$.

The following algorithm given by Sidney (1979) derives from one in Kurisu (1976).

## Parallel Chains Algorithm (PCA)

1. For each chain, combine (composite) jobs into larger composite jobs as much as possible, using Theorem 3.8.
2. Schedule the remaining (composite) jobs using MJR, without further regard for precedence.

Note how two consecutive (composite) jobs in a chain will be combined into one at Step 1, if preference contradicts precedence. Thus, when we reach Step 2 , preference ordering within each chain will be identical with precedence, so scheduling all the remaining (composite) jobs by preference (using MJR) will automatically satisfy precedence requirements.

F2 $\mid$ (perm), groups $\mid C_{\text {max }}$
If we are given a collection of groups of jobs to schedule, we must order the jobs within each group, and simultaneously sequence the groups. The following simple procedure can be shown to be optimal; proof omitted.

## Parallel Groups Algorithm

1. Order each group independently to achieve minimal makespan, using JR. Each group is now a string, which can be consolidated into a composite job.
2. Order the composite jobs using MJR.

## More General Precedence Requirements

As discussed in Sidney (1979), the Parallel Chains Algorithm (PCA) may be used to simplify, and sometimes fully sequence, more complex precedence networks. We simply look for parallel chain subnetworks and sequence them (that is, reduce each to a single chain) using PCA. A parallel chain subnetwork is a set of parallel chains within a network such that the initial jobs of each chain have the same set of predecessors, and the terminal jobs all have the same set of successors. Thus, the single chain generated by PCA can be substituted for the parallel chains, connecting the same predecessors to the same successors. The simplified network thus produced may contain other parallel chain subnetworks, which can be similarly sequenced. In some cases, notably for parallel in-tree, out-tree, and more generally series-parallel networks, such repeated application of PCA will fully sequence the jobs, producing a feasible schedule with minimal makespan (for definitions and further results, see Sidney (1979), Monma (1979), and Monma and Sidney ((1979)).

For completely general precedence structures, Monma (1980) showed that $F 2 \mid$ perm, prec $\mid C_{\max }$ is NP-complete. Hariri and Potts (1984) devise three branch-and-bound algorithms, the best of which can easily handle 50 jobs.

### 3.2.6 Two-Stage Manufacturing Cells as Flow Shops with Lags

Modern manufacturing systems may involve automated storage and retrieval stations, robotic loading and unloading of machines, transportation robots, etc. Many such steps in the production process introduce significant delays, interrelated by precedence, into the scheduling of each job. We wish to consider the sequencing of $n$ jobs in such two-machine robotic cells, to minimize makespan. We first discuss how to model such a facility as a flow shop with lags, and then illustrate our approach with an application presented in Kogan and Levner (1998).

It will simplify our presentation to use the task-on-arc representation of precedence structures in this section. The typical two-machine flow shop with lags, shown in Fig. 3.8(a) in the task-on-node style we have used up to now, is given with tasks on arcs in Fig. 3.8(b). The nodes now represent the times at which the jobs start and end on each machine. Note how the new presentation enables us to differentiate among the four types of lags. Since $\ell_{j}$ is a FS (finishstart) lag, it runs from the end of $a_{j}$ (node 2) to the start of $b_{j}$ (node 3). If
$S S_{j}$ (a start-start lag for $J_{j}$ ) were specified, it would be a task connecting node 1 to node 3 , etc.

Now in a complex shop where the timing of each task is constrained by a variety of delays, some forced to follow others, some constant and some jobrelated, the precedence structure of each job may constitute a quite elaborate network, $N_{j}$, as we try to illustrate in Fig. 3.8(c). In such a case, proceed as follows.

## Lag Algorithm

0 . Let $N_{j}$ be the network of tasks interconnecting the four nodes denoting the start and end of $J_{j}$ on each machine, and set $j:=1$.

1. Let $a_{j}$ equal the length of the longest path in $N_{j}$ between Nodes 1 and 2.
2. Let $b_{j}$ equal the length of the longest path in $N_{j}$ between Nodes 3 and 4.
3. Let $S S_{j}, S F_{j}, F S_{j}$ and $F F_{j}$ equal the length of the longest path in $N_{j}$, if any, between Nodes $1 \& 3,1 \& 4,2 \& 3$, and $2 \& 4$, respectively. In each case, if there are no such paths, set the corresponding lag to $-\infty$.
4. If all four lags are set to $-\infty$, let $\ell_{j}=0$. Otherwise let

$$
\ell_{j}=\max \left\{S S_{j}-a_{j}, F F_{j}-b_{j}, S F_{j}-a_{j}-b_{j}, F S_{j}\right\} .
$$

5. If $j<n$ set $j:=j+1$ and go to Step 1. Otherwise, continue.

6 . The optimal schedule, $S^{\star}=J R(\underline{a}+\underline{\ell}, \underline{b}+\underline{\ell})$.


Fig. 3.8 The two-machine shop, portrayed (a) with tasks on nodes, (b) with tasks on arcs, and (c) with additional delays complicating the scheduling of each job.

All we have done, of course, is to replace a complex network as in Fig. 3.8(c) with an equivalent simplified network as in Fig. 3.8(b), for which the solution is known.

We have made an important assumption: $N_{j}$ and its parameter values, and hence the values we compute for $a_{j}, b_{j}$ and $\ell_{j}$, must be independent of the job sequence. This will cause minor complications at the start and end of schedules.

Example 3.3: A manufacturing cell with two Computer Numerically Controlled (CNC) machines served by robots is analyzed by Kogan and Levner (1998). Robots are also used to serve Automatic Storage/Retrieval Stations
(AS/RS). Each job requires special fixtures and tooling at each machine, for which the setup times may be significant. Jobs are transported on pallets by a circulating conveyor. In a layout shown schematically in Figure 3.9(a), a typical job advances through the production process as follows, where the time required for each operation is given in square brackets.

1. A pallet with $J_{j}$ and any needed fixtures is unloaded from the Input AS/RS by robot $R_{i n}$, and conveyed to $M_{1}\left[t_{A}\right]$.
2. Robot $R_{1}$ loads job and fixture for $M_{1}$ onto $M_{1}\left[l_{1 j}\right]$, and fixture is set up [ $f_{1 j}$ ], simultaneously with tool setup [ $s_{1 j}$ ].
3. $J_{j}$ is processed on $M_{1}\left[p_{1 j}\right]$. In parallel, pallet delivers its fixture for $M_{2}$ to $M_{2}\left[t_{A B}\right]$.
4. Fixture is set up on $M_{2}\left[f_{2 j}\right]$, at the same time as tooling [ $\left.s_{2 j}\right]$.
5. $R_{1}$ unloads $J_{j}$ from $M_{1}\left[u_{1 j}\right]$ (freeing $M_{1}$ for the next job). $J_{j}$ is delivered by conveyor to $M_{2}\left[t_{A B}\right]$.
6. $R_{2}$ loads $J_{j}$ onto $M_{2}\left[l_{2 j}\right]$, and it is processed $\left[p_{2 j}\right]$.
7. $R_{2}$ unloads $J_{j}$ from $M_{2}\left[u_{2 j}\right]$ (freeing $M_{2}$ for the next job). $J_{j}$ is delivered by conveyor to the Output AS/RS where robot $R_{\text {out }}$ stores it $\left[t_{B}\right]$.


Fig. 3.9 (a) Schematic layout and (b) precedence diagram for a two-machine manufacturing cell.

As usual, we seek the sequence of jobs that minimizes makespan. Clearly, the technological constraints limit us to permutation schedules. Of course, to shorten the schedule, the tasks of successive jobs should overlap each other as much as possible. For example, while $M_{1}$ is processing one job, the pallet for the next job is being placed on the conveyor. Also, we remark that the transport times $t_{A}$ and $t_{B}$ could have been divided into two parts: robotic loading/unloading and conveyor transfer. Since they always occur together, we have combined them.

The above analysis of job flow can be portrayed in the precedence diagram given in Fig. 3.9(b), which shows the start of an arbitrary schedule. For simplicity, in the figure only, whichever job is in position $i$ has been indexed $i$, hereafter, we index it $[i] . J_{[2]}$, whose tasks are shown darker, is typical.

Note how the transport times, $t_{A}$ and $t_{B}$, of any job affect, not the job itself, but the two adjacent jobs. Thus, $J_{[2]}$ may be delayed by the transport of the finished $J_{[1]}$ to the output AS/RS, or the conveying of $J_{[3]}$ to $M_{1}$. We can now see that $J_{[1]}$ is not typical; whichever job is scheduled first has different parameter values, since $\operatorname{arc} t_{B}$ is missing. The same is true for the last job, where $t_{A}$ does not appear. Finally, there is an extra time $t_{A}$ that starts the schedule (the leftmost arc in Fig. 3.9(b)), and a time $t_{B}$ at the end, which can be added on to the makespan; they do not affect the sequencing decisions.

Suppose now that two jobs have been singled out for first and last positions. We can sequence the remaining jobs for positions $2,3, \ldots, n-1$ using the Lag Algorithm. With reference to Fig. 3.9(b), we can see that a job in any of these positions has parameters

$$
\begin{aligned}
a_{j} & =\max \left\{t_{A}, \max \left[s_{1 j}, f_{1 j}+l_{1 j}\right]+p_{1 j}+u_{1 j}\right\}, \\
b_{j} & =\max \left\{t_{B}, \max \left[s_{2 j}, f_{2 j}+l_{2 j}\right]+p_{2 j}+u_{2 j}\right\}, \\
\ell_{j} & =\max \left\{S S_{j}-a_{j}, F F_{j}-b_{j}\right\}, \text { where } \\
& S S_{j}=\max \left[s_{1 j}, f_{1 j}+l_{1 j}\right]+t_{A B}, \\
& F F_{j}=t_{A B}+l_{2 j}+p_{2 j}+u_{2 j} .
\end{aligned}
$$

To solve our problem, we now search over all possible choices for first and last jobs. Parameter values for $J_{[1]}\left[J_{[n]}\right]$ are found from the above formulas by setting $t_{B}=0\left[t_{A}=0\right]$. For each choice of "end jobs", order the rest of the jobs using $J R(\underline{a}+\underline{\ell}, \underline{b}+\underline{\ell})$.

Recall that, having sequenced a set of jobs by Johnson's Rule, if some of the jobs are then removed, the order of the remaining jobs is unchanged. Thus, to avoid rerunning MJR for every pair of end jobs, we can apply Johnson's Rule just once, to all $n$ jobs, and then for each selection maintain this order for the $n-2$ interior jobs. Since JR requires a time $O(n \log n)$, the search procedure dominates, giving an $O\left(n^{2}\right)$ algorithm. Finally, it may be mentioned that Kogan and Levner (1998) view this problem somewhat differently, but arrive at the same conclusion.

In Levner et al. (1995), a very similar problem is presented, using a different, more complex approach. It too can be simply solved by the above algorithm.

### 3.3 The Two-Stage Hybrid Flow Shop with Lags

In this section, based largely on Vairaktarakis (1997), we discuss the problem of minimizing makespan in a two-stage flow shop with multiple identical processors in each stage. We use $F\left(k_{1}, k_{2}\right)$ to denote the flow shop with $k_{i}$ identical processors in stage $G_{i}=1,2$. Hence, in the usual three-field notation, we consider the problem $F\left(k_{1}, k_{2}\right)\left|\ell_{j}\right| C_{\max }$. Of course, we will assume that at least one of the two stages has more than 1 processor, otherwise the problem has been treated earlier.

The applications described earlier for a single processor per stage generalize to multiple processors in each stage. For instance, in the consolidator example, the supplier and/or the delivery department may have multiple trucks in their respective fleets.

### 3.3.1 Preliminaries

In the presence of multiple processors per stage we need to be careful about the definition of permutation and reverse-order schedules.

Definition 3.1 In a two-stage hybrid flow shop, a schedule is a permutation schedule iff for every pair of jobs $J_{i}$ and $J_{j}$ where $a_{i}$ begins before $a_{j}$, $b_{i}$ completes before or at the same time as $b_{j}$.

Definition 3.2 In a two-stage hybrid flow shop, a schedule is a reverse order schedule iff for every pair of jobs $J_{i}$ and $J_{j}$ such that $a_{i}$ begins before $a_{j}, b_{i}$ completes after or at the same time as $b_{j}$.

Consider the above definitions in the context of the consolidator example. If the trucks of the supplier and delivery fleets are located near each other, then they follow the same route and the $G_{1}$ and $G_{2}$ orders are the same. If the two fleets are located diametrically opposite to each other, then the processing of $b$-tasks is done in the reverse order of $a$-tasks.

The rest of this section is devoted to the following problems:

- $F\left(k_{1}, k_{2}\right) \mid$ perm, $\ell_{j} \mid C_{\max }$ when the job order cannot change from $G_{1}$ to $G_{2}$;
- $F\left(k_{1}, k_{2}\right) \mid$ rvrs, $\ell_{j} \mid C_{\max }$ when the job order on $G_{2}$ must be the reverse of that on $G_{1}$;
- $F\left(k_{1}, k_{2}\right)\left|\ell_{j}\right| C_{\max }$ when the orders of jobs on each stage are unrelated.

We present heuristics that have been proposed for these problems and are supported by error bound analyses. The heuristics for the multi-processor problems can be described using a generic heuristic $\mathrm{H}=\mathrm{H}\left(S_{1}, S_{2}\right)$, which assigns the tasks at stage $G_{i}(i=1,2)$ to processors, once an ordering $S_{i}$ of tasks has been specified. The same heuristic approach is introduced in Chap. 5, but will be reviewed here. It uses the first available machine (FAM) rule at $G_{1}$. According to FAM, given a sequence $S$ of tasks, the task to be scheduled next on $k$ parallel identical machines (called a $P k$ system)) is assigned to the first machine that becomes available, i.e. the machine that finishes first the tasks (if any) previously assigned to it.

The mirror image of the FAM rule, called the last busy machine (LBM) rule, is used by heuristic H at $G_{2}$. Given a large constant $T>0$ (we can use $\left.T=\Sigma_{i}\left(a_{i}+b_{i}\right)\right)$ and an ordering $S$ of tasks (considered as single-operation jobs, $\left\{J_{j}, j=1, \ldots, n\right\}$ ), to be scheduled on a $P k$ system, LBM schedules the jobs backwards from $T$, as follows.

## LBM rule

1. Set $t_{i}:=T$ for $i=1, \ldots, k$.
2. Let $J_{j}$ be the last unscheduled task of $S$, and $M_{h}$ a processor with $t_{h}=\max _{i} t_{i}$. Schedule $J_{j}$ on $M_{h}$ to finish at time $t_{h}$.
3. Set $t_{h}:=t_{h}-p_{j}$ and $S:=S-\{j\}$. If $S \neq \phi$, then go to step 2;
else, stop.
The value of $t_{i}$ is the time that $M_{i}$ becomes busy. In step 2 we assign $J_{j}$ to a processor with largest $t_{i}$, i.e. the last processor to become busy. Hence this rule is called the last busy machine rule. Also, note that the value of $T$ is only a reference point (an upper bound on the makespan) and has no effect on the allocation of tasks to processors.

We can now describe the generic heuristic H .

## Heuristic H

1. Specify sequences $S_{1}, S_{2}$ of the tasks at each stage.
2. Apply the FAM rule to the $a$-tasks using $S_{1}$.
3. Apply the LBM rule to the $b$-tasks using $S_{2}$.
4. Rearrange some of the $b$-tasks of the resulting schedule.

Of course, Steps 1 and 4 remain to be determined. At Step 1, we must find two task sequences, $S_{1}$ and $S_{2}$, which are calculated to produce a good schedule when operated on by Steps 2 and 3 (which assign tasks to processors), and Step 4 (which makes a final adjustment of task positions). As we will see, for each of the three variants of the hybrid flow shop with lags, these sequences are generated using variations of the traditional 2-machine flow shop, optimally solved by Johnson's Rule.

### 3.3.2 $F\left(k_{1}, k_{2}\right) \mid$ perm, $\ell_{j} \mid C_{\text {max }}$

To determine $S_{1}$ and $S_{2}$ for the permutation shop, we simplify the problem using the concept of "merged machines" (see Sect. 2 of Chap. 8). Consider the Merged Machines Shop (the subscript denotes permutation):
$\mathbf{M M S}_{\mathbf{P}}: F 2 \mid$ perm, $\ell_{j} \mid C_{\max }$, with job parameters $<a_{j} / k_{1}, b_{j} / k_{2} ; \ell_{j}>$
where we replace the machine group at stage $i$ by a single "superserver" that works at $k_{i}$ times the speed of the original servers, as though each job could be split up and simultaneously run on all the parallel machines. The heuristic for the permutation flow shop is the following adaptation of H, with Steps 2 and 3 unchanged:

## Heuristic $\mathbf{H}_{\mathbf{P}}$

1. Solve $\mathrm{MMS}_{\mathrm{P}}$. Let $S=J R\left(\underline{a} / k_{1}+\underline{\ell}, \underline{b} / k_{2}+\underline{\ell}\right)$ be the resulting
sequence. Set $S_{1}=S_{2}:=S$.
2.     - 3. Same as in H.
1. Start all $b$-tasks earlier by the maximum amount of time $t$ that does not violate the permutation and flow shop constraints.

Note that at Step 4, all b-tasks are shifted the same amount of time $t$ earlier, and hence the order of completion times after the shift of Step 4 remains the same as it was at the end of Step 3. As a result, we retain a permutation schedule, as characterized in Definition 1.1. Since $F 2 \mid$ perm, $\ell_{j} \mid C_{\max }$ requires $O(n \log n)$ time, this is also the computational complexity of $\mathrm{H}_{\mathrm{P}}$.

It can be shown that the makespan achieved in $\mathrm{MMS}_{\mathrm{P}}$ is a lower bound on $C^{\star}$, the minimal makespan of $F\left(k_{1}, k_{2}\right) \mid$ perm, $\ell_{j} \mid C_{\max }$. Based on this lower bound, the performance guarantee for $\mathrm{H}_{\mathrm{P}}$ is:

Theorem 3.9 (Vairaktarakis, 1997) For $F\left(k_{1}, k_{2}\right) \mid$ perm, $\ell_{j} \mid C_{\max }$, let $C_{P}$ be the makespan obtained by $H_{P}$. Then:

$$
C_{P} / C^{\star} \leq 2-1 / k, \quad \text { where } k=\max \left\{k_{1}, k_{2}\right\}
$$

and this bound is tight.
Example 3.4: To illustrate $H_{P}$, consider a 2-stage hybrid permutation flow shop with lags, having $k_{1}=k_{2}=2$ and five jobs with processing requirements $\left.J_{1}=\langle 1,3 ; 3\rangle, J_{2}=\langle 4,5 ; 2\rangle, J_{3}=\langle 2,4 ; 5\rangle, J_{4}=<5,5 ; 4\right\rangle$ and $J_{5}=$ $<6,2 ; 2\rangle$. These jobs are already indexed in the order resulting after step 1 of $H_{P}$. The Gantt chart of the schedule produced by $H_{P}$ on this 5-job example is given in Fig. 3.10. Note that all processors at $G_{2}$ finish at time $C_{P}=17$. After applying step 3 of heuristic H , the $b$-tasks are moved earlier (slid to the left as a block) until one of them cannot be further advanced without violating its lag constraint. In our case, this critical job is $J_{4}$ (see Fig. 3.10). The other jobs could start earlier at stage 2 ( $J_{1}$ for example, could start as early as $a_{1}+\ell_{1}=4$, and so has a slack of 4 ), but this would violate the permutation constraint of Definition 3.1 and would not shorten the makespan.


Fig. 3.10
An example for $F(2,2) \mid$ perm, $\ell_{j} \mid C_{\max }$.

### 3.3.3 $F\left(k_{1}, k_{2}\right)\left|r v r s, \ell_{j}\right| C_{\text {max }}$

The solution for the reverse flow shop is closely analogous to that of the permutation flow shop. As before, we merge the machines at each station to define a Merged Machines Shop:
$\mathbf{M M S}_{\mathbf{R}}: F 2 \mid$ rvrs, $\ell_{j} \mid C_{\text {max }}$, with job parameters $<a_{j} / k_{1}, b_{j} / k_{2} ; \ell_{j}>$
recalling that this problem can be solved in $O(n \log n)$ time as described in Sect. 3.2.3. Heuristic H now becomes:

## Heuristic $\mathbf{H}_{\mathbf{R}}$

1. Solve $\mathrm{MMS}_{\mathrm{R}}$, and use the resulting schedule, namely $S_{1}:=\searrow_{j}$, $S_{2}:=\nearrow \ell_{j}$.
2.     - 3. Same as in H.
1. Start all $b$-tasks earlier by the maximum amount of time $t$ that does not violate the order and flow shop constraints.

Steps 3 and 4 of the heuristic $\mathrm{H}_{\mathrm{R}}$ clearly maintain the reverse order of completion times on the two machines. Since $F 2\left|r v r s, \ell_{j}\right| C_{\max }$ requires $O(n \log n)$ time, this is also the computational complexity of $\mathrm{H}_{\mathrm{R}}$.

The makespan of $\mathrm{MMS}_{\mathrm{R}}$ is, again, a lower bound on $C^{\star}$, the optimal makespan of $F\left(k_{1}, k_{2}\right) \mid$ rvrs, $\ell_{j} \mid C_{\max }$. Using it, the following theorem establishes the worst case performance of $\mathrm{H}_{\mathrm{R}}$.

Theorem 3.10 (Vairaktarakis, 1997) For $F\left(k_{1}, k_{2}\right)\left|r v r s, \ell_{j}\right| C_{\max }$, if $C_{R}$ is the makespan obtained by $H_{R}$, then:
$C_{R} / C^{\star} \leq 2-1 / k, \quad$ where $k=\max \left\{k_{1}, k_{2}\right\}$,
and this bound is tight.
Example 3.5: To illustrate $\mathrm{H}_{\mathrm{R}}$, consider the 5-job instance of $F(2,2)$ given in Example 3.4. The job order produced by step 1 of $\mathrm{H}_{\mathrm{R}}$ is $S_{1}=(3,4,1,2,5)$, with $S_{2}$ being the reverse. Note that $J_{2}$ and $J_{5}$ could be interchanged, since they have equal values of $\ell_{j}$. We have broken the tie arbitrarily (this is a heuristic, after all); or we could evaluate all sequences that satisfy $\searrow \ell_{j}$ and choose the best. In this case, the other schedule gives a slightly longer makespan.

The Gantt chart of the application of $\mathrm{H}_{\mathrm{R}}$ on this 5-job example is given in Fig. 3.11. Again, all tasks scheduled on $M_{2}$ are moved earlier as a block until the lag constraint of one or more jobs becomes binding (here, the critical job is $J_{5}$ ), so all stage-2 processors finish at the same time, with makespan of 22 .


Fig. 3.11
An example for $F(2,2)\left|r v r s, \ell_{j}\right| C_{\max }$.

### 3.3.4 $F\left(k_{1}, k_{2}\right)\left|\ell_{j}\right| C_{\text {max }}$

In this section, we consider the two-stage hybrid flow shop with lags, assuming task orders at each stage are unrelated. Heuristic H now becomes:

## Heuristic $\mathbf{H}_{\mathbf{U}}$

1. Set $S_{1}:=J R\left(\underline{a} / k_{1}, \underline{b} / k_{2}\right)\left(S_{2}\right.$ is not needed $)$.
2. Apply the FAM rule to the $a$-tasks using $S_{1}$.
3. Omit this step.
4. Schedule the $b$-tasks across processors as soon as possible, in order of availability, where $J_{j}$ is available $\ell_{j}$ units after it is completed at $G_{1}$.

Since JR requires $O(n \log n)$ time, this is also the computational complexity of $\mathrm{H}_{\mathrm{U}}$.

For a lower bound on the minimal makespan, we define the following quantities:

- $C_{S F}$ is the minimal makespan of the simple flow shop in step 1 of $\mathrm{H}_{\mathrm{U}}$, where parallel machines at each stage have been combined into one, and all lags set to zero.
- $C_{H F}$ is the minimal makespan of the hybrid flow shop with all lags set to zero.
- $C^{\star}$ is the minimal makespan of the original problem, $F\left(k_{1}, k_{2}\right)\left|\ell_{j}\right| C_{\max }$.

Then

$$
C_{S F} \leq C_{H F} \leq C^{\star}
$$

where the left inequality was shown by Lee and Vairaktarakis (1994). It follows:

Theorem 3.11 (Vairaktarakis, 1997)
Let $C_{U}$ be the makespan obtained by $H_{U}$. Then:
$C_{U} / C^{\star} \leq 2$, and this bound is tight.

### 3.3.5 Summary of Performance Ratio and Complexity Results

A summary of the results of this section, along with the special case of two machines ( $k_{1}=k_{2}=1$ ) discussed earlier, is given in Table 1.1 below. For polynomially solvable problems we provide the complexity bound while for NP-complete problems (denoted NPC) we provide the best known performance ratio.

Table 3.1 Summary of performance ratios, with $k=\max \left\{k_{1}, k_{2}\right\}$

|  | Complexity | Performance |
| :--- | :---: | :---: |
| Status | Ratio |  |
| $F 2 \mid$ perm, $\ell_{j} \mid C_{\max }$ | $O(n \log n)$ | 1 |
| $F\left(k_{1}, k_{2}\right) \mid$ perm,$\ell_{j} \mid C_{\max }$ | NPC | $2-1 / k$ |
| $F 2\left\|r v r s, \ell_{j}\right\| C_{\max }$ | $O(n \log n)$ | 1 |
| $F\left(k_{1}, k_{2}\right)\left\|r v r s, \ell_{j}\right\| C_{\max }$ | NPC | $2-1 / k$ |
| $F 2\left\|\ell_{j}\right\| C_{\max }$ | NPC | 2 |
| $F\left(k_{1}, k_{2}\right)\left\|\ell_{j}\right\| C_{\max }$ | NPC | 2 |

### 3.3.6 Computational Experiments

The heuristics $H_{P}, H_{R}$, and $H_{U}$ have been tested on randomly generated problems with $n=30,40,50$ jobs, $k_{1}, k_{2} \in\{2,4\}$, and various workload scenarios between the 2 stages and the lag durations. These heuristics are shown to have average relative gaps from the respective lower bounds (i.e., $100\left(C_{H}-C_{L B}\right) / C_{L B}$ ) of $1.22 \%, 1.24 \%$ and $2.1 \%$ respectively (see Vairaktarakis, 1997). Moreover, the relative gaps tend to decrease as the lag times increase. Also, it is found that average relative gaps increase with increases in the workload at $G_{1}$.

Average relative gaps also increase with the total number $k_{1}+k_{2}$ of processors. As for the relationship between problem parameters and heuristic performance, it is found that the average relative gaps decrease as the ratio $n / k$ increases. Given that, in flow shops, the number of jobs $n$ is usually much larger than the number $k$ of processors in either stage, the three heuristics have a good chance to perform near optimally.

Finally, one might wonder how often the makespan of $\mathrm{H}_{\mathrm{P}}$ outperformed the one obtained by $\mathrm{H}_{\mathrm{R}}$. The answer is that, for problem sizes of 30 or more jobs, $\mathrm{H}_{\mathrm{R}}$ very rarely outperforms $\mathrm{H}_{\mathrm{P}}$. One the other hand, when $n=15$ the outcome is much less predictable.

Note that the optimal makespan for problem $F\left(k_{1}, k_{2}\right)\left|\ell_{j}\right| C_{\max }$ is no worse than the corresponding optimal values for $F\left(k_{1}, k_{2}\right) \mid$ perm, $\ell_{j} \mid C_{\max }$ and $F\left(k_{1}, k_{2}\right)\left|r v r s, \ell_{j}\right| C_{\text {max }}$. However, this relationship may not hold for the three heuristics because none of them guarantees optimality for the respective problem. In fact, even though rare, for $n \geq 30$ it is possible for $H_{P}$ or $H_{R}$ to outperform $\mathrm{H}_{\mathrm{U}}$.

### 3.4 The $m$-Machine Flow Shop with Lags

Up to now, we have considered flow shops with only 2 stages. In the next chapter we will discuss shops with $m>2$ stages, which are significantly more difficult to analyze. In fact, the problem of minimizing makespan in a 3 -machine flow shop with or without lags is strongly NP-complete. Since the discussion of the problem with lags is very similar to the one without lags, we will treat the two problems together in Sect. 4.7.

### 3.5 Related Production Systems

In this section we present problems whose structure is related to problems considered earlier. More specifically, the following problems possess the lag structure explicitly, or can be interpreted using time lags.

### 3.5.1 Master-Slave Systems

A set of $n$ jobs is to be processed by a system of master and slave processors. Each job consists of 3 tasks. The first is a preprocessing task, the second is a slave task, and the third is a postprocessing task, and they must be executed in this order. The corresponding processing time requirements of job $J_{j}$ are $a_{j}, t_{j}$, and $b_{j}$ respectively. The pre- and postprocessing tasks are to be executed on a single master processor $M_{1}$, while slave tasks are processed on $n$ parallel identical processors. Motivated by parallel computing applications, as well as VLSI CAD problems, Sahni (1995) considered the problem of finding a schedule of the $n$ jobs that minimizes the makespan on $M_{1}$.

Evidently, the number of slave processors equals the number of jobs and hence the slave stage can be assumed to have infinite capacity. Thus, the slave tasks act as time lags, and so we will use $\ell_{j}=t_{j}$. Moreover, every pre- and postprocessing pair of tasks can be viewed as two tasks, coupled together by a time lag, that both require processing on $M_{1}$. Letting cpld stand for coupled, we will denote the above described master-slave optimization problem as $1\left|c p l d, \ell_{j}\right| C_{\max }$.

For this problem, Sahni developed optimal polynomial time algorithms for permutation and reverse order schedules. The former are useful when a queue structure is used to store the successive jobs, while the latter requirement is imposed when a stack structure is used to store tasks. A simple interchange argument can verify the following optimality condition.

Property 3.1 For $1\left|c p l d, \ell_{j}\right| C_{\max }$, there exists an optimal schedule, whether permutation, reverse, or unconstrained, where all preprocessing tasks complete on $M_{1}$ before any postprocessing task starts.

The following algorithm produces an optimal permutation schedule of this type. Recall that a sequence denoted $\nearrow x_{j}\left[\searrow x_{j}\right]$ has the jobs in nondecreasing
[nonincreasing] order of parameter $x_{j}$.
Algorithm for $1 \mid c p l d$, perm, $\boldsymbol{\ell}_{\boldsymbol{j}} \mid \boldsymbol{C}_{\text {max }}$ (Sahni, 1995)

1. Jobs with $a_{j}<b_{j}$ are preprocessed first in order $\nearrow\left(a_{j}+\ell_{j}\right)$.
2. Jobs with $a_{j}=b_{j}$ are preprocessed next in any order.
3. Jobs with $a_{j}>b_{j}$ are preprocessed last in order $\searrow\left(b_{j}+\ell_{j}\right)$.
4. Generate any canonical permutation schedule satisfying steps $1-3$.

Clearly, $1 \mid c p l d$, perm, $\ell_{j} \mid C_{\max }$ is solved in $O(n \log n)$ due to the sorting in steps 1 and 3.

The following algorithm produces an optimal reverse order schedule.
Algorithm for $1|c p l d, r v r s| C_{\text {max }}$ (Sahni, 1995)

1. Jobs are preprocessed in order $\searrow \ell_{j}$.
2. Generate any canonical reverse order schedule satisfying step 1 .

For similar reasons, $1 \mid c p l d$, rvrs $\mid C_{\max }$ is solved in $O(n \log n)$ time.
Finally, consider the unconstrained master-slave system, where no (ordering) constraint is imposed on the jobs other than the fact that the preprocessing, slave, and postprocessing tasks of each job must be executed in this order. Clearly such a system can outperform the permutation and the reverse order master-slave systems. Besides having Property 3.1, we can show with another interchange argument that, given a schedule for the preprocessing tasks, the first-come-first-served rule can optimally schedule the postprocessing tasks. Therefore, a complete schedule of the $n$ jobs is determined by an ordering of the preprocessing tasks. Unfortunately, despite all this, $1\left|c p l d, \ell_{j}\right| C_{\max }$ is strongly NP-complete (see Sahni and Vairaktarakis, 1996). A reasonable ordering of the preprocessing tasks is obtained by the following heuristic.

Heuristic $\mathbf{H}_{\mathrm{UNC}}$ for $\mathbf{1}\left|\boldsymbol{c p l d}, \ell_{\boldsymbol{j}}\right| \boldsymbol{C}_{\max }$ (Sahni and Vairaktarakis, 1996)

1. Jobs with $a_{j} \leq b_{j}$ are preprocessed first in order $\nearrow \ell_{j}$
2. Jobs with $a_{j}>b_{j}$ are preprocessed last in order $\searrow \ell_{j}$
3. Generate any canonical schedule satisfying steps 1 and 2 , where the postprocessing tasks are scheduled on a first-come-first-served basis.
4. Compute the makespan $C_{U N C}$ of the generated schedule $S_{U N C}$.

The complexity of $\mathrm{H}_{\mathrm{UNC}}$ is $O(n \log n)$ due to the sorting in steps 1 and 2 . A schedule produced by $\mathrm{H}_{\mathrm{UNC}}$ is accompanied by the following performance guarantee.

Theorem 3.12 (Sahni and Vairaktarakis, 1996)
For $1\left|c p l d, \ell_{j}\right| C_{\max }$, let $C^{\star}$ be the optimal makespan. Then
$C_{U N C} / C^{\star} \leq 3 / 2$, and this bound is tight.

Example 3.6: Consider the 5 -job instance with the following processing times:

| $j$ | $a_{j}$ | $\ell_{j}$ | $b_{j}$ |
| :---: | :---: | :---: | :---: |
| 1 | 3 | 7 | 4 |
| 2 | 6 | 5 | 1 |
| 3 | 7 | 2 | 7 |
| 4 | 3 | 1 | 6 |
| 5 | 8 | 8 | 9 |

Then, the optimal permutation and reverse order sequences are $S_{P}=$ $(4,1,5,3,2)$ and $S_{R}=(5,1,2,3,4)$ respectively, while $\mathrm{H}_{\mathrm{UNC}}$ yields the sequence $S_{U}=(4,3,1,5,2)$. The makespan values of these schedules are 54 , 55 , and 54 respectively.

### 3.5.2 Multiple Master-Slave Processors

The master-slave system has been extended to analyze the case where several master machines are available for processing. Sahni and Vairaktarakis (1996) cite applications of $F m|c p l d| C_{\max }$ from parallel computer scheduling, VLSI CAD problems, fleet scheduling, and scheduling of semiconductor testing operations. They develop heuristics with performance ratio 2 for $F m \mid c p l d$, perm, $\ell_{j} \mid C_{\max }$ and $F m\left|c p l d, \ell_{j}\right| C_{\max }$, and with performance ratio (2-1/m) for $F m\left|c p l d, r v r s, \ell_{j}\right| C_{\max }$.

### 3.6 Conclusions

In the two-machine case, the extension of Johnson's Rule to include transfer lags has considerably extended the range of problems that can be efficiently solved. Beyond the obvious transportation delays between machines, many less obvious applications of the lag concept arise. We have mentioned several in the Examples section, such as setup and teardown times which can be treated as negative lags, compound jobs that can be advanced from machine to machine in lots, etc.

For more than two machines, the concept of nonbottleneck machines is critical. Every sequence of nonbottlenecks is merely a lag between the two adjacent machines. As we shall see in the next chapter, this is a powerful tool, especially for calculating lower bounds.

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## Chapter 4 THE m-MACHINE FLOW SHOP


#### Abstract

Considered to be a very general case of flow shop systems, the $m$-machine flow shop is the most researched system in all of flow shop theory. Beyond solving the problem under a variety of objectives and side constraints, the $m$-machine flow shop serves as a test bed for new methodological tools. Regarding solutions, the research presented in this chapter is rich in lower bounding schemes, dominance properties, heuristic algorithms and computational experiments measuring their success. The models considered not only deal with all the standard regular performance measures, but also application-specific objective functions. A lot of work is also available on problems with multiple objectives. We find that the most successful solutions on problems of practical size are due to metaheuristic implementations including simulated annealing, tabu search and genetic algorithms. In contrast, branch-and-bound algorithms are mostly inadequate.


### 4.1 Examples

The $m$-stage flow shop with one machine per stage is most typically found in industry and reflects the simplest case of multi-stage processing. Approximately $50 \%$ of all U.S. manufacturing utilizes a batch processing layout, often represented by multi-stage flow shop cells (Adam and Ebert, 1992). The following examples highlight the practical importance of the flow shop studied in this chapter.

- Reid and Stark (1982) report the case of a small alloy shop consisting of four machines. Along with a shop foreman, the manager estimates the job processing requirements which become input to a computer program for scheduling the jobs.
- Kim et al. (1996) studied a 3-stage flow shop producing electronic printed circuit boards (PCB's). The 3 stages are surface mounting (SM), automatic
insertion of components (AI) and manual insertion (MI). Each stage may involve more than one station, in all cases arranged in series, resulting in a multi-stage flow shop.
- Bartholdi and Eisenstein (1996) point to textile shops where workers work on an item and hand it to the next crew on the line over multiple stations.


### 4.2 Preliminaries

Simply stated, the $m$-machine flow shop consists of a single processor, always available, in each of $m$ stages. A given set $\mathcal{J}=\left\{J_{1}, J_{2}, \ldots, J_{n}\right\}$ of $n$ independent jobs require processing, each made up of $m$ well-ordered tasks with known requirements. Task $i$ of job $j$ requires processor $i$ for a processing time $p_{i j}$. Each task of a job requires a different machine, no job visits a work station more than once, and the machine sequence is $M_{1}, M_{2}, \ldots, M_{m}$ for all jobs. By $M_{k}$ we indicate the machine in stage $k, k=1, \ldots, m$. Unless specified otherwise, all jobs are available simultaneously at the start (at time zero), and remain available without interruption until all work on them is finished. For shops with up to 3 stages, we generally use the simplified notation $a_{j}, b_{j}, c_{j}$ to indicate the processing times of $J_{j}$ at $M_{1}, M_{2}, M_{3}$, respectively. Vector $\bar{p}_{j}=\left\langle p_{1 j}, p_{2 j}, \ldots, p_{m j}\right\rangle$ indicates the processing times of $J_{j}$ on all machines. Also, recall that throughout this monograph:
$C_{k}(S)\left[C_{k j}(S)\right]=$ completion time of all tasks [of $\left.J_{j}\right]$ on $M_{k}$ in schedule $S$.

### 4.2.1 Dominance given a Partial Schedule

In this chapter, we will continue to look for properties that allow us to narrow the search for an optimal schedule to a dominant subset. In this context, we limit ourselves to permutation schedules. As before, we will continue to seek certain job orderings that can be ruled out. We will also find that certain partial sequences can sometimes be shown to dominate other partial sequences. At this point, we make a few preliminary remarks.

## Job Dominance

In Chap. 1, we introduced the idea of job dominance in permutation schedules: $J_{i}$ dominates $J_{j}$ if there exists at least one optimal schedule in which $J_{i}$ precedes $J_{j}$. We may be able to establish this property only for schedules where the jobs are adjacent (local dominance), or without this limitation (global dominance). Such properties are useful for limiting our search for an optimum to a dominant set: the subset of schedules that meets all dominance requirements we have been able to establish.

Often, our search procedure is branch-and-bound, where a schedule is being built, either forwards or backwards, one job at a time. For the moment, assume a forward algorithm, where at an arbitrary node a partial initial
schedule $\sigma$ exists, with a set $\mathcal{U}$ of $u$ jobs remaining unscheduled. We wish to find dominance relations among the jobs in $\mathcal{U}$.

A common feature of all the results published to date is that, given $\sigma$, dominance always concerns the position in the schedule immediately after $\sigma$. This leads to the following definition, where Any denotes any objective or measure, giving to any schedule $S$ the value $A(S)$.

Definition 4.1. For $F m \mid$ perm $\mid A n y$, with initial partial schedule $\sigma, J_{i}$ dominates $J_{j}$ (written $i \sim_{\sigma} j$ ) if, for any complete schedule $\sigma j \pi$, there exists a complete schedule $\sigma i \pi^{\prime}$ such that $A\left(\sigma i \pi^{\prime}\right) \leq A(\sigma j \pi)$.

That is, there is a schedule starting with $\sigma i$ that is better than any schedule starting with $\sigma j$, so $J_{j}$ need not be considered for the next position after $\sigma$.

The following obvious consequence of this definition suggests how this could be shown (where $\sigma i j \pi_{1} \pi_{2}$ is a complete schedule).

Theorem 4.1. For Fm $\mid$ perm $\mid$ Any, with initial partial schedule $\sigma$,

$$
A\left(\sigma i j \pi_{1} \pi_{2}\right) \leq A\left(\sigma j \pi_{1} i \pi_{2}\right) \forall \pi_{1}, \pi_{2} \Rightarrow i \leadsto \sigma j .
$$

## Sequence Dominance

In this class of dominance results, we compare one partial sequence $\sigma_{1}$ with another, $\sigma_{2}$, which contains the same jobs. We now limit our attention to the large class of regular measures (denoted Reg) with objective function $R(S)$. Roughly (see Chap. 1 for details), regular measures are those that are always improved by reducing job completion times. All the objectives to be discussed in this chapter are regular.

Definition 4.2. For $F m \mid$ perm $\mid$ Reg, let $\sigma_{1}$ and $\sigma_{2}$ be sequences of the same job subset. Then $\sigma_{1}$ strongly dominates $\sigma_{2}$ if
$C_{k}\left(\sigma_{1}\right) \leq C_{k}\left(\sigma_{2}\right), k=1,2, \ldots, m$.
Note that the definition is independent of the particular objective. The following definition is measure-dependent.

Definition 4.3. For $F m \mid$ perm $\mid$ Reg, let $\sigma_{1}$ and $\sigma_{2}$ be sequences of the same job subset. Then $\sigma_{1}$ dominates $\sigma_{2}$ with respect to Reg if
$R\left(\sigma_{1} \pi\right) \leq R\left(\sigma_{2} \pi\right)$ for any job sequence $\pi$.
Connecting these definitions, we have the following
Theorem 4.2. For Fm|perm $\mid$ Reg, let $\sigma_{1}$ and $\sigma_{2}$ be sequences of the same job subset. Then,
$\sigma_{1}$ strongly dominates $\sigma_{2} \Rightarrow \sigma_{1}$ dominates $\sigma_{2}$ with respect to Reg.

Proof: Consider any schedule completion $\pi=(i, j, \ldots, u)$ containing the remaining unscheduled jobs. Since

$$
C_{k}\left(\sigma_{a} i\right)=\max _{h=1, \ldots, k}\left\{C_{h}\left(\sigma_{a}\right)+\sum_{l=h}^{k} p_{l i}\right\}, \text { for } a=1,2, k=1, \ldots, m,
$$

and by hypothesis $C_{k}\left(\sigma_{1}\right) \leq C_{k}\left(\sigma_{2}\right)$, it follows that $C_{k}\left(\sigma_{1} i\right) \leq C_{k}\left(\sigma_{2} i\right)$ for all $k$. Adding another job, it follows similarly that $C_{k}\left(\sigma_{1} i j\right) \leq C_{k}\left(\sigma_{2} i j\right)$; and so on. Thus, for each of the unscheduled jobs $J_{v}, v=i, j, \ldots, u$, $C_{m v}\left(\sigma_{1} \pi\right) \leq C_{m v}\left(\sigma_{2} \pi\right)$. That is, for any completion $\pi$, every job in $\pi$ has an earlier completion time following $\sigma_{1}$ than following $\sigma_{2}$. Hence for any regular measure, $R\left(\sigma_{1} \pi\right) \leq R\left(\sigma_{2} \pi\right)$.
Szwarc (1983a) gives this result for the case $R e g=C_{\text {max }}$.

### 4.2.2 Ordered Flow Shops

Flow shops with special structure are sometimes considered in the literature. One such class of shops that we shall refer to from time to time is the ordered flow shop. The definitions below may be found in Smith et al. (1975).

Definition 4.4. A flow shop is ordered if the processing times satisfy the following conditions:

1. There exists a job indexing such that $p_{k 1} \leq p_{k 2} \leq \ldots \leq p_{k n}$, for all $M_{k}$.
2. There exists a machine indexing s.t. $p_{1 j} \leq p_{2 j} \leq \ldots \leq p_{m j}$, for all $J_{j}$.

Thus, in an ordered shop, we can speak of sequencing the jobs in SPT order without ambiguity, since this order is the same on all machines. That is, SPT on each machine results in a permutation schedule.

### 4.3 Complexity of $\boldsymbol{F} 3 \mid($ perm $) \mid C_{\text {max }}$

As we saw in Theorems 1.1 and 1.2 there exists an optimal makespan minimization schedule with the same job order on $M_{1}$ and $M_{2}$. The same holds for $M_{m-1}$ and $M_{m}$, although for $m>3$ the two orders are not necessarily identical. It follows that there exists an optimal permutation schedule for $F 3 \mid$ (perm) $\mid C_{\text {max }}$. Unfortunately, this simplifying result is not enough to classify $F 3 \mid($ perm $) \mid C_{\text {max }}$ as an "easy problem", as shown in

Theorem 4.1 (Garey et al., 1976)
$F 3 \mid($ perm $) \mid C_{\max }$ is NP-complete in the strong sense.
Outline of Proof: We reduce the strongly NP-complete problem

## 3-Partition

INSTANCE: An integer $V$, and $3 k$ positive integers $v_{i}: i \in \mathcal{T}=\{1,2, \ldots, 3 k\}$ such that $\Sigma_{i \in \mathcal{T}} n_{i}=k V$, and $V / 4<v_{i}<V / 2 \quad \forall i \in \mathcal{T}$.
QUESTION: Can $\mathcal{T}$ be partitioned into mutually disjoint subsets $\mathcal{S}_{1}, \mathcal{S}_{2}, \ldots, \mathcal{S}_{k}$,
with $\left|\mathcal{S}_{j}\right|=3$, and $\Sigma_{i \in \mathcal{S}_{j}} v_{i}=V$ for $j=1,2, \ldots, k$ ?
to the decision version of our problem:
$F 3 \mid($ perm $) \mid C_{\text {max }} \leq B$ ?
INSTANCE: A threshold value $B$, and $n$ jobs $\left\{J_{j}: j=1,2 \ldots, n\right\}$, each with processing requirements $\left\langle a_{j}, b_{j}, c_{j}\right\rangle$, on machines $M_{1}, M_{2}, M_{3}$ of a 3-machine flow shop.
QUESTION: Does there exist a schedule with $C_{\max } \leq B$ ?
For any instance of 3-Partition define an instance of $F 3 \mid($ perm $) \mid C_{\max } \leq B$ ? as follows:

- $B:=(2 k+1) V$,
- $n:=4 k+1$,
- $\left.\left\langle a_{0}, b_{0}, c_{0}\right\rangle:=<0, V, 2 V\right\rangle$,
- $<a_{j}, b_{j}, c_{j}>:=<2 V, V, 2 V>$ for $j=1, \ldots, k-1$,
- $<a_{k}, b_{k}, c_{k}>:=<2 V, V, 0>$,
- $<a_{j}, b_{j}, c_{j}>:=<0, v_{j}, 0>$ for $k+1 \leq j \leq 4 k$.

To see why the two problems will always have the same answer, we note that we can achieve a makespan of $B=(2 k+1) V$ (the answer "yes") if and only if jobs $J_{0}, J_{1}, \ldots, J_{k}$ are scheduled as in Fig. 4.1 with "filler" jobs $J_{j}: k+1 \leq j \leq 4 k M$ inducing a 3 -partition of $v_{j}: j \in \mathcal{T}$.
In a related result, Gonzalez and Sahni (1978) show that $F 3 \| C_{\text {max }}$ is NPcomplete even when every $J_{j} \in \mathcal{J}$ has precisely two tasks with positive processing time requirements. The reduction is from 2-Partition.


Fig. 4.1 Optimal schedule of NP-complete instance of $F 3 \mid$ (perm) $\mid C_{\text {max }}$

### 4.4 Calculation of Makespan for a Given Sequence

If an arbitrary job sequence $S=(1, \ldots, n)$ is given (jobs reindexed to match number to position), the following equations can be used to calculate recursively the completion time of each task in a permutation flow shop. For those familiar with the elements of project scheduling, we are simply applying the Critical Path Method, where completion corresponds to "early finish".

In a specified schedule, for task $T_{k j}$ of $J_{j}$ on $M_{k}$ with completion time $C_{k j}$, the work cannot start until $J_{j}$ leaves $M_{k-1}$ (that is, $C_{k j}-p_{k j} \geq C_{k-1, j}$ ), and
it must also wait until the preceding job vacates $M_{k}\left(C_{k j}-p_{k j} \geq C_{k, j-1}\right)$. These are evidently the only constraints on its timing, so the earliest it can complete is

$$
\begin{equation*}
C_{k j}=\max \left\{C_{k-1, j}, C_{k, j-1}\right\}+p_{k j}, \quad j \in \mathcal{J}, k \in \mathcal{M} \tag{4.1}
\end{equation*}
$$

with boundary conditions $C_{k 0}=C_{0 j}=0$. Starting with $C_{11}$, we can iterate through increasing values of $j$ and $k$, eventually determining the makespan, $C_{m n}$.

### 4.5 An Integer Program for $\operatorname{Fm} \mid$ perm $\mid C_{\text {max }}$

Since our problem is NP-hard, we will be discussing special cases, approximations, and search techniques. Before we begin, we present an integer program that can be used to solve small instances and that summarizes in mathematical form the essential components of the classic $m$-machine flow shop.

Define the variables:

$$
\begin{aligned}
& x_{i j}= \begin{cases}1 & \text { if } J_{j} \text { is in position } i, \\
0 & \text { otherwise },\end{cases} \\
& c_{k i}=\text { completion time on } M_{k} \text { of the job in position i, }
\end{aligned}
$$

not to be confused with $C_{k j}$, the completion time on $M_{k}$ of $J_{j}$. In order to search over different schedules, the binary variables are used to assign jobs to positions, letting us write $p_{k[i]}$ as $\sum_{j=1}^{n} p_{k j} x_{i j}$, where $[i]$ indexes the job in position $i$. Then the following integer program minimizes the makespan $c_{m n}$.

$$
\begin{array}{lll}
\operatorname{minimize} & c_{m n} & \\
\text { subject to } & c_{k i} \geq c_{k-1, i}+\sum_{j=1}^{n} p_{k j} x_{i j}, \quad i \in \mathcal{J}, k \in \mathcal{M} \\
& c_{k i} \geq c_{k, i-1}+\sum_{j=1}^{n} p_{k j} x_{i j}, \quad i \in \mathcal{J}, k \in \mathcal{M} \\
& \sum_{i=1}^{n} x_{i j}=1, \quad j \in \mathcal{J} & \\
& \sum_{j=1}^{n} x_{i j}=1, \quad i \in \mathcal{J} & \\
& x_{i j}=0 \text { or } 1, \quad i, j \in \mathcal{J} & \tag{4.6}
\end{array}
$$

with $c_{k 0}=c_{0 i}=0$; where (4.2) enforces task order within jobs, (4.3) does the same on each machine, and (4.4) - (4.6) ensure that each job is assigned to just one position in the schedule, and each position gets one job. Essentially, with the aid of the integer variables to search over schedules, we have converted (4.1) into an optimization. It will not, of course, solve very large problems.

### 4.6 Polynomially Solvable Cases of $\operatorname{Fm} \mid$ perm $\mid C_{\text {max }}$

In this section, we present special cases of $F m \mid$ perm $\mid C_{\text {max }}$ with $m>2$ that are solvable in polynomial time. We start by giving two rather trivial results. First we suppose that all the task times of a given job are equal.

Theorem 4.3. For $F m\left|(p e r m),(p m t n), p_{i j}=p_{j}\right| C_{\max }$, any permutation schedule is optimal, with

$$
C_{\max }=\sum_{j=1}^{n} p_{j}+(m-1) p_{\max }, \text { where } p_{\max }=\max _{j \in \mathcal{J}} p_{j}
$$

For a proof, see Chapter 11 where we extend this result. A similar theorem holds when all the task times on any one machine are equal, i.e., when all jobs have the same processing requirements.

Theorem 4.4. For $F m\left|(p e r m),(p m t n), p_{i j}=p_{i}\right| C_{\max }$, any permutation schedule is optimal, with

$$
C_{\max }=\sum_{i=1}^{m} p_{i}+(n-1) p_{\max }, \text { where } p_{\max }=\max _{i \in \mathcal{M}} p_{i}
$$

this apparently artificial model has application to lot streaming, where a large number of identical items to be manufactured are divided into equal sublots. Each sublot then acts as a job.

Most other cases with efficient solutions depend on one or more stages behaving like nonbottleneck machines. Recall that a nonbottleneck is a conceptual machine that can process any number of jobs simultaneously: it has infinite capacity, or at least the capacity of $n$ parallel machines. Since each job passes through such a machine without any waiting, a nonbottleneck acts like a transfer lag between the adjacent stages.

For another way to visualize this, recall how our problem can be represented as a directed graph. Fig. 4.2 shows the three-machine case (where permutation schedules are always optimal). Although this problem is NPhard, we saw in Chap. 3 how $F 2 \mid$ perm, $\ell_{j} \mid C_{\max }$, pictured in Fig. 4.2(b), can be easily solved. The difference, of course, is the absence of the arrows enforcing the task sequence on the second machine, call them the $\boldsymbol{M}_{2}$-arcs. In each case, the quantity to be minimized, namely the makespan, corresponds to the critical (i.e., the longest) path through the digraph. Now observe: if the parameters of the three-machine problem are such that, regardless of the job sequence, no path involving $M_{2}$-arcs is ever critical, then we can ignore those arcs and our problem can be solved as a two-machine problem with lags. For example, suppose all the task times on $M_{2}$ are very small, and all the times on $M_{1}$ and $M_{3}$ are very large. Obviously there will never be any congestion on $M_{2}$; the makespan is unaffected whether there is one processor at the second stage or many. Actually, only one of $M_{1}$ and $M_{3}$ need "dominate" $M_{2}$; see Theorem 4.2 below.

Similarly for $m$ machines, as presented in Monma and Rinnooy Kan (1983), any $M_{k}$ may be considered nonbottleneck if, for every job sequence, the critical
path (or at least one of them, if not unique) contains just one task processed on $M_{k}$.

Such stages have the properties:

- Consecutive nonbottlenecks $M_{1}, M_{2}, \ldots, M_{u}$ can be replaced by a release time $\Sigma_{k=1}^{u} p_{j k}$ for each $J_{j}$.
- Consecutive nonbottlenecks $M_{u}, M_{u+1}, \ldots, M_{v}$ can be replaced by one nonbottleneck machine with processing time $\Sigma_{k=u}^{v} p_{j k}$ for each $J_{j}$.
- Consecutive nonbottlenecks $M_{v}, M_{v+1}, \ldots, M_{m}$ can be replaced by a final delay $\Sigma_{k=v}^{m} p_{j k}$ for each $J_{j}$.


Fig. $4.2 \quad$ Directed graph representation of schedule $(1, \ldots, n)$ for $F 3 \mid$ perm $\mid C_{\text {max }}$, (a) in general, and (b) with $M_{2}$ dominated

The following result gives conditions for a nonbottleneck.
Theorem 4.2 (Monma and Rinnooy Kan, 1983)
For Fm|perm|$C_{\max }$, if:
(a) $\min _{j} p_{k-1, j} \geq \max _{j} p_{k j}$, or
(b) $\min _{j} p_{k+1, j} \geq \max _{j} p_{k j}$, or
(c) $\min \left\{p_{k-1, j}, p_{k+1, j}\right\} \geq p_{k j}$, for $j=1,2, \ldots, n$,
then $M_{k}$ can be considered nonbottleneck.
Under these conditions, it is often said that $M_{k}$ is dominated by one or both of the adjacent machines. While conditions (a) and (b) involve all the jobs collectively, note that for condition (c) we test each job separately.

These and other approaches have been used to find polynomially solvable cases of $F m \mid$ perm $\mid C_{\text {max }}$. Unsurprisingly, most of the "easy" cases deal with the 3 -machine flow shop, for which we saw earlier that permutation schedules are always optimal. The following subsection lists these cases.

### 4.6.1 $\mathrm{F} 3 \mid$ (perm) $\mid C_{\text {max }}$

A number of cases where simple solutions exist can be tied together using the concept of machine dominance.

- Machine $M_{2}$ is dominated

With $m=3$, Theorem 4.2 becomes
Theorem 4.3 For $F 3 \mid($ perm $) \mid C_{\max }$, if any of the following conditions holds:
a) $\min _{j} a_{j} \geq \max _{j} b_{j}$, or
b) $\min _{j} c_{j} \geq \max _{j} b_{j}$, or
c) $\min \left\{a_{j}, c_{j}\right\} \geq b_{j}$, for $j=1,2, \ldots, n$,
then $M_{2}$ is dominated, so $M_{2}$ is nonbottleneck, and $S^{\star}=J R(\underline{a}+\underline{b}, \underline{c}+\underline{b})$.
Proof outline: As can easily be seen from Fig. 4.2(a), the important feature that all three conditions of Theorem 4.2 have in common is this: they imply that the longest or critical path through the network, which defines the makespan, never traverses any of the $M_{2}$-arcs. Thus, $M_{2}$ is nonbottleneck, so that we need only consider the network shown in Fig. 4.2(b), where, as discussed in Chap. 3, the critical path is:

$$
\begin{aligned}
C_{\max }(S) & =\max _{j=1, \ldots, n}\left[\sum_{i=1}^{j} a_{i}+b_{j}+\sum_{i=j}^{n} c_{i}\right] \\
& =\max _{j=1, \ldots, n}\left[\sum_{i=1}^{j}\left(a_{i}+b_{i}\right)+\sum_{i=j}^{n}\left(c_{i}+b_{i}\right)-\sum_{i=1}^{n} b_{i}\right]
\end{aligned}
$$

The final sum is sequence-invariant, and can be omitted when optimizing over $S$. We are left with a formula for the critical path through a two-machine flow shop where $J_{j}$ has task times $\left\langle a_{j}+b_{j}, c_{j}+b_{j}\right\rangle$. For this, we know that the schedule given by Johnson's Rule, $J R(\underline{a}+\underline{b}, \underline{c}+\underline{b})$, is optimal.
Parts of Theorem 4.3 originally appeared in Johnson (1954), Burns and Rooker $(1976,1978)$ and Szwarc (1977).

## - Machine $M_{2}$ is dominant

Definition 4.5. $M_{2}$ is dominant if either of the following conditions holds:
a) $\min _{j} b_{j} \geq \max _{j} a_{j}$ (i.e., $M_{2}$ dominates $M_{1}$ ), or
b) $\min _{j} b_{j} \geq \max _{j} c_{j}$ (i.e., $M_{2}$ dominates $M_{3}$ ).

When $M_{2}$ dominates $M_{1}$, Szwarc (1977) showed that an optimal sequence can be found as follows: Compute $J R(\underline{b}, \underline{c})$, and let $J_{[1]}$ be the first job in this sequence. Starting with $J R(\underline{b}, \underline{c})$, create all sequences produced by rescheduling a job with $a_{j} \leq a_{[1]}$ to position 1 , thus obtaining a number of alternative sequences. The best among $J R(\underline{b}, \underline{c})$ and the newly created sequences is optimal. By symmetry, when $M_{2}$ dominates $M_{3}$, the optimal schedule lies among $J R(\underline{a}, \underline{b})$ and all sequences produced from it by moving a job with $c_{j} \leq c_{[n]}$ to position $n$.

- $J R(\underline{a}, \underline{b})=J R(\underline{b}, \underline{c})=J R(\underline{a}, \underline{c})$

When these three schedules are identical, Burns and Rooker (1975) showed that they are optimal for $F 3 \mid($ perm $) \mid C_{\text {max }}$.

- $b_{j}=b(j=1, \ldots, n)$ and $\nearrow a_{j}=\searrow c_{j}$.

Under these conditions, $S^{\star}=\nearrow a_{j}$ for $F 3 \mid($ perm $) \mid C_{\max }$ (Szwarc, 1977).

## - Lower bound condition.

Szwarc (1977) also gave the following result. Let $S_{M}=J R(\underline{a}+\underline{b}, \underline{c}+\underline{b})$ be the modified Johnson sequence, with makespan $C^{\prime \prime}\left(S_{M}\right)$, where the primes are to emphasize that we are scheduling on two machines. Then $C^{\prime \prime}\left(S_{M}\right)-\Sigma_{j} b_{j}$ is the minimal makespan of the two-processor shop with times $\left\langle a_{j}, c_{j}\right\rangle$ and lags $b_{j}$. Since the complete three-processor network has additional paths, $C^{\prime \prime}\left(S_{M}\right)-\Sigma_{j} b_{j}$ is a lower bound on the minimal makespan of the original three-machine problem. Now, let $C^{\prime \prime \prime}\left(S_{M}\right)$ be the makespan of the threeprocessor problem using the sequence $S_{M}$. This is a feasible solution to our problem, so if it achieves the lower bound, it must be optimal. Thus:

Theorem 4.4 For $F 3 \mid($ perm $) \mid C_{\max }$, if $C^{\prime \prime \prime}\left(S_{M}\right)=C^{\prime \prime}\left(S_{M}\right)-\Sigma_{j} b_{j}$, then $S^{\star}=S_{M}=J R(\underline{a}+\underline{b}, \underline{c}+\underline{b})$.

- $\left[\min \left(a_{i}, b_{j}\right)-\min \left(a_{j}, b_{i}\right)\right]\left[\min \left(b_{i}, c_{j}\right)-\min \left(b_{j}, c_{i}\right)\right] \geq 0, \forall i, j$.

Under these conditions, $F 3 \mid$ (perm) $\mid C_{\text {max }}$ is solved in $O\left(n^{2} \log n\right)$ time (Szwarc, 1977).

An interesting computational experiment is presented in Smits and Baker (1981), examining how often a random instance of $F 3 \mid($ perm $) \mid C_{\text {max }}$ is optimally solved by one of the above mentioned special cases. It is found that by far the most successful test is the Lower Bound condition (so called because $C^{\prime \prime}\left(S_{M}\right)-\Sigma_{j} b_{j}$ is an obvious lower bound for $\left.C^{\prime \prime \prime}\left(S_{M}\right)\right)$. In their experiment, Smits and Baker found it to produce an optimum in 341 out of 600 instances with up to $n=50$ jobs. The processing times were either randomly drawn from a uniform distribution, or correlated (i.e., large $a_{j}$ means that $b_{j}, c_{j}$ tend to be large), or when $a_{j}, b_{j}, c_{j}$ are positively correlated with $M_{1}, M_{2}, M_{3}$ respectively; i.e., there is a trend, or when trend and correlated appear simultaneously. It was also found that the special cases presented above were particularly ineffective in solving problems with correlated processing times: only 7 of 50 instances were solved to optimality for $n=50$.

### 4.6.2 $\mathrm{Fm} \| C_{\text {max }}$

As expected, when $m>3$, problem difficulty increases and polynomial special cases become rare. For the ordered flow shop, the following result significantly reduces the search space and allows finding an optimal solution using dynamic programming rather than exhaustive enumeration. The result makes use of the following class of schedules.

Definition 4.6. In permutation ordered flow shops, a schedule is said to be pyramidal when there exists position $k, 1 \leq k \leq n$, such that the jobs in positions $1,2, \ldots, k$ are in SPT order, with the remaining jobs in LPT order.

Then, for ordered processing times as in Definition 4.4 we have:
Theorem 4.5 (Smith et al., 1976) For Fm $\mid$ perm, ordered $\mid C_{\text {max }}$, there exists an optimal schedule which is pyramidal.

A related positive result can be obtained for the proportionate flow shop, in which machines have different speeds and the time required to process a job is in proportion to the machine speed. Specifically, letting $s_{i}$ be the speed of $M_{i}$, the time that any $J_{j}$ requires on $M_{i}$ is given by $p_{i j}=p_{j} / s_{i}$. Then:

Theorem 4.6 (Eck and Pinedo, 1988) For Fm $\mid$ perm, $p_{i j}=p_{j} / s_{i} \mid C_{\max }$, if $M_{1}\left[M_{m}\right]$ is the slowest machine, then $S^{\star}=L P T[S P T]$.

## 4.7 $\boldsymbol{F m} \mid$ perm $\mid C_{\text {max }}$

Given the difficulty and importance of this problem, much effort has gone into finding effective enumerative schemes using branch-and-bound. Another approach, rather than insisting on finding the optimum, is to be content with heuristic solutions. We will briefly discuss aspects of both strategies.

### 4.7.1 Lower Bounds, with and without Lags

Lower bounds, preferably both tight and easily computed, are useful for two purposes. First, to evaluate heuristics, in the absence of optimal values to which the heuristic solution can be compared, a good lower bound on those values can be useful. Second, when searching for the optimum in a branching tree, we need at each node a lower bound on all completions of the partial schedule there.

Our discussion will assume the second case: that a partial schedule $\sigma$ has been determined, which occupies each machine $M_{k}$ up to a time $C_{k}(\sigma)$, and the set $\mathcal{U}$ of unscheduled jobs must be appended. In the first case, it is simple enough to set $\sigma=\phi, C_{k}(\sigma)=0$, and $\mathcal{U}=\mathcal{J}$. In either case, we want a lower bound on the makespans of all permissible schedules.

In addition, since most of the bounds to be discussed introduce artificial lags between stages, it adds very little complication to assume that there are actual transfer lags $\ell_{i j}$ in the original problem, which we shall do; they can always be zeroed out.

Lageweg et al. (1978) present a comprehensive classification scheme that incorporates almost all lower bounds for this problem. The basic idea is to treat some of the machines, initially bottlenecks (ordinary machines capable of processing one job at a time), as nonbottlenecks: machines that are able to process any number of jobs simultaneously, so do not need to be scheduled. Obviously, introducing nonbottlenecks shortens the makespan and so produces lower bounds. A machine $M_{k}$ that is made nonbottleneck will be denoted $N_{k}$. Observe that several consecutive nonbottlenecks, say $N_{1}, N_{2}, N_{3}$, can be considered a single nonbottleneck on which $J_{j}$ has task time $\Sigma_{k=1}^{3} p_{k j}$.

## A Simple bound

The simplest bound is found by picking one machine, say $M_{u}$, to be the unique bottleneck. The preceding nonbottlenecks combine to become a single nonbottleneck we shall call $N_{1 u}=\left\{N_{1}, N_{2}, \ldots, N_{u-1}\right\}$. On it, the processing time of $J_{j}, j \in \mathcal{U}$, incorporating the partial schedule $\sigma$ already fixed, is:

$$
p_{1 u, j}=\max _{k=1, \ldots, u-1}\left\{C_{k}(\sigma)+\sum_{i=k}^{u-1}\left(p_{i j}+\ell_{i j}\right)\right\}
$$

This of course simplifies to $p_{1 u, j}=\sum_{i=1}^{u-1}\left(p_{i j}+\ell_{i j}\right)$ if $\sigma$ is empty. Similarly, the succeeding nonbottlenecks merge to form $N_{u m}=\left\{N_{u+1}, N_{u+2}, \ldots, N_{m}\right\}$, on which each $J_{j}$ requires time

$$
p_{u m, j}=\sum_{i=u+1}^{m}\left(p_{i j}+\ell_{i-1, j}\right)
$$

where we include $\ell_{u j}$ since $J_{j}$ could be last on $M_{u}$, and as always $\ell_{m j}=0$ : there is no lag after the last machine.

Now, how long will it take to process the jobs in $\mathcal{U}$ on the three machines $N_{1 u}, M_{u}, N_{u m}$ ? Note that the first job to be scheduled on $M_{u}$ cannot start until that job has been processed on $N_{1 u}$. Since we do not know which job that is, all we can say for sure is that it will require at least a time

$$
l_{1 u}=\min _{j \in \mathcal{U}} p_{1 u, j}
$$

before $M_{u}$ can start processing. We might think of $l_{1 u}$ as a sort of machine "setup time". After that, $M_{u}$ processes its workload, requiring at least $\Sigma_{j \in \mathcal{U}} p_{u j}$ time units. Finally, whichever job is processed last on $M_{u}$ must be processed on $N_{u m}$, and the smallest time this will take (which might be loosely interpreted as a "teardown time" for $M_{u}$ ) is

$$
l_{u m}=\min _{j \in \mathcal{U}} p_{u m, j}
$$

The sum of these three quantities is a lower bound for the makespan. Since this is true for any $M_{u}$, we can choose the one that gives the greatest lower bound, getting finally:

$$
L B_{1}=\max _{u=1, . ., m}\left\{l_{1 u}+\sum_{j \in \mathcal{U}} p_{u j}+l_{u m}\right\}
$$

This lower bound was first presented in Lomnicki (1965) and in Ingall and Schrage (1965).

Lower bounds of the type $l_{1 u}$ and $l_{u m}$ (we will call them min-job-sum bounds) can always be used for a sequence of nonbottlenecks, independently of all other machines. They are weak bounds, and many researchers have proposed improvements.

## A Tighter Bound

The following bound has already been discussed for the case $m=2$ in Sect. 3.2.4. Still with one bottleneck $M_{u}$, we improve the bound $L B_{1}$ as follows. Since all jobs are processed in parallel on $N_{1 u}$, each $J_{j}$ is available for pro-
cessing on $M_{u}$ at time $p_{1 u, j}$, which can be viewed as an ordinary job release or ready time, $r_{j}$. But to minimize makespan on one machine when each $J_{j}$ arrives at time $r_{j}$, we use Jackson's Rule (Jackson, 1955), sequencing the jobs on $M_{u}$ in nondecreasing order of release times ( $/ r_{j}$ ), as a simple job interchange argument will verify. Thus, if $C_{u}(S)$ denotes the makespan on $M_{u}$ using schedule $S$ when jobs arrive at times $r_{j}$, we have:

$$
L B_{2}=\max _{u=1, \ldots, m}\left\{C_{u}\left(\nearrow r_{j}\right)+l_{u m}\right\}, \text { where } r_{j}=p_{1 u, j} .
$$

Note that we have strengthened the bound at the front of the schedule, but $N_{u m}$ is still represented by the weak min-job-sum bound. Through a timereversal argument, the roles of $N_{1 u}$ and $N_{u m}$ can be interchanged and another similar bound can be found:

$$
L B_{2}^{\prime}=\max _{u=1, \ldots, m}\left\{l_{1 u}+C_{u}\left(\nearrow r_{j}\right)\right\}, \text { where } r_{j}=p_{u m, j} .
$$

Attempts to strengthen the bound at both ends simultaneously result in an NP-complete problem.

## A Two-Bottleneck Bound

An even tighter lower bound can be calculated by defining two bottleneck machines, $M_{u}$ and $M_{v}, 1 \leq u<v \leq m$, with all the rest nonbottleneck. We now base our bound on the simplified system of five machines: $N_{1 u}, M_{u}, N_{u v}, M_{v}, N_{v m}$. As before, the first and last add at least the amount $l_{1 u}$ and $l_{v m}$, respectively, to the makespan. We are left with two ordinary machines, $M_{u}$ and $M_{v}$, with the nonbottleneck $N_{u v}=\left\{N_{u+1}, N_{u+2}, \ldots, N_{v-1}\right\}$ in between. Of course, $N_{u v}=\phi$ if $v=u+1$. Since a nonbottleneck allows each job to proceed at its own pace, each $J_{j} \in \mathcal{U}$ has a delay of

$$
\ell_{u v, j}=\ell_{u j}+\sum_{i=u+1}^{v-1}\left(p_{i j}+\ell_{i j}\right)
$$

which is effectively a transfer lag between being processed for a time $p_{u j}$ on $M_{u}$ and for a time $p_{v j}$ on $M_{v}$. As we argued above (see the proof of Theorem 4.3), the optimal schedule for such a two-machine flow shop with lags is:

$$
J R(\underline{\alpha}, \underline{\beta}), \quad \text { where } \quad \alpha_{j}=p_{u j}+\ell_{u v, j}, \quad \beta_{j}=p_{v j}+\ell_{u v, j}
$$

The length of this schedule, call it $C(u, v)$, is a lower bound on the makespan contribution of the unscheduled jobs on machines $M_{u}, M_{u+1}, \ldots, M_{v}$. We therefore have an overall lower bound of

$$
L B_{3}=\max _{1 \leq u<v \leq m}\left\{l_{1 u}+C(u, v)+l_{v m}\right\}
$$

Computing $L B_{3}$ requires $O\left(m^{2}\right)$ applications of Johnson's $O(n \log n)$ algorithm, so it needs a time $O\left(m^{2} n \log n\right)$, somewhat slow when used repeatedly in a branch-and-bound algorithm. Note that $L B_{3}$ reduces to $L B_{1}$ when $u=v$. The computational complexity of $L B_{3}$ can be reduced by a factor $O(n)$ if we set $u=1$ or $v=m$. Szwarc (1983b) was the first to include lags in $L B_{3}$, but only for the case $u=1, v=m$. A comparison of $L B_{1}$ and $L B_{3}$ without lags is provided in Baker (1975), where $L B_{3}$ is found to be very effective.

## Other Bounds

No bounds have been proposed with more than two bottlenecks, since we cannot efficiently minimize the makespan in flow shops with over two machines. As a shorthand to symbolize the types of bounds that have been proposed, we use $b$ for a bottleneck, and for the nonbottleneck sequences an $l$ denotes the simple min-job-sum, while $L$ is used for a more sophisticated bound (thus, big $L$ for a presumably bigger lower bound).

Using this notation, $L B_{1}, L B_{2}, L B_{3}$ are bounds of type $(l, b, l),(L, b, l)$ and $(l, b, L, b, l)$, respectively. Lageweg et al. (1978), who introduce this systematic approach, go through all the different categories, namely $(x, b, x)$ and $(x, b, x, b, x)$ where each $x$ can be either $l$ or $L$. They point out that certain types of bounds are inherently stronger than, or dominate, other types, meaning that the stronger approach can always produce a tighter bound. Thus, $(L, b, L)$ dominates $(l, b, L)$ or $(L, b, l)$, all of which dominate $(l, b, l)$. Generally, for the same number of bottlenecks ( $b$ 's), the approach that uses fewer min-job-sums ( $l$ 's) dominates. Also, a procedure that uses two $b$ 's dominates a procedure with one $b$ if it uses at least as many L's. Hopefully, these conclusions are easily seen, as they are easily shown. Otherwise, dominance does not hold. For example, an algorithm of type $(L, b, L)$ can do better or worse than a ( $l, b, L, b, l$ ) approach, for different problem instances. For more details, and additional bounds, see Lageweg et al. (1978).

## A New Type of Bound

In all bounds proposed up to now in the literature, as summarized above, all but one or two machines are replaced by nonbottlenecks. We introduce here a new approach to lower-bounding the makespan in the $m$-machine flow shop with or without lags. We give it more extensive coverage than usual, since it has not heretofore appeared in publication. One strength of this technique is that it bounds the $m$-machine makespan problem even when permutation schedules are not required.

## Without a Predetermined Partial Schedule $\sigma$

To keep things simple, we start by proposing a lower bound on the makespan, with no assumption that an initial partial schedule $\sigma$ has already fixed.

Theorem 4.7 For every $u=1,2, \ldots, m-1$, the makespan of

$$
S^{u}=J R\left(\sum_{i=1}^{u} \underline{p}_{i} / u+\underline{\ell}_{u}, \quad \sum_{i=u+1}^{m} \underline{p}_{i} /(m-u)+\underline{\ell}_{u}\right)
$$

where $\underline{p}_{i}=\left(p_{i 1}, \ldots, p_{i n}\right), \quad \underline{\ell}_{u}=\left(\ell_{u 1}, \ldots, \ell_{u n}\right)$, is a lower bound for the optimal makespan of $F m\left|\ell_{i j}\right| C_{\max }$.

That is, replace the first $u$ machines with a single processor on which the processing time of each $J_{j}$ is the average of the $u$ separate times: $\sum_{i=1}^{u} p_{i j} / u$; and similarly group the remaining $m-u$ machines into one machine where task times are the average over those machines. We now have a two-machine
problem, with lags between them equal to the lags between $M_{u}$ and $M_{u+1}$. We assert that its optimal schedule gives a lower bound to our problem.

Proof: We will verify this result by constructing an auxiliary two-machine problem, as follows. First, replace the first $u$ machines by a single superprocessor, call it the "left" machine $M_{L}$, that can do any and all the work of the $u$ machines, and do it $u$ times as fast. That is, $M_{L}$ can process task $T_{k j}$ in time $p_{k j} / u$, for $j \in \mathcal{J}$ and $k=1, \ldots, u$. Similarly, another super-processor, the "right" machine $M_{R}$, will replace the remaining $m-u$ processors, doing their work $m-u$ times faster.

We first show how, for arbitrary schedule $S$ of an instance of $F m\left|\ell_{i j}\right| C_{\max }$, we map the first $u$ processors into the first super-processor $M_{L}$. In Fig. 4.3, we illustrate with a non-permutation $S$, using the 5 -job 4 -machine instance given in Example 4.1 and assuming $u=2$, where for simplicity we set all lags to zero. Note how lags would stretch out $S$ and only reinforce the conclusion that the auxiliary system has a smaller makespan. For given $S$ (see Fig. 4.3(a)), order all the tasks on machines $M_{1}, M_{2}, \ldots, M_{u}$ in nondecreasing order of completion times and process them on $M_{L}$ in that order and without delay. This is shown in Fig. 4.3(b), where each task is half as long as it was in $S$ because $u=2$. The fact that $M_{L}$ is $u$ times faster than each of $M_{1}, M_{2}, \ldots, M_{u}$ ensures that the completion time of a task on $M_{L}$ is no greater than its original completion time. Moreover, the total processing time of $J_{j}$ on $M_{L}$ is $\Sigma_{k=1}^{u} p_{k j} / u$. One can then eliminate preemption of jobs on $M_{L}$ simply by concatenating the tasks $T_{1 j}, \ldots, T_{u j}$ and ordering jobs $j \in \mathcal{J}$ on $M_{L}$ in the order they are processed on $M_{u}$ (see Fig. 4.3(c)).
(a)



(b) $M_{L}$

(d) $M_{R}$
(e) $M_{R}$

|  | 2 | 1 | 1 | 2 | 3 | 4 | 3 | 5 | 4 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | 2 | 1 | 3 |  |  |  | 4 | 5 |  |

Fig. 4.3 Creation of a two-machine problem to give a lower bound for an instance of $F m \| C_{\text {max }}$ with $n=5, m=4$ and $u=2$

A mirror image construction can be employed to replace machines $M_{u+1}$, $\ldots, M_{m}$ by a single super-processor $M_{R}$. On $M_{R}$, tasks $T_{k j}, k=u+1, \ldots, m$, are scheduled backwards from time $C(S)$, in nonincreasing order of their start times on $S$, as in Fig. 4.3(d). Each successive task is scheduled as late as possible so as not to violate precedence and lags. The tasks of a job may then be grouped, and the jobs scheduled on $M_{R}$ in the order they are scheduled on $M_{u+1}$ (see Fig. 4.3(e)). On both $M_{L}$ and $M_{R}$, we could have skipped the intermediate step of scheduling the separate tasks (as in Fig. 4.3(b) and (d)) and simply scheduled the aggregated tasks, but we went through the intermediate step to make clear that each job is completed on $M_{L}$ no later than on $M_{u}$, and is started on $M_{R}$ no earlier than on $M_{u+1}$, so task precedence is satisfied.

Let $S^{\prime}$ be the resulting schedule on $M_{L}, M_{R}$. By construction, this schedule is feasible for $F 2\left|\ell_{u j}\right| C_{\max }$, and has $C\left(S^{\prime}\right)=C(S)$ (indeed, almost always the schedule on $M_{R}$ can be left-shifted to make $C\left(S^{\prime}\right)$ even smaller). From Mitten's algorithm, we know that $C\left(S^{u}\right) \leq C\left(S^{\prime}\right)$, and hence is a lower bound for $F m\left|\ell_{i j}\right| C_{\max }$.

## If a Partial Schedule Already Exists

Suppose now that partial schedule $\sigma$ already exists, occupying each $M_{k}$ up to a time $C_{k}(\sigma)$, with job set $\mathcal{U}$ remaining to be scheduled. We can simply assume that $M_{k}$ is initially occupied by a dummy job requiring time $C_{k}(\sigma)$ (and requiring zero processing on every other machine). We can then add these $m$ jobs to $\mathcal{U}$ and repeat the above construction.

## Combined Bounds

We can now define a more general class of lower bounds by combining this new type of bound with the earlier nonbottleneck bounds. Thus, suppose we partition the set of machines into three groups. Group 1 includes the stations $M_{1}, \ldots, M_{u}$, Group 2 includes the intermediate machines $M_{u+1}, \ldots, M_{v-1}$, and Group 3 the machines $M_{v}, \ldots, M_{m}$. For any pair of values $u, v$ with $1 \leq u<v \leq m$, we calculate the sequence

$$
S^{u, v}=J R\left(\sum_{i=1}^{u} \underline{p}_{i} / u+\underline{\ell}_{u v}, \sum_{i=v}^{m} \underline{p}_{i} /(m-v+1)+\underline{\ell}_{u v}\right)
$$

where

$$
\ell_{u v, j}=\ell_{u j}+\sum_{i=u+1}^{v-1}\left(p_{i j}+\ell_{i j}\right)
$$

That is, we define a two-machine flow shop with lags, in which the data for machines in Groups 1 and 3 provide the processing times (in the spirit of Theorem 4.7), while machines in the middle group are made nonbottleneck and so generate lags (as in the two-bottleneck bound, $L B_{3}$ ). As we know, the optimal permutation schedule for such a shop is given by $S^{u, v}$. Note that, when $v=u+1$, Group 2 disappears, and the lags are simply $\ell_{u j}$. Also, we earlier introduced $S^{u, u+1}$ as $S^{u}$.

At this point, the following theorem should be clear. Since it is a combination of earlier results, the proof combines elements of those arguments and will be omitted.

Theorem 4.8 For each pair of values $(u, v)$, the makespan of $S^{u, v}$ is a lower bound on the minimal makespan of $F m \mid$ perm, $\ell_{i j} \mid C_{\max }$.

The work required to calculate $S^{u, v}$ for all $u, v$ is $O\left(m^{2} n \log n\right)$ because there are $m(m-1) / 2=O\left(m^{2}\right)$ combinations of $(u, v)$ values, for each of which the effort to calculate $S^{u, v}$ is $O(n \log n)$.

## Computational Experiments

As we know, $S^{u, v}$ generalizes two earlier results: $S^{1, m}$ reduces to the sequence obtained in $L B_{3}$, while $S^{u, u+1}$ duplicates the result in Theorem 4.7. We can verify that neither of these two bounds dominates the other. Consider the problem instance with $m=4$, and $n=10$ copies of the job $\left.\left.J=<p_{1}, \ldots, p_{4} ; \ell_{1}, \ell_{2}, \ell_{3}\right\rangle=<p, 10,10, p ; 5,5,5\right\rangle$. For the case $(u, v)=$ $(1,4)$, the corresponding two-machine problem has 10 identical jobs with $p_{1}^{\prime}=p_{2}^{\prime}=p$ and $\ell_{1}^{\prime}=35$, so the makespan $C\left(S^{1,4}\right)=35+11 p$. For $(u, v)=(2,3)$, all jobs have $p_{1}^{\prime}=p_{2}^{\prime}=(10+p) / 2$ and $\ell_{1}^{\prime}=5$, so $C\left(S^{2,3}\right)=5+11(10+p) / 2=60+5.5 p$. It follows that if $p$ is small (say, $p \leq 4$ ), $S^{2,3}$ gives the larger (i.e., tighter) bound, while for $p \geq 5, S^{1,4}$ dominates.

We can vary the values of $n$ and $\ell_{i}$, and observe what situations favor each of the 2 lower bounds. It is easily seen that $S^{2,3}$ is superior (i.e., $C\left(S^{2,3}\right)>$ $\left.C\left(S^{1,4}\right)\right)$ when

- lags are small relative to processing times;
- the number of jobs is large;
- processing times on the intermediate machines are large relative to the early and late machines.
Since neither $S^{u, u+1}$ nor $S^{1, m}$ dominates, we may reasonably ask whether computing $S^{u, v}$ for all values of $u$ and $v$ can produce tighter bounds than either. Below we provide an example with $m=4$ and $n=5$ where $S^{1,3}$ gives the greatest lower bound.

Example 4.1: Consider the following instance of $F m \mid$ perm, $\ell_{i j} \mid C_{\max }$ with 5 jobs and 4 machines.

| $j$ | $p_{1 j}$ | $\ell_{1 j}$ | $p_{2 j}$ | $\ell_{2 j}$ | $p_{3 j}$ | $\ell_{3 j}$ | $p_{4 j}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 6 | 1 | 1 | 0 | 10 | 2 | 1 |
| 2 | 6 | 1 | 2 | 3 | 11 | 1 | 2 |
| 3 | 8 | 1 | 2 | 1 | 15 | 2 | 1 |
| 4 | 7 | 3 | 1 | 2 | 13 | 1 | 2 |
| 5 | 9 | 2 | 3 | 2 | 17 | 1 | 2 |

Table 4.1 reports the calculations made to compute the sequences $S^{u, v}$ for $1 \leq$ $u<v \leq 4$. Note that in this problem instance all time lags are comparable and the processing times in stages 1 and 3 are larger than in stages 2 and 4. As expected, $S^{1,3}$ accurately reflects the bottleneck stages and indeed $C\left(S^{1,3}\right)=52$ outperforms all other lower bounds.

Table 4.1 Lower bound calculations for a 5 -job instance of $F 4\left|p e r m, \ell_{i j}\right| C_{\max }$

| $u v j$ | $\underline{p}_{1}^{\prime}$ | $\underline{\ell}_{u v}$ | $\underline{p}^{\prime}$ | $\underline{\underline{p}_{1}^{\prime}}+\underline{\ell}_{u v} \underline{p}_{2}^{\prime}+\underline{\ell}_{u v}$ |  | $S^{u, v}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 12 | $\underline{p}_{1}$ |  | $\frac{1}{3}\left(\underline{p}_{2}+\underline{p}_{3}+\underline{p}_{4}\right)$ |  |  |  |
| 1 | 6 | 1 | 4 | 7 | 5 |  |
| 2 | 6 | 1 | 5 | 7 | 6 |  |
| 3 | 8 | 1 | 6 | 9 | 7 | $S^{1,2}=(5,4,3,2,1)$ |
| 4 | 7 | 3 | 5.3 | 10 | 8.3 | $C\left(S^{1,2}\right)=41$ |
| 5 | 9 | 2 | 7.3 | 11 | 9.3 |  |
| 13 | $\underline{p}_{1}$ | $\underline{\ell}_{13}$ | $\frac{1}{2}\left(\underline{p}_{3}+\underline{p}_{4}\right)$ |  |  |  |
| 1 | 6 | 2 | 5.5 | 8 | 7.5 |  |
| 2 | 6 | 6 | 6.5 | 12 | 12.5 |  |
| 3 | 8 | 4 | 8 | 12 | 12 | $S^{1,3}=(2,3,4,5,1)$ |
| 4 | 7 | 6 | 7.5 | 13 | 13.5 | $C\left(S^{1,3}\right)=52$ |
| 5 | 9 | 7 | 9.5 | 16 | 16.5 |  |
| 14 | $\underline{p}_{1}$ | $\underline{\ell}_{14}$ | $\underline{p}_{4}$ |  |  |  |
| 1 | 6 | 14 | 1 | 20 | 15 |  |
| 2 | 6 | 18 | 2 | 24 | 20 |  |
| 3 | 8 | 21 | 1 | 29 | 22 | $S^{1,4}=(5,4,3,2,1)$ |
| 4 | 7 | 20 | 2 | 27 | 22 | $C\left(S^{1,4}\right)=51$ |
| 5 | 9 | 25 | 2 | 34 | 27 |  |
| 23 | $\frac{1}{2}\left(\underline{p}_{1}+\underline{p}_{2}\right)$ | $\underline{\ell}_{23}$ | $\frac{1}{2}\left(\underline{p}_{3}+\underline{p}_{4}\right)$ |  |  |  |
| 1 | 3.5 | 0 | 5.5 | 3.5 | 5.5 |  |
| 2 | 4 | 3 | 6.5 | 7 | 9.5 |  |
| 3 | 5 | 1 | 8 | 6 | 9 | $S^{2,3}=(1,3,4,2,5)$ |
| 4 | 4 | 2 | 7.5 | 6 | 9.5 | $C\left(S^{2,3}\right)=41$ |
| 5 | 6 | 2 | 9.5 | 8 | 11.5 |  |
| 24 | $\frac{1}{2}\left(\underline{p}_{1}+\underline{p}_{2}\right)$ | $\underline{\ell}_{24}$ | $\underline{p}_{4}$ |  |  |  |
| 1 | 3.5 | 12 | 1 | 15.5 | 13 |  |
| 2 | 4 | 15 | 2 | 19 | 17 |  |
| 3 | 5 | 18 | 1 | 23 | 19 | $S^{2,4}=(5,3,4,2,1)$ |
| 4 | 4 | 16 | 2 | 20 | 18 | $C\left(S^{2,4}\right)=37$ |
| 5 | 6 | 20 | 2 | 26 | 22 |  |
| 34 | $\frac{1}{3}\left(\underline{p}_{1}+\underline{p}_{2}+\underline{p}_{3}\right)$ | $\underline{\ell}_{34}$ | $\underline{p}_{4}$ |  |  |  |
| 1 | 5.7 | 2 | 1 | 7.7 | 3 |  |
| 2 | 6.3 | 1 | 2 | 7.3 | 3 |  |
| 3 | 8.3 | 2 | 1 | 10.3 | 3 | $S^{3,4}=(1,2,3,4,5)$ |
| 4 | 7 | 1 | 2 | 8 | 3 | $C\left(S^{3,4}\right)=40$ |
| 5 | 9.7 | 1 | 2 | 10.7 | 3 |  |

### 4.8 Dominance Properties for $\operatorname{Fm} \mid$ perm $\mid C_{\text {max }}$

Continuing with our discussion of the makespan objective, on which the research reported in the literature focuses, we now present properties that aim to reduce the solution space of a problem by eliminating sequences that result to suboptimal solutions. Such properties may be characterized as job dominance or sequence dominance. If we specialize Theorem 4.1, recalling that " $J_{i}$ dominates $J_{j}$ following $\sigma$ " is written $j \sim_{\sigma} i$, we get

Corollary 4.1 For $F m \mid$ perm $\mid C_{\max }$ with initial partial sequence $\sigma$,
$C\left(\sigma i j \pi_{1} \pi_{2}\right) \leq C\left(\sigma j \pi_{1} i \pi_{2}\right)$ for all $\pi_{1}, \pi_{2} \Rightarrow i \sim_{\sigma} j$,
where $\sigma i j \pi_{1} \pi_{2}$ is a complete schedule.
Szwarc (1971) uses this as his definition of job dominance.
Sequence dominance was introduced earlier in Definitions 4.2 and 4.3, which we will not repeat here. The work of Gupta (1971) indicates that sequence dominance is much more demanding than job dominance in terms of storage space and CPU time, because there are $\binom{n}{r}$ distinct subsets of $J$ of size $r$, each of which has $r$ ! permutations. Furthermore, sequence dominance does not contribute much to reducing the time required to identify a set of undominated solutions. For these reasons, most of the research devoted to dominance properties for $F m \| C_{\max }$ focuses on job dominance.

The job dominance properties that have appeared in the literature assume permutation schedules, i.e., schedules for $F m \mid$ perm $\mid C_{\max }$. Szwarc (1971) presented a comparison of various job dominance properties. The following two stand out.

Theorem 4.9 (Dudek and Teuton, 1964) For Fm|perm|$C_{\text {max }}$ with initial partial sequence $\sigma$,

$$
C_{k}(\sigma i j) \leq C_{k}(\sigma j i), \text { for } k=2, \ldots, m \Rightarrow i \leadsto_{\sigma} j
$$

For the following theorem, we define

$$
\Delta_{k}=C_{k}(\sigma i j)-C_{k}(\sigma j) \text { for } k=1,2, \ldots, m
$$

Theorem 4.10 (Szwarc, 1971) For Fm|perm $\mid C_{\max }$ with initial partial sequence $\sigma$,

$$
\Delta_{k-1} \leq \Delta_{k} \leq p_{k i} \text { for } k=2, \ldots, m \Rightarrow i \leadsto_{\sigma} j
$$

The latter property subsumes several dominance criteria that have appeared in the literature; see Szwarc (1971), Bagga and Chakravarti (1968), Smith and Dudek (1967) and (1969). Moreover, the following criteria are shown in Szwarc (1971) to be equivalent to the condition of Theorem 4.10:

- $\Delta_{k-1} \leq p_{k i}, \Delta_{k} \leq p_{k i}$ for $k=2, \ldots, m$, or
- $\max \left\{\Delta_{1}, \ldots, \Delta_{m}\right\} \leq p_{k i}$ for $k=2, \ldots, m$, or
- $\Delta_{k} \leq \min \left\{p_{k i} \ldots, p_{m i}\right\}$ for $k=2, \ldots, m$.

Using Theorem 4.10, Baker (1975) developed an elimination algorithm that selects the best among all undominated sequences resulting from Theorem 4.10 and found that elimination followed by branch-and-bound strategies is computationally inefficient compared to a branch-and-bound algorithm that uses solely backtracking and no dominance. A slightly different type of job dominance property is presented next where the focus is on the last job of an optimal sequence.

Theorem 4.11 (Szwarc, 1973)
If $\max \left\{\Delta_{1}, \ldots, \Delta_{m}\right\} \leq p_{m j}$, then $J_{j}$ is never last.
In the last two sections we studied lower bounding schemes and dominance properties as a means of reducing the set of undominated solutions and identify an optimal one by a branch-and-bound algorithm. Towards this end, we are in need of heuristic algorithms which can quickly identify a near-optimal solution for an initial upper bound. Of course, heuristics may also be used (and usually are) as stand-alone. This is the focus of the next section.

### 4.9 Heuristic Algorithms

In this subsection we present constructive and iterative heuristics. In constructive heuristics, also called one-pass or single-shot heuristics, a single permutation is proposed, built one job at a time, or by computing an index for every job and then ordering the jobs by index values. Iterative heuristics, also called improvement heuristics, improve upon an initial solution by job exchanges.

As with most flow shop designs, the literature is largely concerned with permutation sequencing, despite the observation by Potts et al. (1991) that such practice may produce poor solutions in some cases. We begin with two attempts to find good nonpermutation schedules. An obvious extension of Johnson's algorithm is the following heuristic proposed by Gonzalez and Sahni (1978) where for simplicity, we assume that $m$ is an even integer, if necessary adding a dummy stage where each job requires zero time.

Heuristic $\mathbf{H}_{\text {CON }}$ for $F m \| C_{\text {max }}$

1. Compute schedules $S_{2 k-1,2 k}=J R\left(\underline{p}_{2 k-1}, \underline{p}_{2 k}\right)$, for $k=1,2, \ldots, m / 2$.
2. Let $S_{C O N}$ be the concatenation of partial schedules $S_{12}, S_{34}, \ldots, S_{m-1, m}$.

Of course, $S_{C O N}$ is not generally a permutation schedule because different permutations will usually be optimal for different pairs of machines. It will not often produce good schedules, since successive pairs of machines are in no way coordinated. Simple job interchange, say of adjacent pairs, would clearly improve $S_{C O N}$ for many instances.

Despite its simplicity, the worst case makespan of $S_{C O N}$ is the best known to date. Indeed, if $C^{\star}$ is the optimal makespan value for $F m \| C_{\max }$, then
clearly $C\left(S_{2 k-1,2 k}\right) \leq C^{\star}$ for $k=1,2, \ldots, m / 2$, so that $C\left(S_{C O N}\right) \leq m C^{\star} / 2$. The following result incorporates odd values of $m$ :

Theorem 4.12 (Gonzalez and Sahni, 1978) For Fm $\| C_{\max }$ :

$$
C\left(S_{C O N}\right) / C^{\star} \leq\lceil m / 2\rceil, \quad \text { and this bound is tight for } m=3,4
$$

Sevast'janov (1995) presented absolute performance guarantees by applying to the flow shop makespan problem results from the non-strict vector summation problem (NVS) which is stated as follows: Given a norm in the plane, and a finite family of vectors with zero sum and each vector having at most unit length, does there exist a permutation of the vectors such that for any two successive partial sums of the vectors specified by the permutation at least one of them belongs to a given domain $G$ ? For the adaptation to the flow shop makespan problem, the norm used is the Euclidean and upon scaling of the job processing times $G$ is the unit ball. The finite set of vectors is formed using the differences $\sum_{j=1}^{i} p_{\sigma(j), k}-\sum_{j=1}^{i-1} p_{\sigma(j), k+1}$ for given permutation $\sigma, i \in \mathcal{J}, k \in \mathcal{M}$, and the observation that there exists $\sigma$ such that

$$
C(\sigma) \leq \max _{i} P_{i}+\sum_{k=1}^{m-1} \max _{i}\left(\sum_{j=1}^{i} p_{\sigma(j), k}-\sum_{j=1}^{i-1} p_{\sigma(j), k+1}\right)
$$

where $P_{i}=\Sigma_{k} p_{k i}$; see Sevast'janov (1994). Then, Sevast'janov (1995) presented the absolute performance bound $C(\sigma) \leq \max _{i} P_{i}+\beta(m) p_{\text {max }}$, with

$$
m-1+\lceil(m-1) / 2\rceil \leq \beta(m) \leq m^{2}-3 m+3-1 /(m-2)
$$

(note that $\max _{i} P_{i} \leq C^{\star}$ ). In Sevast'janov (1997), the author developed a linear time heuristic with absolute performance guarantee $\beta(3)=3$ for problem $F 3|(p m t n)| C_{\max }$ and an $O(n \log n)$ heuristic with $4 \leq \beta(4) \leq 6$ for problem $F 4|p m t n| C_{\max }$. To date, these appear to be the strongest approximation results available in the literature.

The results considered so far regard permutation schedules. Koulamas's (1998) heuristic, referred to as HFC, explicitly accounts for improvements in makespan by means of nonpermutation schedules. HFC makes use of the following $\binom{m}{2}=O\left(m^{2}\right)$ schedules:

$$
S_{k, k^{\prime}}=J R\left(\underline{p}_{k}, \underline{p}_{k^{\prime}}\right), \text { for } 1 \leq k<k^{\prime} \leq m
$$

## Heuristic HFC for $F m \| C_{\text {max }}$

1. Set $S_{0}:=\phi$ and compute schedules $\mathcal{S}=\left\{S_{k, k^{\prime}}: 1 \leq k<k^{\prime} \leq m\right\}$.
2. Schedule next in $S_{0}$ an unscheduled job with greatest total number of unscheduled followers in all schedules in $\mathcal{S}$.
Repeat until a complete permutation of jobs in $\mathcal{J}$ is obtained.
3. For $i=1$ to $n-1$ do

If for some $l, 2 \leq l<m, J_{i}$ precedes $J_{i+1}$ in $\left\{S_{0}, S_{k, k^{\prime}}: 1 \leq k<k^{\prime} \leq l\right\}$, and $J_{i+1}$ precedes $J_{i}$ on $\left\{S_{k, k^{\prime}}: l+1 \leq k<k^{\prime} \leq m\right\}$,
then let $J_{i}$ pass $J_{i+1}$ after $M_{l}$ on $S_{0}$.
4. Let $S_{H F C}$ be the resulting schedule.

Koulamas's implementation of HFC is shown to take $O\left(m^{2} n^{2}\right)$ time. Surprisingly, as we will see shortly, HFC does not compare favorably with the best known heuristics for $F m \mid$ perm $\mid C_{\text {max }}$. This might be due to the fact that it only allows for a single pass for each job.

The heuristics presented so far resulted in nonpermutation schedules. We focus next on heuristics for $F m \mid$ perm $\mid C_{\text {max }}$. The earliest published heuristic for this problem appears to be Palmer's (1965) algorithm. It seeks to give priority to jobs having the strongest tendency to increase in processing time as they progress through the stages (and, of course, to schedule jobs with the opposite tendency later). To accomplish this, Palmer proposes the schedule $\nearrow s_{j}$, where the slope, $s_{j}$, of $J_{j}$ is defined:

$$
s_{j}=\sum_{i=1}^{m}(m-2 i+1) p_{i j} .
$$

Since the coefficients decrease as $i$ increases, $s_{j}$ is small if early times are small and late ones large; and vice versa. When applied to $F 2 \mid($ perm $) \mid C_{\max }$ the slope heuristic, $\nearrow s_{j}$, sequences jobs in nondecreasing order of $b_{j}-a_{j}$ and is not necessarily optimal. It has complexity $O(n m+n \log n)$ due to calculating all $s_{j}$ 's and subsequently sorting them, and is shown in Nowicki and Smutnicki (1993) to have worst case relative error bound $\mathrm{m} / \sqrt{2}$.

Note that the $s_{j}$ values ignore machine $M_{(m+1) / 2}\left(s_{(m+1) / 2}=0\right)$ when $m$ is odd. Hundal and Rajgopal (1988) created two more schedules to resolve this peculiarity (namely, the schedules associated with slopes $s_{j}^{\prime}=\sum_{i=1}^{m}(m-2 i) p_{i j}$ and $\left.s_{j}^{\prime \prime}=\Sigma_{i=1}^{m}(m-2 i+2) p_{i j}\right)$ and select one with the smallest makespan.

Palmer's slope algorithm is not the only way to express "slope". Gupta's (1971) version of slope is:

$$
f_{j}=\operatorname{sign}\left(p_{1 j}-p_{m j}\right) / \min _{k=1, \ldots, m-1}\left(p_{k j}+p_{k+1, j}\right) .
$$

Nowicki and Smutnicki (1994) established the worst case relative error bound for Hundal and Rajgopal's heuristic to be $m / \sqrt{2}+O(1 / m)$. The same authors showed that Gupta's slope algorithm has worst case relative error bound $m-1$. Alternative slopes are given in Bonney and Gundry (1976) using the cumulative processing times of jobs.

One of the best performing polynomial time heuristics is due to Campbell, Dudek and Smith (1970), often referred to as the CDS heuristic. It solves a series of pseudoproblems using Johnson's algorithm, as follows:

Heuristic CDS for $F m \mid$ perm $\mid C_{\text {max }}$

1. For $r=1,2, \ldots, m-1$ define

$$
\begin{equation*}
a_{j}^{r}=\sum_{k=1}^{r} p_{k j}, \quad b_{j}^{r}=\sum_{k=m-r+1}^{m} p_{k j}, \quad j=1, \ldots, n . \tag{4.7}
\end{equation*}
$$

2. Compute $J R\left(\underline{a}^{r}, \underline{b}^{r}\right)$, and choose the best of the $m-1$ schedules.

In other words, the CDS heuristic solves a series of two-machine pseudoproblems where the processing requirements of $J_{j}$ on $M_{1}\left[M_{2}\right]$ equals the combined processing required on the first [last] $r$ stations of the original flow shop. On
randomly generated problems with up to 8 jobs and 3,5 or 7 machines the optimality gap of the CDS heuristic is reported to be $2.54 \%$. Nowicki and Smutnicki (1989) showed that a worst case relative error bound of the CDS algorithm is $\lceil m / 2\rceil$ but this bound is not necessarily tight.

Page (1961), building on the CDS heuristic, expresses Johnson's Rule as a list schedule, using the priority index $I_{r}(j)=\operatorname{sign}\left(a_{j r}-b_{j r}\right) / \min \left(a_{j r}, b_{j r}\right)$ (see Version 3 of JR in Chap. 2), in terms of which $J R\left(\underline{a}_{r}, \underline{b}_{r}\right)$ becomes $\nearrow I_{r}(j)$. Having chosen the best of these over $r$, the indices are sorted by various methods (e.g., merging, pairing) as well as by 2 -Opt and other job exchanges so as to develop alternative heuristics. Each one of the proposed heuristics produce a single permutation.

Like the CDS algorithm and its extensions, a simple variation of JR was presented in Dannenbring (1977) using linear combinations of the task times for the values $a_{j}, b_{j}$ and running a single iteration of Johnson's algorithm. The heuristic, referred to as rapid access or RA, is as follows:

Heuristic RA for $F m \mid$ perm $\mid C_{\text {max }}$

1. Define

$$
A_{j}=\sum_{i=1}^{m}(m-i+1) p_{i j}, \quad B_{j}=\sum_{i=1}^{m} i p_{i j}, \quad j=1, \ldots, n
$$

2. Compute $J R(\underline{A}, \underline{B})$.

Heuristic RA and is shown in Nowicki and Smutnicki (1991) to have worst case relative error bound $m / \sqrt{2}$ which becomes tight as $m$ approaches infinity. To further improve the RA heuristic, Dannenbring (1977) experimented with pairwise job exchanges. His rapid access with close order search (or RACS) heuristic considers the $n-1$ possible swaps of adjacent jobs, while the rapid access with extended search (or RAES) heuristic allows for unlimited adjacent job swaps as long as they reduce $C_{\max }$. It was found that RAES outperformed all heuristics presented earlier in this section in more than $70 \%$ of the randomly generated instances tested, which were of various sizes with $n, m \leq 50$. Nowicki and Smutnicki (1993) showed, however, that the worst case relative error bound of RAES is the same as for RA, i.e., $m / \sqrt{2}$.

Another well known heuristic algorithm is due to Nawaz, Enscore and Ham (1983) and is referred to as the NEH heuristic. It is based on the premise that a job $J_{j}$ with larger total processing requirement $\Sigma_{i} p_{i j}$ would rather be processed earlier. Using $C(\sigma)$ to represent the makespan of the partial schedule $\sigma$ as it develops, and denoting the total processing time of a job by $P_{j}=\Sigma_{i} p_{i j}$, the steps of this heuristic are as follows:

Heuristic NEH for $F m \mid$ perm $\mid C_{\text {max }}$

1. Let $\searrow P_{j}=\left(J_{[1]}, J_{[2]}, \ldots, J_{[n]}\right)$, and set $\sigma:=\phi$.
2. For $i=1$ to $n$, do
begin
Let $\sigma_{k}$ be $\sigma$ with $J_{[i]}$ added in position $k \quad(k=1, \ldots, i)$.
Find $k^{\star}: C\left(\sigma_{k^{\star}}\right)=\min _{k=1, \ldots, i} C\left(\sigma_{k}\right)$. Set $\sigma:=\sigma_{k^{\star}}$
end
3. Let $S_{N E H}=\sigma$.

In words, find successively the job requiring the most total processing, and insert it into the partial schedule in the position that minimizes the makespan. By careful calculation of the change in the makespan value caused by each insertion, Taillard (1990) showed that the computational complexity of NEH is $O\left(m n^{2}\right)$. The complexity of CDS is $O(m n \log n)$, so NEH is expected to require slightly more CPU time for large $n$.

As we will see shortly, NEH is one of the most commonly used algorithms in flow shops, often used as a seed for other heuristics. This has attracted interest in refining NEH. Rad et al. (2009) developed improved variations of NEH with complexity ranging from $O\left(m^{2} n^{2}\right)$ to much more expensive variants that involve local search. One of these variations, we'll call it NEH1, involves a different ordering of the jobs in Step 1; specifically:

Heuristic NEH1 for $F m \mid$ perm $\mid C_{\text {max }}$

1. List all task times in nondecreasing order: $L_{p}=\left(p_{(1)}, p_{(2)}, \ldots, p_{(m n)}\right)$
2. Let $L_{J}=\left(J_{[1]}, J_{[2]}, \ldots, J_{[m n]}\right)$ be the corresponding job list, where $p_{(l)}$ is the time of one of the tasks of $J_{[l]}=<p_{1[l]}, p_{2[l]}, \ldots, p_{m[l]}>$. Evidently, every $J_{j} \in \mathcal{J}$ appears precisely $m$ times in $L_{J}$. Set $\sigma:=\phi$.
```
3. For }i=1\mathrm{ to mn, do
    begin
        If }\mp@subsup{J}{[i]}{}\mathrm{ is already in }\sigma\mathrm{ , remove it.
        Let }\mp@subsup{\sigma}{k}{}\mathrm{ be }\sigma\mathrm{ with }\mp@subsup{J}{[i]}{}\mathrm{ inserted in position }k(k=1,\ldots,i)
        Find \mp@subsup{k}{}{\star}}:C(\mp@subsup{\sigma}{\mp@subsup{k}{}{\star}}{})=\mp@subsup{\operatorname{min}}{k=1,\ldots,i}{}C(\mp@subsup{\sigma}{k}{}).\quad\mathrm{ Set }\sigma:=\mp@subsup{\sigma}{\mp@subsup{k}{}{\star}}{}
    end
```

4. Let $S_{N E H 1}=\sigma$.

Clearly, Step 3 is repeated $m$ times per job, thus $m n$ times. The complexity of NEH1 is $O\left(m^{2} n^{2}\right)$ because the makespan increase due to an insertion can be found in $O(m n)$ time.

Another improvement on NEH is proposed in Dong et al. (2008). Here, jobs are considered for insertion in the order $\nearrow\left(P_{j} / m+d e v_{j}\right)$ (rather than $\left.\nearrow P_{j}\right)$, where $\operatorname{dev}_{j}=\sqrt{\Sigma_{k}\left(p_{k j}-\Sigma_{j} P_{j} / m\right)^{2}}$. While inserting a job in Step 2 of NEH, if multiple sequences produce the same makespan value $C\left(\sigma_{k^{\star}}\right)$, a tie-breaking rule is proposed that aims in balancing the utilization among machines.

We now consider the relative merits of NEH and CDS. Nawaz et al. (1983) and Park (1981) performed computational experiments on randomly generated problems with up to 30 jobs and 25 machines and reported that the number of times the NEH heuristic produces a solution that is superior to CDS is significantly greater than when the reverse happens. Turner and Booth (1987) reached a similar conclusion for NEH against the RAES heuristic of Dannenbring (1977). Koulamas (1998) showed that NEH and HFC deliver comparable solution quality, but this depends heavily on the test set.

In an interesting experiment, Framinan et al. (2002) demonstrated that the $\searrow P_{j}$ ordering of jobs is not crucial to the success of NEH; in fact, a random initial sequence performs comparably. Instead, the success of NEH is due to the large number of partial sequences tested.

As with RAES, Suliman (2000) developed an iterative heuristic that starts with the CDS schedule and employs adjacent pairwise job exchanges within an ordered search scheme. To limit the number of exchanges, the forward direction is used (i.e., a job and its successors) only for as long as such exchanges improve makespan. If a forward exchange does not improve the makespan, the reverse direction is used for exchanges. The resulting algorithm is found to have performance comparable to NEH which runs faster. In a related local search heuristic, Krone and Steiglitz (1974) employed insertions in the backward direction.

As in NEH1, one may allow jobs already scheduled in $\sigma_{k^{\star}}$ (at Step 2 of NEH1) to further reduce the makespan value by re-inserting jobs to different positions of the partial sequence. The number of permissible insertions affects the CPU requirements and the solution quality produced by the resulting heuristic. Three such local search variations of NEH1 are presented in Rad et al. (2009). The best of the three involves a full local search; below we refer to it as $\mathrm{NEH1}_{L}$.

Another constructive heuristic is presented in Widmer and Hertz (1989) where jobs are thought of as "cities" and the makespan as the length of a traveling salesman tour. The "distance" between $J_{i}$ and $J_{j}$ occupying positions $k-1$ and $k$ of a permutation is captured by

$$
d_{i j}=p_{1 i}+\sum_{k=2}^{m}\left[(m-k)\left|p_{i k}-p_{j, k-1}\right|\right]+p_{j m}
$$

where the summation approximates the idle times inserted in the schedule as jobs $J_{i}$ and $J_{j}$ progress through machines $M_{2}, \ldots, M_{m}$, giving a heavier weight to inefficiencies occurring earlier. Then, the greedy heuristic is used to minimize the "total distance traveled" by visiting next the "city" that adds the least amount to the "tour". Note that the permutations produced by the above heuristics which do not lead to local optima, can be improved by an application of RACS or RAES. Even then, Widmer and Hertz (1989) found that NEH outperforms the local optima produced by Johnson's, Palmer's and Gupta's heuristics as well as CDS and RAES.

The traveling salesman tour idea has also been used in Ho and Chang (1991) with a different distance metric, but the resulting algorithm is not
found to perform very well for the makespan objective and for this reason we omit the details here.

An entirely different approach was used in Sevast'janov (1995) who reduced $F m \mid$ perm $\mid C_{\text {max }}$ to the Vector Summation Problem in Banach spaces. This implies the worst case error $O\left(\left[m^{2}-3 m+3+\frac{1}{m-2}\right] \max _{i j} p_{i j}\right)$. However, in her experiment, Lourenco (1996) showed that NEH outperforms Sevast'janov's algorithm.

The list of heuristics presented in this subsection is by no means exhaustive, but alternate heuristics appear to be dominated by those presented here. Contributors include Sarin and Lefoka (1993), Davoud Pour (2001), Framinan et al. (2003), Gupta (1972), King and Spachis (1980), Stinson and Smith (1982) to name a few. A survey of these works is included in Ruiz and Maroto (2005), who present by far the most comprehensive comparison of constructive and iterative heuristics. They cover 15 algorithms in their study, including most of those presented above and the popular dispatching sequences FCFS, SPT and LPT. For their experiment, the authors used a set of 120 particularly difficult problems presented in Taillard (1990). This set of test problems includes 10 problems for each of the combinations $(n, m) \in(20,5),(20,10),(20,20),(50,5),(50,10),(50,20),(100,5),(100,10)$, $(100,20),(200,10),(200,20),(500,20)$ and can be found in the OR Library: http://mscmga.ms.ic.ac.uk/info.html. Benchmark makespan values are presented in Taillard (1993) for these problems, and subsequently improved in the literature. The findings presented below are based on the average relative percentage error from the best known solution at the date of the experiment (i.e. year 2004).

- Dispatching rules FCFS, SPT and LPT produce poor schedules.
- NEH produces the best schedules, with Suliman's improvement heuristic a close second. Palmer's and Gupta's heuristics are among the worse performers.
- RACS solution quality is similar to that of Ho and Chang's heuristic, while RAES exhibits smaller average relative error than RAES by $7.42 \%$.
- HFC (which allows for passing) delivers average solution quality.
- The relative error increases with $m$ and is independent of $n$.

In regards to the last observation, all 20 problems in the test set corresponding to $n=50,100$ and $m=20$ were open (as of 2004) with respect to the optimal makespan value. Further, Rad et al. (2009) showed that, on average, variation NEH1 improves by about $1.5 \%$ upon the makespan values of the solutions obtained by NEH on the Taillard suite. Further, the local search employed in heuristic $\mathrm{NEH}_{L}$ improves upon NEH1 by another $0.5 \%$ on the Taillard suite.

### 4.10 Metaheuristics

In this section we review metaheuristics developed for $F m \mid$ perm $\mid C_{\text {max }}$. Metaheuristics make extensive use of neighborhood search, as do iterative heuristics. However, they are also equipped with mechanisms that allow the search to deviate from local optima - temporarily accepting inferior solutions - so as to direct the search to other (hopefully more promising) areas of the search space. Metaheuristic algorithms are mimetic in nature and include simulated annealing (SA), tabu search (TS), genetic algorithms (GA), ant colony optimization (ACO) and hybrids thereof.

### 4.10.1 Simulated Annealing

We start our review with metaheuristics based on SA; a methodology that simulates the annealing process used in physics to cool solids slowly until they reach a low energy state (see van Laarhoven and Aarts, 1987). The current benchmark for SA algorithms is the implementation of Osman and Potts (1989). It utilizes two ways to improve upon an incumbent solution: interchanging the positions of a pair of jobs in the permutation, or moving one job in a pair to a position right after the other. Both types of job movements are examples of descent algorithms that seek to reduce the makespan value. We refer to the first as a swap and to the second as an insertion. Swaps and insertions may be done in a predetermined order that captures all $\binom{n}{2}$ possible pair combinations, or randomly. In their experiments on randomly generated problems with $m \leq 20$ and $n \leq 100$, Osman and Potts found that their SA outperforms CDS and NEH when these two do not benefit by a decent method to improve the starting solution. When CDS and NEH are so improved, and the pairs of jobs are chosen in a predetermined order, it is found that the two methods are comparable in solution quality. Moreover, compared to NEH, the Osman and Pott's SA algorithm finds the better solution for $82.5 \%$ of the problems while for the remaining $17.5 \%$ there is a tie - thus giving a definite advantage to SA. It is also found that this SA algorithm performs far better when insertions are used instead of swaps, and surprisingly, when the insertion pairs are chosen randomly. It was also found that, the starting solution used in SA makes a statistically significant difference. Rad et al. (2009) showed that using their $\mathrm{NEH1}_{L}$ solution as the seed, yields on average about $2.3 \%$ better makespan performance for the problems in the Taillard suite compared to using the NEH solution as the seed.

Ogbu and Smith (1990a) developed a similar SA algorithm where the initial schedule is produced by Palmer's slope algorithm, and the RA, RACS and RAES heuristics of Dannenbring (1977). Ogbu and Smith (1990b) compared the performance of their SA algorithm against the one by Osman and Potts giving a slight advantage to the latter. Alternative SA implementations exhibiting robust performance with respect to temperature cooling are
presented by Ishibuchi et al. (1995) who show that the solution quality is comparable to that of Osman and Potts. Zegordi et al. (1995) used a so-called move desirability index for each job within the SA framework, resulting in faster convergence. Still, the overall solution quality is inferior to the one of Osman and Potts for the problems tested.

### 4.10.2 Tabu Search

Tabu search based algorithms have also attracted significant attention. TS starts with a carefully selected initial solution. The incumbent solution is transformed into another solution through appropriate moves like swaps, insertions, etc., chosen from a neighborhood. To avoid cycling through the same or similar solutions, a tabu list is created that includes solutions that are prohibited for a period of time - also a parameter. The tabu list allows TS to exit local minima in hopes of reaching alternative, hopefully better, optima. The algorithm terminates when an iteration or time limit is reached; see Glover (1989, 1990).

Using the traveling-salesman based heuristic of Widmer and Hertz (1989), presented above, as the initial solution, the authors developed a TS implementation referred to as SPIRIT. As we mentioned before, NEH outperforms the local optima produced by Johnson's, Palmer's and Gupta's heuristics as well as CDS and RAES. Of the 500 instances with $n, m \leq 20$ randomly generated to compare SPIRIT with NEH, Widmer and Hertz found that SPIRIT produced a better solution $46.8 \%$ of the time, NEH was better in $18.6 \%$ of the cases, and both methods yielded the same makespan value for the remaining $34.6 \%$. Thus SPIRIT was at least as good as NEH $81.4 \%$ of the time, and its superiority grew as the problem sizes increased.

Taillard (1990) developed an algorithm similar to SPIRIT that uses an improved NEH initial schedule rather than the traveling salesman based solution, and various neighborhood searches. He showed experimentally that the two implementations have comparable performance. Further enhancements in the initial solution of SPIRIT are made by Reeves (1993) who used insertions on the NEH schedule. The ensuing TS implementation is shown to outperform the SA of Osman and Potts for the problems tested. Moccellin (1995) presented yet another variant of SPIRIT using a different metric for the related Traveling Salesman Problem, and a different heuristic to solve it.

A different cohort of TS implementations is based on the following observation. Let $S=(1,2, \ldots, n)$ be the permutation associated with any initial solution and $C(S)$ be the length of the critical path from the start of $T_{11}$ to the end of $T_{m n}$ in a precedence network, as shown in Fig. 4.4 for $m=4, n=7$. Assume that $u_{1}, \ldots, u_{m-1}$ are the indices that maximize the path length:

$$
\begin{equation*}
C(S)=\sum_{j=1}^{u_{1}} p_{1 j}+\sum_{j=u_{1}}^{u_{2}} p_{2 j}+\ldots+\sum_{j=u_{m-1}}^{n} p_{m j} \tag{4.8}
\end{equation*}
$$

for $u_{0}=1 \leq u_{1} \leq u_{2} \leq \ldots \leq u_{m-1} \leq n=u_{m}$. Thus, if the heavy ar-
rows show the critical path in Fig. 4.4, we would have $\left(u_{0}, u_{1}, \ldots, u_{m}\right)=$ $(1,2,2,5,7)$. Let $\mathcal{B}_{k}=\left\{u_{k-1}, \ldots, u_{k}\right\}$ be the corresponding block of jobs (properly, job indices) in $S$ that are processed consecutively on $M_{k}$ for $k=1,2, \ldots, m$. In Fig. 4.4, for example, $\mathcal{B}_{1}=\{1,2\}, \mathcal{B}_{2}=\{2\}, \mathcal{B}_{3}=$ $\{2,3,4,5\}$, and $\mathcal{B}_{4}=\{5,6,7\}$. We will refer to $u_{k-1}$ and $u_{k}$ as the "end" jobs of the block, the rest of $\mathcal{B}_{k}$ being "interior" jobs. Clearly, $C(S)$ does not decrease by swapping two interior jobs (jobs in $\mathcal{B}_{k}-\left\{u_{k-1}, u_{k}\right\}$ ), or by moving an interior job to a different interior position in $\mathcal{B}_{k}$. Therefore (as observed by Grabowski, 1982), noting that end jobs are also members of adjacent blocks, the only way to decrease $C(S)$ with a single swap or insertion is to move a job from one block to another.


Fig. 4.4 Directed graph representation of schedule $S=(1, \ldots, 7)$ for $F 4 \mid$ perm $\mid C_{\text {max }}$

The above result was used by Nowicki and Smutnicki (1996) within the TS framework so as to avoid unpromising swaps and/or insertions. They experimented only with insertions (since the neighborhood size when using swaps is prohibitive) and even then, they only allowed insertions of jobs of one block to a limited number of positions in the adjacent blocks. Specifically, either

- move an interior job in $\mathcal{B}_{k}(k=1, \ldots, m)$ to a position in either

$$
\mathcal{L}_{k}=\left\{u_{k-2}+n_{l}, \ldots, u_{k-1}\right\} \subseteq \mathcal{B}_{k-1}, \text { or } \mathcal{R}_{k}=\left\{u_{k}, \ldots, u_{k+1}-n_{r}\right\} \subseteq \mathcal{B}_{k+1}
$$

or

- move $u_{k}(k=1, \ldots, m-1)$ to a position in either $\mathcal{L}_{k}$ or $\mathcal{R}_{k+1} ;$
where $\mathcal{L}_{1}=\mathcal{R}_{m}=\phi$, and $n_{l}$ and $n_{r}$ are nonnegative integers which control the size of the neighborhood. In their TS implementation, the authors used a tracking mechanism for recording and recovering high quality solutions, referred to as Back Jump Tracking. The initial solution used is obtained by NEH and the resulting implementation is abbreviated as TS Algorithm with Back jump tracking, or TSAB. In the ensuing experiment TSAB improved upon the majority of the benchmark makespan values in Taillard (1993), and the improved values were found much faster than in Taillard (1990).

A related TS implementation referred to as GP, is due to Grabowski and Pempera (2001). Here, insertions are of the form:

- move an interior job in $\mathcal{B}_{l}$ to the interior of $\mathcal{B}_{k},(k, l=1, \ldots, m ; k \neq l)$.
and are made only when they improve the value of the makespan. As a result, the size of the neighborhood in GP is smaller than in TSAB. Again, the NEH heuristic is used for the initial solution. It is shown that GP outperforms TSAB with respect to CPU times and provides comparable solution quality.

Much reduced is the neighborhood used in Grabowski and Wodecki (2004). For $k=1, \ldots, m$, they limit insertions to

- move a head job in $\mathcal{H}_{k} \equiv \mathcal{B}_{k}-\left\{u_{k}\right\}$ to the position just after $u_{k}$,
or
- move a tail job, $j \in \mathcal{T}_{k} \equiv \mathcal{B}_{k}-\left\{u_{k-1}\right\}$ to the position just before $u_{k-1}$,

In the resulting implementation, referred to as GW, further CPU savings are attained by estimating the makespan of $S^{\prime}$. Defining

$$
\delta_{l}(i, j)=p_{l j}-p_{l i}, \text { for } l=1, \ldots, m, i, j \in \mathcal{J} \text { with } \delta_{0}(i, j)=\delta_{m+1}(i, j)=0
$$

then, if we get $S^{\prime}$ from $S$ by inserting head job $J_{j}, j \in \mathcal{H}_{k}$ after $u_{k}$ :

$$
C\left(S^{\prime}\right) \approx C(S)+\Delta_{k R}
$$

where

$$
\Delta_{k R}= \begin{cases}\delta_{k+1}\left(u_{k}, j\right) & \text { if } j \in \mathcal{H}_{k}-\left\{u_{k-1}\right\} \\ \delta_{k+1}\left(u_{k}, j\right)+\delta_{k-1}\left(j, u_{k-1}+1\right) & \text { if } j=u_{k-1}\end{cases}
$$

To see the rationale for this formula, consider block $\mathcal{B}_{3}$ of the critical path (heavy arrows) in Fig. 4.4, with $\mathcal{H}_{3}=\{2,3,4\}$, and suppose (the first case of $\Delta_{k R}$ ) we move $J_{j}, j \in \mathcal{H}_{3}-\left\{u_{2}\right\}=\{3,4\}$, say $j=3$. We move it to just after $J_{5}$ (since $u_{3}=5$ ), so it takes position 5 , while $J_{4}$ and $J_{5}$ move left one place. Being an end job, $J_{u_{3}}$ contributes two task times to the critical path, on $M_{3}$ and $M_{4}$, while the interior job in position 3 or 4 contributes only one. It should now be clear that the path gains $p_{43}$ and loses $p_{45}$, which is precisely $\delta_{4}(5,3)=\delta_{k+1}\left(u_{k}, j\right)$. The same logic applies to the second case of $\Delta_{k R}$, where a trade-off occurs at both ends of $\mathcal{B}_{3}$.

Why is this only an approximation to the new critical path length? First, the same path that was critical for $S$ may no longer be critical for $S^{\prime}$, in which case the new length of that path underestimates $C\left(S^{\prime}\right)$. More importantly (and this is not reflected in the original paper), look again at Fig. 4.4. At the left end of $\mathcal{B}_{3}, J_{2}$ contributes three tasks: $T_{12}, T_{22}, T_{32}$, to the critical path. If we move $J_{2}$ to after $u_{3}$ (the case $j=u_{k-1}$ of $\Delta_{k R}$ ), $J_{3}$ moves left to take its place, producing a trade-off on $M_{1}$ as well as $M_{2}$. In general, besides $\delta_{k-1}\left(j, u_{k-1}+1\right)$, we may need to add (as here) $\delta_{k-2}\left(j, u_{k-1}+1\right)$, and perhaps $\delta_{k-3}\left(j, u_{k-1}+1\right)$, etc., depending on how many vertical arrows lead to the start of $\mathcal{B}_{k}$. The same is true at the right end of $\mathcal{B}_{k}$ where, if the right end job
contributes $t>2$ tasks to the critical path, $\delta_{k+1}\left(u_{k}, j\right)$ should be replaced by $\sum_{x=1}^{t-1} \delta_{k+x}\left(u_{k}, j\right)$ in the formulas for $\Delta_{k R}$.

This modification would improve the accuracy of the estimate, and would be quite feasible to implement. Without it, since each $\delta_{l}(i, j)$ may be either positive or negative, we cannot know that we have a lower bound, as the authors assert.

Everything we have said about moving head jobs, above, can be equivalently said of tail jobs. Thus, by inserting any job $J_{j}, j \in \mathcal{T}_{k}$, before $u_{k-1}$, we get another set of approximations

$$
C\left(S^{\prime}\right) \approx C(S)+\Delta_{k L},
$$

where, as given in Grabowski and Wodecki (2004):

$$
\Delta_{k L}= \begin{cases}\delta_{k-1}\left(u_{k-1}, j\right) & \text { if } j \in \mathcal{T}_{k}-\left\{u_{k}\right\} \\ \delta_{k-1}\left(u_{k-1}, j\right)+\delta_{k+1}\left(j, u_{k}-1\right) & \text { if } j=u_{k}\end{cases}
$$

and where each $\delta$ term can be elaborated by adding additional such terms as indicated. Finally, given $S$, we can test all the different moves and choose the neighbor with the best, i.e., smallest estimate (the one most likely to lead to a small makespan), as measured by $\Delta_{k L}$ or $\Delta_{k R}$.

In experiments with the Taillard (1990) problem set, the authors show that GW delivers the same solution quality as TSAB in $15 \%$ to $20 \%$ of the iterations. Moreover, GW produces makespan values comparable to the best known benchmark values within 5 minutes of CPU time. The algorithm GW, of course, uses the original formulas without the elaborations given above.

Our presentation of TS implementations is not exhaustive, but the ones presented here dominate the competition. Alternative implementations are given in Ben-Daya and Al-Fawzan (1998) and Moccellin and dos Santos (2000). The latter also present an SA implementation and a hybrid of their SA and TS metaheuristics. The hybrid was found to outperform its SA and TS counterparts, suggesting hybrid metaheuristics as a viable direction.

One very successful hybrid is Stützle's Iterative Local Search (or ILS) algorithm (Stützle, 1998b). It starts with an initial solution further improved by local search. The incumbent solution is then modified and further improved by local search. The revised solution becomes the new incumbent only if an acceptance criterion is satisfied, carefully chosen to allow for diversification in the search space and avoid converging to the same local optima. A list of promising solutions is maintained and exploited as in tabu search. It is shown that ILS significantly outperforms Taillard's (1990) TS implementation and (on average) produces better solutions than TSAB.

The critical path concept in Grabowski (1982), was also used in Werner (1993) within an iterative search. Based on the critical path, a limited number of alternate paths is developed, a search neighborhood is defined around them, and then searched to identify a good schedule.

### 4.10.3 Genetic Algorithms

Genetic algorithms (GA) offer yet another alternative for metaheuristics; see Holland (1992). Genetic algorithms belong to the larger class of evolutionary algorithms (EA), which generate solutions to optimization problems using techniques (such as mutation and crossover) inspired by natural evolution.

A genetic algorithm is a search heuristic that mimics the process of natural evolution. This heuristic is routinely used to generate useful solutions to optimization and search problems. After generating a few (preferably) feasible solutions to an optimization problem over permutations, a second generation of heuristic solutions can be produced from those already selected through genetic operators as described below. For each new solution to be produced, a pair of "parent" solutions is selected for breeding from the pool selected previously. They produce a "child" solution which typically shares many of the characteristics of its "parents". New parents are selected for each new child, and the process continues until a new population of solutions of appropriate size is generated.

Two popular types of genetic operators that are used to produce a child in terms of its parents' characteristics are: crossover operators and mutation operators. There are many types of crossover operators. For instance, a single crossover point operator in a scheduling problem may select a single location in the permutation (i.e., the organism). All jobs beyond that point are swapped between the two parent permutations to produce two children permutations.

Similarly, there is a variety of mutation operators. The classic example of a mutation operator involves a probability that an arbitrary position in a permutation will be changed from its original state. A common method of implementing the mutation operator involes generating a random variable for each position in the permutation. These random variables tell whether or not each position is modified through swaps or insertions. Thus, several changes would be made sequentially.

These processes ultimately result in the next generation population of chromosomes (e.g. job permutations) that is different from the previous generation. Generally the average fitness will have increased by this procedure for the population, since only the best organisms are selected for breeding using a "fitness function". In addition, a small proportion of less fit solutions are included, for reasons of diversity so as to avoid getting trapped in local optima.

A GA thus uses crossover and/or mutation operators to produce a sequence of generations evolving towards an optimal solution. An acceptance criterion is used to avoid local minima and search different areas of the search space. Intensification techniques are often used to improve upon new permutations, using neighborhood search.

A nice survey of genetic metaheuristics for $F m \| C_{\max }$ is presented in Ruiz and Maroto (2005) who reviewed the implementations of Chen et al. (2005),

Murata et al. (1996), Ponnambalam et al. (2001), Reeves (1995), Reeves and Yamada (1998), and the hybrid GA/SA algorithm of Wang and Zheng (2003). In their computational experiment - again using Taillards's (1990) set - they included metaheuristics ILS, SPIRIT, Reeve's GA and Osman and Pott's SA implementations. The stopping criterion used was a fixed CPU time limit. It was found that ILS outperforms the other three metaheuristics with Reeve's GA a close second and average \% relative errors $1.65 \%$ and $1.84 \%$ respectively. Again, problem difficulty is found to increase with $m$ and be independent of $n$. Alternative GA implementations are presented in Colin (1995), Aldowaisan and Allahverdi (2003), Etiler (2004) and Ruiz et al. (2006); the latter referred to as RMA. A hybrid GA where local minima are intensified using variable neighborhood search, or VNS, (see Hansen and Mladenovic, 2001) is presented in Zobolas et al. (2009). Their GA/VNS hybrid performs near optimally across all instances of the Taillard suite and produces solutions comparable to RMA. On average, both RMA and GA/NVS produce better solutions than the SA of Osman and Potts, SPIRIT and ILS.

### 4.10.4 Other Metaheuristics

Yet another class of metaheuristics involves Iterated Greedy (IG) search and has been applied successfully on the set covering problem; see Jacobs and Brusco (1995), and Marchiori and Steenbeek (2000). IG algorithms start with an initial permutation, and proceed in two phases: The destruction phase where a subset $\mathcal{D} \subset \mathcal{J}$ of jobs is removed from the initial schedule $S$, and the construction phase where the resulting permutation of $\mathcal{J}-\mathcal{D}$ is extended in a greedy fashion so as to reconstruct a complete schedule. The IG implementation of Ruiz and Stützle (2007), referred to as RS, reconstructs $S$ one job $j \in \mathcal{D}$ at a time, using the insertion phase of NEH. Being a metaheuristic, an acceptance criterion is used to produce solutions that deviate from local minima. Local search improvements may be used after every insertion to obtain algorithm $\mathrm{RS}_{L}$ which, obviously, take more time.

Nearly the entire gamut of GA implementations, RS and $\mathrm{RS}_{L}$, are tested in Ruiz and Stützle (2007). In all, 14 algorithms are compared including ILS and two ant colony optimization metaheuristics referred to as $M-M M A S$ and $P A C O$ in Rajendran and Ziegler (2004). The former is a modified version of a Max-Min Ant System (MMAS) first introduced by Stützle (1998a), while PACO incorporates relative distances between 2 scheduling positions for the same job. Briefly, in MMAS, we first initialize the pheromone matrix $\left\{\tau_{i j}\right\}$ and other parameters, where $\tau_{i j}$ is the desirability of assigning job $i$ to position $j$ of a permutation schedule. Then steps 1 and 2 below are repeated for a fixed number of iterations or fixed amount of CPU time:

1. Construct a solution, improve it with local search, and update the matrix $\left\{\tau_{i j}\right\}$,
2. Store the best solution found so far.

The Max-Min Ant System stipulates upper and lower bounds $\tau_{\min } \leq \tau_{i j} \leq$ $\tau_{\text {max }}$ for the range of values for $\tau_{i j}$.

All 14 algorithms considered in Ruiz and Stützle (2007) are tested on the Taillard set, with fixed CPU time limit as the stopping criterion. It is found that $\mathrm{RS}_{L}$ outperforms all other metaheuristics, with RMA a close second, with average (over the 120 problems in the set) relative errors $0.44 \%$ and $0.57 \%$ respectively. Algorithms M-MMAS, PACO and RS exhibit comparable performance at $0.88 \%, 0.75 \%$ and $0.78 \%$ respectively. Even more interesting is the finding that $\mathrm{RS}_{L}$ solutions deteriorate as $n$ increases with $n=50,100$ and $m=20$ being its "sweet spot". Recall that this subset of problems from the Taillard set has been the most difficult to solve optimally. In contrast, TSAB and GW exhibit robust performance for $n=50,100,200$ and GW outperforms other metaheuristics when $n=200$.

Another ant colony optimization algorithm is presented in Ying and Liao (2004). In the ensuing experiments the authors showed that their implementation outperforms the GA of Collin (1995) on the Tailard set but did not test against the best of the algorithms presented above.

In the quest for simpler, faster, better metaheuristics, the latest class introduced for $F m \mid$ perm $\mid C_{\text {max }}$ is the Particle Swarm Optimization method (PSO) first used to optimize continuous nonlinear functions; see Eberhard and Kennedy (1995) and Kennedy and Eberhard (1995). In PSO, the members of a solution space are called particles. Each particle moves around the search space of all permutations (via neighborhood search techniques) with a specific velocity, and is maintained throughout the search so that it directs it towards an optimal solution. When using a global neighborhood search, the best solution in the swarm is sought. Otherwise local search leads incumbent solutions to the best particle in a restricted neighborhood. Two such implementations are presented in Liu et al. (2007) and in Tasgetiren et al. (2007). In the latter article, two PSO implementations were proposed using different neighborhood searches. Testing on the Taillard suite, the authors found that their algorithms produce better solutions than a generic GA algorithm, but take longer time. Zobolas et al. (2009) showed that the best of the two PSO implementations produces solutions comparable to RMA, GA/VMS which in turn produce slightly better solutions than PACO, M-MMAS and ILS on the Taillard suite. Further, Rad et al. (2009) showed that using their $\mathrm{NEH}_{L}$ solution as the seed, the improvement in the makespan performance of PACO and ILS for problems in the Taillard suite is statistically significant compared to using the NEH solution as the seed.

Tasgetiren et al. (2007) conducted further tests on a more recent benchmark suite presented in Watson et al. (2002) and found that TS implementations outperform their PSO implementation. The benchmark suite of Watson et al. (2002) consists of 14,000 problems with $n=20,50,100$, and 200 jobs, and $m=20$. The problems are divided into two groups: the random group with processing times $p_{k j}$ drawn uniformly from [1,99], and the narrow random group with $p_{k j}$ drawn from [45,55]. A correlation parame-
ter $\alpha \in\{0,0.1,0.2, \ldots, 1\}$ is used across jobs, or across machines, or both - thus resulting to 33 correlation combinations - and 100 instances were generated for each problem size and parameter combination, resulting to $4 \times 33 \times 100=13,200$ test problems in the random group. The narrow random group includes the remaining 800 problems. They can be found at http://www.cs.colostate.edu/sched/generator/.

### 4.11 Exact Algorithms

As expected, the sheer number of permutations and the lack of significant structural properties for $F m \mid$ perm $\mid C_{\text {max }}$ conspire against exact algorithms for problem instances of more than a few jobs. For this reason, we cite without details the branch-and-bound implementations of Ignall and Schrage (1965), Bansal (1977) and Stafford (1988).

### 4.12 Lot Streaming

We will give only a brief introduction to the subject of lot streaming over $m$ machines, referring the reader to Sarin and Jaiprakash (2007) for complete details. Recall, as introduced in Sect. 2.2.6, that lot streaming involves the production of many identical copies of a product, by dividing the entire job lot into several sublots for convenience and efficiency.

In our earlier discussion of the two-machine case, we assumed the job lot was made up of $n$ identical items. Instead, to simplify notation, we will now consider the job lot to be a single job, made up of so many identical parts that it is effectively infinitely divisible. This entire production lot requires a processing time $p_{k}$ on $M_{k}, k=1, \ldots, m$. As before, we wish to partition the job into $v$ sublots to minimize makespan, but now $x_{i}$ is the fraction of the work assigned to sublot $i, i=1, \ldots, v$, with $0<x_{i}<1, \sum_{i=1}^{v} x_{i}=1$. This modification changes nothing, except that we get rid of the multiplier $n$.

We shall again assume consistent sublots. Actually, consistent sublots are always optimal for up to 3 machines, as the following corollary to Theorem 2.1 establishes.

Theorem 4.5. For Fm|lots $\mid C_{\max }$, if $x_{i k}$ is the lot size of sublot $i$ on $M_{k}$, then there exists an optimal schedule in which $x_{i 1}=x_{i 2}$ and $x_{i, m-1}=$ $x_{i m}, i=1, \ldots, v$.
Proof: The proof of Theorem 2.1 carries over for the last two machines, and a time reversal argument extends the same proof to $M_{1}$ and $M_{2}$.

Note that, without consistent sublots (as in Theorem 4.5), the identity of a "job" (i.e., a sublot) is lost as the work advances through the shop, so the concept of a permutation schedule becomes meaningless. With consistency (as assumed hereafter), sublots retain their integrity and schedules are automatically permutation.

### 4.12.1 Two Sublots

For the case $v=2$, we give here a simpler algorithm than the similar one presented in Williams et al. (1997), which builds on Baker and Pyke (1990). It will be convenient to let $\left(x_{1}, x_{2}\right)=(x, 1-x)$. The two sublots are effectively two jobs whose processing times on each machine can be varied by shortening one and simultaneously lengthening the other proportionately. For arbitrary $x, 0<x<1$, the makespan $C_{\text {max }}(x)$ can be expressed as usual as the critical path through the precedence network, as shown for $m=4$ in Figure 4.5(a) where each task is labeled by its processing time. We have:

$$
\begin{aligned}
C_{\max }(x) & =\max _{k=1, \ldots, m}\{x P(1, k)+(1-x) P(k, m)\} \\
& =\max _{k=1, \ldots, m}\{x[P(1, k-1)-P(k+1, m)]+P(k, m)\}
\end{aligned}
$$

where $P(u, v)=\Sigma_{i=u}^{v} p_{i}, 1 \leq u \leq v \leq m$. Let us define

$$
\beta_{k}=P(1, k-1)-P(k+1, m), \quad \alpha_{k}=P(k, m),
$$

so that

$$
\begin{equation*}
C_{\max }(x)=\max _{k=1, \ldots, m} f_{k}(x), \quad \text { where } f_{k}(x)=\alpha_{k}+\beta_{k} x . \tag{4.9}
\end{equation*}
$$


(a)

(b)

(c)

Fig. 4.5 Three ways to visualize the two-sublot problem, illustrated for the case $m=4, \bar{p}=<3,2,7,6>$

Another way to look at our problem is as a linear program, since (4.9) amounts to minimizing $C_{\text {max }}(x)$ subject to $C_{\max }(x) \geq \alpha_{k}+\beta_{k} x, k=1, \ldots, m$ and $0<x<1$. In Figure 4.5(b), which shows instance $m=4, \bar{p}=<3,2,7,6>$, the makespan (as a function of $x$ ) is given by the upper envelope (in heavy lines) of the route lengths $\left\{f_{k}(x), k=1, \ldots, m\right\}$. Note that some of the constraints $f_{k}(x)$ form a part of this envelope, and some do not; we will call them active and inactive, respectively. Note also that at optimality, two of the
constraints are binding: two of the machines are simultaneously critical. This of course is true at all the corner points, or extreme points, of the envelope.

Our strategy will be, starting at the left $(x=0)$, to move from one extreme point to the next until the optimum is reached. As we advance through the active constraints, we check the slope of each. Since $f_{1}(x)$ is always the leftmost active constraint and always has $\beta_{1}<0$, we start there. Since the slopes are increasing in $k$, we stop when we reach the first $\beta_{k} \geq 0$. In the sample instance, we go from $f_{1}$ to $f_{3}$ to $f_{4}$ and stop when $\beta_{4}>0$.

It remains only to specify how to find each successive active function. Thus, in the example, how do we know that $f_{1}$ is followed by $f_{3}$, and not $f_{2}$ or $f_{4}$ ? For very small $x, M_{1}$ is always critical. As $x$ increases, we eventually reach the point, circled in Figure $4.5(\mathrm{~b})$, where a second machine becomes simultaneously critical. At this point, the schedule looks like Figure 4.5(c). Since the first sublot ends at the same time the second sublot starts on both critical machines, $M_{1}$ and $M_{3}$, and there is no slack time on any intervening machines, we must have $p_{2} x+p_{3} x=p_{1}(1-x)+p_{2}(1-x)$, or $x P(2,3)=$ $(1-x) P(1,2)$. In general, if $M_{c}$ is critical, the next machine to become critical as $x$ increases does so at the smallest value of $x$ for which

$$
x P(c+1, k)=(1-x) P(c, k-1)
$$

for some $M_{k}$, from which

$$
x=P(c, k-1) /[P(c, k-1)+P(c+1, k)] .
$$

The above observations lead to the following algorithm, where $M_{c}$ will denote the trial critical machine, and $M_{c^{\prime}}$ its successor.

Two-Sublot Algorithm for $\operatorname{Fm} \mid$ (perm), $\operatorname{lots} \mid C_{\max }$

1. Set $c:=1$.
2. Let

$$
x=\min _{k=c+1, \ldots, m}\{P(c, k-1) /[P(c, k-1)+P(c+1, k)]\},
$$

and let $c^{\prime}$ be the minimizing value of $k$ (the largest such $k$, if more than one). 3. If $\beta_{c^{\prime}}<0$, set $c:=c^{\prime}$ and go to Step 2. Otherwise, continue.
4. $M_{c}$ and $M_{c^{\prime}}$ are critical at optimality, with $x^{\star}=x$ and $C_{\max }^{\star}=\alpha_{c}+\beta_{c} x$.

We note that the discrete problem, where $n$ items make up the job lot, is quickly solved using the above continuous algorithm. The discrete assumption implies that the only acceptable fractions of the lot in the first sublot, instead of any $x, 0<x<1$, are $x=1 / n, 2 / n, \ldots,(n-1) / n$. Since the upper envelope is convex, the optimum is one of the two feasible sublot sizes on either side of the continuous $x^{\star}$. If $i / n \leq x^{\star}<(i+1) / n$, the first sublot should contain either $i$ or $i+1$ items, whichever gives smaller makespan.

The case of three sublots can also be solved optimally in polynomial time, as shown in Williams et al. (1997).

### 4.12.2 Heuristics

We present two of the simpler heuristics that have been proposed for the $m$-machine $v$-sublot lot streaming problem. Both, with refinements, are from Baker and Pyke (1990).

## The Two-Machine Heuristic

The following heuristic combines two earlier results, namely:

- The CDS heuristic due to Campbell et al. (1970) presented in Sect. 4.9. This is a simple but effective procedure that approximates the $m$-machine makespan problem by $m-12$-machine pseudoproblems.
- The solution to the 2-machine lot streaming problem given in Sect. 2.2.6. This is applied to each of the pseudoproblems, from which the best schedule can then be chosen.

Specifically, the algorithm is as follows.

1. For $r=1,2, \ldots, m-1$, define

$$
a(r)=\sum_{k=1}^{r} p_{k}, \quad b(r)=\sum_{k=m-r+1}^{m} p_{k} \quad(\text { as in }(4.7)) .
$$

2. For the $r^{\text {th }}$ pseudoproblem, the 2-machine solution gives sublot sizes

$$
x_{i}(r)=n q^{i-1}(r)(1-q(r)) /\left(1-q^{v}(r)\right), \quad i=1, \ldots, v,
$$

where $q(r)=b(r) / a(r)$ (as in (2.2)). The sublots, processed in numerical order on $m$ machines, can now be considered jobs with task times

$$
p_{k i}(r)=p_{k} x_{i}(r), \quad i=1, \ldots, v, \quad k=1, \ldots, m
$$

They generate a schedule $S(r)$, for which the makespan $C(r)$ can be found using the recursion (4.1) given in Sect. 4.4.
3. Choose $S\left(r^{\star}\right)$ with $C\left(r^{\star}\right)=\min _{r=1, \ldots, m-1} C(r)$.

## The Two-Sublot Heuristic

The solution to the two-sublot problem (the continuous case) finds the optimal fraction, $x^{\star}$, of the job lot to allocate to the first sublot. This gives a ratio $r=\left(1-x^{\star}\right) / x^{\star}$ for the work allocated to the first and second sublots. When $v$ sublots are called for, the heuristic simply maintains this ratio for each successive pair of sublots. Thus, $x_{2}=r x_{1}, x_{3}=r x_{2}=r^{2} x_{1}$, etc., so that

$$
\sum_{i=1}^{v} x_{i}=x_{1} \sum_{i=0}^{v-1} r^{i}=x_{1}\left(1-r^{v}\right) /(1-r)=1
$$

which gives

$$
x_{1}=(1-r) /\left(1-r^{v}\right), \text { and } \quad x_{i}=(1-r) r^{i-1} /\left(1-r^{v}\right), \quad i=2, \ldots, v .
$$

Thus, for the sample instance illustrated in Figure 4.5(b), we have $x^{\star}=$ $7 / 13$, giving $r=6 / 7=0.857$. With $v=4$, we get $\left(x_{1}, x_{2}, x_{3}, x_{4}\right)=$ (0.310, 0.266, 0.228, 0.195).

Baker and Pyke (1990) find that the two-sublot heuristic does better than the two-machine one, and performs well. Additional computational results on
the two-sublot heuristic as well as other improved heuristics are provided in Williams et al. (1997).

## $4.13 \mathrm{Fm} \| \Sigma C_{j}$

The $m$-machine flow shop has received significant attention even for objective functions other than the makespan. The sum of completion times (or flow time) objective is the most popular, but only for permutation schedules. As seen in Chap. 2, $F 2 \mid$ perm $\mid \Sigma C_{j}$ is NP-complete in the strong sense and hence Fm|perm $\mid \Sigma C_{j}$ is at least as difficult.

A formulation for $F m \mid$ perm $\mid \Sigma C_{j}$ can be obtained using expression (4.8), except than now, the completion time of every job, not just the last, is needed. Let $S=(1,2, \ldots, n)$ be an arbitrary sequence, reindexing for simplicity so that the job in position $j$ is $J_{j}$; and let

$$
C_{k j}(S)=\text { the time until } J_{j} \text { is completed on } M_{k}, k=1, \ldots, m, j=1, \ldots, n
$$

That is, $C_{k j}(S)$ is the length of the critical path from $T_{11}$ to $T_{k j}$ in the network representation (see Fig. 4.4). Assume that $1 \leq u_{1}^{j} \leq u_{2}^{j} \leq \ldots \leq u_{k-1}^{j} \leq j$ are the indices maximizing the path length, so that:

$$
\begin{equation*}
C_{k j}(S)=\sum_{i=1}^{u_{1}^{j}} p_{1 i}+\sum_{i=u_{1}^{j}}^{u_{2}^{j}} p_{2 i}+\ldots+\sum_{i=u_{k-1}^{j}}^{j} p_{k i} . \tag{4.10}
\end{equation*}
$$

Now, the total flow time cost associated with $S$ is

$$
F(S)=\sum_{j=1}^{n} C_{m j}(S)
$$

Problem $F m \mid$ perm $\mid \Sigma C_{j}$ does not seem to possess significant structural properties, which hinders the development of efficient solution procedures. The following sufficient condition may prove useful for small $m$.

Theorem 4.13 (Szwarc, 1983a)
Permutation $S_{0}$ is optimal for $F m \mid$ perm $\mid \Sigma C_{j}$ if for some $k, 1 \leq k<m$ :
(a) $S_{0}$ minimizes $F k \mid$ perm $\mid \Sigma C_{j}$,
(b) $C_{m j}\left(S_{0}\right)=C_{k j}\left(S_{0}\right)+\Sigma_{i=k+1}^{m} p_{i j}$ for $j=1,2, \ldots, n$,
where $F k$ refers to the first $k$ of the $m$ machines in the problem instance.
Proof: By definition of $F(S)$, using (b):

$$
F\left(S_{0}\right)=\sum_{j=1}^{n} C_{k j}\left(S_{0}\right)+\sum_{j=1}^{n} \sum_{i=k+1}^{m} p_{i j}
$$

By (a), the first summation is the minimal total flow time over the first $k$ machines. The second summation is clearly the least we can add for each job over the remaining machines. Thus, $S_{0}=S^{\star}$.

As an application, for $k=1$ and $m=2$, conditions (a) and (b) are trivially satisfied when $a_{j} \geq b_{j}$ for all $j$ and $S_{0}$ is the SPT order of $a$-tasks. Hence, $S_{0}=\nearrow a_{j}$ optimally solves $F 2\left|a_{j} \geq b_{j}\right| \Sigma C_{i}$.

For special problem data, of course, simple results may be possible:
Theorem 4.14 (Panwalkar and Khan, 1976)
For Fm|perm, ordered $\mid \Sigma C_{j}, \quad S^{\star}=S P T=\nearrow p_{. j}$.
Recall that, for ordered flow shops the SPT order of processing times is well defined because of condition 1 in Definition 4.4. Without such special structure, one might still consider sequencing in order of total processing times: $\nearrow P_{j}$, where $P_{j}=\Sigma_{i} p_{i j}$. We could call this schedule total SPT, or just SPT. The following result helps evaluate this schedule.

Theorem 4.15 (Gonzalez and Sahni, 1978) For $F m \| \Sigma C_{j}$,
if schedule $\nearrow P_{j}$ yields a total flow time $F_{P}$, then
$F_{P} / F^{\star} \leq m$, and this bound is tight.
Proof: Let $C_{m j}(S)$ be the completion time of the job in position $j$ of schedule $S$. Indexing jobs in $\nearrow P_{j}$ order, we have

$$
C_{m j}\left(\nearrow P_{j}\right) \leq \sum_{i=1}^{j} P_{i}, \quad j=1, \ldots, n
$$

On the other hand, the SPT indexing ensures that, in any schedule, hence in any optimal schedule $S^{\star}$,

$$
C_{m j}\left(S^{\star}\right) \geq \sum_{i=1}^{j} P_{i} / m, \quad j=1, \ldots, n .
$$

The RHS of the last inequality is a lower bound even if the flow shop constraints are removed and $F m$ is replaced by $P m$; namely, $m$ parallel identical processors. The last two inequalities combine to give $C_{m j}\left(S^{\star}\right) \geq$ $C_{m j}\left(\nearrow P_{j}\right) / m$, which, adding over all $j=1, \ldots, n$, yields the desired worst case error bound.

For $m=2$, an improvement is almost always possible for this bound, depending on the data. Specifically,

Theorem 4.16 (Hoogeveen and Kawaguchi, 1999) For $F 2 \mid($ perm $) \mid \Sigma C_{j}$, if $F_{a+b}$ is the total flow time using schedule $\nearrow P_{j}=\nearrow\left(a_{j}+b_{j}\right)$, then

$$
F_{a+b} / F^{\star} \leq 2 \beta /(\alpha+\beta), \text { where }
$$

$\alpha=\min _{j}\left\{\min \left(a_{j}, b_{j}\right)\right\}, \beta=\max _{j}\left\{\max \left(a_{j}, b_{j}\right)\right\}$, and this bound is tight.
Heuristics for the general $F m \mid$ perm $\mid \Sigma C_{j}$ are developed by Miyazaki et al. (1978), Ahmadi and Bagchi (1990) and Karabati and Kouvelis (1993) who also developed a branch-and-bound algorithm. Bansal (1977) extended the lower bound procedure of Ignall and Schrage (1965) for $F 2 \mid$ perm $\mid \Sigma C_{j}$ to the case $m>2$ and used it within a branch-and-bound algorithm as well.

In obtaining tight lower bounds, Karabati and Kouvelis (1993) observed that, in the precedence network for a given schedule, if the time to a job's completion is computed along an arbitrary path, it can only be less than the true time to completion computed along the critical, or longest, path. To put this mathematically, let $\tau_{j}$ be any path from $T_{11}$ to $T_{m j}$, and let $\mathcal{T}_{j}$ be the set of all such paths. The path is defined independently of the particular
tasks that occupy the positions along the path; that is, independently of the schedule $S$. It represents one sequence of tasks that constrain the completion of the job in position $i$, which we call $J_{[i]}$.

Earlier, we defined $C_{m j}(S)$ as the length of the critical path defining the completion time of $J_{[j]}$. Now, let $C_{m j}\left(S, \tau_{j}\right)$ be the length of path $\tau_{j}$ in $S$, so that $C_{m j}(S)=\max _{\tau_{j} \in \mathcal{T}_{j}} C_{m j}\left(S, \tau_{j}\right)$. We can now express the observation of Karabati and Kouvelis (1993):

$$
\min _{S} \sum_{j=1}^{n} C_{m j}\left(S, \tau_{j}\right) \leq \min _{S} \sum_{j=1}^{n} C_{m j}(S)=\min _{S} F(S)
$$

indicating that, for any choice of paths $\left\{\tau_{j}, j=1, \ldots, n\right\}$, the LHS in the above expression is a lower bound on the optimal flow time value $F^{\star}=\min _{S} F(S)$. Since the inequality holds for any choice of path vector $\tau=\left(\tau_{1}, \tau_{2}, \ldots, \tau_{n}\right) \in$ $\mathcal{T}=\left(\mathcal{T}_{1} \times \mathcal{T}_{2} \times \ldots \times \mathcal{T}_{n}\right)$, it holds for the maximizing $\tau$, so

$$
\max _{\tau} \min _{S} \sum_{j=1}^{n} C_{m j}\left(S, \tau_{j}\right) \leq F^{\star}=\min _{S} \max _{\tau} \sum_{j=1}^{n} C_{m j}\left(S, \tau_{j}\right)
$$

Therefore, one can develop lower bounds by minimizing $\Sigma_{j=1}^{n} C_{m j}\left(S, \tau_{j}\right)$ over permutations $S$ while maximizing over $\mathcal{T}$, the set of all paths (i.e., all task sequences) from the start of processing to the completion times of all jobs.

The following zero-one integer program yields just such a lower bound on $F^{\star}$. Let an arbitrary path $\tau_{j}$ to the completion of the job in position $j, J_{[j]}$, be determined by the indices $1=v_{0}^{j} \leq v_{1}^{j} \leq \ldots \leq v_{m-1}^{j} \leq v_{m}^{j}=j$ (we write $\left.\tau_{j}=\left(v_{1}^{j}, \ldots, v_{m-1}^{j}\right)\right)$, so that, for $S=([1],[2], \ldots,[n])$ :

$$
C_{m j}\left(S, \tau_{j}\right)=\sum_{i=1}^{v_{1}^{j}} p_{1[i]}+\sum_{i=v_{1}^{j}}^{v_{2}^{j}} p_{2[i]}+\ldots+\sum_{i=v_{m-1}^{j}}^{j} p_{m[i]}=\sum_{k=1}^{m} \sum_{i=v_{k-1}^{j}}^{v_{k}^{j}} p_{k[i]} .
$$

In order to search over different schedules, we use the following binary variables to assign jobs to positions:

$$
x_{i j}= \begin{cases}1 & \text { if } J_{j} \text { is in position } i \\ 0 & \text { otherwise }\end{cases}
$$

Now, instead of specifying $S$, we can write $p_{k[i]}$ as $\sum_{h=1}^{n} p_{k h} x_{i h}$. The following formulation yields our lower bound on $F^{*}$ :

$$
\begin{align*}
& \text { LB minimize } M \\
& \text { subject to } \quad \sum_{j=1}^{n} C_{m j}\left(\tau_{j}\right) \leq M, \quad \tau \in \mathcal{T} \text {, }  \tag{4.11}\\
& \sum_{i=1}^{n} x_{i j}=1, \quad j \in \mathcal{J} \text {, }  \tag{4.12}\\
& \sum_{j=1}^{n} x_{i j}=1, \quad i \in \mathcal{J},  \tag{4.13}\\
& x_{i j}=0 \text { or } 1, \quad i, j \in \mathcal{J}, \tag{4.14}
\end{align*}
$$

where

$$
C_{m j}\left(\tau_{j}\right)=\sum_{k=1}^{m} \sum_{i=v_{k-1}^{j}}^{v_{k}^{j}} \sum_{h=1}^{n} p_{k h} x_{i h} .
$$

The difficulty with this formulation is that the path vectors $\tau \in \mathcal{T}$ are factorially many. By restricting the set to just a few vectors $\mathcal{T}^{\prime} \subset \mathcal{T}$, and replacing the LHS of inequalities (4.11) by a convex combination, Karabati and Kouvelis (1993) proposed approximating $\mathbf{L B}$ by an assignment problem

```
minimize \(\quad \sum_{j=1}^{n} \sum_{\tau \in T^{\prime}} \lambda_{\tau} C_{m j}(\tau)\)
subject to \(\sum_{\tau \in T^{\prime}} \lambda_{\tau}=1\), and (4.12), (4.13), and (4.14),
```

where $\left\{\lambda_{\tau}: \tau \in T^{\prime}\right\}$ are appropriately selected surrogate multipliers. The added benefit of this formulation is that it yields a permutation and hence an assignment-based heuristic. In the ensuing experiments on problems with 12 jobs and 6 machines, it is shown that the assignment-based heuristic performs favorably against the heuristic of Ahmadi and Bagchi (1990), while its relative error from the lower bound is about $16.5 \%$ less than the corresponding error exhibited by the heuristic of Miyazaki et al. (1978). For randomly generated problems of size up to $n=30$ jobs, the heuristic in Karabati and Kouvelis (1993) produced near optimal schedules for problem instances where processing times exhibited correlation, trend, or both. The authors also incorporated the above lower bounding scheme within a branch-and-bound algorithm and were able to solve problems of size up to $n=10$.

Various heuristics have been developed for $F m \mid$ perm $\mid \Sigma C_{j}$, often imitating heuristics for $F m \mid$ perm $\mid C_{\max }$. Ho (1995), for example, presented a "slope" heuristic where jobs $j \in \mathcal{J}$ are initially ordered in nondecreasing order of the indices

$$
s_{j}=\sum_{k=1}^{m}(m-k+1) p_{k j},
$$

then improved by pairwise interchanges and insertions. For $\operatorname{Fm} \mid$ perm $\mid \Sigma C_{j}$, similarly, Rajendran and Ziegler (1997) examined schedules based on the nondecreasing order of

$$
s_{j}^{l}=\sum_{k=l}^{m}(m-k+1) p_{k j},
$$

and generated alternative sequences using $l=1,2, \ldots, m$. Also, they experimented with sequential insertion heuristics. Liu and Reeves (2001) developed heuristics using an index function for unscheduled jobs, whose value depends on the incumbent partial schedule. Pairwise forward (backward) interchanges are used to improve upon the current best solution where a job exchanges positions with its successors (predecessors). Wang et al. (1997) presented heuristics where the next job scheduled is chosen so as to minimize the number of machines where idle time is inserted, or to minimize a "distance" between the partial schedule and unscheduled jobs. Liu and Reeves (2001) tested their heuristics against those of Ho (1995), Rajendran and Ziegler (1997), Wang et al. (1997) and Woo and Yim (1998) on the Taillard suite with the flow time objective - recall that the Taillard set was developed for $F m \mid$ perm $\mid C_{\max }$, not $F m \mid$ perm $\mid \Sigma C_{j}$. It is found that the heuristics of Woo and Yim (1998)
and Liu and Reeves (2001) compare favorably against the heuristics in this experiment.

Interestingly, Framinan et al. (2002) showed that the NEH heuristic with the $\nearrow P_{j}$ order of jobs replaced by $\searrow P_{j}$ performs very well, comparably to the heuristic of Woo and Yim (1998). This means that NEH can be adapted to solve not just for the $C_{\max }$ objective but also for $\Sigma C_{j}$.

With minor changes, metaheuristics for $F m \mid$ perm $\mid C_{\text {max }}$ can be adapted for $F m \mid$ perm $\mid \Sigma C_{j}$. Rajendran and Ziegler (2004) evaluated their ant colony optimization implementations M-MMAS and PACO mentioned in Sect. 4.10 against the best value obtained by the algorithms tested in Liu and Reeves (2001), on the Taillard set with $n=20,50,100$ and $m=5,10,20$ (i.e., 90 of the 120 problem instances). Let "L\&R Best" denote the best values. The starting solution used for M-MMAS and PACO is Rajendran's (1993) heuristic, improved by local search. It is found that both implementations outperform L\&R Best, with PACO outperforming M-MMAS for $n=50,100$ and $m=10,20$ (i.e., large problems) and the other way around for $n=20$ or $m=5$ (i.e., small problems). Moreover, for the 90 problems considered from the Taillard set, the authors established 83 improved benchmark total flow time values using M-MMAS and PACO.

Similarly, Tasgetiren et al. (2007) adapted for $F m \mid$ perm $\mid \Sigma C_{j}$ the better of their two PSO implementations introduced in Sect. 4.10, and tested them against M-MMAS, PACO and the L\&R Best values, on the Taillard suite. PSO produced improved benchmark values for 57 of the 90 problems in the Taillard set (excluding problems with $n=200$ and $m=5,10,20$ ), on average outperformed M-MMAS and PACO in 7 of the 9 problem sets tested, with an overall average of only $0.34 \%$ off the best value amongst L\&R Best, M-MMAS and PACO.

### 4.13.1 $\mathrm{Fm}|\mathrm{perm}| \Sigma\left(C_{i}-\bar{C}\right)^{2}$

A nonlinear objective function involving flow times is the completion time variance of jobs, or

$$
C T V=\sum_{i=1}^{n}\left(C_{i}-\bar{C}\right)^{2}, \text { where } \bar{C}=\sum_{i} C_{i} / n
$$

first introduced by Gowrishankar et al. (2001). They note that CTV aims at schedules that minimize variations in resource consumption and machine utilization. Further, if $\bar{C}$ is a common due-date for all jobs, then CTV leads to min-sum-square deviations from it.

A branch-and-bound algorithm is developed for $\operatorname{Fm}|\operatorname{perm}| \Sigma\left(C_{i}-\bar{C}\right)^{2}$ and the following GRS (the authors' initials) heuristic. It uses the same set of auxiliary two-machine problems as were introduced by Campbell et al. (1970) for the makespan objective in the CDS heuristic (see (4.7) in Sect. 4.9).

## Heuristic GRS

1. For $r=1,2, \ldots, m-1$, do :
(a) Define an auxiliary 2-machine problem with

$$
a_{j r}=\sum_{k=1}^{r} p_{k j}, b_{j r}=\sum_{k=m-r+1}^{m} p_{k j}, j=1, \ldots, n
$$

(b) Reindex the jobs in $\searrow t_{j r}$ order, where $t_{j r}=a_{j r}+b_{j r}$.
(c) Define the V-shaped sequence $S_{r}=(1,3,5, \ldots, 6,4,2)$.
2. Evaluate each $S_{r}, r=1, \ldots, m-1$ with respect to the CTV objective applied to the original $m$-machine instance, and pick the best.
3. Improve the winning sequence by an insertion scheme followed by pairwise interchanges.

A new ACO implementation, referred to as NACO, is presented by Gajpal and Rajendran (2006), as well as adaptations of MMAS and PACO for $F m \mid$ perm $\mid \Sigma\left(C_{i}-\bar{C}\right)^{2}$. The ensuing computational experiments on the Taillard suite with $n=20,50,100$ (i.e., 90 of the 120 problems in total) indicate that all 3 ACO implementations outperform GRS, MMAS outperforms PACO and NACO for $n=20$, and NACO outperforms PACO in 4 of the 6 remaining $(n, m)$ combinations. Benchmark values are provided for all 90 problem instances.

### 4.14 Fm $\| L_{\text {max }}$

In the next few sections we present results relating to lateness criteria: $L_{\max }$, $\Sigma U_{j}$, and $\Sigma w_{j} T_{j}$. We will start with the criterion of maximal lateness. In a note by Grabowski et al. (1983) it is shown that $F m\left|r_{j}, p e r m\right| L_{\max }$ is equivalent to a makespan minimization problem. To verify this, consider a flow shop with $m+2$ processors: $M_{0}, M_{1}, \ldots, M_{m+1}$. Further, assume $M_{0}$ and $M_{m+1}$ have infinite capacity. For every job we have $p_{0 j}=r_{j}$, and task $T_{0 j}$ starts processing at time 0 without interruption. Finally, assume that $p_{m+1, j}=\max _{i} d_{i}-d_{j}$. By construction, if $S$ is a permutation schedule on $M_{1}, \ldots, M_{m}$ where the lateness of $J_{j}$ is $L_{m j}(S)$, then the completion time of $J_{j}$ in the same schedule evaluated on $M_{0}, M_{1}, \ldots, M_{m+1}$ is $C_{m+1, j}^{+}(S)=C_{m j}(S)+p_{m+1, j}=L_{m j}(S)+\max _{i} d_{i}$, where $C^{+}$denotes completion times in the system with extra machines. It follows that $C_{\max }^{+}(S) \equiv \max _{j} C_{m+1, j}^{+}(S)=L_{\max }(S)+\max _{i} d_{i}$. Thus, minimizing $L_{\max }$ on $M_{1}, \ldots, M_{m}$ is equivalent to minimizing the makespan on $M_{0}, M_{1}, \ldots, M_{m+1}$, and hence $F m\left|r_{j}, \operatorname{perm}\right| L_{\max }$ is equivalent to the above makespan optimization problem.

Using the block structure $\mathcal{B}_{1}, \ldots, \mathcal{B}_{m}$ based on (4.8), Grabowski et al. (1983) adapted to $F m \mid$ perm $\mid C_{\text {max }}$ the algorithm first developed in Grabowski (1980) for problem $F 2 \mid$ perm $\mid C_{\text {max }}$. The resulting branch-and-bound algorithm uses lower bounds amongst those derived in Lageweg et al. (1978). Computational results are presented for up to 50 jobs, and up to 5 machines. The stopping criterion for the branch-and-bound algorithm is 1 minute of

CPU time. It is found that problem difficulty does not increase dramatically with $n$, but it does so with $m$. This is to be expected, because the number of blocks $B_{k}=\left\{u_{k-1}, \ldots, u_{k}\right\}$ induced by (4.8) increases with $m$. Subsequently, the authors highlight how to use their procedure so as to solve $F m\left|r_{j}\right| L_{\max }$; namely, by allowing for different job permutations in each block $B_{k}$.

### 4.15 $\mathrm{Fm} \| \Sigma \boldsymbol{U}_{\boldsymbol{j}}$

Very little has been done with respect to objective $\Sigma U_{j}$. As we saw in Chap. 2 problem $F 2 \mid($ perm $),(p m t n) \mid \Sigma U_{j}$ is ordinary NP-complete. It is not difficult to observe that $F m \| \Sigma U_{j}$ is no easier than $F m \| C_{\max }$. Simply, for any instance $I$ of $F m \| C_{\max }$ and given value $K>0$, construct instance $I^{\prime}$ of $F m \| \Sigma U_{j}$ obtained from $I$ with $d_{j}=K$ for all $j$. Then, there exists a solution for $I^{\prime}$ with $\Sigma U_{j}=0$ if and only if there exists a solution for $I$ with $C_{\max } \leq K$. Therefore, Theorem 7.2 implies that $F 3 \| \Sigma U_{j}$ is strongly NP-complete.

The above results indicate that, for the $\Sigma U_{j}$ objective, the best hope for the development of efficient lower bounds and heuristics is to make use of results for problem $1 \| \Sigma U_{j}$ where a single machine $M$ is available and every job is specified by its processing time $p_{j}$ and due-date $d_{j}$. This problem is solved in $O(n \log n)$ time by Moore's Algorithm (first presented in Sect. 2.11.3, repeated here for convenience). At any time, $\sigma$ is the partial schedule of early jobs (by "early" we mean non-tardy, i.e., early or precisely on time), always maintained in $\mathrm{EDD}=\nearrow d_{j}$ order, which starts empty and evolves by adding the next $J_{j}$ to the end $(\sigma:=\sigma j)$ and then, when necessary, removing the largest $J_{k}$ from anywhere in $\sigma(\sigma:=\sigma-k)$. The removed jobs accumulate in $\mathcal{L}$, the set of late (that is, tardy) jobs, which can be appended to the end of the schedule.

Moore's Algorithm (MA) (Moore, 1968) for $1\left|\mid \Sigma U_{j}\right.$
0 . Index all jobs in $\mathcal{J}$ in $\nearrow d_{j}$ order. Set $\sigma=\mathcal{L}=\phi$.

1. For $j=1$ to $n$ do:
(a) Set $\sigma:=\sigma j$, and let $C_{j}$ be the completion time of $J_{j}$.
(b) If $d_{j}<C_{j}$, then let $J_{k}$ be a job with $p_{k}=\max _{i \in \sigma} p_{i}$, and set $\sigma:=\sigma-k$ and $\mathcal{L}:=\mathcal{L}+k$;
else continue.
2. $S^{\star}=\left(\sigma, \sigma_{L}\right)$, where $\sigma_{L}$ is any sequence of the jobs in $\mathcal{L}$.

Hariri and Potts (1989) adapted MA to heuristically solve $F m \| \Sigma U_{j}$. With $P_{j}=\Sigma_{k} p_{k j}$, they replace Step 1(b) by:
(b) If $d_{j}<C_{j}$, then let $J_{k}$ be a job with $P_{k}=\max _{i \in \sigma} P_{i}$, and if $\sigma-k$ has no tardy jobs, set $\sigma:=\sigma-k$ and $\mathcal{L}:=\mathcal{L}+k$; else set $\sigma:=\sigma-j$ and $\mathcal{L}:=\mathcal{L}+j$.
else continue.

Of course, in Step 2 we can no longer claim an optimal schedule. Instead, calling the heuristic $\mathrm{H}_{\mathrm{HP}}$, we get $S_{H P}=\left(\sigma, \sigma_{L}\right)$. This adaptation takes $O\left(m n^{2}\right)$ required to compute the $C_{k j}$ values for $k=1, \ldots, m$ and $j \in \mathcal{J}$.

Similarly, MA can be adapted to obtain lower bounds for $F m \| \Sigma U_{j}$ in polynomial time, as described next. Let SMk denote the single machine subproblem on machine $M_{k}, k=1,2, \ldots, m$ where job due dates are revised as

$$
d_{j}^{k}=d_{j}-\sum_{r=1}^{k-1} p_{r j}-\sum_{r=k+1}^{m} p_{r j}=d_{j}-P_{j}+p_{k j} .
$$

Thus, we subtract the total processing required for $J_{j}$ on machines other than $M_{k}$. This is a somewhat smaller due date, giving a tighter bound than the one in Hariri and Potts (1989). Let us define
$n_{k}(S)=$ the number of tardy jobs in SMk, using any schedule $S$.
$n_{0}(S)=$ the number tardy in the original problem, $F m \| \Sigma U_{j}$, using $S$.
Note that, for any schedule, every job that is tardy in SMk is also tardy in the original problem, i.e., $n_{k}(S) \leq n_{0}(S)$, because, in going from one machine to $m$ machines, every due date is increased by $P_{j}-p_{k j}$ while every completion time increases by at least this much. Since the schedule $S_{M A}$ produced by MA is optimal for $\mathrm{SMk}, n_{k}\left(S_{M A}\right) \leq n_{k}(S)$. It follows that $n_{k}\left(S_{M A}\right)$, and so

$$
L B=\max _{k=1, \ldots, m} n_{k}\left(S_{M A}\right)
$$

is a lower bound for $F m \| \Sigma U_{j}$ and is computed in $O(m n \log n)$ time.
Hariri and Potts (1989) go on, using the concept of consistent early and late sets (jobs that must be early or late in every SMk), to potentially improve this bound by at most one. Their procedure requires $O\left(m n^{2} \log n\right)$ time. They also develop an alternative but related bounding scheme that takes $O\left(m n^{3}\right)$ time. However, both are shown to be inefficient for use within a branch-and-bound framework. Not surprisingly, their implementation provides a satisfactory solution method for problems with up to $n=15$ jobs and $m=5$ machines, and problems with $n=20$ and $m=2$ or 3 .

### 4.16Fm\| $\boldsymbol{\Sigma} w_{j} T_{j}$

Du and Leung (1990) have shown that $F m \| \Sigma T_{j}$ is strongly NP-complete even when $m=2$, and hence so is the more general problem $F m \| \Sigma w_{j} T_{j}$. Interestingly, there is a significant body of literature for $F m|\operatorname{perm}| \Sigma T_{j}$ and $F m \mid$ perm $\mid \Sigma w_{j} T_{j}$ which involves lower bounding schemes, exact algorithms, evaluation of dispatching rules, constructive and improvement heuristics, and metaheuristics.

### 4.16.1 Lower Bounds for $\operatorname{Fm}|\operatorname{perm}| \Sigma T_{j}$

As usual, we present the bounds in the context of a branch-and-bound algorithm, where a partial schedule $\sigma$ already occupies each $M_{k}$ up to time $C_{k}(\sigma)$.

Let $u$ denote the number of jobs remaining to be scheduled, and hereafter, when we refer to jobs or tasks, we mean unscheduled ones, and the lower bounds we compute are for those jobs only. For bounds on the total schedule, the tardiness sum of the jobs in $\sigma$ are a sunk cost that should be added. Of course, if there is no $\sigma$, let all $C_{k}(\sigma)=0$.

## A Simple bound

Simple lower bounds are presented in Kim (1995). In a permutation schedule, the task $T_{k,[i]}$ processed in position $i$ of $M_{k}$ (let us call this position $k i$ ) is preceded on $M_{k}$ by $i-1$ tasks, and they in turn must be preceded by at least one task on each of $M_{1}, \ldots, M_{k-1}$, machines which are already occupied by the jobs in $\sigma$. Similarly, $T_{k,[i]}$ is followed by at least one task on each of $M_{k+1}, \ldots, M_{m}$. It will be convenient to define:
$q_{k i}=$ the $i^{\text {th }}$ smallest task time on $M_{k}$, with $Q_{k i}=\sum_{j=1}^{i} q_{k j}, 1=1, \ldots, u$.
Then, accounting for all possible $k$ 's, the completion time $C_{m,[i]}$ of task $T_{m,[i]}$ is at least

$$
t_{[i]}=\max _{k=1, \ldots, m}\left\{\max _{j=1, \ldots, k}\left[C_{j}(\sigma)+\sum_{h=j}^{k-1} q_{h 1}\right]+Q_{k i}+\sum_{h=k+1}^{m} q_{h 1}\right\} .
$$

We note that $\sigma$ is omitted in Kim (1995), so the first term is simpler. Also, the last summation can be strengthened (increased) by noting that one job, whichever occupies position $k i$, must be processed consecutively through the remaining machines, so we can replace $\sum_{h=k+1}^{m} q_{h 1}$ by $\min _{j} \Sigma_{h=k+1}^{m} p_{h j}$. A similar argument can be made to strengthen the first summation.

Reindex jobs in $\nearrow d_{j}$ order. Then, for any permutation schedule,

$$
\begin{equation*}
\sum_{i=1}^{u} T_{i}=\sum_{i=1}^{u}\left(C_{m,[i]}-d_{[i]}\right)^{+} \geq \sum_{i=1}^{u}\left(t_{[i]}-d_{i}\right)^{+} \tag{4.15}
\end{equation*}
$$

The final $d_{i}$ in place of $d_{[i]}$ needs explanation. Note that $t_{[i]}$ is the completion time (or rather a lower bound on it; lets call it the early completion time) of the job in position $i$; we do not know which job that will be. Which due date should be paired with each $t_{[i]}$ ? Given two sets of $n$ numbers, in this case $t_{[i]}$ and $d_{i}$, to minimize $\sum_{i=1}^{u}\left(t_{[i]}-d_{\pi(i)}\right)^{+}$for some permutation $\pi$, a simple interchange argument shows that the two sets should be paired off smaller-to-smaller. Thus, $t_{[1]}$ should be paired with $\min _{i} d_{i}$, etc., to make sure the resulting sum is a lower bound.

## A Tighter Bound

Extending the above ideas, a more efficient lower bound is presented in Chung et al. (2006). They start by computing recursively
$s_{k i}=$ a lower bound on the start time of the job in position $k i$,
for $i=1, \ldots, u$ and $k=1, \ldots, m$. The basic idea is to schedule the tasks on each machine in SPT order, regardless of their job affiliations. In doing this, we present slightly tighter bounds than they give. Recall, we defined
$q_{k i}=$ the $i^{t h}$ smallest task time on $M_{k}$, with $Q_{k i}=\sum_{j=1}^{i} q_{k j}$.
First, for $M_{1}$ :

$$
s_{1 i}=C_{1}(\sigma)+Q_{1, i-1}, \text { with } s_{11}=C_{1}(\sigma) .
$$

Second, computing iteratively for $k=2, \ldots, m$ :

$$
s_{k 1}=\max \left\{C_{k}(\sigma), \max _{r=1, \ldots, k-1}\left\{s_{r 1}+\min _{j} \sum_{h=r}^{k-1} p_{h j}\right\}\right\} .
$$

Then, iteratively for $k=2, \ldots, m$, and for each $k$, iteratively for $i=2, \ldots, u$ :

$$
\begin{equation*}
s_{k i}=\max \left\{s_{k, i-1}+q_{k, i-1}, \max _{r=1, \ldots, k-1}\left\{s_{r i}+\min _{j} \sum_{h=r}^{k-1} p_{h j}\right\}\right\} \tag{4.16}
\end{equation*}
$$

We can now develop, for each $M_{k}$, a lower bound on the total tardiness. Given the start time $s_{k i}$ for position $k i$, a lower bound on the tardiness of any job $J_{j}$ scheduled in that position is $\left(s_{k i}+\sum_{h=k}^{m} p_{h j}-d_{j}\right)^{+}$. Define $\delta_{k j} \equiv d_{j}-\sum_{h=k}^{m} p_{h j}$, a sort of modified due date. Then, in the spirit of (4.15), we must index jobs in increasing order of $\delta_{k j}$ ( $k$ fixed) to be sure that $\sum_{i=1}^{n}\left(s_{k i}-\delta_{k i}\right)^{+}$is a valid lower bound. Finally, to summarize our result:

Theorem 4.17 (Chung et al., 2006) For Fm $\mid$ perm $\mid \Sigma T_{j}$,
$\max _{k=1, \ldots, m} \sum_{i=1}^{u}\left(s_{k i}-\delta_{k i}\right)^{+}$
is a lower bound on $\Sigma_{i} T_{i}$, where

- $s_{k i}$ is a lower bound on the start time of the job in position ki; see (4.16);
- $\delta_{k i}=d_{i}-\Sigma_{h=k}^{m} p_{h i}$, indexed for each $k$ in $/ \delta_{\cdot j}$ order.


### 4.16.2 Dominance Property

Chung et al. (2006) also developed the following sequence dominance property. Let $T T(\sigma)$ denote the total tardiness of jobs in a partial schedule $\sigma$. Recall Definition 4.3, which becomes in this case:

Definition 4.7. Given two partial schedules $\sigma_{1}$ and $\sigma_{2}$ of the same set of jobs, with the set $\mathcal{U}$ unscheduled, $\sigma_{1}$ dominates $\sigma_{2}$ with respect to total tardiness if $T T\left(\sigma_{1} \pi\right) \leq T T\left(\sigma_{2} \pi\right)$, for every permutation $\pi$ of $\mathcal{U}$.

Theorem 4.18 (Chung et al., 2006) For $F m \mid$ perm $\mid \Sigma T_{j}$, let $\sigma_{1}$ and $\sigma_{2}$ be two partial schedules of the same subset $\mathcal{S}$ of jobs, leaving $u$ jobs in $\mathcal{U}=\mathcal{J}-\mathcal{S}$ to be scheduled. If

$$
\begin{equation*}
T T\left(\sigma_{2}\right)-T T\left(\sigma_{1}\right) \geq u \max _{k}\left\{C_{k}\left(\sigma_{1}\right)-C_{k}\left(\sigma_{2}\right)\right\}^{+}, \tag{4.17}
\end{equation*}
$$

then $\sigma_{1}$ dominates $\sigma_{2}$. The dominance is strict if (4.17) is a strict inequality.
The above result is due to the observation that, if $\pi$ is any sequence of jobs, then

$$
C_{m}\left(\sigma_{1} \pi\right)-C_{m}\left(\sigma_{2} \pi\right) \leq \max _{k}\left\{C_{k}\left(\sigma_{1}\right)-C_{k}\left(\sigma_{2}\right)\right\}
$$

### 4.16.3 Branch-and-Bound Algorithms

Using the simple lower bound on $\Sigma T_{i}$ given above, Kim (1995) developed a branch-and-bound algorithm that uses depth-first search and is able to solve problems of size up to $(n, m)=(20,4)$ or $(12,10)$ within 1 hour of CPU time.

Chung et al. (2006) also used their bound (Theorem 4.17) and dominance criterion (Theorem 4.18) in a branch-and-bound procedure. The algorithm used depth-first search and is shown to solve problems with up to $(n, m)=$ $(20,8)$ when the stopping criterion is a limit of 4 million nodes on the search tree. In comparisons with Kim's algorithm, it is found that the one of Chung et al. solved the same problems 14 times faster.

### 4.16.4 Heuristics

The above results indicate that we are unlikely to be able to optimally solve large problems in short time. Hence, a significant body of literature has been devoted to heuristics. Surprisingly, some research is available on absolute performance guarantees of heuristics. These are bounds on the difference $C_{H}-C^{\star}$. A heuristic is said to provide the absolute performance guarantee $\beta$ if for any problem instance,

$$
\begin{equation*}
C_{H}-C^{\star} \leq \beta \tag{4.18}
\end{equation*}
$$

It makes sense to find bounds where $\beta$ is not a multiple of $C^{\star}$; rather, a quantity that hopefully is much smaller than $C^{\star}$. Hence, in most cases $\beta$ is a quantity which is data dependent, such as $p_{\max }=\max _{i}\left\{a_{i}, b_{i}\right\}$.

Sevast'janov (1997) observed a relationship between the non-strict vector summation (NVS) problem on a plane to problem $F m \| C_{\text {max }}$. Problem NVS involves finding a permutation of $n$ given vectors (in our case the vectors $\left(p_{1 j}, \ldots, p_{m j}\right)$ for $\left.j=1, \ldots, n\right)$ so that any two successive partial sums $\left(\Sigma_{j=1}^{i} p_{1 j}, \ldots, \Sigma_{j=1}^{i} p_{m j}\right)$ of vectors in this permutation belong to a given domain $G$ (e.g. a unit-circle in the plane). The solution constructed in Sevast'janov (1997) for NVS yields a permutation $\pi_{S}$ for $F 3 \| C_{\max }$ found in $O(n)$ time for which

$$
C\left(\pi_{S}\right) \leq C^{\star}+3 p_{\max }
$$

and a permutation $\pi_{S}^{\prime}$ for $F 4 \| C_{\max }$ found in $O(n \log n)$ time for which

$$
C\left(\pi_{S}^{\prime}\right) \leq C^{\star}+6 p_{\max }
$$

Except for the above elegant results, a host of priority rules have been tested for $F m|\operatorname{perm}| \Sigma w_{j} T_{j}$ including the following (as usual, $P_{j}=\Sigma_{k} p_{k j}$ ):
$\mathrm{EDD}: \nearrow d_{j}$,
EWDD: $\nearrow w_{j} / d_{j}$,
EDDP: $\nearrow d_{j} / P_{j}$,
$\mathrm{SPT}: \nearrow P_{j}$,
LPT: $\searrow P_{j}$,

Slack (LS): $\nearrow\left(d_{j}-C(\sigma j)\right)$,
Slack per Remaining Work (S/RW): $\nearrow\left(d_{j}-C(\sigma j)\right) / P_{j}$,
where $C(\sigma j)$ denotes the completion time of unscheduled $J_{j}$ when appended at the end of a partial schedule $\sigma$. Unfortunately, all such rules considered to date exhibit poor performance as shown in Vallada et al. (2008). In the same article, the authors test all algorithms given in the literature for problem $F m|\operatorname{perm}| \Sigma w_{j} T_{j}$. Their experiment is quite exhaustive and encompasses all the above dispatching rules, 10 constructive and 8 improvement heuristics, and 17 metaheuristic implementations involving tabu search, simulated annealing, genetic algorithms, differential evolution, and hybrids thereof. Test are performed on problems with $n=50,150,250,350$ and $m=10,30,50$ and random integer processing times drawn uniformly from [1, 99]. Due dates are random integers generated as in Potts and Van Wassenhove (1982) for problem $1 \| \Sigma w_{j} T_{j}$, drawn uniformly from $[P(1-T-R / 2), P(1-T+R / 2)]$ where $T=0.2,0.4$, or 0.6 is a tardiness factor, $R=0.2,0.6$, or 1 is a due date range parameter, and $P$ is the lower bound proposed by Taillard (1993) for problem $F m \mid$ perm $\mid C_{\text {max }}$. For every $(n, m, T, R)$ combination, 5 problem instances are generated, thus resulting to a total of 540 test problems and can be found at http://soa.iti.es under Problem instances - Benchmark for flow shops and due dates.

The quality of the solutions produced by the various algorithms is evaluated by

Relative Deviation Index (RDI) $=\frac{\text { Method }_{\text {sol }}-\text { Best }_{\text {sol }}}{\text { Worst }_{\text {sol }}-\text { Best }_{\text {sol }}} 100 \%$
where Best $_{\text {sol }}$ and $W_{\text {orst }}^{\text {sol }}$ are the best and worse objective values obtained by the methods tested (whose values are denoted by Method ${ }_{\text {sol }}$ ).

In terms of dispatching rules and improvement heuristics, the following is due to Kim et al. (1996) and outperforms all others.

## Extensive Neighborhood Search (ENS) Heuristic

1. Apply NEH where jobs are considered in EDD order.
2. Perform improving insertions as long as possible.

Recall that in NEH jobs are ordered according to $\searrow P_{j}$; different from the EDD order in step 1 of ENS. In the experiment of Vallada et al. (2008), algorithm ENS exhibits $R D I=1.02 \%$.

In terms of metaheuristics, the SA implementations referred to as SAH and SRH in Parthasarathy and Rajendran (1997) and Hasija and Rajendran (2004) respectively, exhibit the RDI values $5.27 \%$ and $2.55 \%$; lowest amongst the 17 metaheuristics tested. SAH starts with the sequence produced by the EWDD rule, improved by local search, while SRH starts with a custom rule improved by a perturbation scheme.

### 4.17 Multiple Objectives: Heuristic Approaches

As discussed in Chap. 2, when we wish to advance more than one objective (this always means two objectives), we can formulate a single criterion (i.e., mathematical function) that combines them, as with hierarchical objectives (e.g., minimize $\Sigma C_{j}$ subject to maintaining the smallest possible makespan, written $\Sigma C_{j} \mid \min C_{\max }$ ) or compound objectives (like $\alpha \Sigma C_{j}+\beta C_{\max }$ ); or we can consider the two objectives as a pair of outcomes, the bicriteria approach, and seek solutions that do well for both. Ultimately, we would like to find the set of efficient or undominated solutions: those for which no other solution is better on both counts. For more details on bicriteria analysis, see Sect. 2.11.

No optimization algorithms have been proposed for this class of problems. We discuss in this section some of the heuristics that have been published.

### 4.17.1 $C_{\max }$ and $\Sigma C_{j}$

Intuitively, the objectives $C_{\max }$ and $\Sigma C_{j}$ are not in conflict: they both benefit by minimizing the makespan of all initial partial schedules. Rajendran (1995) developed an improvement heuristic (referred to as R95) intended to produce good bicriteria schedules for $C_{\max }$ and $\Sigma C_{j}$. Heuristic R95 starts with the CDS schedule, strengthened by adjacent pairwise interchanges. He tested it against the heuristic of Ho and Chang (1991) originally developed for the makespan objective on randomly generated problems with $n \leq 50$ jobs and $m \leq 30$ machines, with mixed results. The same heuristic with the starting schedule replaced by NEH is presented in Rajendran (1994) (referred to as R94).

Explicitly accounting for both objectives, Sridhar and Rajendran (1996) developed a genetic algorithm (GA) that uses equal weights for makespan, flowtime and machine idle time, and initial sequences NEH and RC.

As we saw earlier, Framinan et al. (2002) demonstrated that NEH with respect to the initial sequence $\searrow P_{j}\left[\nearrow P_{j}\right]$ performs well for $F m \mid$ perm $\mid C_{\max }$ $\left[F m \mid\right.$ perm $\left.\mid \Sigma C_{j}\right]$. It is also shown that NEH with initial $\searrow P_{j}$ offers good performance with respect to either objective. Motivated by this observation, two heuristics are developed for $F m \mid$ perm $\mid\left(C_{\max }, \Sigma C_{j}\right)$ by Framinan et al. (2002), referred to as a priori (call it Prior) and a posteriori or Post heuristics. They both use NEH with starting sequence $\searrow P_{j}$ or $\nearrow P_{j}$. Various weight values $\alpha$ are used to evaluate the objective $\alpha(n / 2) C_{\max }+(1-\alpha) \Sigma C_{j}$ (the factor $n / 2$ is added to equalize the magnitudes of the two criteria). Undominated heuristic solutions are stored to form the efficient frontier. Post requires about 20\% more CPU time than Prior because it examines a lot more subsequences. Prior is found to perform better than R94 and R95 and slightly worse than the GA of Sridhar and Rajendran (1996). Post generally outperforms this GA and is much faster.

### 4.17.2 $C_{\text {max }}$ and $T_{\max }$

Several publications have considered the joint objectives $C_{\max }$ and $T_{\max }$. Daniels and Chambers (1990) propose the following heuristic for the bicriteria problem $F m \mid$ perm $\mid\left(C_{\text {max }}, T_{\max }\right)$, already discussed for the case $m=2$ in Chap. 2. We seek to minimize $C_{\max }$ subject to $T_{\max } \leq B$, as $B$ is varied over an appropriate range of values. For each $B$, we make $C_{\max }$ as small as possible using a heuristic (we'll call it the DC heuristic) that builds a permutation schedule backwards, from position $n$ to position 1 . Let $\mathcal{U}$ be the set of unscheduled jobs (note, they will make up the start of the schedule) in iteration $n-i+1$ of the algorithm, when we choose the job $J_{[i]}$ to place in position $i$. To limit its tardiness, this job is chosen from the eligible set

$$
\mathcal{E}_{i}=\left\{j \in \mathcal{U}: \sum_{h \in \mathcal{U}} p_{1 h}+\sum_{k=2}^{m} p_{k j}-d_{j} \leq B\right\}
$$

where the two summations reflect the least amount of time required to complete all jobs in $\mathcal{U}$ when job $j$ is scheduled last amongst them. For given $B$, a feasible schedule is produced when a job is assigned to every position according to the following steps:

Heuristic DC for $F m|\operatorname{perm}|\left(C_{\max } \mid T_{\max } \leq B\right)$
0. Set $\mathcal{U}:=\mathcal{J}$;

1. For $i=n$ down to 1 do
(a)Calculate $\mathcal{E}_{i}$;
(b) Apply NEH heuristic to $\mathcal{U}$ and let $S_{i}$ be the resulting schedule ;
(c) Let $J_{[i]}$ attain $\max _{j \in \mathcal{E}_{i}} C_{m j}\left(S_{i}\right)$ and set $\mathcal{U}:=\mathcal{U}-\{[i]\}$.

Evidently, the range of $T_{\max }$ values considered in the DC heuristic is the maximal tardiness incurred by the NEH schedule or less. The above loop fails to produce a complete solution when $\mathcal{E}_{i}=\phi$. When this happens, heuristic DC backtracks to use an untried job in $\mathcal{E}_{i+1}, \ldots, \mathcal{E}_{n}$ in this order, as long as one such job exists. Otherwise a larger value $B$ is tried. It is also possible that the above steps produce a permutation which, upon evaluation of $T_{\max }$, turns out to be infeasible. Then, the DC heuristic backtracks to the last scheduled position for which the tardiness limit is exceeded. We start by finding $J R$ unconstrained, and letting the resulting maximal job tardiness be our initial value for $B$. By decrementing $B$ iteratively over a suitable range, all efficient schedules can be generated.

In a computational experiment with $n=10$ jobs and $m=2,5$ or 10 machines, Daniels and Chambers (1990) observed that the number of efficient schedules for $F m \mid$ perm $\mid\left(C_{\max } \mid T_{\max }\right)$ is small, increases with $m$, and their heuristic produces about $50 \%$ of them. Moreover, feasible efficient solutions are found to be within $3 \%$ from the optimal, on average.

A variation of the DC heuristic is developed by Framinan and Leistein (2006). Their heuristic (referred to as FL) starts with the least slack order $L S: \nearrow\left(d_{j}-P_{j}\right)$, or its inverse $\searrow\left(d_{j}-P_{j}\right)$, and $T_{\max }$ is computed. Suppose
$T_{\text {max }}$ is attained at position $i$. For all preceding jobs, the DC heuristic steps are employed, augmented by job exchanges used to reduce the makespan without increasing the value $T_{\max }$. The FL implementation is tested on a problem suite by Demirkol et al. (1998) originally designed for the lateness objective. This suite includes 160 flow shop problems with $n \leq 50$ and $m \leq$ 25. It is shown that the FL implementation significantly outperforms the DC heuristic for $F m \mid$ perm $\mid\left(C_{\max } \mid T_{\max } \leq B\right)$.

Another objective considered in the literature that involves $C_{\max }$ and $T_{\max }$ is the convex combination of the two objectives as in $\operatorname{Fm} \mid$ perm $\mid\left(\lambda C_{\max }+(1-\right.$ $\left.\lambda) T_{\max } \mid T_{\max } \leq B\right)$ with $\lambda \in[0,1]$. A simulated annealing metaheuristic is developed in Chakravarthy and Rajendran (1999) for this problem, which reduces to $F m|\operatorname{perm}|\left(C_{\max } \mid T_{\max } \leq B\right)$ when $\lambda=1$. The starting solution for this metaheuristic (referred to as CR) is the best amongst the LS and EDD sequences, and the sequence produced by the NEH heuristic. Trial $T_{\max }$ values are no greater than

$$
\max _{j}\left(\sum_{k=1}^{m} p_{k j}+\frac{n-1}{n} \sum_{l=1}^{n} p_{m l}\right)-\min _{j} d_{j}
$$

which might be tight for some instances. Even though CR outperforms the DC and FL heuristics for problems with less than 12 jobs, it is shown that it may fail to produce feasible solutions (see Framinan and Leistein, 2006).

Allahverdi (2004) proposed a neighborhood search heuristic (referred to as AH) that starts with the best of the LS, EDD and NEH sequences and employs insertions of job pairs in a framework similar to that of the NEH heuristic. As with FL and CR, heuristic AH is found to outperform all of DC, FL and CR for $F m \mid$ perm $\mid\left(\lambda C_{\max }+(1-\lambda) T_{\max } \mid T_{\max } \leq B\right)$, and takes very little CPU time on problems with $n \leq 150$ and $m \leq 20$. Moreover, AH performs well across different values of $m$ and $\lambda$. The performance of AH is also evaluated on $F m \mid$ perm $\mid\left(\lambda C_{\max }+(1-\lambda) T_{\max }\right)$ and compared against the corresponding solutions obtained by DC, FL and CR. The findings are similar to those for the constrained objective.

Finally, a genetic algorithm is developed by Ruiz and Allahverdi (2009) that is strengthened by local search. On randomly generated problems with $n \leq 350$ and $m \leq 50$, this genetic algorithm (referred to as RA) produces better solutions than AH and FL and exhibits robust performance across different values of $\lambda$.

### 4.17.3 $\Sigma w_{j} C_{j}$ and $\Sigma u_{j} T_{j}$

For problem $F m \mid$ perm $\mid \Sigma\left(w_{j} C_{j}+u_{j} T_{j}\right)$ Gelders and Sambandam (1978) developed four heuristics by appending jobs to a partial schedule on the basis of machine idle times, makespan lower bound, and the $d_{j}, w_{j}, u_{j}$ values of each job. The resulting schedules are then improved by adjacent pairwise interchanges.

For the same objective, Rajendran and Ziegler (1999) developed a constructive heuristic that selects the best amongst the following $2 m$ job permu-
tations. Defining:

$$
f(j, r)=\frac{\sum_{k=r}^{m}(m-k+1) p_{k j}}{w_{j}+u_{j}}, \quad \text { and } \quad g(j, r)=\frac{d_{j} \sum_{k=1}^{r} p_{k j}}{\left(w_{j}+u_{j}\right) \sum_{k=1}^{m} p_{k j}},
$$

they consider, for each $r(r=1,2, \ldots, m)$, the sequences $\nearrow f(j, \cdot)$ and $\nearrow g(j, \cdot)$. Even though the best amongst these permutations does not perform well, improvements based on job insertions yield significantly improved solutions shown to outperform those obtained in Gelders and Sambandam (1978). Using these heuristic solutions as seeds within the simulated annealing framework developed in Ishibuchi et al.(1995) (originally developed for Fm $\mid$ perm $\mid C_{\text {max }}$ ), is shown to yield further improvements within reasonable CPU time.

### 4.18 Setups and Teardowns: $F m \mid$ perm, $s_{i j}, t_{i j} \mid A n y$

Often, before a machine is ready to process a certain job, it must be prepared in some way: it may need to be tested, adjusted, tooled, supplied with materials, etc. The time needed for such preparation is called setup time. Similarly, after a job is completed, the machine may be tied up an additional teardown time (sometimes referred to as removal time) for taking the job off the machine, cleaning up, putting away tools and materials, etc.

We say that the setup and teardown of $J_{j}$ are sequence-independent if they depend only on $J_{j}$, not on the jobs that precede or follow it. We call them separable or anticipatory if the job does not have to be physically present: the setup for a job on $M_{i+1}$ and its teardown on $M_{i-1}$ can proceed at the same time that it is being processed on $M_{i}$. Under these two assumptions, let us define $s_{i j}, p_{i j}$, and $t_{i j}$ as the nonnegative setup, processing, and teardown times of $J_{j}$ on $M_{i}$.

As noted among our introductory examples (see Sect. 3.1.1), setups (and the same is true of teardowns) that are separable and sequence-independent can be treated as negative lags. In fact, even sequence-dependent setup times can be so handled provided they are additive. In general, the setup time on a machine is sequence-dependent if it depends on the job that was just completed as well as the job to be done next. Thus, if $J_{k}$ follows $J_{j}$ on $M_{i}$, the setup time for $J_{k}$ is $s_{i j k}$, which may be an arbitrary function of $j$ and $k$. An additive function has the form $s_{i j k}=u_{i j}+v_{i k}$. Now, of course, we can interpret this as the time to tear down $J_{j}$ and then set up $J_{k}$.

Observe that, although the three parts of a job are separable, it can never help to insert idle time between them: if such idle time exists, we are free to move the setup later and/or the teardown earlier to remove it. We therefore define $p_{i j}^{\prime}=s_{i j}+p_{i j}+t_{i j}$ as the effective processing time. However, this effective processing of $J_{j}$ can begin on $M_{i+1}$ a time $t_{i j}+s_{i+1, j}$ before it effectively completes on $M_{i}$, because the true processing ends $t_{i j}$ earlier on
$M_{i}$, and starts $s_{i+1, j}$ later on $M_{i+1}$, than the effective processing (see Fig. 3.5 for the case with setups but no teardowns). We conclude:

Theorem 4.19 Fm $\mid$ perm, $s_{i j}, t_{i j} \mid$ Any is equivalent to Fm $\mid$ perm, $\ell_{i j} \mid$ Any with parameters
$p_{i j}^{\prime}=s_{i j}+p_{i j}+t_{i j}, i=1, \ldots, m, j=1, . ., n$, and
$\ell_{i j}=-\left(t_{i j}+s_{i+1, j}\right), i=1, \ldots, m-1, j=1, . ., n$.
For example, in the two-machine case (now each job has only one lag, $\ell_{1 j}=\ell_{j}$ ) with makespan objective, as Yoshida and Hitomi (1979) showed for setups only, and Sule (1982) extended to include teardowns, we have:

Corollary 4.1. For $F 2 \mid$ perm, $s_{i j}, t_{i j} \mid C_{\max }$,

$$
S^{\star}=J R\left(\underline{P}_{1}+\underline{\ell}, \underline{P}_{2}+\underline{\ell}\right)=J R\left(\underline{p}_{1}+\underline{s}_{1}-\underline{s}_{2}, \underline{p}_{2}+\underline{t}_{2}-\underline{t}_{1}\right)
$$

Note that this is effectively a two-machine flow shop with lags, and as we have seen, even for the makespan objective, the best solution is not always a permutation schedule. Thus, only by limiting our search to permutation schedules can we solve the problem efficiently. Also, observe that the calculation of the job parameters, $P_{1 j}+\ell_{j}$ and $P_{2 j}+\ell_{j}$, may well produce negative values for these "processing times". This may seem bizarre, but does not invalidate Johnson's Rule, which finds the optimal schedule. Of course, to find the length of that schedule, original data must be used.

### 4.19 Cyclic Scheduling

No efficient algorithm is likely to minimize the cycle time in a repetitive manufacturing setting, as McCormick et al. (1987) have shown that Fm|perm, cyclic|CT is strongly NP-complete. The extension of the mixed integer program given in Sect. 4.5 for $F m \mid$ perm $\mid C_{\text {max }}$ to the cyclic case is a straightforward application of the ideas introduced in Sect. 1.7.1. With $x_{i j}$ a $0-1$ variable indicating that $J_{j}$ is in position $i$, and $c_{k i}$ the completion time on $M_{k}$ of the job in position $i$, the following mixed integer program finds the minimal cycle time $Z$.

$$
\begin{array}{ll}
\text { minimize } & Z \\
\text { subject to } & Z \geq c_{k n}-c_{k 1}+\sum_{j=1}^{n} p_{k j} x_{1 j}, \quad k \in \mathcal{M} \\
& c_{k i} \geq c_{k-1, i}+\sum_{j=1}^{n} p_{k j} x_{i j}, \quad i \in \mathcal{J}, \quad k \in \mathcal{M} \\
& c_{k i} \geq c_{k, i-1}+\sum_{j=1}^{n} p_{k j} x_{i j}, \quad i \in \mathcal{J}, \quad k \in \mathcal{M} \\
& \sum_{i=1}^{n} x_{i j}=1, \quad j \in \mathcal{J} \\
& \sum_{j=1}^{n} x_{i j}=1, \quad i \in \mathcal{J} \\
& x_{i j}=0 \text { or } 1, \quad i, j \in \mathcal{J} \tag{4.24}
\end{array}
$$

with $c_{k 0}=c_{0 i}=0$; where (4.19) ensures that cycle time $Z=\max _{k}\left\{c_{k n}-\right.$ $\left.s_{k 1}\right\}=\max _{k}\left\{c_{k n}-\left(c_{k 1}-p_{k[1]}\right)\right\}$. As before, (4.20) enforces task order within
jobs, (4.21) does the same on each machine, and (4.22) - (4.24) ensure that each job is assigned to just one position in the schedule, and each position gets one job.

### 4.19.1 The Minimal Cycle Time for a Given Sequence

A straightforward procedure can be given to find the minimal cycle time, $C T^{\star}(S)$, for a given sequence $S$; apparently it has not previously been noted. As usual, reindex jobs so that $S=(1, \ldots, n)$. We first find the completion (or early finish) times, $C_{k j}$, of all tasks, using (4.1). With makespan fixed, we then perform a backward pass to find the latest (or late finish) time, $C_{k j}^{\prime}$, each task can be completed without increasing the makespan. Any task $T_{k j}$ that is on a critical path (there may be more than one such path) will have $C_{k j}=C_{k j}^{\prime}$. Now, to make the work as compact as possible on each $M_{k}$, schedule all tasks that precede the earliest critical task as late as possible, and schedule tasks that follow the last critical task as early as possible. A more formal statement of this concept follows.

## Shortest Cycle Time (SCT(S)) Algorithm for Fm|perm, cyclic|CT(S)

0 . Specify the job sequence, $S:=(1, \ldots, n)$, reindexing jobs as needed.

1. Compute the early finish times of all tasks, using:

$$
\begin{equation*}
C_{k j}=\max \left\{C_{k-1, j}, C_{k, j-1}\right\}+p_{k j}, \tag{4.1}
\end{equation*}
$$

with $C_{0 j}=C_{k 0}=0$; for $k=1, \ldots, m$, and for each $k$, for $j=1, \ldots, n$.
2. Compute the late finish times of all tasks:

$$
C_{k j}^{\prime}=\min \left\{C_{k+1, j}^{\prime}-p_{k+1, j}, C_{k, j+1}^{\prime}-p_{k, j+1}\right\},
$$

with $C_{m n}^{\prime}=C_{m n}$ and $C_{m+1, j}=C_{k, n+1}=p_{m+1, j}=p_{k, n+1}=0$. Iterate backwards, for $k=m, \ldots, 1$, and for each $k$, for $j=n, \ldots, 1$.
3. Now,

$$
C T^{\star}(S)=\max _{k=1, \ldots, m}\left\{C_{k n}-C_{k 1}^{\prime}+p_{k 1}\right\}
$$

In Fig. 4.6, we show how the steps progress for an instance with $m=n=4$,


Fig. 4.6 Steps in determining the shortest cycle time for a schedule
where the processing times are $\bar{p}_{1}=<4,4,11,7>, \bar{p}_{2}=<8,9,9,5>, \bar{p}_{3}=$ $<11,5,8,11>$, and $\bar{p}_{4}=<10,7,5,3>$. Given $S=(1,2,3,4)$, Fig. 4.6(a) shows the usual early start schedule found in Step 1, giving makespan of 52.

With makespan fixed, Step 2 places each task as late as possible, as shown in Fig. 4.6(b). Critical tasks (dark shading) are unmoved. The compact schedule shown in Fig. 4.6(c) has pre-critical tasks scheduled late, and post-critical tasks early, giving minimal machine occupancies of $33,34,35$, and 26 , so that $C T^{\star}=35$.

### 4.20 Concluding Remarks

The $m$-machine permutation flow shop with $m>2$ is found to be one of the most extensively researched productions systems. A rich literature exists for all the popular scheduling objectives as well as bicriteria. In contrast, nonpermutation scheduled are mostly neglected. Results on exact algorithms have been disappointing and the problems we can optimally solve to date are not of realistic size. A notable exception is the $F 3 \mid$ perm $\mid C_{\max }$ problem for which adaptations of Johnson's rule often provide optimal solutions.

The most celebrated results for the $m$-machine flow shop are the NEH and CDS heuristics. The NEH heuristic is a creative insertion-type heuristic, while Jonson's rule is in the core of the CDS as well as in nearly every lower bounding scheme for the makespan objective.

There is a plethora of metaheuristics that have been implemented on the $m$-machine flow shop. In many cases, results have been impressive and nearoptimal solutions are produced for fairly large problems. This is done by creating and evaluating a large number of promising sequences even though substantive structural properties of the $m$-machine flow shop are still lacking.

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## Chapter 5 THE HYBRID FLOW SHOP


#### Abstract

In this chapter we organize the literature on the hybrid flow shop scheduling problem that has appeared since the late 1950's. We see a number of interesting and diverse industrial applications of this system, and find that the majority of research focuses on the makespan objective. Our coverage of results is exhaustive and categorized along concepts such as complexity, error-bound analyses, computational experiments and choice of objective. Several new results are included that have not appeared in the literature before. Surprisingly, existing research does not focus on the deterministic version of the problem alone, but also on the case of stochastic processing times.


### 5.1 Preliminaries

The flow shops we have considered up to now have consisted of a single processor per stage. In this chapter, we introduce production systems where at least one stage consists of more than 1 processor, and where every task can be executed on any one of the processors in the corresponding stage. The existence of multiple parallel machines in a stage offers significant processing flexibility but at the same time forces the decision maker to make wise use of the additional resources. The case where all processors at a stage are parallel identical machines has received significant attention in the literature. The system with $m$ stages and $k_{i}$ machines at stage $G_{i}(1=1,2, \ldots, m)$ is denoted $F\left(k_{1}, k_{2}, \ldots, k_{m}\right)$ and referred to as the hybrid, compound, flexible, or multiprocessor flow shop by various authors. We will adopt the term hybrid flow shop which reflects the fact that $F\left(k_{1}, k_{2}, \ldots, k_{m}\right)$ is a hybrid of two production systems: a single station with multiple parallel identical machines, and the simple flow shop Fm . If the number of stage is not specified by a subscript, we may add it in front, as in $\operatorname{Fm}(2,2, \ldots, 2)$. When there are only two or three stages, we will simplify the notation as usual by writing
$p_{1 j}:=a_{j}, p_{2 j}:=b_{j}, p_{3 j}:=c_{j}$ and referring to these tasks as $a$-tasks, $b$-tasks and $c$-tasks, respectively. All job parameters are assumed to be integers.

Most of the research on hybrid flow shops has focused on minimizing the makespan, so this chapter emphasizes that objective. In Section 5.7, several other objectives are briefly discussed.

### 5.1.1 Applications

Hybrid flow shops are almost always present when the production system consists of a fixed sequence of stages. If each stage of production is viewed as a single work center, then the entire system is approximated by a flow shop with a single processor per stage. Finer analysis of the production system can be achieved by explicitly considering all of the resources in each stage. The resulting system is a hybrid flow shop. Alternatively, traditional flow shops evolve to become hybrid by adding processors to alleviate the demands placed on bottleneck stages. Below we describe some specific real-world applications.

- We have come across a related manufacturing process of a major label manufacturer located in Painesville, Ohio. This company is a global manufacturer of labels that decorate and provide appropriate information on bottles and packages, stickers for medical products, etc. This manufacturing process consists of two major steps, coating and slitting. Coating ensures that the raw label sheets are single, double, or triple coated as appropriate to ensure the right sticking strength, the number of times that the label can be stuck/unstuck (i.e., in film processing envelopes), and good performance under various conditions of humidity, electrostatic fields, etc. Slitting is done by a number of slitters each equipped with several knives that can be adjusted to cut a roll into several smaller rolls of required lengths. The particular facility considered here is equipped with 4 coaters and 5 slitters, thus resulting to an $F(4,5)$ system.
- A 5-stage hybrid flow shop is described in Paul (1979) for an application from the glass-container industry. Here, a furnace feeds molten glass into a number of glass-forming machines that shape the finished product. The formed product is then passed on to the annealing stage which consists of multiple parallel identical kilns that cool and harden each product over the necessary amount of time. The annealed containers proceed to the inspection stage and then on to packing and storage. In this application the production process consists of five distinct stages: molting, forming, annealing, inspection, and packaging.
- The loading/unloading operations in cargo ships have been modeled by Bish et al. (2005) and Li and Vairaktarakis (2004). For every ship, a quay crane is available to unload containers from inside the ship to trucks available to transport each container to the appropriate position of the berth yard. The crane requires time $a_{i}$ to unload container $i$ from the ship and load it onto a truck. Since $a_{i}$ is usually small, the truck has to be available
throughout the $a_{i}$ units of time required to unload container $i$. Then, the truck takes the container to the appropriate berth location and returns to the quay; this takes time $b_{i}$. Upon unloading all containers from inside the ship, other containers are transported by trucks from the yard to the ship. Let $b_{j}$ be the time required by a truck to transport container $j$ for loading into the ship. Each truck is considered to be busy during time $c_{j}$ : the time required by the quay crane to load container $j$ from the truck onto the ship.
This application can be modeled as a 3 stage flow shop $F(1, k, 1)$ with multiprocessor tasks. Specifically, stages 1 and 3 have a single processor; the quay crane. Stage 2 consists of $k$ trucks. Unloading job $i$ has the form $\left(a_{i}, b_{i}, 0\right)$ while loading job $j$ has the form $\left(0, b_{j}, c_{j}\right)$. The $a$ - and $c$-tasks are multiprocessor because both the quay crane and the corresponding truck have to be available when loading/unloading a container. The makespan minimization objective is considered in Bish et al. (2005) and Li and Vairaktarakis (2004) who present heuristic algorithms for the general problem supported by worst-case analyses, and optimal algorithms for special cases of the problem.
- In all previous applications, the multiple processors of each stage are physically located in the same facility. Salvador (1973) described applications from the polymer, chemical, and petrochemical industries where jobs can practically be processed at any one of multiple plants at each stage of processing.


## 5.2 $F\left(k_{1}, k_{2}, \ldots, k_{m}\right) \| C_{\text {max }}$ : Basic Results

We first present such results as are known for the general makespan problem: a mathematical program, and some special properties that can be used in search algorithms. In the following section, we look at some special cases.

### 5.2.1 A Mixed-Integer Programming Formulation

Several mathematical programs have been proposed for the hybrid flow shop with makespan objective. We present one due to Guinet et al. (1996). Recalling that, for $i, j=1, \ldots, n, h=1, \ldots, k_{r}$, and $r=1, \ldots, m$ :

- $C_{r j}$ : the completion time of $J_{j}$ at $G_{r}$,
- $M_{h r}$ : machine $h$ at $G_{r}$,
we define the following variables:

$$
\begin{aligned}
& x_{i j h r}= \begin{cases}1 & \text { if } J_{i} \text { immediately precedes } J_{j} \text { on } M_{h r} \\
0 & \text { otherwise }\end{cases} \\
& x_{0 j h r}= \begin{cases}1 & \text { if } J_{j} \text { is the first job processed on } M_{h r} \\
0 & \text { otherwise }\end{cases}
\end{aligned}
$$

$$
x_{i 0 h r}= \begin{cases}1 & \text { if } J_{i} \text { is the last job processed on } M_{h r} \\ 0 & \text { otherwise }\end{cases}
$$

Using these variables, the following mixed-integer program minimizes the makespan, $Z$ :

$$
\begin{align*}
& \text { HFS minimize } Z \\
& \text { subject to } \quad \sum_{i=1}^{n} \sum_{h=1}^{k_{r}} x_{i j h r}=1 \quad \forall j, r  \tag{5.1}\\
& \sum_{j=0}^{n} x_{i j h r} \leq 1 \quad \forall i, h, r  \tag{5.2}\\
& \sum_{i=0, i \neq j}^{n} x_{i j h r}-\sum_{j=0, i \neq j}^{n} x_{i j h r}=0 \quad \forall i, h, r  \tag{5.3}\\
& C_{r i}+\sum_{h=1}^{k_{r}} x_{i j h r} p_{r j}+\left(\sum_{h=1}^{k_{r}} x_{i j h r}-1\right) M \leq C_{r j} \forall i, j, r  \tag{5.4}\\
& C_{r j} \geq C_{r-1, j}+p_{r j} \forall j, r  \tag{5.5}\\
& 0<C_{r j} \leq Z \quad \forall j, r  \tag{5.6}\\
& x_{i j h r} \in\{0,1\} \quad \forall i, j, h, r \tag{5.7}
\end{align*}
$$

Constraints (5.1) ensure that, in each stage, every job is processed by only one processor. The fact that no machine can process more than one job at a time is captured in (5.2). Constraints (5.3) ensure that, in each stage and each machine within a stage, every job has a single predecessor (which is a dummy job for the first job of each machine). Job completion times are computed via (5.4), where $M$ is an arbitrary large number. Constraints (5.5) enforce the precedence constraints. The makespan value is captured via (5.6).

### 5.2.2 Some Useful Observations

Several preliminary observations will be helpful in the following development.

## Property 5.1: Reversibility

Consider the "reverse" production problem to the one just formulated. That is, suppose that the jobs require processing in stages $m, m-1, \ldots, 2,1$ in that order. The processing requirement of each job in each stage remains the same. For the makespan minimization objective the hybrid flow shop and its reverse counterpart have a mirror image relationship. Specifically, if $S$ is a schedule for the hybrid flow shop that has makespan $C_{S}$, and the completion time of $J_{j}$ at $G_{r}$ is $C_{r j}$, then one can create an equivalent schedule $S_{R}$ for the reverse problem, as follows: Assign all tasks to be processed on the same machines as in $S$, but now the completion time of $J_{j}$ at $G_{r}$ is changed to $C_{r j}^{\prime}=C_{S}-C_{r j}+p_{r j}$. By symmetry, the flow shop constraints for the reverse problem are satisfied in $S_{R}$. Moreover, the makespan $C_{R}$ of $S_{R}$ is $C_{R}=C_{S}$. This reversibility property is helpful in analyses because it allows one to work with the reverse system and draw conclusions for the original hybrid flow shop.

## Property 5.2: FCFS Scheduling for the Last Stage

A simple interchange argument shows that, for a given assignment of tasks
in stages $G_{1}, G_{2}, \ldots, G_{m-1}$, it is optimal for problem $F\left(k_{1}, k_{2}, \ldots, k_{m}\right) \| C_{\max }$ to schedule the $G_{m}$-tasks in nondecreasing order of completion times of the corresponding $G_{m-1}$-tasks, or equivalently, in first-come-first-served(FCFS) order.

One can apply Property 5.2 to the reverse instance of a problem. That is, given a partial schedule for stages $m, m-1, \ldots, 2$, the $G_{1}$-tasks can be scheduled optimally (for $F\left(k_{1}, k_{2}, \ldots, k_{m}\right) \| C_{\max }$ ) by ordering them in nonincreasing order of $C_{2 j}$ values, and subsequently scheduling them (in this order) on a machine at $G_{1}$ that allows the latest (or, in reverse time, earliest) possible start.

## Approximation using Merged Machines

The technique of merging machines, also introduced in Sect. 2 of Chap. 8, is used to approximate the more complex hybrid by a simple $m$-machine flow shop. Suppose that at each stage $G_{r}$ we replace the machines $M_{i r}(i=$ $1,2, \ldots, k_{r}$ ) by a single dummy machine $M_{r}$, that will do the work $k_{r}$ times faster. To reflect this, we replace the processing time requirement of $p_{r j}$ for $J_{j}$ at $G_{r}$, by the requirement of $p_{r j} / k_{r}$ time units on $M_{r}(j=1,2, \ldots, n, r=$ $1,2, \ldots, m)$. Its as though, with the original machine group, each job can be divided into $k_{r}$ equal parts which are simultaneously processed on all machines. As we will see later, this simple flow shop has proven instrumental in identifying lower bounds for the makespan minimization objective in the hybrid shop whose capacity it closely approximates.

### 5.3 The Simplest Hybrid Systems

In most of this section we study the 2-stage hybrid flow shop with a single processor in one of the two stages. According to our notation such a system is denoted as either $F(k, 1)$ or $F(1, k)$. By the reversibility property, these two systems are equivalent when minimizing makespan; we will speak of either interchangeably.

We begin with some elementary results that arise when the second of two stages has unlimited servers. If, on the other hand, the number of servers is finite, the following subsection establishes the NP-completeness of even the simplest hybrid flow shop. After that, some approximation results are given. Dominance properties and lower bounds are presented in the next subsection. Then, we present some key error bound results that have appeared in the literature followed by comments on computational experiments. At the end of the section we consider the system with $m-1$ single-processor stages.

### 5.3.1 Some Results for $\boldsymbol{F}(k, n)$

We first mention some very simple results that arise when the second stage is nonbottleneck. A nonbottleneck station has such abundant facilities that
no job ever has to wait. There may be an unlimited number ( $n$ or more) of machines, or one large machine that can simultaneously process any number of jobs (e.g., an oven for heat treatment), or $G_{2}$ may simply be a place where the jobs can be left for paint to dry, or some other chemical process to take place. We denote any of these possibilities as an $n$-machine stage.

Nonbottleneck stages can simplify analysis. We give here, without proof, the most elementary results.

- For $F(1, n) \| C_{\max }, ~ \searrow b_{j}$ is optimal (Gupta and Tunc, 1991).
- For $F(1, n) \| L_{\max }$, or $T_{\max }, ~ \searrow\left(b_{j}-d_{j}\right)$ is optimal.
- For $F(1, n) \| \Sigma w_{j} C_{j}, \nearrow a_{j} / w_{j}$ is optimal.
- For $F(k, n) \| \Sigma C_{j}, \nearrow a_{j}$ is optimal,
where $\nearrow a_{j}$ (that is, SPT) on parallel processors, means: assign jobs in SPT order to the first available machine at $G_{1}$.


### 5.3.2 Complexity of $F(2,1) \mid($ pmtn $) \mid C_{\max }$

Without preemption, this problem is clearly NP-complete; the single-stage problem of minimizing makespan on two parallel identical machines is already NP-complete. With preemption, its complexity status remained open for some time, until it was finally established in the following result.

Theorem 5.1 (Hoogeveen, Lenstra and Veltman, 1996)
$F(2,1)|p m t n| C_{\max }$ is strongly NP-complete.
Proof Outline: Here, we present only the authors' proof that the problem is ordinary NP-complete. They also give a reduction from the 3-Partition problem to prove strong NP-completeness, which we omit.

The reduction is from the NP-complete problem:

## PARTITION

INSTANCE: An integer $V$, and $k$ positive integers $v_{i}: i \in \mathcal{T}=\{1,2, \ldots, k\}$ such that $\Sigma_{i \in \mathcal{T}} v_{i}=2 V$.
QUESTION: Can $\mathcal{T}$ be partitioned into disjoint subsets $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ such that $\Sigma_{i \in \mathcal{S}_{1}} v_{i}=\Sigma_{i \in \mathcal{S}_{2}} v_{i}=V ?$

Given an instance of Partition, construct an instance of the decision problem $F(2,1)|p m t n| C_{\max } \leq B$ ? as follows:

- $n:=k+4$;
- $\left\langle a_{j}, b_{j}\right\rangle:=\left\langle\alpha v_{j}, v_{j}\right\rangle, \alpha>1, j=1,2, \ldots, k$;
- $\left\langle a_{k+1}, b_{k+1}\right\rangle:=<0, \alpha V>$
- $<a_{k+2}, b_{k+2}>:=<\alpha V+V, \alpha V>$
- $<a_{k+3}, b_{k+3}>:=<\alpha V+2 V, 0>$
- $\left\langle a_{k+4}, b_{k+4}>:=<V, 0>\right.$
- $B:=2 V(\alpha+1)$.

Now, suppose a partition of $\mathcal{T}$ exists, and let $\mathcal{S}_{1}$ be a set of jobs (technically, the set of indices of those jobs) with $\Sigma_{j \in \mathcal{S}_{1}} b_{j}=\Sigma_{i \in \mathcal{S}_{1}} v_{i}=V$. In this case, the schedule in Fig. 5.1 has length $2(\alpha V+V)$, which is the minimum possible since there is no idle time on any machine. Note that, in the figure, we denote by $\mathcal{S}_{i}, i=1,2$ the jobs in $\mathcal{S}_{i}$ sequenced in any order. Also, observe that this schedule is achieved without preemption, but obviously could not be improved with preemption.

To prove the reverse implication, suppose that a schedule of length $2 V(\alpha+1)$ exists. Property 2 allows us to apply the FCFS rule on $M_{2}$ without preemption. As we have seen, a schedule of this length must have all machines fully occupied over the scheduled interval. It follows that $J_{k+1}$ must be first on $M_{2}$, since only it, requiring no processing on stage $G_{1}$, can start on $M_{2}$ at time zero. Similarly, $J_{k+3}$ and $J_{k+4}$ must be last on $M_{11}$ and $M_{21}$ since, requiring no time on $M_{2}$, they can occupy $G_{1}$ to the end. $J_{k+2}$ will not fit on $M_{11}$. To see that it must be scheduled first on $M_{21}$, assume that it starts at time $\Delta$ for $\Delta \geq 0$. Then $M_{2}$ will have to process the jobs that were completed on $G_{1}$ before the start of $a_{k+2}$ and $a_{k+3}$, and do it before $b_{k+2}$ starts, that is, in the interval $[\alpha V, \alpha V+V+\Delta]$, of length $V+\Delta$. The total time available in $G_{1}$ prior to $a_{k+1}$ and $a_{k+2}$ is $\alpha V+\Delta$. Even if the $a$-tasks processed there are all complete, the corresponding $b$-tasks will require $V+\Delta / \alpha$ time (because $b_{j}=a_{j} / \alpha$ for $\left.j=1, \ldots, k\right)$, which will underfill the available time $V+\Delta$ and leave idle time on $M_{2}$, unless $\Delta=0$. If any $a$-task is incomplete, i.e. preempted, the corresponding $b$-task cannot be processed until later, again leaving idle time on $M_{2}$.


Fig. 5.1 Optimal schedule of constructed instance of $F(2,1)|p m t n| C_{\text {max }}$
We conclude that $S^{\star}$ must be as in Fig. 5.1 and the subsets $\mathcal{S}_{1}, \mathcal{S}_{2}$ define a partition. This completes the demonstration that a partition exists if and only if a schedule of length $2 V(\alpha+1)$ exists.

Since $F(2,1)$ and $F(1,2)$ (equivalent, by reversibility) are the simplest possible 2-stage flow shops with a multiprocessor stage, Theorem 5.1 implies that any flow shop system with more than one machine in a stage is strongly NP-complete as well. Therefore, all the scheduling problems with the objective of minimizing makespan that are presented in the rest of this chapter are strongly NP-complete.

### 5.3.3 Approximation Results

Approximation algorithms for $F(k, 1)$ use dispatching rules to schedule tasks on the multiprocessor stage. Prior to making these assignments, it is necessary to sequence the tasks in the order they will be scheduled. For now, we defer consideration of how this sequence $S$ is specified; just assume we have it.

The most popular dispatching rule for $F(k, 1)$ is the First Available Machine (FAM) rule, where the next task in the given sequence $S$ is scheduled to the machine that first completes all the tasks previously assigned to it. The symmetry between $F(k, 1)$ and $F(1, k)$ suggests that for the latter system we might use the mirror image of FAM, which is referred to as the Last Busy Machine ( $L B M$ ) rule. The LBM rule schedules the tasks in reverse order of $S$ to the second stage of $F(1, k)$, as follows.

## LBM rule

1. Set $t_{i}:=T$ where $T=\Sigma_{j}\left(a_{j}+b_{j}\right)$ for $i=1, \ldots, k$.
2. Let $J_{j}$ be the last job in $S$ that has not been scheduled yet on $G_{2}$, and $M_{h}$ a machine with $t_{h}=\max _{i} t_{i}$. Schedule $J_{j}$ on $M_{h}$ to finish at time $t_{h}$.
3. Set $t_{h}:=t_{h}-b_{j}$ and $S:=S-\{j\}$. If $S \neq \phi$ then go to Step 2; else, stop.

Evidently, LBM is the mirror image of FAM where the start time 0 is replaced by the common finish time $T$ for all identical processors.

We can now state approximation results for $F(1, k) \| C_{\max }$. We begin with a simple heuristic that does not use the LBM rule.

Heuristic $\mathbf{H}_{\mathbf{S}}$ for $F(1, k) \| C_{\text {max }}$

1. Specify sequence $S$ of the $n$ jobs.
2. Schedule $a$-tasks according to $S$.
3. Apply the FCFS rule to the $b$-tasks of $S$.
4. Compute the makespan $C_{S}$ of the resulting schedule $S_{S}$.

Theorem 5.2 (Sriskandarajah and Sethi, 1989)
For $F(1, k) \| C_{\max }$, if $C^{\star}$ is the minimal makespan:
$C_{S} / C^{\star} \leq 3-1 / k$, and this bound is tight.
In this bound, $(2-1 / k) C^{\star}$ is contributed by the FCFS rule for $b$-tasks, and $C^{\star}$ is contributed by the $a$-tasks scheduled in Step 2 , since $\Sigma_{j} a_{j} \leq C^{\star}$. Related results are presented by Chen (1995) that use an arbitrary sequence in Step 2 of $\mathrm{H}_{\mathrm{S}}$. For instance, if Step 3 uses the LBM rule for $b$-tasks, the worst case error bound remains tight at $3-1 / k$. In fact, Chen (1995) considers the set of heuristics that replace Step 3 of $\mathrm{H}_{\mathrm{S}}$ by dispatching rules where
i) all machines at $G_{2}$ finish at the same time,
ii) no idle time exists between $b$-tasks, and
iii) the completion time of task $b^{\prime}$ with smallest start time is less than or equal to the start time of all $G_{2}$-machines (other than the one that executes $\left.b^{\prime}\right)$.

Chen (1995) shows that any such heuristic has tight worst case error bound equal to $3-1 / k$. For ease of presentation, this observation is translated here in light of the reversibility property; in his paper Chen (1995) states the result for the system $F(k, 1)$.

Heuristic class $\mathcal{H}_{\mathbf{C}}$ for $F(1, k) \| C_{\text {max }}$

1. Specify sequence $S$ of the $n$ jobs.
2. Schedule $b$-tasks using any list scheduling rule of ordered list $S$ that satisfies conditions i) - iii).
3. Schedule $a$-tasks in nondecreasing order of start times of the corresponding $b$-tasks.
4. Compute the makespan $C_{C}$ of the resulting schedule $S_{C}$.

Note that every heuristic $\mathrm{H}_{\mathrm{C}} \in \mathcal{H}_{\mathrm{C}}$ uses the sequence $S$ to schedule tasks in the multiprocessor stage, unlike $\mathrm{H}_{\mathrm{S}}$ that uses $S$ for the single processor stage. According to Property 5.2 (the reversibility property), the single processor schedule is the best possible for the given assignment of $b$-tasks, for $H_{C} \in \mathcal{H}_{\mathrm{C}}$. Therefore, class $\mathcal{H}_{\mathrm{C}}$ and heuristic $\mathrm{H}_{\mathrm{S}}$ are conceptually very different. The following result is known for the class $\mathcal{H}_{\mathrm{C}}$ of heuristics.

Theorem 5.3 (Chen, 1995)
For $F(1, k) \| C_{\max }$, and for every heuristic $H_{C} \in \mathcal{H}_{C}$ we have:
$C_{C} / C^{\star} \leq 2$, and this bound is tight.
One might think that the error bound of 2 can be further improved by appropriately selecting sequence $S$. Chen (1995) showed that if $S=J R(\underline{a}, \underline{b})$ the bound of 2 remains tight. This means that sequence $J R(\underline{a}, \underline{b})$ can perform as badly as any arbitrary sequence. Theorem 5.3 was first established in Sriskandarajah and Sethi (1989) for the special case $k=2$; a looser bound was obtained for $k \geq 3$.

### 5.3.4 Dominance Properties, Lower Bounds and Computational Experiments for $F(k, 1) \| C_{\max }$

The literature indicates that researchers have tried hard to discover some type of Johnson order as the building block of optimal or near optimal solutions. This is indicated by the fact that the sequence $S$ used to dispatch tasks to machines often has features similar to the Johnson sequence. However, Johnson-type dominance properties are not possible. Still, the following has the flavor of JR, since it puts earlier the job that is smaller at $G_{1}$ and longer at $G_{2}$ :

Property 5.3 (Gupta, Hariri and Potts, 1997)
If the $a$-tasks of $J_{i}$ and $J_{j}$ are scheduled on the same $G_{1}$-processor, and if $a_{i} \leq a_{j}$ and $b_{i} \geq b_{j}$, then there is an optimal schedule for $F(k, 1) \| C_{\max }$ where $a_{i}$ precedes $a_{j}$.

Properties 5.2 and 5.3 are often used in branch-and-bound implementations to solve $F(k, 1) \| C_{\max }$. The lower bounds used determine to a large extent the efficiency of these algorithms. Such bounds are included in Gupta (1988), Gupta et al. (1997), Lee and Vairaktarakis (1994) and Haouari and M'Hallah (1997). The next two lower bounds are drawn from Gupta et al. (1997).

To obtain the first lower bound $L B_{1}$ for $F(k, 1) \| C_{\max }$, observe that for a given schedule, the completion times of $a$-tasks act as release dates for $b$ -tasks. Since the completion time of $a_{j}$ is $C_{1 j} \geq a_{j}$, a lower bound can be obtained by scheduling $b$-tasks on a single processor subject to release dates $r_{j}=a_{j}$ for $j=1, \ldots, n$. This problem is optimally solved if we schedule the $b$-tasks in $\nearrow r_{j}$ order. Let $\pi$ be the resulting order. Then,

$$
L B_{1}=\max _{j=1, \ldots, n}\left\{a_{\pi(j)}+\sum_{i=j}^{n} b_{\pi(i)}\right\}
$$

A better lower bound is obtained if we know a subset of jobs, whose indices make up the set $\mathcal{N}_{0} \subset \mathcal{N}=\{1,2, \ldots, n\}$, containing at least one job assigned to every first-stage machine in an optimal schedule $S^{\star}$. Let (1), $(2), \ldots,(k) \in \mathcal{N}_{0}$ index jobs that are assigned to machines $M_{11}, M_{21}, \ldots, M_{k 1}$ respectively. We assume that these jobs have the additional property that $(h)$ is the last job index in $\mathcal{N}_{0}$ assigned to $M_{h 1}$ and that the completion times, $C_{(h)}$, of $a_{(h)}$ in $S^{\star}$ are such that $C_{(1)} \leq C_{(2)} \leq \ldots \leq C_{(k)}$. Furthermore, let $\mathcal{N}_{0}^{\prime}$ be the set of indices of the jobs preceding $J_{(1)}, J_{(2)}, \ldots, J_{(k)}$ on the $G_{1}$-machines that are not in $\mathcal{N}_{0}$, and $\overline{\mathcal{N}}_{0}=\mathcal{N}-\mathcal{N}_{0}-\mathcal{N}_{0}^{\prime}$. Then, in $S^{\star}$ we have

$$
C^{\star} \geq C_{(1)}+\sum_{r=1}^{k} b_{(r)}+\sum_{i \in \bar{N}_{0}} b_{i}
$$

and

$$
C^{\star} \geq C_{(h)}+\sum_{r=h}^{k} b_{(r)} \quad \text { for } \quad h=2, \ldots, k
$$

because without loss of generality $b$-tasks are assigned in $G_{2}$-machines according to the FCFS rule (according to Property 5.2). Averaging these $k$ inequalities, we get

$$
C^{\star} \geq \sum_{h=1}^{k} C_{(h)} / k+\sum_{h=1}^{k} \sum_{r=h}^{k} b_{(r)} / k+\sum_{i \in \overline{\mathcal{N}}_{0}} b_{i} / k
$$

which yields

$$
C^{\star} \geq \sum_{i \in \mathcal{N}_{0}} a_{i} / k+\sum_{i \in \mathcal{N}_{0}^{\prime}} a_{i} / k+\sum_{r=1}^{k} r b_{(r)} / k+\sum_{i \in \overline{\mathcal{N}}_{0}} b_{i} / k .
$$

If $J_{[1]}, \ldots, J_{\left[n_{0}\right]}$ is an ordering of jobs in $\mathcal{N}_{0}$ such that $b_{[1]} \leq \ldots \leq b_{\left[n_{0}\right]}$ we get that $C^{\star} \geq L B_{2}\left(\mathcal{N}_{0}\right)$, where

$$
L B_{2}\left(\mathcal{N}_{0}\right)=\left\lceil\sum_{i \in \mathcal{N}_{0}} a_{i} / k+\sum_{i \in \mathcal{N}-\mathcal{N}_{0}} \min \left\{a_{i}, b_{i}\right\} / k+\sum_{r=1}^{k}(k-r+1) b_{[r]} / k\right\rceil
$$

where we round up because when the processing times are integer, $C^{\star}$ is also an integer. A trivial implementation of $L B_{2}$ is to assume that $\mathcal{N}_{0}=$
$\{1, \ldots, n\}$. A tighter implementation exploits the observation that, in $S^{\star}$, all $G_{1}$-machines are processing $a$-tasks at least until time

$$
T=\sum_{j=1}^{n} a_{j} / k-(1-1 / k) \cdot \max _{j=1, \ldots, n} a_{j}
$$

since otherwise the FAM rule could be used to start some $a$-tasks earlier. Then, one can initialize $\mathcal{N}_{0}=\{1, \ldots, n\}$ (so $\mathcal{N}-\mathcal{N}_{0}:=\phi$ ) and iteratively select a job, $J_{v}, v \in \mathcal{N}_{0}$, with $b_{v}=\min _{j \in \mathcal{N}_{0}} b_{j}$. If $a_{v}+\Sigma_{i \in \mathcal{N}-\mathcal{N}_{0}} a_{i}<T$, then let $\mathcal{N}-\mathcal{N}_{0}:=\mathcal{N}-\mathcal{N}_{0}+\{v\}$ and $\mathcal{N}_{0}:=\mathcal{N}_{0}-\{v\}$, and repeat; otherwise terminate. The resulting set $\mathcal{N}_{0}$ yields the improved lower bound $L B_{2}\left(\mathcal{N}_{0}\right)$.

Gupta et al. (1977) perform a computational experiment that shows that, in most cases, the trivial lower bound

$$
L B=\max \left\{\min _{1 \leq j \leq n} a_{j}+\sum_{j=1}^{n} b_{j},\left\lceil\sum_{j=1}^{n} a_{j} / k\right\rceil+\min _{1 \leq j \leq n} b_{j}\right\}
$$

outperforms the more sophisticated bounds presented above. They also experiment with a number of heuristics where
i) a sequence $S$ is chosen,
ii) $a$-tasks are scheduled according to the FAM rule, and
iii) $b$-tasks are scheduled by the FCFS rule (see Property 5.2).

For Step i), they found that $S=J R(\underline{a} / k, \underline{b})$ consistently outperformed $J R(\underline{a}, \underline{b})$ in their experiments. They also demonstrated that solving the equivalent reverse problem (described in Property 5.1) often yields a better schedule than the one obtained for the original problem. Further, they tested a number of improvements for Step ii), and one of their algorithms (which is based on pairwise interchanges) was found to be optimal for most problem instances and near optimal for the rest, compiling an average relative gap of about $0.1 \%$. This indicates that problem $F(k, 1) \| C_{\max }$ is well solved for all practical purposes. A limited experiment for the same problem is provided in Chen (1995).

### 5.3.5 $\operatorname{Fm}(1,1, \ldots, 1, k) \| C_{\text {max }}$

For this problem Sriskandarajah and Sethi (1989) extend the heuristic $H_{S}$ presented in Sect. 5.3.3 to the $\operatorname{Fm}(1,1, \ldots, 1, k)$ system. Specifically, in Step 2 of $H_{S}$ the next task in $S$ is scheduled to all single-processor stages, not just the first. It is shown that the error bound is tight and equals

$$
C_{S} / C^{\star} \leq m+1-\frac{1}{k}
$$

## $5.4 F\left(k_{1}, k_{2}\right) \| C_{\text {max }}$

Before considering the general two-stage hybrid flow shop, we look at the case where $k_{1}=k_{2}$.

### 5.4.1 $F(k, k) \| C_{\text {max }}$

When each of a flow shop's two stages has the same number of processors, we can view the system as $k$ parallel flow lines. Error analyses that view $F m(k, \ldots, k)$ as such are presented in Sriskandarajah and Sethi (1989). Since simple flow shops in parallel are treated in Chap. 4, we relegate these results to that chapter. Here, we consider results that do not treat $F(k, k)$ as a system of 2-machine flow lines. Consider the following heuristic due to Langston (1987).

Heuristic $\mathbf{H}_{\mathbf{L}}$ for $F(k, k) \| C_{\text {max }}$

1. $S:=\searrow b_{j}$.
2. Apply the FAM rule to the a-tasks in the sequence $S$.
3. Apply the FCFS rule to the b-tasks.
4. Compute the makespan $C_{L}$ of the resulting schedule $S_{L}$.

Theorem 5.4 (Langston, 1987)
For $F(k, k) \| C_{\max }$, if $C^{\star}$ is the optimal makespan:
$C_{L} / C^{\star} \leq 2$, and this bound is tight.
Langston (1987) also investigates a heuristic $\mathrm{H}_{\mathrm{L}^{\prime}}$ that only differs from $\mathrm{H}_{\mathrm{L}}$ in that $S=\nearrow a_{j}$. One might assume that, due to the reversibility property, $\mathrm{H}_{\mathrm{L}^{\prime}}$ should exhibit the same worst-case performance ratio. To see that this is not the case, consider an instance of $F(k, k) \| C_{\max }$ that consists of $n=k_{0} k+1$ (for $k_{0}>1$, integer) jobs with the following parameters: $\left(a_{1}, b_{1}\right)=\left(x, k_{0} x\right)$ for $x>0,\left(a_{2}, b_{2}\right)=\ldots=\left(a_{k+1}, b_{k+1}\right)=\left(x, \frac{1}{2} k_{0} x\right)$, and $\left(a_{j}, b_{j}\right)=(x, 0)$ for $j=k+2, \ldots, n$. Note that every permutation of these jobs qualifies as the sequence $S$ in $\mathrm{H}_{\mathrm{L}^{\prime}}$. If we choose $S=(1,2, \ldots, n)$, we obtain an optimal schedule with $C^{\star}=\left(k_{0}+1\right) x$. However, for $S=(n, n-1, \ldots, 1), C_{L^{\prime}}=2.5 k_{0} x$ and hence the ratio $C_{L^{\prime}} / C^{\star}$ approaches the value 2.5 asymptotically. Note that the LPT order of the $b$-tasks in the original problem corresponds to the SPT order of $a$-tasks in the reverse problem. The above observation indicates that, for the purpose of worst-case analysis, the original problem is not completely symmetric to its reverse counterpart. Moreover, the analysis of heuristics $\mathrm{H}_{\mathrm{L}}, \mathrm{H}_{\mathrm{L}^{\prime}}$ indicates that a Johnson-type sequence of the jobs might yield a better heuristic.

### 5.4.2 $F\left(k_{1}, k_{2}\right) \| C_{\text {max }}$

We now consider the general two-stage hybrid flow shop.

## Lower Bounds on $C_{\text {max }}$

The next lemma provides an intuitive lower bound for $F\left(k_{1}, k_{2}\right) \| C_{\max }$ often used in algorithmic analysis. This lower bound uses the "merged machines" shop MMS (where multiple machines at a station are replaced by a single
faster one, as described in Sect. 1) and was first presented in Buten and Shen (1973). Simpler proofs are provided in Lee and Vairaktarakis (1994) and Chen (1995). The proof given below is from Lee and Vairaktarakis (1994).

Lemma 5.1 (Buten and Shen, 1973) If $C_{M}$ is the makespan of the optimal schedule $S_{M}=J R\left(\underline{a} / k_{1}, \underline{b} / k_{2}\right)$ for the MMS problem, and if $C^{\star}$ is the optimal makespan value for $F\left(k_{1}, k_{2}\right) \| C_{\max }$, then, $C_{M} \leq C^{\star}$.

Proof: Consider an optimal solution $S^{\star}$ for $F\left(k_{1}, k_{2}\right) \| C_{\max }$. Let $S=$ $\left(J_{1}, J_{2}, \ldots, J_{n}\right)$ be the sequence of jobs reindexed in nondecreasing order of completion times of the $a$-tasks in $S^{\star}$. Consider the partial schedule of $S^{\star}$ consisting of the first $i a$-tasks of $S$ and the last $n-i+1 b$-tasks of $S$. Since in $S^{\star}$ the last $n-i+1 b$-tasks of $S$ start no earlier than the completion time of the first $i a$-tasks of $S$ (due to the flow shop constraints), we have that

$$
C^{\star} \geq \sum_{j=1}^{i} a_{j} / k_{1}+\sum_{j=i}^{n} b_{j} / k_{2} \text { for every } i=1, \ldots, n
$$

For the MMS problem (where the processing time requirements for $J_{j}$ are $<a_{j} / k_{1}, b_{j} / k_{2}>$ ), schedule the jobs according to sequence $S$. Let $C_{S}$ be the resulting makespan. Then it is clear that

$$
C_{S}=\sum_{j=1}^{i_{0}} a_{j} / k_{1}+\sum_{j=i_{0}}^{n} b_{j} / k_{2}
$$

where $J_{i_{0}}$ is the last job whose $b$-task starts immediately after the completion of the corresponding $a$-task (note that such a task always exists, since $J_{1}$ satisfies this property).

Combining the last two expressions we get that $C_{S} \leq C^{\star}$. However, the sequence $S$ is not necessarily optimal for the MMS system and hence $C_{M} \leq C_{S}$. The last two relations establish that $C_{M} \leq C^{\star}$.

In the optimal solution $S^{\star}$ for $F\left(k_{1}, k_{2}\right) \| C_{\max }, G_{2}$-machines incur idle time due to the flow shop constraints for the first job of each $G_{1}$-machine. The lower bound in Lemma 5.1 can be improved by carefully calculating the inserted idle time due to these jobs. We distinguish the following two cases:

Case i: $k_{1} \geq k_{2}$
Suppose that $S=\nearrow a_{j}$. Using $S$, it is clear that because of the flow shop constraints there will be in $G_{2}$ a machine with initial idle time no less than $a_{1}$, a machine with idle time no less than $a_{2}, \ldots$, a machine with idle time no less than $a_{k_{2}}$. As a result, $C^{\star}$ cannot be less than the average workload of the machines at $G_{2}$ plus the necessary initial idle time there, i.e.

$$
C^{\star} \geq\left[\sum_{k=1}^{k_{2}} a_{k}+\sum_{j=1}^{n} b_{j}\right] / k_{2}
$$

Case ii: $k_{1}<k_{2}$
Similarly, in this case there must be at $G_{2}$ a machine with idle time no less than $a_{1}, \ldots$, and a machine with idle time no less than $a_{k_{1}}$. The remaining $k_{2}-k_{1} G_{2}$-machines cannot begin operating until at least two tasks have
been processed at $G_{1}$. Since the shortest $k_{1} a$-tasks are already considered, there must exist at $G_{2}$ a machine with idle time no less than $a_{k_{1}+1}+a_{1}, \ldots$ , a machine with idle time no less than $a_{k_{2}}+a_{1}$. Therefore,

$$
C^{\star} \geq\left[\sum_{k=1}^{k_{1}} a_{k}+\left(k_{2}-k_{1}\right) a_{1}+\sum_{j=1}^{n} b_{j}\right] / k_{2}
$$

To simplify, we define $A_{k}\left[B_{k}\right]$ as the sum of the $k$ shortest $a$-tasks [ $b$-tasks]. Then we have the following lower bounds on $C^{\star}$ :

$$
\begin{array}{ll}
L B_{1}=\max \left\{\left[A_{k_{2}}+B_{n}\right] / k_{2}, A_{1}+\max _{j} b_{j}\right\}, & \text { if } k_{1} \geq k_{2} \\
L B_{2}=\max \left\{\left[A_{k_{1}}+\left(k_{2}-k_{1}\right) A_{1}+B_{n}\right] / k_{2}, A_{1}+\max _{j} b_{j}\right\}, & \text { if } k_{1}<k_{2}
\end{array}
$$

The reversibility property can be employed to obtain the symmetric lower bounds

$$
\begin{array}{ll}
L B_{3}=\max \left\{\left[B_{k_{1}}+\left(k_{1}-k_{2}\right) B_{1}+A_{n}\right] / k_{1}, B_{1}+\max _{j} a_{j}\right\}, & \text { if } k_{1} \geq k_{2} \\
L B_{4} & =\max \left\{\left[B_{k_{2}}+A_{n}\right] / k_{1}, B_{1}+\max _{j} a_{j}\right\}, \\
\text { if } k_{1}<k_{2}
\end{array}
$$

It is not difficult to see that none of $C_{L B}, L B_{1}, L B_{2}, L B_{3}, L B_{4}$ dominates the others.

## Heuristics and Worst-Case Error Bound Analyses

In what follows we present heuristic algorithms for the 2 stage hybrid flowshop with an arbitrary number of processors per stage. Error bound analyses of heuristics for $F\left(k_{1}, k_{2}\right) \| C_{\max }$ has attracted significant attention since the seminal article of Buten and Shen (1973). Many of these results are organized and presented here followed by a literature survey of heuristics that exhibit excellent performance. Most heuristics that have appeared in the literature follow the following four generic steps.

Heuristic $\mathbf{H}$ for $F\left(k_{1}, k_{2}\right) \| C_{\text {max }}$

1. Specify a sequence $S$ for the $n$ jobs.
2. Apply the FAM rule to the $a$-tasks in the sequence $S$.
3. Apply the FCFS rule to the $b$-tasks.
4. Compute the makespan $C_{H}$ of the resulting schedule $S_{H}$.

Note that, according to Property 5.2, Step 3 results in an optimal allocation of $b$-tasks. Hence, the important decision differentiating the various heuristics is made in Step 1. The complexity of Steps 2 through 4 of H is $O(n)$. If $S$ is chosen arbitrarily, then $O(n)$ is also the complexity of H . In this case, we have the following result.

Theorem 5.5 (Buten and Shen, 1973)
For $F\left(k_{1}, k_{2}\right) \| C_{\max }$, using an arbitrary sequence $S$ :
$C_{H} / C^{\star} \leq 3-1 / \max \left\{k_{1}, k_{2}\right\}$ and this bound is tight.

Except for the actual value of the error, the importance of Theorem 5.5 lies in the fact that the heuristic error is bounded for any sequence $S$, as opposed to the possibility of the error being arbitrarily large for large values of $k_{1}$ or $k_{2}$. Knowing this fact, one would like to carefully choose the sequence $S$ so as to achieve improved worst-case performance guarantee. The following result provides such improvement.
Theorem 5.6 (Lee and Vairaktarakis, 1994 and Chen, 1995)
For $F\left(k_{1}, k_{2}\right) \| C_{\max }$, using $S=J R\left(\underline{a} / k_{1}, \underline{b} / k_{2}\right)$ :
$C_{H} / C^{\star} \leq 2-1 / \max \left\{k_{1}, k_{2}\right\}$, and this bound is tight.
When $a_{j}+b_{j} \leq C_{M}$ for all $J_{j}$, the above result was proved in Buten and Shen (1973). Recall that $C_{M}$ is the makespan of the merged 2-machine flow shop corresponding to $F\left(k_{1}, k_{2}\right)$. Therefore, the results of Buten and Shen hold for problem instances where no individual job requires more processing time than $C_{M}$. It turns out that, as Theorem 5.6 says, the $2-1 / \max \left\{k_{1}, k_{2}\right\}$ bound holds even when the loading constraint $a_{j}+b_{j} \leq C_{M}$ is violated. The difficulty in proving the $2-1 / \max \left\{k_{1}, k_{2}\right\}$ bound without a loading constraint is due to the fact that Step 3 of H , although optimal for stage 2 tasks, significantly complicates mathematical analysis. The following variation of H simplifies analysis, and guarantees the $2-1 / \max \left\{k_{1}, k_{2}\right\}$ bound. We call it $\mathrm{H}_{\mathrm{U}}$ since, once the $b$-tasks have been assigned to machines in Step 3, they are unrestricted; i.e., they are scheduled FCFS.

Heuristic $\mathbf{H}_{\mathbf{U}}$ for $F\left(k_{1}, k_{2}\right) \| C_{\text {max }}$

1. Let $S=J R\left(\underline{a} / k_{1}, \underline{b} / k_{2}\right)$.
2. Apply the FAM rule to the $a$-tasks of $S$.
3. Apply the LBM rule to the $b$-tasks of $S$.
4. On each $G_{2}$-machine, resequence the $b$-tasks assigned to it in the order they become available from $G_{1}$, and start each as early as possible.
5. Compute the makespan $C_{U}$ of the resulting schedule $S_{U}$.

Note that, according to Property 5.2, step 3 of $H_{U}$ is suboptimal compared to step 3 of H. However, as we will prove shortly, both heuristics achieve the same worst case error bound. Also observe that, due to Step 1, unnecessary idle time is inserted in $G_{2}$-machines, which is then eliminated in Step 4 of $\mathrm{H}_{\mathrm{U}}$. We can now prove Theorem 5.6.

Proof of Theorem 5.6: Reindexing the jobs, let $S=(1,2, \ldots, n)$ be the order obtained at Step 1 of $H_{U}$. After the scheduling of $a$-tasks at Step 2, the completion time of the $a$-task of any $J_{j}$, say $C_{j}$, is

$$
C_{j} \leq \sum_{i=1}^{j-1} a_{i} / k_{1}+a_{j}=\sum_{i=1}^{j} a_{i} / k_{1}+\left(k_{1}-1\right) a_{j} / k_{1}, \quad j=1, \ldots, n
$$

because the FAM rule is used for $a$-tasks.
At Step 3 of $\mathrm{H}_{\mathrm{U}}$, suppose that the LBM rule is applied on the $b$-tasks, scheduling backwards from $T=\Sigma_{i}\left(a_{i}+b_{i}\right)$, and that $S_{j}$ is the start time of
$b_{j}$ in $G_{2}$, upon completion of Step 3. Then, since FAM and LBM are mirror images of one another, we have

$$
T-S_{j} \leq \sum_{i=j}^{n} b_{i} / k_{2}+\left(k_{2}-1\right) b_{j} / k_{2}, \quad j=1, \ldots, n
$$

After eliminating the unnecessary idle time in Step 4, there must be a job, say $J_{i_{0}}$, whose $b$-task starts at the same time that $a_{i_{0}}$ completes (if no such task existed we would be able to left-shift the $b$-tasks indefinitely). Then, the makespan $C_{U} \leq C_{i_{0}}+\left(T-S_{i_{0}}\right)$. Therefore,

$$
\begin{equation*}
C_{U} \leq \frac{1}{k_{1}} \sum_{i=1}^{i_{0}} a_{i}+\left(1-\frac{1}{k_{1}}\right) a_{i_{0}}+\frac{1}{k_{2}} \sum_{i=i_{0}}^{n} b_{i}+\left(1-\frac{1}{k_{2}}\right) b_{i_{0}} . \tag{5.8}
\end{equation*}
$$

Note that

$$
\sum_{i=1}^{j} a_{i} / k_{1}+\sum_{i=j}^{n} b_{i} / k_{2} \leq C_{A} \leq C^{\star}, \quad j=1, \ldots, n
$$

according to Lemma 5.1. Also,

$$
\left(1-\frac{1}{k_{1}}\right) a_{i_{0}}+\left(1-\frac{1}{k_{2}}\right) b_{i_{0}} \leq\left(1-\frac{1}{k_{\max }}\right)\left(a_{i_{0}}+b_{i_{0}}\right) \leq\left(1-\frac{1}{k_{\max }}\right) C^{\star}
$$

where $k_{\max }=\max \left\{k_{1}, k_{2}\right\}$. Observe that both H and $\mathrm{H}_{\mathrm{U}}$ share Steps 1 and 2, and that Step 3 of $\mathrm{H}_{\mathrm{U}}$ is suboptimal compared to Step 3 of H (according to Property 5.2). Hence the performance ratio of both H and $\mathrm{H}_{\mathrm{U}}$ is bounded by $\left(2-1 / k_{\max }\right) C^{\star}$.

To see that the bound of $2-1 / k_{\max }$ is tight for H (and hence $\mathrm{H}_{\mathrm{U}}$ ), consider the case where $b_{j}=0$ for all jobs. Then the heuristic H reduces to the list scheduling heuristic, RDM, for a single stage of parallel identical machines studied by Graham (1966), where the FAM rule is used in conjunction with an arbitrary permutation of jobs to minimize makespan. The algorithm has a tight worst case error bound of $2-1 / k_{1}$. In a problem where $b_{j}=0$ for all jobs and $k_{1} \geq k_{2}$ the error bound of $2-1 / k_{\max }$ is tight for the heuristic H . This completes the proof of the theorem.

## Absolute Performance Guarantees

So far we have presented results where the measure of performance is a bound $\alpha$ on the ratio $C_{H} / C^{\star}$, i.e.,

$$
\begin{equation*}
C_{H} \leq \alpha C^{\star} \tag{5.9}
\end{equation*}
$$

Alternatively, heuristics can be evaluated by computing an upper bound $\beta$ for the difference $C_{H}-C^{\star}$. A heuristic is said to provide the absolute performance guarantee $\beta$ if for any problem instance,

$$
\begin{equation*}
C_{H}-C^{\star} \leq \beta \tag{5.10}
\end{equation*}
$$

Such performance guarantees sometimes (but not always) are a by-product of error bound analyses for the ratio $C_{H} / C^{\star}$. Conversely, if $C_{H}-C^{\star} \leq \beta$, then $C_{H} / C^{\star} \leq 1+\beta$. By comparing (5.9) and (5.10), it makes sense to find
bounds where $\beta$ is not a multiple of $C^{\star}$; rather, a quantity that hopefully is much smaller than $C^{\star}$. Hence, in most cases $\beta$ is a quantity which is data dependent, such as $p_{\text {max }}=\max _{i}\left\{a_{i}, b_{i}\right\}$.

Chen (1994) presented a $O(n \log n)$ heuristic for $F\left(k_{1}, k_{2}\right) \| C_{\max }$ whose performance guarantee $\beta=\left(2-2 / k_{\max }\right) p_{\max }$. The following heuristic provides a potentially better absolute error bound and is presented in Koulamas and Kyparisis (2000).

Heuristic $\mathbf{H}_{\mathbf{K}}$ for $F\left(k_{1}, k_{2}\right) \| C_{\text {max }}$

1. Let $S=\left(S_{1}, S_{2}\right)$ where $S_{1}$ is an arbitrary permutation of the subset
$\mathcal{F}=\left\{j: a_{j} / k_{1} \leq b_{j} / k_{2}\right\}$, and $S_{2}$ is an arbitrary permutation of
$\mathcal{L}=\left\{j: a_{j} / k_{1}>b_{j} / k_{2}\right\}=\{1,2, \ldots, n\}-\mathcal{F}$.
2.     - 3. Same as in $\mathrm{H}_{\mathrm{U}}$.
1. Compute the makespan $C_{K}$ and the resulting schedule $S_{K}$.

Note that Step 1 requires $O(n)$ time and hence the complexity of $\mathrm{H}_{\mathrm{K}}$ is also $O(n)$.

Theorem 5.7 (Koulamas and Kyparisis, 2000)
For $F\left(k_{1}, k_{2}\right) \| C_{\max }$, if $C^{\star}$ is the optimal makespan:

$$
C_{K}-C^{\star} \leq\left(2-1 / \min \left\{k_{1}, k_{2}\right\}\right) p_{\max }, \text { where } p_{\max }=\max _{j}\left\{a_{j}, b_{j}\right\}
$$

Proof: If we repeat the analysis in Theorem 5.6 using the sequence $S=$ $\left(S_{1}, S_{2}\right)$ rather than $S=J R\left(\underline{a} / k_{1}, \underline{b} / k_{2}\right)$, we obtain an inequality analogous to (5.8):

$$
\begin{equation*}
C_{K} \leq \sum_{j=1}^{j_{0}} \frac{a_{j}^{\prime}}{k_{1}}+\sum_{j=j_{0}}^{n} \frac{b_{j}^{\prime}}{k_{2}}+\left(1-\frac{1}{k_{1}}\right) a_{j_{0}}^{\prime}+\left(1-\frac{1}{k_{2}}\right) b_{j_{0}}^{\prime} \tag{5.11}
\end{equation*}
$$

where $a_{j}^{\prime}, b_{j}^{\prime}$ are the processing time requirements of the $j^{t h}$ job in sequence $S=\left(S_{1}, S_{2}\right)$, and $j_{0}$ is the critical job in the schedule obtained by $H_{K}$. Note that jobs in $\mathcal{F}$ (the "first" jobs) are indexed before jobs in $\mathcal{L}$ (the "last"). By rearranging terms in (5.11) we get

$$
\begin{align*}
C_{K} & \leq \sum_{j=1}^{j_{0}}\left(\frac{a_{j}^{\prime}}{k_{1}}-\frac{b_{j}^{\prime}}{k_{2}}\right)+\sum_{j=1}^{n} \frac{b_{j}^{\prime}}{k_{2}}+\left(1-\frac{1}{k_{1}}\right) a_{j_{0}}^{\prime}+b_{j_{0}}^{\prime}  \tag{5.12}\\
& =\sum_{j=1}^{j_{0}-1}\left(\frac{a_{j}^{\prime}}{k_{1}}-\frac{b_{j}^{\prime}}{k_{2}}\right)+\sum_{j=1}^{n} \frac{b_{j}^{\prime}}{k_{2}}+a_{j_{0}}^{\prime}+\left(1-\frac{1}{k_{2}}\right) b_{j_{0}}^{\prime} \tag{5.13}
\end{align*}
$$

Now consider two cases.

$$
\text { 1. } \sum_{j=1}^{n}\left(a_{j}^{\prime} / k_{1}-b_{j}^{\prime} / k_{2}\right) \leq 0
$$

Note that, by definition, $\mathcal{F}$ contains all jobs with $a_{j}^{\prime} / k_{1}-b_{j}^{\prime} / k_{2} \leq 0$, and these are the first-indexed jobs. Hence, if $\Sigma_{j=1}^{n}\left(a_{j}^{\prime} / k_{1}-b_{j}^{\prime} / k_{2}\right) \leq 0$, it must be true that $\sum_{j=1}^{r}\left(a_{j}^{\prime} / k_{1}-b_{j}^{\prime} / k_{2}\right) \leq 0$ for every $r=1, \ldots, n$. In this case, the
first summations in (5.12) and (5.13) can be discarded, and, if we replace $a_{j_{0}}^{\prime}$ and $b_{j_{0}}^{\prime}$ by $p_{\text {max }}=\max _{j}\left\{a_{j}, b_{j}\right\}$, we get:

$$
C_{K} \leq \sum_{j=1}^{n} \frac{b_{j}^{\prime}}{k_{2}}+\min \left\{2-\frac{1}{k_{1}}, 2-\frac{1}{k_{2}}\right\} p_{\max } \leq C^{\star}+\left(2-\frac{1}{\min \left\{k_{1}, k_{2}\right\}}\right) p_{\max }
$$

2. $\sum_{j=1}^{n}\left(a_{j}^{\prime} / k_{1}-b_{j}^{\prime} / k_{2}\right) \geq 0$

Now, the same absolute bound can be obtained by applying the above analysis to the reverse problem and employing the reversibility property. In either case the theorem holds.

The absolute bound $\beta_{K}=\left(2-1 / \min \left\{k_{1}, k_{2}\right\}\right) p_{\max }$ for $\mathrm{H}_{\mathrm{K}}$ is the best known for $F\left(k_{1}, k_{2}\right) \| C_{\max }$ obtained for a linear time heuristic. For another bound, observe that, based on (5.8),

$$
\begin{aligned}
C_{U} & \leq C^{\star}+\left(1-\frac{1}{k_{1}}\right) a_{i_{0}}+\left(1-\frac{1}{k_{2}}\right) b_{i_{0}} \\
& \leq C^{\star}+\left(2-\frac{1}{k_{1}}-\frac{1}{k_{2}}\right) p_{\max }
\end{aligned}
$$

which is the best known absolute bound for $F\left(k_{1}, k_{2}\right) \| C_{\max }$ for any heuristic with $O(n \log n)$ complexity.

## Computational Experiments

Lee and Vairaktarakis (1994) report the average relative gap $\left(C_{U}-L B\right) / L B$ (where $L B=\min \left\{L B_{1}, L B_{2}, L B_{3}, L B_{4}\right\}$ ) for hybrid shops $F\left(k_{1}, k_{2}\right)$ with $\left(k_{1}, k_{2}\right)=(2,4),(4,4),(4,2)$, and $n=30,40$ or 50 jobs. The processing time ratios $\Sigma_{j} a_{j} / \Sigma_{j} b_{j}$ reported are $2: 4,4: 4$ and $4: 2$. A ratio $2: 4$ means that the processing time of $a_{j}$ is chosen randomly from a uniform distribution on $[1,20]$ and the processing time of $b_{j}$ is chosen randomly from a uniform distribution on $[1,40]$. This experiment considers 27 ratio/size combinations. For each combination the average relative gap over 50 randomly selected problems is reported. It is found that the average relative gap for $F(2,4)$ is $1.2 \%$. For $F(4,2)$, heuristic $\mathrm{H}_{\mathrm{U}}$ compiles an average relative gap of $2.2 \%$ for the workload ratio $4: 2$ where $G_{2}$ is underloaded, while the corresponding gap for the ratios 2:4 and 4:4 is negligible. For $F(4,4)$, the value of the average relative gap is $2.4 \%$. The average relative gap over all 27 combinations is $1.6 \%$. These gaps suggest that the performance of $\mathrm{H}_{\mathrm{U}}$ deteriorates as the total number of machines increases. When the workload of $G_{1}$ is at least as big as that of $G_{2}$, it is found that the average relative gaps assume greater values which reflects suboptimal $G_{1}$ scheduling. This can be explained from the fact that significant suboptimality due to $G_{1}$ scheduling can be magnified by suboptimality of the scheduling of $G_{2}$ tasks. Also, the average relative gap exhibited by $\mathrm{H}_{\mathrm{U}}$ decreases as the number of jobs increase.

Guinet et al. (1996) test a number of heuristics based on shortest processing time (SPT) sequences and longest processing time (LPT) sequences. The major difference between these heuristics and the ones we have presented up to now, is that they do not apply the steps 2 and 3 of H . Specifically, the generic form of a heuristic presented in Guinet et al. (1996) is as follows.

Heuristic $\mathbf{H}_{\mathbf{G}}$ for $F\left(k_{1}, k_{2}\right) \| C_{\text {max }}$

1. Specify sequence $S$ of the $n$ jobs.
2. Select the first unscheduled job in $S$, say $J_{j}$. Assign $a_{j}, b_{j}$ so that the resulting makespan is the smallest possible. In case there is more than one such assignment, assign $a_{j}\left(b_{j}\right)$ to a machine that becomes free last. Remove $J_{j}$ from $S$ and repeat this step until $S$ is empty.
3. Compute the makespan $C_{G}$ of the resulting schedule $S_{G}$.

Based on Property 2, the assignment of $b$-tasks is suboptimal. As for the $a$-tasks, the tiebreaking rule used in Step 2 of $\mathrm{H}_{\mathrm{G}}$ allows subsequent long $a$ tasks to start earlier than they would if the FAM rule was used. The following sequences are tested in Step 1 of $\mathrm{H}_{\mathrm{G}}: S_{1}=\nearrow a_{j}, S_{2}=\nearrow b_{j}, S_{3}=\nearrow\left(a_{j}+b_{j}\right)$, $S_{4}=\searrow a_{j}, S_{5}=\searrow b_{j}, S_{6}=\searrow\left(a_{j}+b_{j}\right)$, and $S_{7}=J R(\underline{a}, \underline{b})$.

The performance of the seven heuristics is measured based on the relative percentage deviations from the lower bound

$$
L B=\max \left\{A_{1}+\max \left\{B_{n} / k_{2}, \max _{j} b_{j}\right\}, B_{1}+\max \left\{A_{n} / k_{1}, \max _{j} a_{j}\right\}\right\}
$$

which is slightly weaker than $\max \left\{L B_{1}, L B_{4}\right\}$. A very extensive computational experiment is performed with $n=50,100,150,200,250$, and 300 jobs, $k_{i}=2,3$, and 4 machines per stage (in all combinations), and two processing time scenarios; namely, where both $a_{j}$ and $b_{j}$ are drawn uniformly from $[10,30]$ or both from $[10,100]$. For each combination of parameters 32 problem instances are generated. It is found that using $S_{7}$ in Step 1 gives the best results, with overall average percentage deviation of $0.67 \%$. The next best performance was exhibited using sequence $S_{5}$ with overall average percentage deviation $1.55 \% . S_{2}$ and $S_{4}$ are the worse performers with deviations in excess of $6 \%$. All heuristics tend to perform better for larger values of $n$ given fixed $\left(k_{1}, k_{2}\right)$.

## $5.5 F\left(k_{1}, k_{2}, k_{3}\right) \| C_{\text {max }}$

Sevast'janov (1997), developed a heuristic SEV for this problem with absolute performance

$$
C_{S E V} \leq \max \left\{\sum_{j} a_{j} / k_{1}, \sum_{j} b_{j} / k_{2}, \sum_{j} c_{j} / k_{3}\right\}+\left(m^{2}-1\right) p_{\max }
$$

Koulamas and Kyparisis (2000) proposed a heuristic for the same problem based on the merged machines flow shop (MMS) approximation (see Sect.
5.2). To obtain their performance bound the authors also make use of another developed in Sevast'janov (1997) for $F 3 \| C_{\max }$ with absolute performance

$$
C_{S E V 1} \leq \max \left\{\sum_{j} a_{j}, \sum_{j} b_{j}, \sum_{j} c_{j}\right\}+3 p_{\max }
$$

This bound has been presented in more detail in Chap. 4. With this background, we have:

Heuristic $\mathbf{H}_{\mathbf{K K}}$ for $F\left(k_{1}, k_{2}, k_{3}\right) \| C_{\text {max }}$

1. Let $S_{S E V}$ be Sevast'janov's sequence for the instance of $F 3 \| C_{\max }$ with $\bar{p}_{j}=<a_{j} / k_{1}, b_{j} / k_{2}, c_{j} / k_{3}>, \quad j=1, \ldots, n$.
2. Apply the FAM rule to the $a$-tasks of $S$.
3. Apply the FAM rule to the $b$-tasks of $S$, starting at time $t=\Sigma_{j} a_{j}$.
4. Apply the LBM rule to the $c$-tasks of $S$, starting at time $t=\Sigma_{j}\left(a_{j}+b_{j}+c_{j}\right)$.
5. Shift $b$-tasks early as much as possible, and subsequently, shift $c$-tasks as early as possible.
6. Compute the makespan $C_{K K}$ of the resulting schedule $S_{K K}$.

Theorem 5.8 (Koulamas and Kyparisis, 2000)
For $F\left(k_{1}, k_{2}, k_{3}\right) \| C_{\text {max }}$ :
$C_{K K}-C^{\star} \leq\left(4+3 / \min \left\{k_{1}, k_{2}\right\}-1 / k_{1}-2 / k_{2}-1 / k_{3}\right) p_{\max }$,
where $p_{\text {max }}=\max _{j}\left\{a_{j}, b_{j}, c_{j}\right\}$.

### 5.5.1 $F(k, 1, k) \| C_{\text {max }}$

Langston (1987) considered the problem $F(k, 1, k) \| C_{\max }$ where the middle stage consists of a single transport that ferries work from $G_{1}$ to $G_{3}$. The following heuristic algorithm, referred to as a compound heuristic, addresses this application.

## Heuristic $\mathbf{H}_{\text {LA }}$

1. Let $S$ be a fixed arbitrary sequence of jobs.
2. For every $h=1,2, \ldots, k$, repeat Steps $3-6$.
3. Apply the FAM rule to $h$ parallel identical machines $M_{1}, \ldots, M_{h}$, with respect to $p_{j}=a_{j}+c_{j}, j=1, \ldots, n$.
4. Let $\mathcal{J}_{l}$ be the subset of jobs allocated to $M_{l}, l=1, \ldots, h$, where the machines are indexed in nondecreasing order of $\Sigma_{j \in \mathcal{J}_{l}} a_{j}$.
5. Assign all $a$-tasks [ $c$-tasks] of jobs in $\mathcal{J}_{l}$ to the $l^{t h}$ machine of $G_{1}\left[G_{3}\right]$, $l=1, \ldots, h$.
6. Direct the transport to execute all jobs in $\mathcal{J}_{1}, \mathcal{J}_{2}, \ldots, \mathcal{J}_{h}$ in this order.
7. Let $S_{L A}$ be the shortest among the $k$ schedules produced in Steps 1-6, and $C_{L A}$ be the associated makespan.

Heuristic $\mathrm{H}_{\mathrm{LA}}$ starts with an arbitrary sequence and generates $k$ different schedules each utilizing precisely $h$ machines in $G_{1}$ and $G_{3}, h=1, \ldots, k$. In iteration $h$ of Steps $3-6$, the set $\{1,2, \ldots, n\}$ is partitioned into $h$ subsets. All jobs in a subset are executed on the same pair of $G_{1}$ and $G_{3}$ machines as in Step 4. Finally, the transport is directed to execute all jobs of subset $S_{l}$ before proceeding to jobs in $S_{l+1}$. The winning schedule is the shortest amongst the $k$ generated by the heuristic; this is the compound nature of the heuristic. Steps 3,5 , and 6 of $\mathrm{H}_{\mathrm{LA}}$ require $O(n)$ time. Step 4 takes $O(k \log k)$ time. Hence, each iteration requires $O(\max \{n, k \log k\})$ which results to $O(k \max \{n, k \log k\})$ complexity for $\mathrm{H}_{\mathrm{LA}}$. The performance of $\mathrm{H}_{\mathrm{LA}}$ is described next.

Theorem 5.9 (Langston, 1987)
For $F(k, 1, k) \| C_{\max }$, if $C^{\star}$ is the optimal makespan:
$C_{L A}-C^{\star} \leq 3-1 / k$, and this bound is tight.
This error bound highlights the apparent inefficiencies in $\mathrm{H}_{\mathrm{LA}}$. In particular, transporting jobs as in Step 6 of $\mathrm{H}_{\mathrm{LA}}$ forces many jobs in $S_{2}, \ldots, S_{h}$ that are finished early in $G_{1}$ to wait until all jobs in $S_{1}$ have been transported. However, no studies have appeared in the literature that improve upon this transport dispatching rule.

## $5.6 \boldsymbol{F}\left(k_{1}, k_{2}, \ldots, k_{m}\right) \| C_{\max }$

Results for this problem in full generality are sparse. A worst case analysis of the only published heuristic, and a discussion of a stochastic version of the problem, are presented. We also briefly mention a branch-and-bound formulation.

### 5.6.1 Heuristic and Worst Case Error Bound Analysis

We give next a heuristic algorithm $\mathrm{H}_{\mathrm{LV}}$ for $F\left(k_{1}, k_{2}, \ldots, k_{m}\right) \| C_{\text {max }}$, due to Lee and Vairaktarakis (1994). It utilizes $\mathrm{H}_{\mathrm{U}}$ (discussed in Sect. 5.4.2), which is designed for a two-station shop. $\mathrm{H}_{\mathrm{LV}}$ assumes that the number $m$ of stages is even; if not, we can introduce a dummy stage with zero machines.

Heuristic $\mathbf{H}_{\mathbf{L V}}$ for $F\left(k_{1}, k_{2}, \ldots, k_{m}\right) \| C_{\text {max }}$

1. Apply $\mathrm{H}_{\mathrm{U}}$ on stages $2 r-1,2 r$ and let $S_{r}$ be the resulting schedule, for $r=1,2, \ldots, m / 2$.
2. Concatenate the schedules $S_{r}$, for $r=1,2, \ldots, m / 2$. Eliminate all idle time between $a$-tasks. Eliminate unnecessary idle time between $b$-tasks.
3. Compute the makespan $C_{L V}$ of the resulting schedule $S_{L V}$.

The complexity of $\mathrm{H}_{\mathrm{LV}}$ is $O(m n \log n)$. The next result provides a worst case performance ratio for $\mathrm{H}_{\mathrm{LV}}$.

Theorem 5.10 (Lee and Vairaktarakis, 1994)
For $F\left(k_{1}, k_{2}, \ldots, k_{m}\right) \| C_{\text {max }}$ :

$$
C_{L V} / C^{\star} \leq m-\sum_{r=1}^{m / 2}\left[1 / \max \left\{k_{2 r-1}, k_{2 r}\right\}\right]
$$

Step 2 of heuristic $\mathrm{H}_{\mathrm{LV}}$ is highly inefficient as it forces significant idle time between stages. Its merit is that it facilitates the analysis of the above theorem. If $k_{1}=k_{2}=\ldots=k_{m}=1$ then the hybrid flow shop reduces to the traditional $m$-machine flow shop. In this case, $C_{L V} / C^{\star} \leq\lceil m / 2\rceil$ coincides with the bound of Gonzalez and Sahni (1978), which is the best known error bound for $\operatorname{Fm}(1,1, \ldots, 1) \| C_{\max }$. Several cases where the error bound of $H_{L V}$ is tight are known. Tight instances of $F(1,1,1) \| C_{\max }$ and $F(1,1,1,1) \| C_{\max }$ are given in Gonzalez and Sahni (1978). In case the hybrid flow shop consists of only two stages, tight examples were given earlier for $F\left(k_{1}, k_{2}\right) \| C_{\max }$.

### 5.6.2 Stochastic Processing Times

Koulamas and Kyparisis (2000) presented asymptotic results based on absolute performance guarantees. Suppose that task processing times are stochastic. Let us use $P_{i j}(i=1, \ldots, m, j=1, \ldots, n)$, rather than $p_{i j}$, to denote the time to process task $i$ of $J_{j}$, to emphasize its random nature. Let $P_{i j}$ be independent, identically distributed uniform random variables drawn from the interval $[A, B], 0 \leq A<B$. Then:

Theorem 5.11 (Koulamas and Kyparisis, 2000)
For every heuristic $H$ with
$C_{H} \leq C^{\star}+f(m) P_{\max }$,
where $P_{\max }=\max _{i, j} P_{i j}$ and $f(m)$ is bounded, the following hold:
i) $E\left(C_{H} / C^{\star}\right) \leq 1+2 m f(m) k_{\max } /(m n-2)$, where $k_{\max }=\max _{r} k_{r}$,
ii) $\operatorname{Pr}\left(C_{H} / C^{\star}-1>2 m f(m) k_{\max } /[x(m n-1)]\right) \leq\left(x e^{1-x}\right)^{m n-1}$, for every $0<x<1$.

Proof: By hypothesis, we have $C_{H} / C^{\star} \leq 1+f(m) P_{\max } / C^{\star}$. Observe that $C^{\star} \geq \Sigma_{i, j} P_{i j} / m k_{\max }$ since the right hand side corresponds to the average workload of $m k_{\max }$ (which is no less than $k_{1}+k_{2}+\ldots+k_{m}$ ) parallel identical machines with preemption allowed and no precedence constraints amongst tasks. Therefore,

$$
\begin{equation*}
C_{H} / C^{\star} \leq 1+m f(m) k_{\max } P_{\max } / \sum_{i, j} P_{i j} \tag{5.14}
\end{equation*}
$$

Coffman and Gilbert (1985) give the following result: Let $X_{(n)}$ be the $n^{\text {th }}$ order statistic of $n$ i.i.d. uniform random variables $X_{j}, j=1,2, \ldots, n$. Then,

$$
E\left(X_{(n)} / \sum_{j=1}^{n} X_{j}\right) \leq 2 /(n-2), \quad n>2
$$

Since $P_{\text {max }}$ is the $m n^{t h}$ order statistic of the $m n$ random variables $P_{i j}$, this
implies that $E\left(P_{\max } / \Sigma_{i, j} P_{i j}\right) \leq 2 /(m n-2), \quad m n>2$. By taking the expectation on both sides of (5.14) and using the latter inequality, we obtain i).

To prove ii), note that

$$
\begin{aligned}
P\left(\frac{C_{H}}{C^{\star}}-1>\frac{2 m f(m) k_{\max }}{x(m n-1)}\right) & \leq P\left(\frac{m f(m) k_{\max } P_{\max }}{\sum_{i, j} P_{i j}}>\frac{2 m f(m) k_{\max }}{x(m n-1)}\right) \\
& =P\left(\frac{P_{\max }}{\sum_{i, j} P_{i j}}>\frac{2}{x(m n-1)}\right)
\end{aligned}
$$

due to 5.14. Coffman and Gilbert (1985) showed that

$$
P\left(X_{(n)} / \sum_{j=1}^{n} X_{j}>2 / x(n-1)\right) \leq\left(x e^{1-x}\right)^{n-1}, \quad \forall 0<x<1
$$

which implies

$$
P\left(P_{\max } / \sum_{i, j} P_{i j}>2 / x(m n-1)\right) \leq\left(x e^{1-x}\right)^{m n-1}, \quad \forall 0<x<1
$$

This completes the proof of ii).
Inequality i) shows that, as $n$ approaches infinity, the expected performance of every heuristic that satisfies $C_{H} \leq C^{\star}+f(m) P_{\max }$ is asymptotically optimal. Moreover, inequality ii) shows that the probability of the performance ratio exceeding 1 by more than $\epsilon=2 m f(m) k_{\text {max }} / x(m n-1)$ approaches 0 exponentially fast as $n$ approaches infinity. The existence of at least one heuristic $H$ for which $C_{H} \leq C^{\star}+f(m) p_{\max }$ is demonstrated in Sevast'janov (1995) where $f(m)=m^{2}-1$ for any number $m$ of stages. For two or three stages, all of the heuristics $H_{K}, H_{L V}, H_{S}$, and $H_{K K}$ satisfy the absolute performance condition.

Approximability results for $F\left(k_{1}, k_{2}, \ldots, k_{m}\right) \| C_{\max }$ are tabulated next, together with the corresponding references. PTAS indicates that the corresponding variant accepts a polynomial-time approximation scheme.

|  | Number of machines/stage: | $k_{r}(r=1,2, \ldots, m)$ |  |
| :--- | :---: | :---: | :---: |
| $\#$ of stages | $k_{r}=1$ | $k_{r}$ fixed | $k_{r}$ arbitrary |
| $m=2$ | $O(n \log n)$ | PTAS, [16] | PTAS, [27] |
| $m \geq 3$ (fixed) | PTAS, [16] | PTAS, [16] | Open |
| $m$ arbitrary | PTAS, [32] | PTAS, [32] | PTAS, [32] |

As we see, when $m \geq 3$ is constant and $k_{1}, k_{2}, \ldots, k_{m}$ is part of the input, the approximability of $F\left(k_{1}, k_{2}, \ldots, k_{m}\right) \| C_{\max }$ remains open. When the number $m$ of stages is part of the input, Williamson et al. (1997) show that no PTAS with worst case performance ratio less than $5 / 4$ exists unless $\mathcal{P}=\mathcal{N} \mathcal{P}$.

### 5.6.3 Branch-and-Bound Implementations

Brah and Hunsucker (1991) present a branch-and-bound algorithm for $F\left(k_{1}, k_{2}, \ldots, k_{m}\right) \| C_{\max }$ and report computational experiments with up to 8 jobs and up to 5 stages. The number of visited nodes in the search tree indicates high exponential growth of the search space. A more efficent branch-and-bound algorithm for $F\left(k_{1}, k_{2}, \ldots, k_{m}\right) \mid$ perm $\mid C_{\text {max }}$ is presented in Rajendran and Chaudhuri (1992). Here, a schedule is assumed to be a permutation schedule if, whenever $J_{i}$ precedes $J_{j}$ on any machine, $J_{i}$ precedes $J_{j}$ on every machine that executes both jobs. Regrettably, this algorithm does not greatly enlarge the set of solvable problems; experiments are reported for $n \leq 8$ and $m \leq 10$.

### 5.7 Other Objectives

All of the above developments focus on the makespan objective. Beyond makespan, the literature is very sparse.

### 5.7.1 $F\left(k_{1}, \ldots, k_{m}\right)\left|d_{j}\right| T_{\max }$

Algorithms for $F\left(k_{1}, \ldots, k_{m}\right)\left|d_{j}\right| T_{\max }$ have been developed by Guinet and Solomon (1996). The mixed integer program HFS from Subsect. 5.2.1 can be adapted to this problem by replacing the set of constraints (5.6) by the following:

$$
C_{r j}-d_{j} \leq Z, \quad r=1, \ldots, m, \quad j=1, \ldots, n .
$$

Since the tardiness of $J_{j}$ is given by $T_{j}=\max \left\{C_{r j}-d_{j}, 0\right\}$, the optimal objective value of the program for $F\left(k_{1}, \ldots, k_{m}\right)\left|d_{j}\right| T_{\max }$ is $T^{\star}=Z$ if $Z>0$; $T^{\star}=0$ otherwise.

The heuristics presented in Guinet and Solomon (1996) follow the same generic steps as the heuristic $\mathrm{H}_{\mathrm{G}}$ in Subsect. 5.4.2. In Step 1 of $\mathrm{H}_{\mathrm{G}}$, a number of sequences are identified for the $m$-stage auxiliary flow shop based on the algorithms of Campbell et al. (1970), Nawaz et al. (1983), and Townsend (1977) for $F m \| C_{\text {max }}$. In addition, priority sequences are tested based on the SPT, EDD, and MST (minimal slack time) rules. In Step 2 of $\mathrm{H}_{\mathrm{G}}$, two different rules are tested both of which try to minimize the completion time of a newly inserted job while minimizing the idle time that has to be inserted as a result; an idea similar in spirit to the 2-stage version of $\mathrm{H}_{\mathrm{G}}$.

The lower bound used by Guinet and Solomon is $L B_{V}=\min _{r=1}^{m} V_{r}$, where

$$
V_{r}=\min _{j} \sum_{i=1}^{r-1} p_{i j}+\max \left\{\max _{j} p_{r j}, \sum_{j} p_{r j} / k_{r}\right\}+\min _{j} \sum_{i=r+1}^{m} p_{i j}-\max _{j} d_{j}
$$

The first term of $V_{r}$ is a lower bound on the start time of the $G_{r}$-task that starts the earliest. The second term is a lower bound on the additional pro-
cessing time on $G_{r}$ (note that $G_{r}$ is replaced here by an equivalent merged machine). The third term is a lower bound on the additional time required to complete any $J_{j}$ beyond $C_{r j}$. Hence, the first 3 terms provide a lower bound on $C_{m j}$. Subtracting $\max _{j} d_{j}$ therefore provides a lower bound on the maximum tardiness.

Extensive computational experiments with up to 30 jobs, up to 5 flow shop stages, and up to 5 machines per stage on randomly generated problems where processing times are uniformly drawn from $[4,20]$ indicate that the schedules produced using the sequence of Nawaz et al. often outperform schedules that use popular dispatching rules. Even so, the resulting schedules have an average deviation from $L B_{V}$ of $20.9 \%$, leaving a lot of room for improvement. Similar experiments are performed for the makespan objective; an overall average gap of $8.5 \%$ is recorded in this case.

### 5.7.2 Total Cost

Chang and Liao (1994) considered a hybrid flow shop problem where the production system is $F\left(k_{1}, \ldots, k_{m}\right)$, finite buffers are available to store jobs between stages, and limited overtime capacity is available in each production stage at a cost. Jobs are batches of part types in known volumes. The objective function is the minimization of the combined cost of work-in-process, finished good inventory, total earliness/tardiness, plus the cost of overtime capacity. Let us denote this problem by $F\left(k_{1}, \ldots, k_{m}\right)\left|d_{j}, B_{r}, O_{r}\right| f\left(C_{r j}, O_{r}\right)$, where $B_{r}$ is the capacity of the buffer in front of stage $r, O_{r}$ is the overtime capacity available for stage $r$ over the entire production horizon, and $f\left(C_{r j}, O_{r}\right)$ is a cost function of the task completion times and the overtime capacity allowances. This problem is found in a make-to-order hybrid flow shop that manufactures a medium variety of discrete parts having their own due date and demanded quantity, at medium volume.

For this problem, the authors developed a mixed integer program for deciding the number of parts of each type that are processed in each stage during each period of the production horizon, as well as the overtime capacity required to process these parts. The difference between this model and the ones presented so far, is in the level of detail involved in managing the machines within each stage. Here, the total capacity of all the machines in a stage is used as a proxy for stage capacity. Instead, the earlier models consider explicitly the scheduling problem in each and every stage machine. Evidently, the total stage capacity is an approximation of the actual capacity of each stage due to the idle time that has to be inserted to satisfy the flow shop and limited buffer constraints. Using a Langrangean relaxation based heuristic for solving $F\left(k_{1}, \ldots, k_{m}\right)\left|d_{j}\right| T_{\max }$, Chang and Liao present limited computational experiments on real and randomly generated data with up to 14 stages and up to 20 part types. They find that the deviation of their heuristic from the Lagrangean lower bound is less than $1.5 \%$.

### 5.7.3 Conclusions

We saw in this chapter that the overwhelming majority of research activity is concentrated on the 2 -stage system and the makespan objective. Within this space, there is a large body of literature concentrating on error bound analysis. The approaches that have appeared, seem to emulate techniques first developed for $P m \| C_{\text {max }}$. Excellent lower bounds based on the merged-machines construct demonstrate that quick heuristic solutions for $F\left(k_{1}, k_{2}\right) \| C_{\text {max }}$ are near optimal. In contrast, branch-and-bound algorithms seem to be hopeless due to lack of structure in the problem and hence lack of major optimality and dominance properties. Despite the achievements, a lot more is desired. For 3 or more stages results are lacking, and solutions (heuristic or otherwise) are unavailable even for the makespan objective. Moreover, the gamut of objectives considered in the literature is quite narrow.

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## Chapter 6 THE NO-WAIT FLOW SHOP


#### Abstract

After describing some real-world examples of flow shops with no waiting, we demonstrate the equivalence of no-wait and blocking in the shop with $m=2$. For the no-wait shop, some research is available on the flow time objective while the majority of research focuses on the makespan objective. Hence, we start with a detailed discussion of an $O(n \log n)$ algorithm for $F 2|n w t| C_{\max }$. For the makespan objective and $m>2$ machines, we present a plethora of polynomially solvable special cases of the problem. Most interesting is the case with semi-ordered processing time matrices. In this case, a so called pyramidal schedule provides an optimal solution in $O\left(n^{2}\right)$ time. Heuristic algorithms for $m>2$ and the makespan objective show that the problem is intimately related to the traveling salesman problem. We survey the state of the art on metaheuristics. The problem is analyzed in the presence of lot streaming, with separable setup and teardown times, and in assemblytype and hybrid flow shops.


### 6.1 Introduction

In the flow shop systems studied so far, the default assumption has been that there is unlimited buffer storage available between stages, so that jobs completing at one stage can wait for their turn at the next stage without causing congestion. Another possibility, to be considered in Chap. 7, is that there is finite waiting capacity between stages (it may be none at all), leading to blocking: a job finishing at a stage $G_{k}$ and finding the ensuing stage (processors and buffer capacity) fully occupied remains on that stage- $k$ machine, forcing it to remain idle until space becomes available.

In this chapter we study systems where, for technological or other extraneous reasons, a job, once started, must flow through every stage to completion without any delay. Such systems are commonly referred to as nondelay or
no-wait flow shops. Note that such systems are bufferless, since no job is permitted to utilize a buffer.

Clearly, preemption is never allowed in a no-wait shop, since it would break up the uninterrupted flow we require. Also, in the simple flow shop, one might think that jobs cannot overtake other jobs: only permutation schedules are possible. However, this will not be true if (a) tasks of zero duration may occur; and (b) whenever $p_{i j}=0$ for some $i, j, J_{j}$ can skip over $M_{i}$, regardless of its occupancy, and start immediately on $M_{i+1}$. The alternative assumption would be that every job has $m$ tasks, even if some of them have zero length, and so it must visit every stage and find an idle processor to occupy before moving on, even if that occupancy is only instantaneous. The latter assumption is rather artificial, but is sometimes made for analytic convenience. If it is, or if all processing times are strictly positive, then only permutation schedules are feasible in the simple flow shop, and this is shown by adding (perm) in the problem specification.

This chapter will deal almost exclusively with the makespan objective; few results exist for other criteria. In broadest generality, the system to be considered is denoted $F\left(k_{1}, k_{2}, \ldots, k_{m}\right)|n w t| C_{\max }$ in the standard 3-field notation, where nwt stands for no-wait. Evidently, this system differs from the hybrid flow shop considered in Chap. 5 only in the no-wait requirement.

The outline of the rest of the chapter is as follows. In Sect. 6.2, we note some real-world applications of the model. In Sect. 6.3, we study problem $F 2|n w t| C_{\max }$ and provide a polynomial algorithm. Sect. 6.4 discusses the $m$ machine no-wait shop, still with makespan objective. Some systems similar to the no-wait flow shop are covered in Sect. 6.5, while some other objectives are briefly mentioned in Sect. 6.6. Concluding remarks are made in Sect. 6.7.

### 6.2 Applications

Obvious applications of no-wait flow shops are found when the storage space available between stages is null. Less obvious are applications where the nowait constraint is a result of a process requirement or is due to unavailability of resources. Various such examples are given below. See also the survey paper by Hall and Sriskandarajah (1996) for more discussion of no-wait scheduling and its uses.

- In steel manufacturing, the alloy undergoes heating and forming. The molten steel is left in position until the temperature drops to meet specifications. Only when conditions are just right can finishing operations commence. Thus, heating, forming and finishing form a 3 -stage no-wait flow shop imposed by temperature requirements. Besides the steel industry, operations like melting into ingots, unmolding, reheating, soaking, and preliminary rolling are common in plastic molding and silverware industries.
- In an application of $F 2|n w t| C_{\max }$ arising in steel slitting, sheets of steel are sliced into different widths using a slitting machine. Before the slitting of a job (or order) can be done, the cutting head of the machine must be adjusted according to the widths required by the job. The cutting head is detachable and hence width adjustments can be made off-line. Two such cutting heads are available. While a job is processed using the first head, the second is set up off-line for the next job. Thus, the first stage of processing a job is the cutting head setup, and the second is the actual slitting. Given a set of $n$ jobs with specified setup and processing time requirements $s_{i}$ and $p_{i}$ respectively, $i=1,2, \ldots, n$, the objective is to minimize the makespan of the slitting operations. Note that the setup of $J_{j+1}$ (jobs indexed in order of processing) cannot begin until the setup of $J_{j}$ is completed (there is only one setup operator) and until the slitting of $J_{j-1}$ is done (freeing up the slitting head). Thus, although there is no technological requirement that there be no wait between setup and slitting of a job, setups can always be delayed as necessary to effect this without affecting the makespan.
- Many no-wait flow shop applications are found in chemical processes when reactants take a given amount of time to complete a chemical reaction (e.g., corrosion, catalysis, etc.) following the completion of the upstream operation. The mathematical formulation of such applications is considered in Reddi and Ramamoorthy (1972). Ball and Magazine (1988) consider a related application that arises in the context of optimal insertion of chips on a printed circuit board.
- In food processing, canning operations must immediately follow cooking to ensure freshness.
- In-line robotic cells used in automobile assembly or spot-molding, together with the supporting transportation systems that remove parts from the robot arms often form no-wait flow shops. Such applications are described in Logendran and Sriskandarajah (1996), Kise et al. (1991) and Sethi et al. (1992).
- Another typical application of no-wait flow shops is found in workforce planning of synchronous assembly lines. A synchronous or paced assembly line consists of $m$ stations arranged in series. A set of jobs is to be processed so that each job visits every station in the same order, say 1 to $m$. Every station has the same production cycle of $c$ units of time, referred to as a period. Hence, in a paced assembly line, jobs move in unison one station downstream at the end of every period. At that time, a new job enters the first station and a completed job exits the last station. Given that the labor requirements of each job in each station are different, the production manager must assign the necessary number of workers to each workstation in each period (assuming complete worker flexibility) so that each operation is completed within that period. Let $\left(W_{1 j}, W_{2 j}, \ldots, W_{m j}\right)$ be the work requirements associated with job $J_{j}, j=1, \ldots, n$, where $W_{i j}$ is the number of workers necessary to complete task $i$ of $J_{j}$ within one pe-
riod, $i=1, \ldots, m$. The problem is to determine a permutation of the jobs so that the resulting workforce schedule satisfies some overall objective relating to the number of workers necessary to process all jobs. Here, the no-wait requirement is imposed by the common production cycle among all stations, which in turn is the result of labor agreements. Such paced assembly lines found in fire-engine manufacturing are studied by Lee and Vairaktarakis (1997), Vairaktarakis and Winch (1999), and Vairaktarakis and Cai (2002). Abadi et al. (2000) considered the problem of minimizing the length of the cycle time of a minimal part set on the $m$-machine flow shop with blocking, which as we see below bears some similarity with the no-wait system.


### 6.3 F2|nwt, (perm) $\mid C_{\text {max }}$ and $\operatorname{F2|block,(\text {perm})|C_{\text {max}}}$

We start with the simplest case of the no-wait flow shop, having two stages with one machine each. Sahni and Cho (1979) showed that $F 2|n w t| C_{\max }$ is strongly NP-complete when the job set includes jobs that consist of just one task. The reduction used is from 3-Partition. Since it is almost identical to the reduction given in Sect. 2.4 of Chap. 3, we omit details here. Unless otherwise stated, we assume hereafter that all jobs have two distinct operations, each of which must be scheduled even if it requires zero time. We call this requirement that no job can skip a machine the no-skip provision. In the no-wait shop, this clearly implies that only permutation schedules are feasible, so we write (perm). Keep in mind that this is only true because we are assuming no-skip behavior; otherwise, non-permutation schedules exist and may be optimal. With the no-skip provision, as we shall see, the problem is not NP-complete.

An arbitrary schedule for a typical five-job instance is shown in Fig. 6.1.


Fig. 6.1
An example of a no-wait schedule

Note how the two operations of each job are locked together by the no-wait requirement. It is easy to see that, given an arbitrary schedule $S=(1, \ldots, n)$, the completion time of any $J_{j}$ is

$$
C_{2 j}(S)=a_{1}+\sum_{i=2}^{j} \max \left(a_{i}, b_{i-1}\right)+b_{j}, \quad j=1, \ldots, n
$$

Consider now how $J_{1}$ and $J_{2}$ overlap. Because $a_{2}>b_{1}$, there is idle time of $a_{2}-b_{1}$ on $M_{2}$. On the other hand, between $J_{2}$ and $J_{3}$ the idle time is $b_{2}-a_{3}$ since $b_{2}>a_{3}$, and it occurs on $M_{1}$. It is now easy to see that each
adjacent pair of jobs, say $J_{j}$ and $J_{j+1}$, introduces idle time of $\left|a_{j+1}-b_{j}\right|$ on one machine or the other.

Now note that, if we define

$$
I_{i}=\text { total idle time on } M_{i} \quad(i=1,2),
$$

we can write the makespan as

$$
C_{\max }=\sum_{j=1}^{n} a_{j}+I_{1}=\sum_{j=1}^{n} b_{j}+I_{2}
$$

so that minimizing makespan is equivalent to minimizing $I_{1}$, or $I_{2}$, or $I_{1}+I_{2}$.
With reference to Fig. 6.1, we are ready to say: to minimize $C_{\max }$, find the schedule $S=(1,2, \ldots, n)$ that minimizes $\sum_{j=0}^{n}\left|a_{j+1}-b_{j}\right|$, provided we define $b_{0}=a_{n+1}=0$. What we need is a dummy job $J_{0}$ with $\left.<a_{0}, b_{0}\right\rangle=<0,0>$, appended at both ends. Thus, in the following discussion, it should be understood that $a_{n+1}=a_{0}$. You can picture a cyclic schedule indefinitely repeated, each iteration starting with $J_{0}$, or a schedule that cycles back on itself so $J_{n+1}=J_{0}$. The advantage of the latter viewpoint is that each schedule of $n$ jobs can be represented by a tour that visits each of $n$ locations, making makespan minimization into a Traveling Salesman problem. We define this formulation below, and show how it leads to a polynomial solution to our problem.

First, however, we introduce another scheduling problem, and show that, for $m=2$ and no-skip, it is equivalent to the no-wait problem.

### 6.3.1 Equivalence of No-Wait and Blocking when $m=2$

We say a flow shop has blocking when no intermediate storage is provided for the jobs to wait between stages. As a result, at the completion of any task, if the downstream machine is busy, a job must wait at the upstream machine, occupying it and blocking the next job until the downstream processor clears and allows flow to resume. Clearly, this production protocol is related to the no-wait flow shop where the waiting time, rather than the waiting space, is required to be zero. In the abbreviated job description, the presence of blocking will be denoted block.

There is a separate literature on the flow shop with blocking, which we cover in Chap. 7. However, with two machines, $F 2 \mid$ block, $($ perm $) \mid A n y$ is equivalent to $F 2 \mid n w t,($ perm $) \mid A n y$. This is because, with blocking, any job $J_{j}$ on $M_{1}$ that is blocked stays on $M_{1}$ until it can start on $M_{2}$, so $a_{j}$ can always be delayed so as to complete just as $b_{j}$ begins, without affecting any job completion time. Similarly, in no-wait schedules, blocking never arises, and if the no-wait requirement is relaxed, blocking will not permit tasks on $M_{2}$ to be processed earlier. In Fig. 6.1, the same schedule with blocking rather than no -wait would have $a_{3}$ and $a_{5}$ shifted left, but no other task would move.

The rest of our discussion of $F 2 \mid n w t,($ perm $) \mid A n y$ will therefore apply equally to $F 2 \mid b l o c k,($ perm $) \mid A n y$. Since there are two separate bodies of literature for the problems with $m \geq 2$, results for $m=2$ may
appear in either one. We bring them together here, calling the problem $F 2 \mid n w t$ or block, (perm) $\mid$ Any, and always remembering that we are making the no-skip assumption.

### 6.3.2 Simple Cases

The following observations are made in Dutta and Cunningham (1975) for some special cases of the makespan problem.

Theorem 6.1 If the sequence $S=(1,2, \ldots, n)$ is such that $a_{i}=b_{i-1}$ for $2 \leq i \leq n$ and $a_{1}=\min _{i} a_{i}, b_{n}=\min _{i} b_{i}$, then $S$ is optimal for F2|nwt or block, (perm) $\mid C_{\max }$.

Theorem 6.2 Let $t=\min _{j}\left[\min \left(a_{j}, b_{j}\right)\right]$. There exists an optimal sequence for $F 2 \mid$ nwt or block, $($ perm $) \mid C_{\max }$ where if $a_{j}=t\left[b_{j}=t\right]$, $J_{j}$ is scheduled first [last].

Theorem 6.3 If $a_{j}=b_{j}$ for $j=1, \ldots, n$, then optimal sequences for $F 2 \mid$ nwt or block, (perm) $\mid C_{\max }$ include $\nearrow a_{j}$ and $\searrow a_{j}$.

### 6.3.3 The Traveling Salesman Problem

Consider $n$ cities $C_{1}, C_{2}, \ldots, C_{n}$. The geographic location of each city $C_{i}$ is specified by the coordinates $\left(a_{i}, b_{i}\right)$ for $i=1,2, \ldots, n$, and let $c_{i j}$ be the distance between $C_{i}$ and $C_{j}$. The problem of finding a tour that visits each city exactly once, starting and ending at the same location, and minimizes the total distance traveled, is known as the Traveling Salesman Problem (TSP). Note that the inter-city distances need not be defined by the usual straight-line geographical (commonly referred to as Euclidean) measure: $c_{i j}=$ $\sqrt{\left(a_{i}-a_{j}\right)^{2}+\left(b_{i}-b_{j}\right)^{2}}$; any other definition of $c_{i j}$ can be used as the need arises, provided it depends only on the parameters of $C_{i}$ and $C_{j}$, and not on the other cities' locations nor on the order in which they are visited. The TSP is one of the most studied problems in combinatorial optimization. Some of the most important results for TSP are found in the well-known book by Lawler et al. (1985). The problem is strongly NP-complete for a variety of distance measures including Euclidean, rectilinear, $L_{\infty}$, etc. (see Garey and Johnson, 1979).

### 6.3.4 F2|nwt or block, (perm) $\mid C_{\max }$ as a TSP

We have shown that, for $F 2 \mid n w t$ or block, (perm) $\mid C_{\max }$, finding the optimal sequence is equivalent to finding the optimal tour in a TSP with $n+1$ locations representing the jobs $J_{0}, J_{1}, \ldots, J_{n}$, with distance metric $c_{i j}=\left|a_{j}-b_{i}\right|$. This distance measure is a special case of the Gilmore-Gomory metric

$$
c_{i j}^{\prime}= \begin{cases}\int_{b_{i}}^{a_{j}} f(x) d x & \text { if } a_{j} \geq b_{i}  \tag{6.1}\\ \int_{a_{j}}^{b_{i}} g(x) d x & \text { if } a_{j}<b_{i}\end{cases}
$$

where the functions $f(\cdot)$ and $g(\cdot)$ are required to be integrable, with $f(x)+$ $g(x) \geq 0$. The latter condition implies that there is no gain to be obtained by cycling between cities. The metric $c_{i j}$ is the special case of $c_{i j}^{\prime}$ where $f(x)=g(x)=1$. Gilmore and Gomory (1964) found that, with the metric $c_{i j}^{\prime}$, TSP is polynomially solvable, and they presented an optimal algorithm with complexity $O(n \log n)$. A variation of the Gilmore-Gomory algorithm is presented in Lawler et al. (1985), together with other solvable cases of the traveling salesman problem.

Vairaktarakis (2003) gives another algorithm for TSP with $c_{i j}^{\prime}$ metric, of the same complexity $O(n \log n)$, which is easier to implement and describe. We now present it, specialized to the simpler metric $c_{i j}$ of our scheduling problem, and from now on we speak of jobs, task times and sequences, rather than cities, locations and tours.

### 6.3.5 F2|nwtorblock, (perm) $\mid C_{\text {max }}$ as Bipartite Graph

It will be convenient to perform a monotone reindexing of the task times, separately on each machine, so that $a_{1} \leq a_{2} \leq \ldots \leq a_{n}$ and $b_{1} \leq b_{2} \leq \ldots \leq$ $b_{n}$. The dummy job may appropriately retain its zero index, since its task times are the smallest of all. Note that now the two times of a given job may have different indices. Let the jobs be indexed according to their $a$-tasks, and let $\pi$ be the permutation of the jobs ordered in nondecreasing order of $b_{j}$, so that the task times of $J_{j}$ are $<a_{j}, b_{\pi(j)}>$ for $j=0,1,2, \ldots, n$.

We now formulate the 2-machine no-wait flow shop as an $(n+1) \times(n+1)$ complete bipartite graph, $\mathcal{B}$. That is, $\mathcal{B}$ has two sets of $n+1$ nodes, one set corresponding to the $a$-tasks and the other set to the $b$-tasks, with arcs connecting $a$-tasks to $b$-tasks. Since each job $J_{j}$ is represented by the pair $<a_{j}, b_{\pi(j)}>$, we introduce the set of $\operatorname{arcs} \mathcal{M}_{J}=\left\{\left(a_{j}, b_{\pi(j)}\right): j=0,1,2, \ldots, n\right\}$ as the job matching. Clearly $\mathcal{M}_{J}$ is a matching of $\mathcal{B}$ because no two arcs have a vertex in common. It will help visualization to consider these arcs directed from $a$-tasks to $b$-tasks, and to call this an $a$-to-b matching.

Example 6.1: Consider the instance of $F 2 \mid$ nwt or block, $($ perm $) \mid C_{\max }$ with six jobs, as follows:

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a_{j}$ | 3 | 4 | 5 | 9 | 12 | 13 |
| $b_{j}$ | 2 | 12 | 6 | 5 | 10 | 12 |

After monotone reindexing and adding a dummy job $J_{0}$, the tasks $a_{i}, b_{i}$, $i=0,1,2, \ldots, 6$, form the node set of the bipartite graph $\mathcal{B}$ as shown in Fig. 6.2(a) or (b), where the processing times have been added beside each node. The job matching $\mathcal{M}_{J}$ is depicted by the solid arcs (the dashed arcs will
be discussed shortly). Throughout our description we will use this example to illustrate an algorithm for $F 2 \mid$ nwt or block, (perm) $\mid C_{\text {max }}$.

(a) A Hamiltonian cycle $\mathcal{M}_{J} \cup \mathcal{M}(S)$

(b) The starting solution $\mathcal{M}_{J} \cup \mathcal{M}_{0}$

Fig. 6.2 Bipartite graph for a sample instance of $F 2 \mid n w t$ or block, $($ perm $) \mid C_{\text {max }}$

Now note that a solution, i.e., a sequence of jobs, can be interpreted as a matching of the $b$-tasks to the $a$-tasks, each job's $b$-task being matched to the $a$-task of the job immediately following it. To accommodate any job sequence, arcs may be defined from each $b$-task to every $a$-task. For instance, the schedule $S^{\#}=(0,1,2, \ldots, n)$ requires an arc from $b_{\pi(j)}$ to $a_{j+1}, j=$ $0,1, \ldots, n$, where $a_{n+1}=a_{0}$. This matching is shown as dashed arcs in Fig. $6.2(\mathrm{a})$. In general, let the $b$-to- $a$ matching corresponding to any job sequence $S$ be the sequence matching, $\mathcal{M}(S)$. Thus, $\mathcal{M}\left(S^{\#}\right)=\left\{\left(b_{\pi(j)}, a_{j+1}\right), j=\right.$ $0,1, \ldots, n\}$. If we associate a cost $\left|a_{j}-b_{i}\right|$ with any arc $\left(b_{i}, a_{j}\right)$ in $\mathcal{M}(S)$, our problem becomes: find the least-cost sequence matching.

### 6.3.6 Finding the Least-Cost Cycle in $\mathcal{B}$

Combining the two arc sets $\mathcal{M}_{J}$ and $\mathcal{M}(S)$ for any $S$, we see that they make up a Hamiltonian cycle: a tour of the graph that visits every node once. Thus for $S^{\#}$, the cycle is $\left(a_{0}, b_{\pi(0)}, a_{1}, b_{\pi(1)}, a_{2}, \ldots, a_{n}, b_{\pi(n)}, a_{0}\right)$, as shown in Fig. 6.2a. If we give the arcs in $\mathcal{M}_{J}$ zero cost, we can now define our problem as seeking the least-cost cycle in $\mathcal{B}$. We leave it to the reader to trace the cycle in the figure, and to verify that the cost of this schedule is $\sum_{j=0}^{n}\left|a_{j+1}-b_{\pi(j)}\right|=37$.

Of course, not every $b$-to- $a$ matching, $\mathcal{M}_{b a}$, is a valid sequence matching, only those for which $\mathcal{M}_{J} \cup \mathcal{M}_{b a}$ form a Hamiltonian cycle. That is, a matching $\mathcal{M}_{b a}$ does not correspond to a sequence if $\mathcal{M}_{J} \cup \mathcal{M}_{b a}$ contains subcycles. Our strategy will be to start with the lowest-cost matching, and then modify it if and as necessary to eliminate subcycles and make it a sequence matching. So, if subcycles were of no concern, what would the least-cost b-to-a matching
be? It turns out to be the identity matching, $\mathcal{M}_{0}$, that matches $b_{0}$ to $a_{0}, b_{1}$ to $a_{1}$, etc. Stated more formally:

Theorem 6.4 If $\Pi(n)$ denotes the set of permutations of $\{1,2, \ldots, n\}$ :

$$
\min _{\sigma \in \Pi(n)} \sum_{i=1}^{n} c_{i, \sigma(i)}=\sum_{i=1}^{n} c_{i i}=\sum_{i=1}^{n}\left|a_{i}-b_{i}\right|
$$

The proof of the above theorem follows from an interchange argument that shows that $c_{i i}+c_{j j}$ is no greater than $c_{i j}+c_{j i}$ for any $1 \leq i \neq j \leq n$.

We start, then, with $\operatorname{arcs} \mathcal{M}_{J} \cup \mathcal{M}_{0}$, as shown for Example 6.1 in Fig. 6.2b, where the matching $\mathcal{M}_{0}$ is depicted by dotted arcs. $\mathcal{M}_{J} \cup \mathcal{M}_{0}$ will form the union of subcycles $\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{m}$, where $2 \leq m \leq n+1$. One of them will always be the trivial subcycle $\left(a_{0}, b_{0}, a_{0}\right)$, while the other $n$ jobs could all make up one subcycle, or at the other extreme could each be a separate subcycle $\left(a_{j}, b_{j}, a_{j}\right), j=1, \ldots, n$. For our example, there are five subcycles, also indicated in Fig. 6.2b.

Our goal is to replace $\mathcal{M}_{0}$ by a modified mapping, adding and deleting arcs so as to interconnect the subcycles at minimal cost. To determine how best to do this, it will be useful to construct a graph $\mathcal{G}$ with vertex set $\left\{\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{m}\right\}$. (To help distinguish the two graphs, we refer to "vertices" and "edges" here, rather than the "nodes" and "arcs" of $\mathcal{B}$.) The edge set of $\mathcal{G}$ is constructed as follows. If the nodes $b_{i}$ and $a_{i+1}, i=0,1,2, \ldots, n-1$, lie in different subcycles of $\mathcal{M}_{J} \cup \mathcal{M}_{0}$, say $\mathcal{C}$ and $\mathcal{C}^{\prime}$, then add an (undirected) edge labeled $e_{i, i+1}$ between $\mathcal{C}$ and $\mathcal{C}^{\prime}$ in $\mathcal{G}$.

Evidently, $\mathcal{G}$ is connected, so that a spanning tree exists with $m-1$ edges. In Fig. 6.3 we depict $\mathcal{G}$ for our example. Each edge is labeled by the appropriate $e_{i, i+1}$. In addition, we associate with each edge the cost $w_{i, i+1}$ of merging the corresponding 2 subcycles. Consider for example the subcycles $\left(a_{0}, b_{0}, a_{0}\right)$ and ( $a_{1}, b_{1}, a_{1}$ ) in Fig. 6.2b. To merge them, we must remove the $\operatorname{arcs}\left(b_{0}, a_{0}\right)$ and $\left(b_{1}, a_{1}\right)$, replacing them with $\left(b_{0}, a_{1}\right)$ and $\left(b_{1}, a_{0}\right)$. It follows that the cost of merging adjacent subcycles is

$$
w_{i, i+1}=-\left(c_{i i}+c_{i+1, i+1}\right)+c_{i, i+1}+c_{i+1, i}
$$

where $c_{i j}=\left|b_{i}-a_{j}\right|$. The numerical value of $w_{i, i+1}$ is given below each arc in Fig. 6.3.


Fig. 6.3 The graph $\mathcal{G}$ for the sample instance of $F 2 \mid$ nwt or block, $($ perm $) \mid C_{\max }$, with the minimal spanning tree $\mathcal{T}$ shown in heavy lines

Let $\mathcal{T}$ be a minimal spanning tree of $\mathcal{G}$, as shown for our example by heavy lines. Edges in $\mathcal{T}$ reflect the lowest-cost set of changes that must be made in $\mathcal{M}_{0}$ in order to merge all subcycles of $\mathcal{M}_{J} \cup \mathcal{M}_{0}$. For this reason, for every $e_{i, i+1} \in \mathcal{T}$, we drop from the identity matching $\mathcal{M}_{0}$ the $\operatorname{arcs}\left(b_{i}, a_{i}\right)$ and $\left(b_{i+1}, a_{i+1}\right)$. However, we cannot simply replace them with $\left(b_{i}, a_{i+1}\right)$ and $\left(b_{i+1}, a_{i}\right)$, as we did in defining $w_{i, i+1}$. Consider, for instance, three consecutive trivial subcycles, say $\mathcal{C}_{i}=\left(a_{i}, b_{i}, a_{i}\right), i=1,2,3$. Having made the substitutions to merge $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$, it will not be possible to do the same thing for $\mathcal{C}_{2}$ and $\mathcal{C}_{3}$. Nevertheless, it is shown (see Vairaktarakis 2003) that the edges of $\mathcal{T}$ always point to the set, $\mathcal{R}$, of arcs in $\mathcal{M}_{0}$ that must be discarded. Thus, in Fig. 6.3, $\mathcal{T}=\left\{e_{01}, e_{12}, e_{23}, e_{56}\right\}$, and hence we drop from $\mathcal{M}_{0}$ the $\operatorname{arcs} \mathcal{R}=\left\{\left(b_{0}, a_{0}\right),\left(b_{1}, a_{1}\right),\left(b_{2}, a_{2}\right),\left(b_{3}, a_{3}\right),\left(b_{5}, a_{5}\right),\left(b_{6}, a_{6}\right)\right\}$. It remains to be decided which new arcs to add, to produce the minimal cost tour.

To do this, we first partition the set $\mathcal{R}$ into subsets of consecutive arcs. Let $\mathcal{B}_{1}, \mathcal{B}_{2}, \ldots, \mathcal{B}_{k}$ be the blocks of consecutive $\left(b_{i}, a_{i}\right) \operatorname{arcs}$ in $\mathcal{R}$. All dropped arcs must be replaced by new ones, and to do this our algorithm is applied separately and independently to each block. Since the arcs are gone, we will refer to $\left(b_{i}, a_{i}\right)$ as a node pair henceforth. Fig. 6.4a shows $\mathcal{M}_{J} \cup\left(\mathcal{M}_{0} \backslash \mathcal{R}\right)$ for the sample instance, with the two blocks of node pairs with missing arcs indicated. Block $\mathcal{B}_{1}$ consists of 4 pairs: $\left(b_{0}, a_{0}\right),\left(b_{1}, a_{1}\right),\left(b_{2}, a_{2}\right)$ and $\left(b_{3}, a_{3}\right)$, while block $\mathcal{B}_{2}$ consists of $\left(b_{5}, a_{5}\right)$ and $\left(b_{6}, a_{6}\right)$.


Fig. 6.4 (a) The block structure and (b) the consequent solution for the sample instance of $F 2 \mid n w t$ or block, $($ perm $) \mid C_{\text {max }}$

Now, how to choose replacement arcs for an arbitrary block $\mathcal{B}_{l}, 1 \leq l \leq k$ ? To start with, if $\mathcal{B}_{l}$ consists of precisely two pairs, say $\left(b_{i}, a_{i}\right)$ and $\left(b_{i+1}, a_{i+1}\right)$, then the 2 subcycles traversed by these pairs are merged optimally by introducing the arcs $\left(b_{i}, a_{i+1}\right)$ and $\left(b_{i+1}, a_{i}\right)$. In Example 6.1, block $\mathcal{B}_{2}$ consists of precisely 2 pairs and hence the corresponding subcycles are merged by using the edges $\left(b_{5}, a_{6}\right)$ and $\left(b_{6}, a_{5}\right)$ (see Fig. 6.4b).

It remains to describe the algorithm for the case where $\mathcal{B}_{l}$ consists of three or more pairs $\left(b_{i}, a_{i}\right)$. Suppose (without loss of generality) that $\mathcal{B}_{l}$ consists of the pairs $\left(b_{1}, a_{1}\right),\left(b_{2}, a_{2}\right), \ldots,\left(b_{r}, a_{r}\right)$, where $r \geq 3$. The first step will
be to label each pair as an A-pair or B-pair depending on whether $a_{i} \geq b_{i}$ or $a_{i}<b_{i}$, respectively. It is then convenient to change, if necessary, the labels of the first and last pairs, to make them the opposite of their immediate neighbor's. Interestingly, having labeled the pairs, the actual processing times of each node will no longer be needed. Perhaps even more surprisingly, the job matching is also irrelevant.

In Example 6.1, the string of labels associated with block $\mathcal{B}_{1}$ has the form $\mathrm{B}, \mathrm{A}, \mathrm{B}, \mathrm{A}$ because the pair $\left(b_{2}, a_{2}\right)=(2,3)$ is of type $\mathrm{A},\left(b_{3}, a_{3}\right)=(5,4)$ is of type $B$, and the first and last labels are chosen to differ from their neighbors. This too is shown in Fig. 6.4b, where we have distinguished A-pairs from B-pairs by using circles and squares, respectively.

To provide a more challenging block of node pairs with which to illustrate the algorithm, we introduce

Example 6.2: Let $\mathcal{B}_{l}$ have 14 pairs, and suppose the values of $\left(b_{j}, a_{j}\right), j=$ $1, \ldots, 14$ generate the string of labels: $A, B, B, B, B, A, A, A, B, A, B, B, B, A$. This block is shown in Fig. 6.5, with the nodes of the A-pairs shown as circles and the B-pair nodes as squares.

The arcs shown are those produced by the following algorithm:

## Arc Generation Algorithm

1. Link the first (smallest) unmatched $b$-node of a B-pair to the first unmatched $a$-node of a larger (i.e., to the right) B-pair. Repeat until all but one $b$-nodes of B-pairs are linked. Link the last $b$-node of a B-pair to the $a$-node of the last A-pair.
2. Now link the A-pairs, similarly but in reverse; to wit:

Link the last unmatched $b$-node of an A-pair to the last unmatched $a$-node of a smaller (i.e., to the left) A-pair. Repeat until all but one $b$-nodes of A-pairs are linked. Link the last $b$-node of an A-pair to the only remaining $a$-node (which is of a B-pair).

Applying this to $\mathcal{B}_{1}$ of Example 6.1, Step 1 produces the two heavy dashed arrows, and Step 2 fills in the dot-dashed arcs in Fig. 6.4b. We can now read the optimal schedule from this graph, starting from $a_{0}$ and following the sequence of arrows. Recall that the solid arrows give the job matching, with $J_{j}$ associated with $\left(a_{j}, b_{\pi(j)}\right)$. Thus, $\left(a_{0}, b_{0}\right)$ followed by $\left(a_{2}, b_{5}\right)$, then $\left(a_{6}, b_{6}\right)$, etc., means $J_{0}$ followed by $J_{2}$, then $J_{6}$, etc., so that the optimal schedule for Example 1 can be seen to be

$$
S^{\star}=(2,6,5,4,3,1)
$$

To gain a little more insight into the algorithm, consider the solution to Example 6.2, as shown in Fig. 6.5. Note how Step 1, which dealt with the B-pairs, produces the dashed "forward-leaning" arrows (from $b_{2}$ to $a_{3}, b_{3}$ to $a_{4}$, etc.). To help explain why this is desirable, we observe:

- We prefer to link pairs to adjacent pairs (either $b_{i}$ to $a_{i+1}$ or $b_{i+1}$ to $a_{i}$ ), or at least to nearby pairs, because, having arranged the $a_{i}$ 's and $b_{i}$ 's in numerical order, nearer pairs will give lower cost, $\left|a_{i}-b_{j}\right|$.
- For two adjacent B-pairs, say $\left(a_{i}, b_{i}\right)$ and $\left(a_{i+1}, b_{i+1}\right)$, we have

$$
a_{i} \leq a_{i+1}, \quad b_{i} \leq b_{i+1}, \quad a_{i}<b_{i}, \quad a_{i+1}<b_{i+1}
$$

It follows that

$$
\left|a_{i+1}-b_{i}\right|<\left|a_{i}-b_{i+1}\right|,
$$

and this is why, when connecting adjacent or nearby B-pairs, forwardleaning arcs (connecting $b_{i}$ to $a_{i+1}$ ) are preferred.


Fig. 6.5 Arc generation for a block with labels ABBBBAAABABBBA

By the same token, we prefer to link sequences of A-pairs by backwardleaning arcs, as we see in the figure. It only remains then to tie up the loose ends with one more arc at each end of the block, to connect the A-pairs and B-pairs together, completing the cycle. Of course, none of this discussion constitutes a formal proof of the algorithm's optimality, which will not be given here and can be found in Vairaktarakis (2003).

### 6.3.7 Summary of the Algorithm

We now summarize the procedure for solving $F 2 \mid n w t$ or $b l o c k,($ perm $) \mid C_{\max }$. It is not hard to check that this variation by Vairaktarakis (2003) produces the same solution as the original algorithm of Gilmore and Gomory (1964). Thus, we will call it

## The GGV Algorithm

Input : $a_{1} \leq a_{2} \leq \ldots \leq a_{n}, b_{1} \leq b_{2} \leq \ldots \leq b_{n}$, viewed as the nodes of a bipartite graph $\mathcal{B}$, and an $a$-to- $b$ matching $\mathcal{M}_{J}=\left\{\left(a_{i}, b_{\pi(i)}\right): i=1, \ldots, n\right\}$. Output : A $b$-to- $a$ matching $\mathcal{M}^{\star}$ that, along with $\mathcal{M}_{J}$, gives a minimal cost tour of the bipartite graph, assuming arcs in $\mathcal{M}_{J}$ have zero cost, and arcs from $b_{i}$ to $a_{j}$ have cost $c_{i j}$ satisfying the Gilmore-Gomory metric.

1. Given $\mathcal{B}$ with arc set $\mathcal{M}_{J} \cup \mathcal{M}_{0}$, where $\mathcal{M}_{0}=\left\{\left(b_{i}, a_{i}\right): i=1, \ldots, n\right\}$, find
cycles and construct graph $\mathcal{G}$.
2. Assign to each edge $e_{i, i+1} \in \mathcal{G}$ a cost
$w_{i, i+1}=c_{i, i+1}+c_{i+1, i}-c_{i i}-c_{i+1, i+1}$,
and find the minimal spanning tree $\mathcal{T}$.
3. Let $\mathcal{R}=\left\{\left(b_{i}, a_{i}\right): e_{i, i+1}\right.$ or $\left.e_{i-1, i} \in \mathcal{T}\right\}$, and let $\mathcal{M}=\mathcal{M}_{0} \backslash \mathcal{R}$.
4. Let $\mathcal{B}_{1}, \mathcal{B}_{2}, \ldots, \mathcal{B}_{k}$ be the subsets of consecutive $\left(b_{i}, a_{i}\right)$ pairs in $\mathcal{R}$.
5. For $l=1$ to $k$ do

If $\mathcal{B}_{l}$ consists of precisely 2 pairs, say $\left(b_{i}, a_{i}\right)$ and $\left(b_{i+1}, a_{i+1}\right)$
then $\mathcal{M}:=\mathcal{M} \cup\left\{\left(b_{i}, a_{i+1}\right),\left(b_{i+1}, a_{i}\right)\right\}$ else
begin

- Assign to each pair $\left(b_{i}, a_{i}\right)$ in $\mathcal{B}_{l}$ the label A [B] if $a_{i} \geq b_{i}\left[a_{i}<b_{i}\right]$.
- Adjust, if necessary, the first and last labels, to make them different from their immediate neighbors.
- Apply the Arc Generation Algorithm, adding the new arcs to $\mathcal{M}$. end

6. Let $\mathcal{M}^{\star}$ be the resulting matching.

The optimality of GGV is established in the following theorem.
Theorem 6.5 (Vairaktarakis, 2003)
(a) The GGV Algorithm solves the TSP optimally.
(b) If $J_{i}=\left\langle a_{i}, b_{\pi(i)}>\right.$ and $\left.c_{i j}=\right| b_{i}-a_{j} \mid$ for $i, j=1, \ldots, n$, then the GGV Algorithm solves $F 2 \mid$ nwt or block, $($ perm $) \mid C_{\max }$ optimally, where $\left(b_{i}, a_{j}\right) \in \mathcal{M}^{\star}$ indicates that $J_{j}$ immediately follows $J_{i}$ in $S^{\star}$.

The working space required to find an optimal tour for TSP is precisely $n$ elements which is the minimum possible if we consider that a tour is a permutation of $n$ cities. Steps 1 and 2 of the GGV algorithm can be executed using a vector whose $i^{t h}$ element indicates the subcycle of $\mathcal{M}_{J} \cup \mathcal{M}_{0}$ traversing the pair $\left(b_{i}, a_{i}\right), i=0,1, \ldots, n$. In our example, this vector is $\left(C_{1}, C_{2}, C_{3}, C_{4}, C_{3}, C_{3}, C_{5}\right)$ (see Fig. 6.2(b)). By looking at this vector we know that $\mathcal{G}$ consists of the edges $e_{01}, e_{12}, e_{23}, e_{34}$ and $e_{56}$ because the pairs $\left(b_{i}, a_{i}\right)$ associated with these edges are traversed by different subcycles. The minimal spanning tree $\mathcal{T}$ consists of the edges $e_{01}, e_{12}, e_{23}$ and $e_{56}$ and hence $\left(b_{4}, a_{4}\right)$ is the only pair not involved with $\mathcal{T}$. Thus, we initialize $\mathcal{M}=\left(\cdot, \cdot, \cdot, \cdot,\left(b_{4}, a_{4}\right), \cdot, \cdot\right)$ in Step 3. This vector shows that there are two blocks to be considered; $\mathcal{B}_{1}$ consisting of 4 pairs, and $\mathcal{B}_{2}$ consisting of 2 pairs of nodes. Applying Step 5 to each block, we get $\mathcal{M}^{\star}=\left(\left(b_{0}, a_{2}\right),\left(b_{1}, a_{o}\right),\left(b_{2}, a_{3}\right),\left(b_{3}, a_{2}\right),\left(b_{4}, a_{4}\right),\left(b_{5}, a_{6}\right),\left(b_{6}, a_{5}\right)\right)$.

### 6.3.8 Generalizations of $F 2 \mid n w t$ or block, $($ perm $) \mid C_{\text {max }}$

Strusevich (1990) generalized $F 2 \mid n w t$ or block, $($ perm $) \mid C_{\text {max }}$ to include job setup and teardown (or removal) times that are detached: either can be done
on one machine at the same time the job is in process on the other. The resulting problem is solvable by the GGV Algorithm.

Röck (1984) considered F2|nwt or block, (perm), $p_{j k}=1$, res $\mid C_{\text {max }}$ when there is an additional single resource res available (e.g., an operator) and each task may or may not require this resource for processing.

Espinouse et al. (1999) considered the problem with a single interval of unavailability for $M_{1}$ and showed that the problem becomes NP-complete. They also developed a heuristic, based on GGV, that yields a worst case error bound of 2 .

### 6.3.9 Other Objectives

Few other objectives (including non-traditional ones) have been investigated for the 2 -machine no-wait flow shop. As might be expected, the flow time objective is amongst them. The special case when all processing times are either zero or one, $F 2 \mid n w t$ or block, $($ perm $), p_{i j} \in\{0,1\} \mid \Sigma C_{j}$, has been considered by Sriskandarajah and Ladet (1986). Their result can easily be generalized to allow processing times to be either of two arbitrary values, say $x$ and $y, 0 \leq x<y$. We partition the jobs into four sets: $\mathcal{J}_{x x}, \mathcal{J}_{x y}, \mathcal{J}_{y x}$, and $\mathcal{J}_{y y}$, where of course $\mathcal{J}_{x y}$ denotes the set of all jobs with task times $\langle x, y\rangle$, etc. Then $S^{\star}=\left(\mathcal{J}_{x x}, \mathcal{J}_{x y}\right.$ and $\mathcal{J}_{y x}$ alternating as far as possible, $\left.\mathcal{J}_{y y}\right)$. By "alternating as far as possible" we mean: (1) Schedule a job from $\mathcal{J}_{x y}$ (if any); (2) Schedule alternate jobs from $\mathcal{J}_{y x}$ and $\mathcal{J}_{x y}$ until one set is empty; (3) Schedule the rest of the other set. Of course, if $x=0, \mathcal{J}_{x x}$ is empty. $S^{\star}$ minimizes both $\Sigma C_{j}$ and $C_{\text {max }}$.

Both F2|nwt or block, (perm) $\mid L_{\max }$ and $F 2 \mid$ nwt or block, $($ perm $) \mid \Sigma C_{j}$ are strongly NP-complete, as was shown by Röck (1984).

Strusevich (1995) considered the 2-machine no-wait flow shop with controllable processing speeds $v_{1}, v_{2}$ on $M_{1}, M_{2}$ respectively. Let $C\left(v_{1}, v_{2}\right)$ be the minimal makespan for given speeds $v_{1}, v_{2}$. For given constants $c_{0}, c_{1}, c_{2} \geq 0$ and given positive integers $c_{3}, c_{4}$, the author considered the problem of minimizing objective $Z\left(v_{1}, v_{2}\right)=c_{0} C\left(v_{1}, v_{2}\right)^{c_{3}}+c_{1} v_{i}^{c_{4}}+c_{2} v_{2}^{c 4}$. It is shown that there exists an optimal solution where speeds $v_{1}^{\star}, v_{2}^{\star}$ may take on one of $O\left(n^{2}\right)$ candidate values for $v_{1}^{\star} / v_{2}^{\star}$. For each such pair, algorithm GGV is adapted to solve $F 2 \mid n w t$ or block, $($ perm $) \mid Z\left(v_{1}, v_{2}\right)$ in $O\left(n^{3}\right)$ time.

### 6.4 Fm|nwt, (perm) $\mid C_{\text {max }}$

We shall now consider problems with more than two machines. Thus, combining blocking with no-wait is no longer possible, and we speak only of the no-wait flow shop. In this section we review results for the $m$-stage flow shop with a single machine in each stage. As in the previous section, we assume that zero-length tasks, if any, must be scheduled (the no-skip assumption), so that only permutation schedules are possible. The special case where $m=2$,
studied earlier, had a polynomial solution, but Lenstra et al. (1977) showed that the problem is strongly NP-complete for arbitrary $m$, while Röck (1984a) showed strong NP-completeness for fixed $m \geq 3$. As usual, $p_{r j}$ is the processing time requirement of $J_{j}$ on $M_{r}$ where $r=1, \ldots, m$ and $j=1, \ldots, n$. A TSP formulation for problem $F m \mid n w t$, (perm) $\mid C_{\text {max }}$ was originally conceived by Piehler (1960) for $m=3$, generalized by Wismer (1972), then modified by Emmons and Mathur (1995) as follows. Consider a complete graph with $n+1$ nodes numbered $0,1, \ldots, n$ where node 0 corresponds to the beginning and the end, and nodes $1, \ldots, n$ to the $n$ jobs. We define the distances between nodes as

$$
\begin{aligned}
d_{i j}= & \text { the additional time required to schedule } J_{j} \text { immediately } \\
& \text { after } J_{i} .
\end{aligned}
$$

They are given by

$$
\begin{aligned}
d_{0 j} & =\sum_{r=1}^{m} p_{r j} \quad \text { for } \quad j=1,2, \ldots, n \\
d_{i 0} & =d_{i i}=0 \quad \text { for } \quad i=1,2, \ldots, n \\
d_{i j} & =\max _{k=1,2, \ldots, m}\left\{p_{k i}+\sum_{r=k}^{m}\left(p_{r j}-p_{r i}\right)\right\}, \quad i, j=1, \ldots, n, i \neq j
\end{aligned}
$$

To understand the formula for $d_{i j}$, see Fig. 6.6, where the maximum is achieved at $k=2$, giving $d_{i j}=\Sigma_{r=2}^{4} p_{r j}-\Sigma_{r=3}^{4} p_{r i}$. Since the no-wait feature locks the tasks of $J_{i}$ together, presenting to the next job a "profile" independent of the earlier jobs, $d_{i j}$ depends only on the characteristics of $J_{i}$ and $J_{j}$. Note that formulating $F m|n w t| C_{\max }$ as a TSP is not possible if jobs may require processing on only a subset of processors; this case was shown to be strongly NP-complete by Sahni and Cho (1979).


Fig. 6.6
The increase in makespan when $J_{j}$ follows $J_{i}$

### 6.4.1 Polynomially Solvable Cases

Sriskandarajah and Ladet (1986) presented an $O(n \log n)$ algorithm for $F 3 \mid n w t,($ perm $), p_{j k} \in\{0,1\} \mid C_{\text {max }}$, i.e., when all task times are either null or 1 . However, when the number $m$ of machines is variable, Gonzalez (1982) proved that $F \mid n w t,($ perm $), p_{j k} \in\{0,1\} \mid C_{\text {max }}$ is strongly NP-complete. Solvable cases with general processing times are provided next.

Smith et al. (1975) report no-wait applications in case processing times are semi-ordered, as defined next:

Definition 6.1 A processing time matrix $\mathbf{P}=\left(p_{i j}\right)$ is said to be semiordered iff there exists an indexing of the jobs such that

$$
p_{k 1} \leq p_{k 2} \leq \ldots \leq p_{k n}, \quad \text { for } \quad k=1, \ldots, m
$$

The permutation in Definition 6.1 induces an SPT order on every processor, and hence we may refer to it as an SPT permutation of the jobs. Henceforth assume jobs are so indexed. We have the following result:

Theorem 6.6 (Panwalkar and Woolam, 1979) For Fm|nwt, (perm) $\mid C_{\max }$ having a semi-ordered processing time matrix with the largest processing times on $M_{m}$, i.e., $p_{k j} \leq p_{m j}$ for all $j=1, \ldots, n, k=1, \ldots, m, S^{\star}=S P T$.

Arora and Rana (1980) proved that, when jobs correspond to cities, the TSP formulation for semi-ordered matrices has a distance matrix that satisfies the property

$$
d_{i j}+d_{i^{\prime} j^{\prime}} \leq d_{i j^{\prime}}+d_{i^{\prime} j} \text { for all } i<i^{\prime} \text { and } j<j^{\prime}
$$

Such distance matrices are known as distribution matrices. The TSP on distribution distance matrices is solvable in $O\left(n^{2}\right)$ time (see Gilmore et al., 1985).

Closely related to semi-ordered processing time matrices is the concept of pyramidal schedules:

Definition 6.2 For a semi-ordered processing time matrix $\mathbf{P}=\left(p_{i j}\right)$, a schedule is said to be pyramidal when all jobs prior to $J_{n}$ (the job with largest task times) are ordered in nondecreasing order of processing times, while the remaining jobs follow $J_{n}$ in nonincreasing order of processing times.

We have the following major result:
Theorem 6.7 (Arora and Rana, 1980) For Fm|nwt, (perm) $\mid C_{\max }$ with a semi-ordered processing time matrix, there exists an optimal schedule which is pyramidal.

Arora and Rana (1980) presented an $O\left(n^{3}\right)$ algorithm that finds an optimal pyramidal schedule, which was later improved in Axsater (1982) who developed the following $O\left(n^{2}\right)$ dynamic program. Observe that a pyramidal schedule can be obtained by building out from the middle, adding jobs in LPT order. Given SPT indexing, for any partial schedule of jobs $\left\{J_{j+1}, \ldots, J_{n}\right\}$, consider adding $J_{j}$ either to the beginning or to the end of the partial schedule. As we have argued, the resulting increase in makespan only depends on the parameters of $J_{j}$ and the job, say $J_{i}$, adjacent to it. If $J_{j}$ is added at the end, following $J_{i}$, we again have:

$$
d_{i j}=\max _{k=1,2, \ldots, m}\left\{p_{k i}+\sum_{r=k}^{m}\left(p_{r j}-p_{r i}\right)\right\}
$$

By symmetry, if $J_{j}$ is added at the start, immediately before $J_{i}$, then the increase in makespan is:

$$
d_{j i}^{\prime}=\max _{k=1,2, \ldots, m}\left\{p_{k j}+\sum_{r=1}^{k}\left(p_{r i}-p_{r j}\right)\right\} .
$$

Define the following state space variables:
$f_{j}^{1}(k)=$ minimal makespan for $J_{j}, J_{j+1}, \ldots, J_{n}$ when $J_{j}$ is scheduled first and $J_{k}(k>j)$ is last, and
$f_{j}^{2}(k)=$ minimal makespan for $J_{j}, J_{j+1}, \ldots, J_{n}$ when $J_{j}$ is scheduled last and $J_{k}(k>j)$ is first.

Then, the following implicit enumeration, adding jobs of decreasing size to either end of the schedule, yields an optimal pyramidal schedule:

$$
\begin{aligned}
& f_{j}^{1}(k)= \begin{cases}\min _{i>j+1}\left\{f_{j+1}^{2}(i)+d_{j i}^{\prime}\right\}, & \text { if } k=j+1 \\
f_{j+1}^{1}(k)+d_{j, j+1}^{\prime}, & \text { if } k>j+1\end{cases} \\
& f_{j}^{2}(k)= \begin{cases}\min _{i>j+1}\left\{f_{j+1}^{1}(i)+d_{i j}\right\}, & \text { if } k=j+1 \\
f_{j+1}^{2}(k)+d_{j, j+1}, & \text { if } k>j+1\end{cases}
\end{aligned}
$$

where the initial conditions are:

$$
\begin{aligned}
& f_{n-1}^{1}(n)=d_{n-1, n}^{\prime}+\sum_{k=1}^{m} p_{k n} \\
& f_{n-1}^{2}(n)=d_{n, n-1}+\sum_{k=1}^{m} p_{k n}
\end{aligned}
$$

The optimal solution is then $C^{\star}=\min _{k=2, \ldots, n}\left\{f_{1}^{1}(k), f_{1}^{2}(k)\right\}$. Evidently, the above dynamic program has $O\left(n^{2}\right)$ states, each computed in $O(1)$ time.

Theorem 6.8 (van der Veen and van Dal, 1991) When the processing times are fixed on all but two machines, $M_{l}$ and $M_{u}, l<u$ (i.e., $p_{k j}=c_{k}$ for $j=1,2, \ldots, n$ and $k=1,2, \ldots, m, k \neq l, u)$, and one of the following holds:
(i) $u=l+1$, or (ii) $l=1$, or (iii) $u=m$, then $F m \mid n w t,($ perm $) \mid C_{\max }$ is polynomially solvable.

The missing case where $1<l<u-1<m-1$ remains open; however, the authors conjecture that the resulting version of the problem is NP-complete.

### 6.4.2 Heuristic Algorithms

The majority of heuristics that have appeared in the literature for problem $F m \mid n w t$, perm $\mid C_{\max }$ employ local search using simulated annealing, genetic algorithms, tabu search, etc. Rajendran (1994) developed a heuristic (referred to as RAJ) based on the NEH heuristic of Nawaz et al. (1983) for Fm|perm $\mid C_{\text {max }}$. It is found that RAJ outperforms the heuristics presented earlier in Gangadharan and Rajendran (1993) and hence RAJ is used as
benchmark to measure the performance of local search algorithms that appeared in the literature after 1994.

Aldowaisan and Allahverdi (2003) developed genetic and simulated annealing algorithms for $F m \mid n w t$, perm $\mid C_{\max }$ while Schuster and Framinan (2003) presented a hybrid of the two, referred to as GASA. They also presented a variable neighborhood search algorithm VNS. The authors showed that VNS outperforms GASA while both outperform RAJ in solution quality.

Grabowski and Pempera (2005) introduced the idea of multimoves. Given a permutation $\pi$ of jobs, consider all insertions $(x, y)$ where $J_{[y]}$ is placed immediately after $J_{[x]}$ when $x<y$. If $x>y$, then $J_{[y]}$ is placed immediately before $J_{[x]}$. Among all such insertions, consider the set that improve makespan. This collection of moves, say $\mathcal{A}(\pi)$, is said to be a multimove. Grabowski and Pempera developed a tabu search algorithm TS and 2 variations of it that utilize mutimoves. Algorithm TS+M applies a multimove in every iteration, and TS+MP applies a multimove periodically and a simple insertion in all other iterations. Also, the authors implemented a greedy algorithm based on simple insertions (referred to as Descent Search algorithm, DS ), and its variation $\mathrm{DS}+\mathrm{M}$ where a multimove instead of a simple insertion is used in each iteration.

The above algorithms are tested on the 30 test problems presented in Heller (1960) for problem $F m \mid$ perm $\mid C_{\text {max }}$ and later used in Reeves (1995). Through their experiments, Grabowski and Pempera (2005) showed that 1000 iterations of their TS heuristic and its variations significantly outperform the GASA heuristic (especially as $n$ increases) of Schuster and Framinan (2003), and less so heuristic VNS. For the 30 instances tested, the average percentage improvement obtained by TS, TS +M , TS+MP over the RAJ heuristic is $6.50 \%, 6.59 \%$ and $6.56 \%$ respectively while the CPU effort is less than 4 seconds for a computer running at $1 \mathrm{GHz}, m=10$ machines and $n=100$ jobs. Heuristics DS and DS +M are inferior to their tabu counterparts, significantly outperform RAJ and GASA, but on average are inferior to VNS. Moreover, multimoves do not seem to make a notable difference in DS +M compared to DS.

### 6.4.3 Fm|nwt, (perm) $\mid C_{\text {max }}$ with Setup and Teardown Times

We have seen how $F m \mid n w t,($ perm $) \mid C_{\text {max }}$ can be formulated as a TSP, due to the no-wait feature which locks the parts of a job rigidly together so that the profile a job presents to the adjacent jobs, both before and after it in a schedule, are predetermined and fixed. This property carries over to $F m \mid n w t,($ perm $), s_{k j}, t_{k j} \mid C_{\max }$, by which we mean the no-wait flow shop with separable setup times, $s_{k j}$, and teardown times, $t_{k j}$, for each $J_{j}$ on each $M_{k}$. We illustrate such a system in Fig. 6.7, where two adjacent jobs are shown in a three-machine system.


Fig. 6.7
Two jobs in an $F 3\left|n w t, s_{k j}, t_{k j}\right| C_{\text {max }}$ system

In the TSP, as before, the distance $d_{i j}$ from $J_{i}$ to $J_{j}$ is the extra time required to process $J_{j}$ if it immediately follows $J_{i}$. Incidentally, we assume a job is completed only when its final teardown is finished. It is convenient to define the dummy job $J_{0}$, with all its parameters set to zero, as the start and end point of the tour (and the schedule). We also introduce the variables (as shown in Fig. 6.7):
$B_{k}=$ the start time of the setup of $J_{k}$ on $M_{1}$ (which may not be the earliest this job occupies a machine, as illustrated by $J_{i}$ );
$C_{k}=$ the completion time of $J_{k}$, i.e. of its final teardown (which may not be on $M_{m}$, as shown for $J_{j}$ in the figure).

Then, the distances are:

$$
\begin{aligned}
& d_{i 0}=d_{i i}=0, \quad \text { for } i=1,2, \ldots, n \\
& d_{i j}=C_{j}-C_{i}=\left(C_{j}-B_{j}\right)+\left(B_{j}-B_{i}\right)-\left(C_{i}-B_{i}\right), \\
& \\
& \quad i=0,1, \ldots, n, \quad j=1, \ldots, n, i \neq j
\end{aligned}
$$

where:

$$
\begin{aligned}
C_{j}-B_{j} & =s_{1 j}+\max _{k=1 \ldots m}\left\{\sum_{r=1}^{k} p_{r j}+t_{k j}\right\}, \\
B_{j}-B_{i} & =s_{1 i}-s_{1 j}+\max _{k=1 \ldots m}\left\{\sum_{r=1}^{k}\left(p_{r i}-p_{r j}\right)+t_{k i}+s_{k j}+p_{k j}\right\}
\end{aligned}
$$

For example, in Fig. 6.7,

$$
\begin{aligned}
& C_{i}-B_{i}=s_{1 i}+\sum_{r=1}^{3} p_{r i}+t_{3 i}, \\
& C_{j}-B_{j}=s_{1 j}+\sum_{r=1}^{2} p_{r j}+t_{2 j}, \\
& B_{j}-B_{i}=s_{1 i}+\sum_{r=1}^{2} p_{r j}+t_{2 i}+s_{2 j}-p_{1 j}-s_{1 j}
\end{aligned}
$$

so that

$$
d_{i j}=-\left(p_{3 i}+t_{3 i}\right)+t_{2 i}+s_{2 j}+p_{2 j}+t_{2 j} .
$$

This sequence of intervals can be easily traced in the figure.

### 6.4.4 Identical Jobs in Fm|nwt, (perm) $\mid C_{\text {max }}$

Since the principal tool for scheduling nondelay flow shops is the TSP algorithm, we have difficulty handling very large numbers of jobs. Frequently
in such cases the jobs fall into a relatively small number of classes or types, each type having the same time requirements at every stage. They may be identical copies of the same product, or similar jobs with the same processing requirements. If the $n$ jobs to be scheduled can be classified into $u$ types ( $u \ll n$ ), each type having identical time requirements, they can often be grouped into batches for simplified scheduling.

It should be noted that, with no-wait scheduling, the kind of batching we are now concerned with is not the same as lot streaming, to be considered later. The latter involves batches or sublots that are transferred together from one machine to the next. Thus, the first job in a sublot must await the completion of all the other jobs in the group before starting its next operation, hence it is delayed, although the entire sublot may require no-wait service. For now, we continue to assume that each separate job must be scheduled without delay. Since all jobs of a type are indistinguishable, $d_{i j}$ is now the additional time of sequencing a job of type $j$ immediately after a type- $i$ job.

## Scheduling Identical Jobs in Large Batches

We first present some conditions under which large numbers of identical jobs should be scheduled consecutively.

Theorem 6.9 (Rothkopf, 1966) If for some type $k, d_{k k} \leq d_{i k}+d_{k j}-d_{i j}$ for all types $i$ and $j, i, j=1, \ldots, u$ and $i, j \neq k$, then there exists an optimal schedule where all jobs of type $k$ are processed consecutively.

Intuitively, Theorem 6.9 states that, if the makespan increase of scheduling 2 jobs of type $k$ next to each other is smaller than scheduling $k$ between $i$ and $j$, then there is an optimal schedule where all jobs of type $k$ are batched together.

If we define a $\boldsymbol{k}$-batch to be two or more jobs of type $k$ scheduled consecutively, we have:

Theorem 6.10 (Emmons and Mathur, 1995) Let $S^{\star}$ be an optimal schedule for the instance $I=\left(n_{1}, n_{2}, \ldots, n_{u}\right)$, where $n_{k}=$ the number of jobs of type $k$ to be scheduled. Then,
(a) if $S^{\star}$ includes a $k$-batch, then an optimal schedule for $I^{-}=\left(n_{1}, \ldots\right.$, $\left.n_{k}-1, \ldots, n_{u}\right)$ can be constructed from $S^{\star}$ by eliminating one job from the $k$-batch.
(b) If $n_{k} \geq u$ and $S^{\star}$ includes a batch of every job type, then an optimal schedule for $I^{+}=\left(n_{1}, \ldots, n_{k}+1, \ldots, n_{u}\right)$ can be constructed from $S^{\star}$ by adding one more job to a $k$-batch.

Suppose, for example, a company that manufactures four types of jobs (say $a, b, c$, and $d)$ has an order for $\left(n_{a}, n_{b}, n_{c}, n_{d}\right)=(10,8,12,5)$. Using the processing times to compute the matrix $\left[d_{j}\right]$, suppose we find that type $a$ and type $c$ each satisfy Theorem 6.9. Thus, the two types can each be grouped as a single long job, and the TSP to be solved has only 16 instead of 36
nodes, resulting in an optimal sequence of $\left(c^{12}, b^{7}, d^{4}, a^{10}, b, d\right)$, where $b^{7}$ denotes seven consecutive jobs of type $b$. Suppose now a much larger order arrives, for $(110,208,312,405)$. Since the solution for the smaller order satisfies Theorem 6.10b, we immediately know the schedule for the large order: $\left(c^{312}, b^{207}, d^{404}, a^{110}, b, d\right)$.

## A Heuristic for $F m \mid n w t,($ perm $) \mid C_{\text {max }}$ with Multiple Products

When scheduling a large number of jobs of a few types or products, it will not always happen that, as the number of jobs of each type grows, a batch of each type develops. Instead, we may find that repeated cycles of a subset of products appear, such as ( $\ldots, a, b, c, a, b, c, a, b, c, \ldots)$, which we may write $\left(\ldots,(a b c)^{3}, \ldots\right)$. Agnetis (1997) presents a heuristic algorithm to find such schedules. Of course, Theorems 6.9 and 6.10 can be used as a preliminary step to simplify the problem. We present here a modified version of the Agnetis algorithm, that is slightly simpler and builds on the earlier development of this chapter.

Our problem, then, is to manufacture, in a no-wait $m$-machine flow shop, $n_{j}$ jobs of type $j, j=1, \ldots, u$ in the shortest time, when $p_{i j}$ is the time required by a job of type $j$ on $M_{i}$. We start with the TSP formulation given at the start of Sect. 6.4 We add a single dummy job $J_{0}$ with all zero times to mark the start of the schedule. The distances between cities (or jobs), $d_{i j}$, are now defined as the additional time required to schedule a type- $j$ job immediately after a type- $i$ job:

$$
\begin{equation*}
d_{i j}=\max _{k=1,2, \ldots, m}\left\{p_{k i}+\sum_{r=k}^{m}\left(p_{r j}-p_{r i}\right)\right\}, \quad i, j=0,1, \ldots, u \tag{6.2}
\end{equation*}
$$

as shown in Fig. 6.6. Note that we no longer define $d_{j j}=0$ : while a single job $J_{j}$ could not follow itself, two or more jobs of type $j$ may well follow each other, at a cost that simplifies to $d_{j j}=\max _{k} p_{k j}$. We omit the special formulas for $d_{0 j}$ and $d_{i 0}$ given in Sect. 6.4; they were never necessary, as the general formula (6.2) works for them, too.

Agnetis (1997) starts by solving the following closely related transportation problem $\mathbf{T}$, portrayed as a bipartite graph $\mathcal{B}$ with supply nodes $\mathcal{L}=$ $\{0,1,2, \ldots, u\}$, demand nodes $\mathcal{L}^{\prime}=\{0,1,2, \ldots, u\}$, and edge set $\{(i, j): i \in$ $\left.\mathcal{L}, j \in \mathcal{L}^{\prime}\right\}$ with unit costs $d_{i j}$. The supply at node $j \in \mathcal{L}$, and the demand at $j \in \mathcal{L}^{\prime}$, are both $n_{j}$. Note how each unit shipped from $i \in \mathcal{L}$ to $j \in \mathcal{L}^{\prime}$ corresponds to a job of type $i$ preceding a job of type $j$, contributing cost (or additional time) $d_{i j}$. The total cost of a solution to $\mathbf{T}$ therefore equals the makespan of the corresponding schedule, provided that the solution to $\mathbf{T}$ corresponds to a schedule at all. Instead of giving us a tour of the cities that solves the TSP and the flow shop problem, the transportation shipments may translate into several disconnected subtours that are not a feasible solution to our problem. This will be clarified by the following example.

Example 3: Quantities of four products (call them $a, b, c$, and $d$ ) are to be manufactured in a three-machine no-wait flow shop. Processing times $p_{k j}$ for
an item of product $j$ on $M_{k}$, and the required quantity $n_{j}$ of each product $j$, are given in Table 6.1a. We first use (6.2) to calculate the costs, $d_{i j}$, (i.e., the increments in schedule length when a type- $j$ job follows a type- $i$ job) and these are presented in Table 6.1b.

Note the addition of a single job of type $o$, the dummy job with all zero costs that will indicate the start of the schedule. All the entries in Table 6.1 b come from (6.2) except $d_{o o}$, which is set to an arbitrary large value (denoted by $K$ ) to block shipment over this route; at least this subtour can be eliminated from the start.

The solution to the transportation problem is presented in Fig. 6.8, in two ways. On the left, a bipartite graph shows the optimal shipments from supply nodes to demand nodes. On the right, the directed graph $\mathcal{G}$ is essentially the same graph, with the two nodes for each product superimposed. Interpreting

| $j$ | $M_{1}$ | $M_{2}$ | $M_{3}$ | $n_{j}$ |
| :---: | :---: | :---: | :---: | :---: |
| $o$ | 0 | 0 | 0 | 1 |
| $a$ | 4 | 6 | 7 | 20 |
| $b$ | 8 | 12 | 15 | 20 |
| $c$ | 13 | 7 | 9 | 15 |
| $d$ | 5 | 10 | 17 | 25 |

(a)

| $j$ | $o$ | $a$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $o$ | $K$ | 17 | 35 | 29 | 32 |
| $a$ | 0 | 7 | 22 | 16 | 20 |
| $b$ | 0 | 7 | 15 | 9 | 17 |
| $c$ | 0 | 7 | 19 | 13 | 18 |
| $d$ | 0 | 7 | 15 | 9 | 17 |

(b)

Table 6.1 (a) processing times and quantities, and (b) costs $d_{i j}$, for sample products
$\mathcal{G}$ from the scheduling perspective, an arc labeled $x$ directed from node $i$ to node $j$ represents $x$ occurrences of a type- $j$ job following a type- $i$ job. We may think of it as representing $x$ separate arcs, each of which must be traversed once. Thus, both graphs are multigraphs, having multiple arcs connecting node pairs.


Fig. 6.8 Two graphs showing the optimum for the transportation problem

For the sample instance, either graph clearly shows a multigraph with two components, call them $\sigma_{o a}$ and $\sigma_{b c d}$. The total cost of this optimal solution to
$\mathbf{T}$ is $T^{\star}=1025$. Note that this will always be a lower bound on the makespan, because the solution set of the TSP (hence the scheduling problem which is equivalent) is a subset of T's solution set. Focusing on $\mathcal{G}$, observe that each component is Eulerian: since supply equals demand for each product, the number of incoming and outgoing arcs are equal at each node. Therefore, Eulerian walks traversing the two components yield partial tours: $\left(o, a^{20}, o\right)$ for $\sigma_{o a}$, and $\left(b^{5}, c, d^{11}, b,(c d b)^{14}\right)$ for $\sigma_{b c d}$. Observe how the second does not translate into a partial schedule, since it lacks a starting point $o$ (we started arbitrarily at $b$ ).

Merging the two partial job schedules into a feasible solution for $F m \mid$ nowait, (perm) $\mid C_{\max }$ is equivalent to patching components $\sigma_{o a}$ and $\sigma_{b c d}$ into a single Eulerian walk; preferably, at minimal cost. Agnetis (1997) developed such a heuristic patching procedure with cost

$$
C_{H} \leq C^{\star}\left(1+(m+1) T_{1}^{\star} / 2 T^{\star}\right)
$$

where $T_{1}^{\star}$ is the optimal cost for problem $\mathbf{T}$ with unit lot sizes, i.e., $n_{j}=1, j=$ $1, \ldots, u$. For the example, the patched Eulerian tour is given in Fig. 6.9. We can now trace the complete tour, starting and ending at $o$, generating the heuristic schedule $S_{H}=\left(a^{20}, d^{11}, b^{6},(c d b)^{14}, c\right)$, with cost $C_{H}=1027$ (which happens to be optimal).


Fig. $6.9 \quad$ Solution to $\mathbf{T}$ patched to connect the subtours

### 6.5 Related No-Wait Systems

In this section we consider other systems related to the no-wait flow shop.

### 6.5.1 The No-Wait Assembly Shop

Consider the 2-stage assembly shop with $m$ machines in stage $G_{1}$ and a single assembly machine in $G_{2}$. Job $J_{j}$ consists of operations $T_{1 j}, T_{2 j}, \ldots, T_{m+1, j}$ where $T_{k j}$ must be performed on machine $M_{k}(k=1,2, \ldots, m+1)$ for $p_{k j}$ units of time. The $m$ processors in $G_{1}$ are all different and work independently in parallel, each producing a separate component of the job. Upon completion of the $m G_{1}$-operations, the final assembly operation $T_{m+1, j}$ is performed by
the single machine $M_{m+1}$ at $G_{2}$. The no-wait requirement means that all the $m$ tasks of a job at $G_{1}$ must complete simultaneously, and assembly must then start without delay. Rather than Fm, we will use Am to denote this system.

For each job, we will assume that the assembly task, and at least one of the $m G_{1}$-tasks, require positive processing time, or if not, that the zero -time stage does not have to be scheduled, so that all no-wait schedules are necessarily permutation. Thus, the optimum is always a permutation schedule, whether we require it or not; which as usual we denote (perm).

The problem of minimizing makespan in an assembly shop subject to setup and removal times has been considered in Gupta et al. (1997). Associated with each task $T_{k j}$ are a prespecified setup time $s_{k j}$ and a teardown or removal time $t_{k j}$, both separable. The start of a typical schedule, $S=(i, j, \ldots)$, for an instance with $m=3$, is illustrated in Fig. 6.10.


Fig. 6.10 Start of schedule $S=(i, j, \ldots)$ for an assembly shop with $m=3$

After proving that $A m \mid n w t,($ perm $), s_{k j}, t_{k j} \mid C_{\max }$ is strongly NP-complete, the authors show that this problem, like most no-wait flow shops, has a TSP formulation with cities $J_{0}, J_{1}, \ldots, J_{n}$ where the dummy job $J_{0}$ that indicates the start and end of the schedule has $s_{k 0}=p_{k, 0}=t_{k 0}=0$ for $k=1, \ldots, m+1$. The distance added when $J_{j}$ follows $J_{i}, d_{i j}$, is measured between assembly start times, as shown in Fig. 6.10. Clearly, for $i \neq j$ :

$$
d_{i j}=\max \left\{\max _{k=1, \ldots, m}\left\{t_{k i}+s_{k j}+p_{k j}\right\}, p_{m+1, i}+t_{m+1, i}+s_{m+1, j}\right\}
$$

To simplify notation, let $u_{i j k}=t_{k i}+s_{k i}+p_{k j}$ and $v_{i j}=p_{m+1, i}+t_{m+1, i}+$ $s_{m+1, j}$. We now have:

$$
\begin{aligned}
d_{i j} & =\max \left\{\max _{k=1, \ldots, m} u_{i j k}, v_{i j}\right\} \\
& =v_{i j}+\max \left\{\max _{k=1, \ldots, m}\left\{u_{i j k}-v_{i j}\right\}, 0\right\} \\
& =v_{i j}+\max _{k=1, \ldots, m}\left\{u_{i j k}-v_{i j}\right\}^{+}
\end{aligned}
$$

where $x^{+} \equiv \max \{x, 0\}$. Since our objective is to minimize the total distance over all tours, the first term always sums to the same constant and can be omitted. Finally, regrouping terms, we can use as our distance metric

$$
d_{i j}=\max _{k=1, \ldots, m}\left\{a_{k j}-b_{k i}\right\}^{+} \text {for } i \neq j
$$

where $a_{k j}=s_{k j}+p_{k j}-s_{m+1, j}, b_{k j}=p_{m+1, j}+t_{m+1, j}-t_{k j}$.
The authors also identify two polynomially solvable special cases as follows.
Theorem 6.11 (Gupta et al., 1997)
(a) If $t_{k j}=t_{j}$ for all $j, k$, so that $b_{k j}=b_{j}=p_{m+1, j}+t_{m+1, j}-t_{j}$, then Am|no-wait, (perm), $s_{j k}, t_{j k} \mid C_{\max }$ is equivalent to $F 2 \mid$ no-wait, (perm) $\mid C_{\max }$ with job processing times $\left.\left\langle a_{j}, b_{j}\right\rangle=<\max _{k=1, \ldots, m} a_{k j}, b_{j}\right\rangle$.
(b) If $s_{k j}+p_{k j}=s_{j}+p_{j}$ for all $j, k$, so $a_{k j}=a_{j}=s_{j}+p_{j}-s_{m+1, j}$ then Am|no-wait, (perm), $s_{k j}, t_{k j} \mid C_{\max }$ is equivalent to $F 2 \mid$ no-wait, (perm) $\mid C_{\max }$ with job processing times $\left.\left\langle a_{j}, b_{j}\right\rangle=<a_{j}, \min _{k=1, \ldots, m} b_{k i}\right\rangle$.

When both conditions in the above two theorems hold, then the resulting TSP is equivalent to the two-machine no-wait flow shop with job processing times $\left.\left\langle a_{j}, b_{j}\right\rangle=<s_{j}+p_{j}-s_{m+1, j}, p_{m+1, j}+t_{m+1, j}-t_{j}\right\rangle$.

### 6.5.2 The No-Wait Hybrid Flow Shop

Ramudhin and Ratliff (1992) considered the problem of selecting the jobs that should complete within an 8 -hour shift in a hybrid 2 -stage no-wait flow shop so as to maximize the weighted sum of selected orders. Setting the due date of all jobs to $d=8$ hours, the problem is equivalent to $F\left(k_{1}, k_{2}\right) \mid$ no-wait, $d_{j}=$ $d \mid \Sigma_{j} w_{j} U_{j}$. The authors formulate the problem using a mathematical program the Lagrangean relaxation of which results in subproblems solved by the longest path algorithm. The resulting paths are then converted to a feasible schedule by appropriate user-driven changes.

Heuristic algorithms supported by worst case error bound analysis have appeared in the literature for the 2-stage hybrid no-wait flow shop. These algorithms apply the FCFS policy on a given permutation (or list) of jobs, i.e., the $a$-task of the next unscheduled job in the list is scheduled on the first available machine at stage $G_{1}$, followed by the corresponding $b$-task on the first available $G_{2}$ machine. This is a logical extension of permutation scheduling in simple job shops, and we denote it perm. Generically, the resulting algorithm is referred to as list heuristic. We have the following results for the completion time $C_{L}$ of the schedule produced by the list heuristic using list $L$.

Theorem 6.12 (Sriskandarajah, 1993) For $F(1, k) \mid$ no-wait, perm $\mid C_{\max }$,

$$
C_{L} / C^{\star} \leq \begin{cases}3-1 / k & \text { if } L=\text { arbitrary } \\ 2 & \text { if } L=\searrow b_{j}\end{cases}
$$

and these bounds are tight.
Clearly, the case $\left(k_{1}, k_{2}\right)=(k, 1)$ is symmetric; simply interchange the role of $a$ - and $b$-tasks, and assume that jobs enter stage 2 and proceed on stage 1 . For the case where $k_{1}=k_{2}=k$, we have

Theorem 6.13 (Sriskandarajah and Sethi, 1989)
For $F(k, k) \mid$ no-wait, perm $\mid C_{\max }$,

$$
C_{L} / C^{\star} \leq \begin{cases}3-1 / k & \text { if } L \text { is obtained by } G G V \\ r & \text { if } L=\searrow\left(a_{j}+b_{j}\right)\end{cases}
$$

The first bound is tight, while $\frac{7}{3}-\frac{2}{3 k} \leq r \leq 3-\frac{1}{k}$.
That is, for the second case, $3-1 / k$ is a proven bound but not known to be tight, while an example exists that has $C_{L} / C^{\star}=(7 k-2) / 3 k$.

### 6.5.3 No-Wait Lot Streaming

Frequently, in nondelay flow shops, many copies of the same or similar products must be produced repetitively. For instance, automobile bumpers of many shapes and sizes may require the same immersion times in a series of tanks for chemical treatments (clean, deoxidize, electroplate, rinse, etc.) for chrome plating. The process is no-wait because the chemical processes, once started, cannot be interrupted. When processing times are the same for all items at each stage, the items can be treated as identical for scheduling purposes, and we will consider them a single product.

The total number of copies of the product to be scheduled is the job lot or process batch, usually predetermined by a planning process based on customer orders and stocking requirements. Now it may happen that each copy need not be separately scheduled, as is usual with no-wait jobs, but can be batched and sent through the shop as a group. In the manufacturing example above, a number of items may be attached to a rack and simultaneously dipped in the successive baths. The time to chemically treat a rack, or sublot, or transfer batch, depends on the surface area, hence is proportional to the number of items in the sublot.

We are now ready to introduce the concept of no-wait lot streaming. A small number of products (we'll start by considering just one) is to be produced, each in quantity. Lot streaming is the splitting of the total job lot of a product into several sublots, each of which is transferred from one machine to the next only when all the items of the sublot are completed. Smaller sublots result in better machine utilization, more overlap on consecutive machines, less work-in-process inventory, and shorter makespan. Indeed, the complete subdivision of the lot into one-unit sublots minimizes makespan. However, practical considerations of material handling, tracking and controlling generally limit the number of sublots. In this context, as we have said, the no-wait feature applies to the entire sublot, rather than each item. Our code for this will be nwt lots.

## No-Wait Lot Streaming for a Single Product

Consider $F m \mid n w t$ lots, $($ perm $) \mid C_{\text {max }}$ where $n$ copies of a single product are to be manufactured. Each item requires a time $p_{i}$ on $M_{i}, i=1, \ldots, m$. This sim-
plest case has received significant attention. An instance taken from Hall et al. (2003) is shown in Fig. 6.11, where 12 units of a product, having $<p_{1}, p_{2}, p_{3}>=<1,3,2>$, must be partitioned into at most three sublots for processing on 3 machines. If the 12 jobs are processed in a single lot, the makespan is $12(1+3+2)=72$. If the jobs are split into 3 sublots with 2,6 , and 4 jobs respectively, then the makespan is only 46 as shown in the figure, and this is optimal.


Fig. 6.11 A simple instance of $F 3 \mid$ nwt lots, $($ perm $) \mid C_{\max }$ wti

Our goal, then, is to find the partition $\left(x_{1}, x_{2}, \ldots, x_{v}\right)$ of the job lot of $n$ items into $v$ sublots, each with $x_{i}$ items (so $\Sigma_{i=1}^{v} x_{i}=n$ ), that minimizes makespan. In reality, the $x_{i}$ are integers. Since this discrete version of the problem may be difficult to solve, Sriskandrajah and Wagneur (1999) considered the version of $F m \mid n w t$ lots, $($ perm $), s_{k} \mid C_{\max }$ where the number of jobs in each lot is allowed to be fractional. Note that they have added setup times $s_{k}$ on each $M_{k}$ at the start of the production run, which does not make the problem more difficult. This relaxation of the integer restriction may give an integer solution anyway, or we can round the values up or down for a good solution, especially if the sublots are large.

With reference to the last machine, $M_{m}$, the makespan can be written:

$$
C_{\max }=s_{m}+\sum_{i=1}^{v} x_{i} p_{m}+\sum_{i=1}^{v} \Delta_{i}=s_{m}+n p_{m}+\sum_{i=1}^{v} \Delta_{i}
$$

where $s_{m}+n p_{m}$ is the total busy time on $M_{m}$ (a constant that can be ignored), and $\Delta_{i}$ is the idle time on $M_{m}$ prior to the start of sublot $i$, as illustrated in Fig. 6.12 for $m=3$.


Fig. 6.12 The general case of $F 3 \mid n w t$ lots, $($ perm $), s_{k} \mid C_{\max }$

We can now formulate the problem as a linear program to minimize $\sum_{i=1}^{v} \Delta_{i}$. We relate the new variables $\Delta_{i}$ to the original $x_{i}$ with the same logic we have used several times now. Thus, with $m=3$ :

$$
\begin{aligned}
& \Delta_{1}=\max \left\{s_{1}+x_{1}\left(p_{1}+p_{2}\right)-s_{3}, s_{2}+x_{1} p_{2}-s_{3}, 0\right\} \\
& \Delta_{i}=\max \left\{x_{i}\left(p_{1}+p_{2}\right)-x_{i-1}\left(p_{2}+p_{3}\right), x_{i} p_{2}-x_{i-1} p_{3}, 0\right\}, i=2, \ldots, v
\end{aligned}
$$

You can trace the sequence of tasks in Fig. 6.12 to check that $\Delta_{1}=s_{2}+$ $x_{1} p_{2}-s_{3}, \Delta_{2}=x_{2}\left(p_{1}+p_{2}\right)-x_{1}\left(p_{2}+p_{3}\right)$, and $\Delta_{v}=x_{v} p_{2}-x_{v-1} p_{3}$.

In the same way, for general $m$, the expressions are:

$$
\begin{aligned}
\Delta_{1} & =\max _{k=1, \ldots, m-1}\left\{s_{k}+x_{1} \sum_{r=k}^{m-1} p_{r}-s_{m}\right\}^{+} \\
\Delta_{i} & =\max _{k=1, \ldots, m-1}\left\{x_{i} \sum_{r=k}^{m-1} p_{r}-x_{i-1} \sum_{r=k+1}^{m} p_{r}\right\}^{+}, \quad i=2, \ldots, v
\end{aligned}
$$

Converting equations of the type $\Delta=\max \{a, b, c\}$ into constraints $\Delta \geq$ $a, \Delta \geq b, \Delta \geq c$, we get the following linear program (Sriskandrajah and Wagneur, 1999), finding an optimal partition of the jobs into fractional lots:

$$
\begin{aligned}
& \text { SW minimize } \quad \sum_{i=1}^{v} \Delta_{i} \\
& \text { subject to } \quad \sum_{i=1}^{v} x_{i}=n \\
& \Delta_{1} \geq s_{k}+x_{1} \sum_{r=k}^{m-1} p_{r}-s_{m}, \quad k=1, \ldots, m-1 \\
& \Delta_{i} \geq x_{i} \sum_{r=k}^{m-1} p_{r}-x_{i-1} \sum_{r=k+1}^{m} p_{r}, \quad i=2, \ldots, v, \quad k=1, \ldots, m-1 \\
& x_{i}, \Delta_{i} \geq 0 \quad i=1, \ldots, v
\end{aligned}
$$

where the cases with $k=m$ are omitted because they amount to nonnegativity constraints.

This formulation is utilized in Kumar et al. (2000) to produce optimal integral lots by assigning to each lot the smallest integer number of units, say $y_{i}$ that exceeds the fractional allocation found in $\mathbf{S W}$. In this fashion, the total number of units $\Sigma_{i} y_{i}$ may exceed $n$. The excess $\Sigma_{i} y_{i}-n$ is eliminated as necessary by finding the lots with the largest $y_{i}-x_{i}$ values and replacing the corresponding $y_{i}$ values by $y_{i}-1$.

## Multiproduct No-Wait Lot Streaming

Hall et al. (2003) considered the multiproduct problem where all the sublots of a product are produced consecutively, with setups required between products on each machine. We must determine simultaneously the sequence of products and, for each product, the breakdown into sublots. The integer sublot vectors for a product are generated by the technique in Kumar et al. (2000), which then represent cities in a Generalized Traveling Salesman problem, defined as follows: a number of countries are to be toured, by visiting precisely one of many cities available in each country, so as to minimize the total travel distance. Here, cities correspond to sublot partitions, and countries correspond to products. The resulting problem is solvable by existing algorithms for the generalized TSP.

For the special case where $m=2$, if the $n$ jobs can be partitioned into $s$ lots each consisting of identical jobs, then Agnetis (1989) developed an optimal algorithm that runs in time $O\left(s \log s+\min \left\{s^{2}, n\right\}\right)$.

### 6.6 Other Objectives

Scheduling objectives other than makespan have not received nearly as much attention. For the flow time objective, $F m \mid n w t$, perm $\mid \Sigma C_{j}$ is NP-complete for variable $m$ (see Lenstra et al., 1977) and for fixed $m \geq 2$ (see Röck, 1984). The corresponding TSP formulation has objective

$$
\operatorname{Min}_{\pi} Z=\sum_{j=0}^{n-1}(n-j) d_{\pi(j), \pi(j+1)}
$$

where $\pi$ is a permutation of cities $J_{0}, J_{1}, \ldots, J_{n}$. The equivalence of this objective to the total flow times is easily seen:

$$
Z=n C_{\pi(1)}+(n-1)\left(C_{\pi(2)}-C_{\pi(1)}\right)+\ldots+\left(C_{\pi(n)}-C_{\pi(n-1)}\right)=\sum_{j} C_{\pi(j)}
$$

The resulting TSP is known as the minimal latency problem (see Afrati et al., 1986). For variable $m$ and processing times zero or one, Gonzalez (1982) showed that $F \mid n w t$, perm, $p_{i j} \in\{0,1\} \mid \Sigma C_{j}$ is NP-complete.

One special case where total flow times are easily minimized is when processing times are semi-ordered; that is (see Definition 6.1):

$$
p_{k j} \leq p_{k, j+1}, \text { for } j=1, \ldots, n-1, \quad k=1, \ldots, m
$$

Theorem 6.14 (van der Veen and van Dal, 1991) For Fm|nwt, perm $\mid \Sigma_{j} C_{j}$ with semi-ordered processing times, the SPT schedule is optimal.

### 6.7 Conclusions

Our review of the research on the no-wait flow shop has highlighted some of the most celebrated results in all of flow shop research, as well as gaping holes on the subject. The elegant GGV algorithm is a special type of the TSP and forms a building block in other more complicated scheduling problems with TSP structure. Also surprising is to see that a simple adjustment on the lot streaming problem with a single product and fractional lots provides a solution to the case when lots are integral. The optimality gaps of the heuristics and metaheuristics that have appeared in the literature for $F m \mid n w t$, perm $\mid C_{\text {max }}$ are acceptable, but not of the levels we saw for $F m \mid$ perm $\mid C_{\text {max }}$. Moreover, the non-permutation version of the problem has not been considered at all. With respect to objectives other than makespan, the literature is surprisingly thin with only minor consideration of the flow time objective. Systems other than the traditional $m$-machine flow shop have been considered - like the 2-stage hybrid flow shop and the assembly flow shop - but not to the extent one would desire, considering the large number of industry applications of the no-wait paradigm.

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## Chapter 7 <br> BLOCKING OR LIMITED BUFFERS IN FLOW SHOPS


#### Abstract

We begin by reviewing many manufacturing and other applications of the flow shop with limited or no interstage storage. For two machines, we show the equivalence of no storage (blocking) and no waiting, which have a polynomial solution; for $m>2$, we establish that the flow shop with blocking is NP-complete. An integer program is given to find the minimum makespan. A variety of bounds are presented. Branch-and-bound algorithms using these bounds are given and evaluated. Various heuristics and metaheuristics are presented and compared. Two different precedence graphs are introduced and used in the above developments. The total tardiness objective is also discussed.


### 7.1 Introduction

We say a flow shop has blocking when no intermediate storage is provided for the jobs to wait between stages. As a result, at the completion of any task, if the downstream machine is busy, a job must wait at the upstream machine, occupying it and blocking the next job until the downstream processor clears and allows flow to resume. Clearly, this production protocol is related to the no-wait flow shop where the waiting time, rather than the waiting space, is required to be zero. In the abbreviated job description, the presence of blocking will be denoted block, while the no-wait requirement is abbreviated $n w t$.

Of course, since there is never any blocking in a no-wait schedule, the optimal schedule with blocking is either the optimal no-wait schedule or something better. This is true for any number of machines and for any objective. Thus, if $A_{\text {block }}^{\star}$ and $A_{n w t}^{\star}$ are the optimal objective function values for the two cases under any measure $A(S)$ of a schedule $S$, then $A_{b l o c k}^{\star} \leq A_{n w t}^{\star}$.

At first glance, it may seem that, under both assumptions, by their very nature, only permutation schedules are possible. Technically, this is only true if
each job must visit each machine; call this the no-skip requirement. Of course, if all task times are strictly positive, this follows automatically. However, if a job finishes processing on $M_{k-1}$ and need not visit $M_{k}$, it can skip straight to $M_{k+1}$ thereby passing an earlier job in process on $M_{k}$. Thus, without the no-skip requirement, non-permutation schedules are possible, and may be optimal.

Throughout this chapter, our search for the optimum will be limited to permutation schedules. We now see that with the no-skip assumption there is no loss of optimality. Also, with $m=2$, the optimal schedule is always permutation, unconditionally. As we know, when the permutation requirement is no real restriction we write (perm); otherwise, perm.

In our usual notation, we wish to schedule a given set $\mathcal{J}=\left\{J_{1}, J_{2}, \ldots, J_{n}\right\}$ of $n$ independent jobs in an $m$-machine flow shop, with $J_{j}$ requiring time $p_{i j}$ on $M_{i}$. For simplicity, when defining a job, we will write $J_{j}=\left\langle p_{1 j}, \ldots, p_{m j}\right\rangle$. Also recall:
$C_{k}(S)\left[C_{k j}(S)\right]=$ completion time of all tasks $\left[\right.$ of $\left.J_{j}\right]$ on $M_{k}$ in schedule $S$.
Now, with blocking, when a job completes processing on a machine, it may have to remain there, occupying that machine for additional time. We will find it more convenient to use the variables:
$D_{k j}(S)=$ departure time of $J_{j}$ from $M_{k}$ in schedule $S$.
Of course, on the last machine, $D_{m j}(S)=C_{m j}(S)$. Also note that, by the nature of blocking,

$$
D_{k j}(S)=\text { start time of } J_{j} \text { on } M_{k+1} \text { in schedule } S
$$

### 7.2 Flow Shops with Limited Interstage Storage

Initially, one might think that a flow shop with limited (rather than null) buffer storage between stages is a more complicated generalization of the shop with blocking. On the contrary, at least for permutation schedules, it can be subsumed as a special case, as follows. Suppose limited storage space for $\beta_{k} \geq 0$ jobs is available between $M_{k-1}$ and $M_{k}, k=2, \ldots, m$, with at least one $\beta_{k}>0$; and let $\beta=\min _{k} \beta_{k}$. That is, a queue of maximal length $\beta_{k}$ can form in front of $M_{k}$. If $\beta \geq n-2$, then of course we have effectively a simple (unlimited-storage) shop, since, if a job requires such a long time on, say, $M_{k}$ that all $n-1$ other jobs queue up behind it, then at worst the final job is on $M_{k-1}$ and there can be no upstream job to be blocked.

Now suppose $\beta<n-2$. Let us introduce $\beta_{k}$ new machines in front of $M_{k}, k=2, \ldots, m$. Each new machine will act as a storage device for a single job. Jobs require zero processing times on the new "storage machines", hence occupy them only as long as they are blocked. In this way, one may convert any limited-storage system to a flow shop with no storage, i.e. with blocking, where the number of machines is no more than $m+\Sigma_{k} \beta_{k}$.

As we observed earlier, in a blocking flow shop with the no-skip requirement, every feasible solution is a permutation schedule. This is not necessarily the case with limited storage, because when $\beta_{k}>0$ passing is possible among operations waiting for $M_{k}$. However, to date published results have been limited to finding the best no-skip permutation schedule, in which even the "storage machines" cannot be skipped, so that each buffer behaves as a well-disciplined FCFS queue. Our short-hand notation for the limited-buffer shop will be ltd, with the buffer size(s) added in parentheses if needed.

Although, with these assumptions, limited buffering is a special case of blocking, authors often deal with the buffered shop, partly because adding storage machines makes the problem unwieldy, and also because they often wish to assume that all $p_{i j}>0$. We end this section with one insightful formula for the limited-buffer shop: a recursion for computing departure times, and ultimately the makespan value, for an arbitrary job sequence $S=(1,2, \ldots, n)$ in an instance of $F m|l t d, p e r m| C_{\text {max }}$. For simplicity, we will write $D_{k j}(S)$ as $D_{k j}$. This can be computed using

$$
\begin{equation*}
D_{k j}=\max \left\{\max \left[D_{k, j-1}, D_{k-1, j}\right]+p_{k j}, D_{k+1, j-\beta_{k+1}-1}\right\} \tag{7.1}
\end{equation*}
$$

for $j=1, \ldots, n$, and for each $j$, for $k=1, \ldots, m$, where $D_{k 0}=D_{0 j}=$ $D_{m+1, j}=0$. To understand this, note that the first term, $\max \left[D_{k, j-1}, D_{k-1, j}\right]$ $+p_{k j}$, is the time $J_{j}$ completes on $M_{k}$. Either this is its departure time, or it is blocked, which means that all $\beta_{k+1}$ buffer positions plus $M_{k+1}$ are occupied. In this case, $J_{j}$ will depart from $M_{k}$ when the job $\beta_{k+1}+1$ positions ahead, i.e., $J_{j-\beta_{k+1}-1}$, vacates $M_{k+1}$, allowing all waiting jobs to advance one position. This recursion is easily adapted to $F m \mid$ block, $($ perm $) \mid C_{\max }$, where $\beta_{k+1}=0$.

### 7.3 Applications

Considerations of cost and storage space limitations lead to the design of flow shops with little or no buffer storage between stages. Without such incentive, in-process inventory tends to accumulate due to poor shop floor management which can be expensive (Dobson and Yano, 1994), and may lead to quality deterioration. The popularity of just-in-time systems has rendered limitedbuffer and blocking flow shops of great importance. Indeed, Kanban systems stipulate limited storage space for work in process. There is research evidence that limited storage is advantageous to timely production; see Schonberger (1982), Zangwill (1987) and Matsuo (1990).

The following examples highlight the practical importance of the flow shop studied in this chapter.

- In steel manufacturing, molten steel undergoes molding, reheating, soaking and rolling. The cost of storing and maintaining the temperature at the appropriate levels for each operation render storage either infeasible or uneconomical. Here, the production technology causes blocking in the flow shop.
- Rippin (1981) presents applications of blocking flow shops due to the instability of intermediate chemical products.
- Dekhici and Belkadi (2010) present a hospital application of a 2-stage hybrid flow shop (see Chapter 5) with blocking, with three operating rooms in stage 1 and eleven beds for post-anesthesia care in stage 2 . With 10 surgical interventions daily, the blocking constraint reflects situations where no bed is available in stage 2. In this case, the patient remains in the operating room until a bed becomes available.
- Other application areas mentioned in the survey paper of Hall and Sriskandarajh (1996) include plastics, pharmaceuticals, silverware production, food canning, and anodizing aluminum products.


### 7.3.1 The Two-Machine Case

Schedules with blocking are distinct from those with no waiting; we have devoted a separate chapter to each. However, this is not true when $m=2$.

Theorem 7.1 F2|block, (perm)|Any is equivalent to F2|nwt, (perm)|Any.
Proof: With blocking, any job $J_{j}$ on $M_{1}$ that is blocked stays on $M_{1}$ until it can start on $M_{2}$, so $a_{j}$ can always be delayed so as to complete just as $b_{j}$ begins, without affecting any job completion time.

Similarly, in no-wait schedules, there is no possibility of blocking, and if the no-wait requirement is relaxed, blocking will only permit tasks on $M_{1}$ to be processed earlier.

For an example, see the no-wait schedule in Figure 6.1. If instead the jobs were scheduled with blocking, the only change would be that $a_{3}$ and $a_{5}$ (not $a_{4}$ ) would move earlier.

However, the two scheduling protocols give different results when $m>2$. To see this, consider the following example.

Example 7.1: We wish to schedule jobs $J_{1}=\langle 4,6,5\rangle, J_{2}=\langle 4,8,12\rangle$, $J_{3}=\langle 10,5,6\rangle, J_{4}=\langle 7,5,7\rangle, J_{5}=\langle 7,10,5\rangle$ and $J_{6}=\langle 4,2,3\rangle$ on three machines, so as to minimize the makespan. With blocking, schedule $(1,2, \ldots, 7)$ is shown in Figure 7.1, where brick walls show where machines are blocked. The makespan is 55 . With no waiting, the makespan of the same schedule becomes 57 . With enough waiting room $\left(\beta_{2}=\beta_{3}=1\right.$ is sufficient), the unrestricted schedule has a makespan of 51 .

Since the two-machine problem with blocking duplicates the one with no waiting, we have combined all results on either type of shop in the no-wait chapter. Thus, for full coverage of our problem when $m=2$, see Chapter 6 .


Fig. 7.1 A schedule for the sample instance of F3|block, (perm) $\mid C_{\max }$

### 7.4 Complexity of $F 3 \mid$ block, perm $\mid C_{\text {max }}$

As noted above, $F 2 \mid$ block, $($ perm $) \mid C_{\max }$ is equivalent to $F 2 \mid n w t$, (perm) $\mid C_{\max }$ and hence solvable in $O(n \log n)$ time (see Chapter 6). When $\beta=1$, problem $F 2 \mid l t d$, perm $\mid C_{\text {max }}$ is converted to the equivalent $F 3 \mid$ block, perm $\mid C_{\max }$ by introducing a single storage processor. We have the following result:

Theorem 7.2 (Papadimitriou and Kanellakis, 1980)
$F 2 \mid l t d(1)$, perm $\mid C_{\max }$ is NP-complete in the strong sense.
Outline of proof: We reduce the strongly NP-complete problem

## 3-WAY MATCHING OF INTEGERS, OR 3MI

INSTANCE: Two sets $\mathcal{U}=\left\{u_{1}, u_{2}, \ldots, u_{k}\right\}$ and $\mathcal{V}=\left\{v_{1}, v_{2}, \ldots, v_{2 k}\right\}$ of positive integers.
QUESTION: Can $\mathcal{V}$ be partitioned into pairs $\mathcal{P}_{i}=\left\{v_{i_{1}}, v_{i_{2}}\right\}$ such that $u_{i}+v_{i_{1}}+v_{i_{2}}=c$, where $c=\left(\Sigma_{j} u_{j}+\Sigma_{j} v_{j}\right) / k$, for $i=1,2, \ldots, k$ ?
to the decision version of our problem:
F2|ltd(1), perm $\mid C_{\text {max }} \leq B$ ?
INSTANCE: A threshold value $B$, $n$ jobs $\left\{J_{j}: j=1,2 \ldots, n\right\}$, each with processing requirements $\left\langle a_{j}, b_{j}\right\rangle$ in a 2-machine flow shop with interstage storage buffer of size $\beta=1$.
QUESTION: Does there exist a feasible schedule with $C_{\max } \leq B$ ?
Although we have no direct reference to the complexity status of 3MI, letting $x_{i}=c-u_{i}$ renders 3MI equivalent to the following problem which is known to be strongly NP-hard (Garey and Johnson, 1979, p. 224).

## NUMERICAL MATCHING WITH TARGET SUMS

INSTANCE: Two sets $\mathcal{X}=\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ and $\mathcal{V}=\left\{v_{1}, v_{2}, \ldots, v_{2 k}\right\}$ of positive integers.
QUESTION: Can $\mathcal{V}$ be partitioned into pairs $\mathcal{P}_{i}=\left\{v_{i_{1}}, v_{i_{2}}\right\}$ such that $v_{i_{1}}+v_{i_{2}}=x_{i}$, for $i=1,2, \ldots, k$ ?

We now proceed with the reduction from 3MI. For any instance of 3MI, the following parameters define an instance of $F 2 \mid l \operatorname{td}(1)$, perm $\mid C_{\max } \leq B$ ? having $4 k+1$ jobs of which $k$ involve a $u_{i}, 2 k$ involve a $v_{i}$, and $k+1$ are "filler" jobs.

- $B:=2 k(c+1)$;
- $n:=4 k+1$;
- Job $U_{j}:=<c / 2, u_{j}+c>, j=1, \ldots, k$;
- Job $V_{j}:=<1, v_{j}>, j=1, \ldots, 2 k$;
- Job $F_{0}:=<0,2>$;
- Job $F_{j}:=<3 c / 2,2>, j=1, \ldots, k-1$;
- Job $F_{k}:=<3 c / 2,0>$.

To see why the two problems will always have the same answer, note that the total processing time of all $4 k+1$ jobs on each machine is $B$. Hence, a schedule with $C_{\max }=B$ exists if and only if there is no idle time on either $M_{1}$ or $M_{2}$. This observation together with the buffer requirement $\beta=1$ imply the following: $F_{0}$ must be scheduled first, and $F_{k}$ last. After $F_{0}$ must come the following four-job block: two jobs of type $V_{j}$ to fill the first two units on $M_{1}$ (it can be assumed without loss of generality that all $u_{i}$ and $v_{i}$ are larger than 2), then a job of type $U_{j}$, and then an $F_{j}$, as in Figure 7.2.

It should be clear that, given the choice of $U_{j}$, the two preceding jobs should be $V_{j 1}$ and $V_{j 2}$ so that the four jobs present a profile with a stagger of two, just right for another similar four-job block to fit into. Let the $i^{\text {th }}$ block $(i=1, \ldots, k)$ be $B_{i}=\left(V_{i 1}, V_{i 2}, U_{i}, F_{i}\right)$. Then the schedule $\left(F_{0}, B_{1}, \ldots, B_{k}\right)$, as illustrated, achieves the desired makespan, and can do so only if 3MI has the answer "yes". It only remains to verify that no more than one job waits for $M_{2}$ at any point in the schedule.


Fig. 7.2 A schedule for the sample instance of $F 3 \mid$ block, (perm) $\mid C_{\text {max }}$

### 7.5 Fm|block, (perm) $\mid C_{\text {max }}$

First, as we know, with the no-skip assumption, blocking implies no passing so only permutation schedules are feasible. Let $S=(1,2, \ldots, n)$ be an arbitrary permutation of the jobs, and $C_{m}(S)$ the associated makespan evaluated for problem $F m \mid$ block, $($ perm $) \mid C_{\text {max }}$. It is not hard to observe that $C_{m}(S)$ equals the length of the critical path from the start of $T_{11}$ to the end of $T_{m n}$ in the precedence graph $\mathcal{G}$ shown in Figure 7.3, for an instance with $m=3, n=6$. In fact, the path shown with heavy arrows is the critical one for the schedule shown in Figure 7.1. Note that, contrary to our usual practice of representing tasks as nodes (the "activity-on-node" diagram), here it is more convenient
to represent tasks by arrows ("activity-on-arc"), with each node representing the instant that the last of the preceding tasks (arrows entering the node) is completed, and so the following tasks (arrows leaving) can start.

Vertex $k j$ corresponds to the start time of $T_{k j}$ for every $j=1, \ldots, n$ and $k=1, \ldots, m$. Each task is represented by a vertical arrow, labeled with its processing time. The dashed arrows are dummy tasks with task times of zero, to enforce the task ordering imposed by blocking. For example, none of the tasks $T_{33}, T_{24}$, and $T_{15}$ can start until $T_{32}$ has finished, due to the dummy arcs rising diagonally from node 42 in the figure. This triple blockage can be seen in Figure 7.1, where $c_{2}$ determines when $c_{3}, b_{4}$, and $a_{5}$ can start.


Fig. 7.3 Precedence graph $\mathcal{G}$ representing $S=(1, \ldots, 6)$ for $F 3 \mid$ block, $($ perm $) \mid C_{\text {max }}$

Let $\tau$ be a path from node $T_{11}$ to node $T_{m n}$ on graph $\mathcal{G}$. If $v_{1}, \ldots, v_{n-1}$ are the indices associated with directed path $\tau$, then the length of $\tau$ is:

$$
\begin{equation*}
L(\tau)=\sum_{k=1}^{v_{1}} p_{k 1}+\sum_{k=v_{1}}^{v_{2}} p_{k 2}+\ldots+\sum_{k=v_{n-1}}^{m} p_{k n}, \tag{7.2}
\end{equation*}
$$

for $1 \leq v_{1} \leq m$, and $\max \left\{1, v_{j-1}-1\right\} \leq v_{j} \leq m$ for $j=2,3, \ldots, n-1$. For example, the heavy arrows in Figure 7.3 define the path $\tau$ with $\left(v_{0}, v_{1}, \ldots, v_{6}\right)=$ $(1,2,3,2,1,3,3)$. Unlike similar formulae in other chapters, where we summed the tasks on each machine, here each summation totals one job's contribution to the path.

Let $\mathcal{T}$ be the set of all directed paths from node $T_{11}$ to node $T_{m n}$ on $\mathcal{G}$. Then, maximizing over all $\tau \in \mathcal{T}$ we get

$$
C_{m}(S)=\max _{\tau \in \mathcal{T}} L(\tau)
$$

To find the optimal makespan for $F m \mid$ block, $($ perm $) \mid C_{\max }$, one minimizes over all permutation schedules $S$, i.e.,

$$
\begin{equation*}
C^{\star}=\min _{S} \max _{\tau \in \mathcal{T}} L(\tau) . \tag{7.3}
\end{equation*}
$$

### 7.5.1 Formulation as an Integer Program

Under the blocking restriction and a given schedule $S=(1,2, \ldots, n)$, the departure time $D_{k j}(S)$ of $J_{j}$ from $M_{k}$, which we write simply $D_{k j}$, can be
computed recursively as follows:

$$
\begin{equation*}
D_{k j}=\max \left\{D_{k-1, j}+p_{k j}, D_{k+1, j-1}\right\} \tag{7.4}
\end{equation*}
$$

with $D_{k 0}=D_{m+1, j}=0$ and $D_{0 j}=D_{1, j-1}$, for $j=1, \ldots, n$ and for each $j$, for $k=1, \ldots, m$. This comes from (7.1) by setting $\beta_{k+1}=$ 0 , and then noting that the first term can be simplified since, with no buffers, $\max \left\{D_{k, j-1}, D_{k-1, j}\right\}$ is always $D_{k-1, j}$ (note how (7.4) implies $D_{k j} \geq$ $D_{k+1, j-1}$ which is the same thing), except for $k=0$, where the boundary value $D_{0 j}=D_{1, j-1}$ is needed.

We can convert this expression into an integer program, after noting that it assigns $D_{k j}$ the smallest value that simultaneously satisfies $D_{k j} \geq D_{k-1, j}+$ $p_{k j}$, and $D_{k j} \geq D_{k+1, j-1}$.

Define the variables:

$$
\begin{aligned}
& x_{i j}= \begin{cases}1 & \text { if } J_{j} \text { is in position } i, \\
0 & \text { otherwise, }\end{cases} \\
& d_{k i}=\text { departure time from } M_{k} \text { of the job in position } i .
\end{aligned}
$$

In order to search over different schedules, the binary variables are used to assign jobs to positions. Instead of specifying $S$, we write $p_{k[i]}$ as $\sum_{j=1}^{n} p_{k j} x_{i j}$, where $[i]$ indexes the job in position $i$. Then the following integer program minimizes the makespan $d_{m n}$.
$\mathbf{B}$ minimize $d_{m n}$

$$
\begin{array}{ll}
\text { subject to } & d_{k i} \geq d_{k-1, i}+\sum_{j=1}^{n} p_{k j} x_{i j}, \quad i \in \mathcal{J}, \quad k \in \mathcal{M} \\
& d_{k i} \geq d_{k+1, i-1}, \quad i \in \mathcal{J}, k \in \mathcal{M} \\
& \sum_{i=1}^{n} x_{i j}=1, \quad j \in \mathcal{J} \\
& \sum_{j=1}^{n} x_{i j}=1, \quad i \in \mathcal{J} \\
& x_{i j}=0 \text { or } 1, \quad i, j \in \mathcal{J} \\
& d_{k 0}=d_{m+1, j}=0, d_{0 i}=d_{1, i-1}, \quad i, j \in \mathcal{J}, \quad k \in \mathcal{M} \tag{7.10}
\end{array}
$$

### 7.5.2 Lower Bounds

A variety of lower bounds on the makespan have been proposed for the flow shop with blocking. We now present a selection of them.

## Bounds based on Lagrangean Relaxation

In the online supplement accompanying Abadi et al. (2000) one can find details of two Lagrangean decomposition schemes that yield good lower bounds. Both schemes are based on program $\mathbf{B}$ provided earlier, when the variables $x_{i j}$ are replaced by $x_{k i j}$ to indicate that $J_{j}$ is the $i^{\text {th }}$ job scheduled on $M_{k}$. As a result, the revised formulation would allow for passing. Constraints

$$
\begin{equation*}
x_{k i j}=x_{k+1, i, j} \tag{7.11}
\end{equation*}
$$

for $i=1, \ldots, n, k=1, \ldots, m-1$ are used to restrict search to permutation schedules. By relaxing constraints (7.5), (7.6), (7.7) and (7.11), the resulting problem is decomposed into $m$ separable problems, one for each machine $M_{k}$. Each of these $m$ subproblems is equivalent to a cost minimization assignment subproblem solved by a simple dynamic program. Summing over all $k=1, \ldots, m$ and with appropriate subgradient Lagrangean multiplier updates we obtain good lower bounds. Alternatively, the original Lagrangean relaxation may be decomposed into pairs of consecutive processors (since $F 2 \mid b l o c k,($ perm $) \mid C_{\max }$ is polynomially solvable) and the partial makespans of these subproblems are summed up. These decomposition schemes have been used to evaluate a heuristic algorithm developed by Abadi et al. (2000) and presented later in this chapter.

## Bound when Partial Schedule is Given

The following lower bound is presented in the context of a branch-andbound algorithm, where a partial schedule $\sigma$ already occupies each $M_{k}$ up to $D_{k}(\sigma)$, the departure time of the last job in $\sigma$. Of course, if there is no $\sigma$, let all $D_{k}(\sigma)=0$. Let $s=|\sigma|$, and without loss of generality suppose $\sigma=(1,2, \ldots, s)$. When blocking is involved, each $J_{j}$ departs from processor $M_{k}$ at time $D_{k j} \geq C_{k j}$, and for the given partial schedule $\sigma$ these times can be computed using (7.4) for $j=1, \ldots, s$ and $k=1, \ldots, m$, giving finally $D_{k}(\sigma)=D_{k s}$.

Consider now a given permutation, say $\sigma^{\prime}=(s+1, s+2, \ldots, n)$, of the unscheduled jobs $\mathcal{U}=\mathcal{J}-\sigma$. Ronconi (2005) observes that, for any $M_{k}$ :

$$
\left.\left.\begin{array}{rl}
L B_{1}(k, & \sigma^{\prime}
\end{array}\right)=\max \left\{D_{k}(\sigma)+p_{k, s+1}, D_{k+1}(\sigma)\right\}\right\}_{m} .
$$

is a lower bound on the makespan associated with $M_{k}$. Specifically, the first term in $L B_{1}\left(k, \sigma^{\prime}\right)$ bounds the departure time $D_{k, s+1}$ of $J_{s+1}$ from $M_{k}$; the second term bounds the additional time required on $M_{k}$ by $J_{s+2}, \ldots, J_{n}$ (each job occupies $M_{k}$ either for its own processing time or until the next machine clears); and the third term gives the time for $J_{n}$ to be completed on the remaining machines.

Since the sequence of the unscheduled jobs is not yet specified, we want a bound independent of $\sigma^{\prime}$. Towards this end, we define the following values:
$t_{i}^{g}:=$ the $g^{t h}$ smallest task time on $M_{i}$ among the unscheduled jobs, $i=1, \ldots, m$. For $M_{k+1}$ only, replace $t_{k+1}^{1}$ by $D_{k+1}(\sigma)-D_{k}(\sigma)$.
Then,
Theorem 7.3 (Ronconi, 2005) For Fm|block, perm $\mid C_{\max }$ with given partial schedule $\sigma, L_{1}=\max _{k} L_{1}(k)$ is a lower bound on the makespan, where

$$
L_{1}(k)=D_{k}(\sigma)+\sum_{g=1}^{n-s} \max \left\{t_{k}^{g}, t_{k+1}^{g}\right\}+\sum_{i=k+1}^{m} t_{i}^{1} .
$$

Brief explanation : $D_{k}(\sigma)$ is the departure time of $J_{s}$ from $M_{k}$; the second term bounds $\max \left\{D_{k+1}(\sigma)-D_{k}(\sigma), p_{k, s+1}\right\}+\sum_{r=s+1}^{n-1} \max \left\{p_{k, r+1}, p_{k+1, r}\right\}$; and the last term bounds $\sum_{i=k+1}^{m} p_{i n}$. Accounting for all values $k=1, \ldots, m$ yields bound $L_{1}$.

## Bound based on Assignment Problem

An alternate lower bound is developed in Karabati et al. (1992) based on (7.3) and the observation that the minimal makespan $C^{*}$ satisfies

$$
C^{*}=\min _{S} \max _{\tau \in \mathcal{T}} L(\tau) \geq \max _{\tau \in \mathcal{T}} \min _{S} L(\tau)
$$

In other words, if for each path $\tau \in \mathcal{T}$ in (7.4) we minimize its length $L(\tau)$ over all permutations of $n$ jobs, then maximizing over all $\tau \in \mathcal{T}$ yields a lower bound for $C^{\star}$. Two observations are in order. First, solving $\min _{S} L(\tau)$ for given $\tau \in \mathcal{T}$ is equivalent to a linear assignment problem (LAP) with coefficient $a_{i j}^{\tau}$ for $J_{j}$ scheduled in position $i$ of a permutation, $1 \leq i, j \leq n$, where

$$
a_{i j}^{\tau}=\sum_{k=v_{i-1}}^{v_{i}} p_{k j}
$$

and $1=v_{0}, v_{1}, \ldots, v_{n-1}, v_{n}=m$ are the indices associated with $\tau$ (see (7.2)). Second, rather than executing LAP for all $\tau \in \mathcal{T}$, lower bounds may be obtained for a subset of directed paths in $\mathcal{T}$.

Karabati et al. (1992) present a computational experiment for randomly generated instances with $n=12, m=6$ and a total of 4 buffers (recall that such a problem is equivalent to $F 10 \mid$ block, $($ perm $\left.) \mid C_{\max }\right)$. For each instance, a total of 1200 paths in $\mathcal{T}$ are tested and the best lower bound is found to be within an average of $2.33 \%$ from the best known heuristic solution found for the corresponding instance.

### 7.5.3 Branch-and-Bound Algorithms

A branch-and-bound algorithm was developed in Ronconi (2005) using depthfirst search and the $L_{1}$ bound given in Theorem 7.3 to fathom branches in the search tree. Though capable of eventually finding the optimum, the program was actually used as a heuristic, with computation terminating after one hour of CPU time, or when the gap (that is, the difference between the lower bound (the least value of $L_{1}$ at any node) and the upper bound (the best-yet solution)) dropped below $0.5 \%$. It was tested on the Taillard suite of problems originally developed for $F m \| C_{\max }$, where problem sizes ranged from $(20,5)$ (i.e., 20 jobs on 5 machines) to $(500,20)$ - see the OR library at http://mscmga.ms.ic.ac.uk/info.html for more details. The effectiveness of the resulting heuristic is questionable. Results were rather insensitive to job number, with gaps from $0.5 \%$ (rare) to about $20 \%$ for instances with 5 machines and about $30 \%$ to $50 \%$ with 20 machines.

### 7.5.4 Heuristic Algorithms

A number of approximation results are available for $F m \mid b l o c k$, perm $\mid C_{\max }$. We present a selection. Some are for the special case $F 2 \mid l t d(\beta)$, perm| $C_{\max }$; recall that this is equivalent to $F(2+\beta) \mid b l o c k$, perm $\mid C_{\max }$. As usual when building a schedule job by job, $\sigma$ will denote a partial schedule of $s<n$ jobs and $\mathcal{U}$ the set of remaining unscheduled jobs (or, as convenient, job indices).

## Buffering Approximated by Blocking, for $m=2$

A result that stands out expresses the makespan for $F 2 \mid \operatorname{ltd}(\beta)$, perm| $C_{\max }$ as a fraction of the makespan for $F 2 \mid n w t$ or block, (perm) $\mid C_{\max }$ (which is polynomially solvable; see Chapter 6). Let $S_{\beta}^{\star}$, $S_{0}^{\star}$ be the corresponding optimal schedules, producing makespans $C_{\beta}^{\star}, C_{0}^{\star}$. If the optimal blocking schedule is used in the buffered flow shop as an approximation, we write the makespan $C_{\beta}\left(S_{0}^{\star}\right)$. We have the following result.

Theorem 7.4 (Papadimitriou and Kanellakis, 1980)

$$
\frac{C_{0}^{\star}}{C_{\beta}^{\star}} \leq \frac{2 \beta+1}{\beta+1}
$$

and this bound is asymptotically tight for $\beta=1$.
The above result indicates that the use of intermediate storage of size $\beta$ can save up to $\beta /(2 \beta+1)$ of the time needed to execute all jobs in $F 2 \mid n w t$ or block, (perm) $\mid C_{\max }$. For $\beta=1$, the theorem says that $C_{0}^{\star} \leq(3 / 2) C_{1}^{\star}$ : giving up the buffer may increase the makespan by up to $50 \%$.

To assess the value of $S_{0}^{\star}$ as a heuristic solution for $F 2 \mid \beta$, perm $\mid C_{\max }$, we note that the makespan it achieves satisfies $C_{\beta}\left(S_{0}^{\star}\right) \leq C_{0}^{\star}$, so

$$
C_{\beta}\left(S_{0}^{\star}\right) \leq \frac{2 \beta+1}{\beta+1} C_{\beta}^{\star} .
$$

## Successive Approximations (SA)

Another heuristic developed for $F 2 \mid l t d$, perm $\mid C_{\text {max }}$ is presented in Dutta and Cunningham (1975). First, based on (7.1), they develop and test a forward dynamic program referred to as DP, unfortunately without success for large problems due to exponential computational and state space. Their idea then is to sequentially apply DP to $p$-job subproblems, where $p$ is small enough to be tractable (indeed, if $p \leq \beta+2$, we effectively have infinite storage and can use Johnson's Rule). They compute an optimal schedule for the first $p$ (randomly chosen) jobs, fix the first job in position 1, add another job to the remaining set, and repeat. A formal statement of this successive approximations heuristic, referred to as SA, follows.

## Heuristic SA for $\boldsymbol{F 2} \mid \boldsymbol{l t d}$, perm| $\boldsymbol{C}_{\text {max }}$

1. Set $k:=1$, and choose an integer $p: 1<p<n$, and a schedule $S=(S(1), S(2), \ldots, S(n))$ with makespan $C_{m}(S)=C$.
2. Using DP or JR, solve the subproblem $F 2 \mid l t d$, perm $\mid C_{\max }$
for the job subset $\left\{S(k), \ldots, S\left(k^{\prime}\right)\right\}$ where $k^{\prime}=\min \{k+p-1, n\}$;
let $\left(\sigma(k), \ldots, \sigma\left(k^{\prime}\right)\right)$ be the resulting subsequence.
3. Set $S(j):=\sigma(j)$ for $j=k, \ldots, k^{\prime}$.
4. Set $k:=k+1$. If $k<n$ then go to Step 2.
5. If $C_{m}(S)<C$ then set $C:=C_{m}(S), k:=1$, and go to Step 2.
6. The heuristic has generated schedule $S_{S A}=S$.

As expected, the CPU requirements of heuristic SA increase with $p$. Dutta and Cunningham (1975) carried out experiments for instances with known optimal makespan, $n=20$ jobs, and $p=3$ or 5 and reported that the performance of SA is satisfactory.

## A Simple Greedy Heuristic

Several heuristics of the "greedy" type, that build a schedule one job at a time according to a criterion that minimizes an immediate cost without regard for future consequences, have been proposed for $F m \mid$ block, $($ perm $) \mid C_{\max }$. In Suhami and Mah (1981), $\sigma$ is extended by the $j \in \mathcal{U}$ that minimizes the makespan of $\sigma j$, iterating until all jobs are scheduled. This sequence is used as an initial upper bound in a branch-and-bound algorithm. Unfortunately, the search tree quickly explodes in size and cannot deal with problems with more than a few jobs.

## Profile Fitting (PF)

The following greedy heuristic developed by McCormick et al. (1989) constructs a permutation schedule one job at a time. At each iteration, the partial schedule $\sigma$ presents to the remaining jobs a profile determined by the departure time from each machine of the last job scheduled. We seek to schedule next a job that best fits this profile; that is, a $J_{j}, j \in \mathcal{U}$, whose time $p_{k j}$ on each $M_{k}$ is as close as possible to the time, $D_{k+1}(\sigma)-D_{k}(\sigma)$, until the next machine becomes available, thus minimizing blocking and idle times.

Heuristic PF for $F m \mid$ block, perm $\mid C_{\text {max }}$

1. Initialize $\sigma:=\phi, \mathcal{U}:=\{1, \ldots, n\}$.
2. Let $D_{k}(\sigma)$ be the departure time of the last job in $\sigma$ on $M_{k}, k=1, \ldots, m$, updated each iteration using (7.4).
3. For every $J_{j}, j \in \mathcal{U}$ compute $F_{j}=\sum_{k=1}^{m-1}\left|p_{k j}-D_{k+1}(\sigma)+D_{k}(\sigma)\right|$.
4. Find $J_{j *}$ with $F_{j *}=\min _{j \in \mathcal{U}} F_{j}$. Update $\sigma:=\sigma j^{*}, \mathcal{U}:=\mathcal{U}-\{j *\}$.
5. If $\mathcal{U} \neq \phi$, go to Step 2. If $\mathcal{U}=\phi$, stop: $S_{P F}=\sigma$.

To explain the penalty function $F_{j}$, it should be enough to note that if $J_{j}$ is scheduled next, then on $M_{k}$ :

- If $p_{j k}<D_{k+1}(\sigma)-D_{k}(\sigma), M_{k}$ is blocked for the difference;
- If $p_{j k}>D_{k+1}(\sigma)-D_{k}(\sigma), M_{k+1}$ is idled for the difference.

Note that the initial job chosen will be $J_{j}$ with smallest $\Sigma_{k=1}^{m-1} p_{k j}$; as good a choice as any. Or we could run the algorithm several times with different randomly chosen jobs put first, and choose the best. McCormick et al. (1989) tested their PF heuristic on the cyclic version of the problem. For small problems with $m=9$ and $n=8$, exhaustive enumeration was employed to identify an optimal schedule and it was found that the heuristic provides near optimal solutions. Less convincing performance is exhibited for larger problems with $m=10$ and $n=80$.

## Modified Profile Fitting (MM)

Ronconi (2004) developed a heuristic referred to as MinMax or MM and is similar to PF with the following exceptions:

- A job with $\min _{j} p_{1 j}$ is scheduled first;
- A job with $\min _{j} p_{m j}$ is scheduled last;
- Lines 1-4 of PF remain the same except that $F_{j}$ is replaced with

$$
\alpha \sum_{k=1}^{m-1}\left|p_{k j}-p_{k+1, i}\right|+(1-\alpha) P_{j}
$$

where $i$ is the last job in $\sigma$ and $\alpha$ is a given parameter used to negotiate the contributions of $P_{j}$ versus an estimate of the total additional blocking and idle time between $\sigma$ and $\sigma j$.

## Leisten's Greedy Algorithm (LGE) for Two Machines

An alternative notion of "greediness" is presented by Leisten (1990) for $F 2 \mid l t d$, perm $\mid C_{\max }$. The idea is to minimize blocking on $M_{1}$ for the next job to be scheduled by testing whether the job $\beta+1$ positions earlier has completed on $M_{2}$. Thus, if $[i]$ indexes the job in position $i$, then with $s$ jobs already scheduled $J_{[s+1]}$ is chosen from $\mathcal{U}$ to minimize $\left|C_{2,[s-\beta]}-C_{1,[s+1]}\right|$, where $C_{2,[s-\beta]}=0$ if $s-\beta \leq 0$. The completion times can be found using (7.1) with $C_{2 j}=D_{2 j}$ and $C_{1 j}=D_{1, j-1}+p_{1 j}$; they are easily updated as each job is positioned. We will refer to this as Leisten's greedy, or LG, heuristic.

Exploiting the reversibility property presented in Chapter 5, LG can be extended to report the best of two sequences: i) jobs scheduled in the order $1,2, \ldots, n$ from their start on $M_{1}$ to their finish on $M_{2}$, and ii) with time reversed, jobs scheduled in inverse sequence $n, n-1, \ldots, 1$ from their start on $M_{2}$ to their finish on $M_{1}$ (note that, in the reverse schedule, the completion time of a task $C_{k j}$ is really its start time in the final schedule). A final extension of LG, referred to as LGE, constructs a sequence by performing steps i) and ii) simultaneously in the spirit of Johnson's Algorithm. Thus, at any point, with partial schedules $\sigma$ at the start and $\sigma^{\prime}$ at the end, we schedule next $J_{j}, j \in \mathcal{U}$, that has minimal cost either following $\sigma$ or preceding $\sigma^{\prime}$. More specifically, with $s=|\sigma|$ and $s^{\prime}=\left|\sigma^{\prime}\right|$ :

Heuristic LGE for $F 2 \mid l t d$, perm $\mid C_{\text {max }}$

1. Initialize $\sigma:=\sigma^{\prime}:=\phi, s:=s^{\prime}:=0, \mathcal{U}:=\{1, \ldots, n\}$.
2. For all $J_{j}, j \in \mathcal{U}$,
with $j=[s+1]$, compute $v(j)=\left|C_{2,[s-\beta]}-C_{1,[s+1]}\right|$, and
with $j=\left[n-s^{\prime}\right]$, compute $v^{\prime}(j)=\left|C_{1,\left[n-s^{\prime}+\beta+1\right]}-C_{2,\left[n-s^{\prime}\right]}\right|$.
3. Let $\delta=\min _{j \in \mathcal{U}}\left\{v(j), v^{\prime}(j)\right\}$.
4. If $\delta=v\left(j^{*}\right)$ for some $j^{*}$, set $\sigma:=\sigma j^{*}, s:=s+1$.
5. If $\delta=v^{\prime}\left(j^{*}\right)$ for some $j^{*}$, set $\sigma^{\prime}:=j^{*} \sigma^{\prime}, s^{\prime}:=s^{\prime}+1$.
6. Set $\mathcal{U}:=\mathcal{U}-\left\{j^{*}\right\}$. If $\mathcal{U} \neq \phi$, go to Step 2. If $\mathcal{U}=\phi$, stop: $S_{L G E}=\left(\sigma, \sigma^{\prime}\right)$.

It was found that LGE performed better when the buffer capacity was multiplied by the factors 0.5 or 10 (i.e., $\beta$ is replaced in LGE by $0.5 \beta$ or $10 \beta$ ).

## Augmented Task Times

The idea of augmenting task times is introduced in Abadi et al. (1997, 2000) for cyclic schedules. Given a blocking schedule, each processing time is lengthened to fill in the blocking delay, thus converting the blocking schedule to a no-wait schedule, for which good heuristics exist. This approach is fully discussed in Sect. 7.8.2. We mention it here simply to point out that the same concept could be applied to static scheduling. A simple linear program NF(S) is given there to compute the augmentations for each task that will fill the blocking time, a calculation that is as pertinent to static as to cyclic scheduling.

## Computational Experiments

We have already mentioned some limited experimental results for the SA, PF, and PA heuristics. We now present some more extensive computational comparisons of proposed heuristic algorithms. Leisten (1990) tested a variety of heuristics developed for $F m \mid$ perm $\mid C_{\max }$, including LPT, NEH, CDS, Dannenbring's (1977) RA heuristic, and Palmer's slope heuristic. LPT simply orders jobs in nonincreasing order of $P_{j}=\Sigma_{k} p_{k j}$. NEH, CDS, RA and Palmer's heuristics are described in detail in Chapter 4 for problem $F m \mid$ perm $\mid C_{\max }$. He also included his two-machine LGE heuristic and the schedule $S_{0}^{\star}$ (see discussion following Theorem 7.4), both designed for two machines. He extended them to problems with $m>2$ machines by solving $m-1$ two-machine problems in which the first $k$ and the last $m-k$ machine processing times are added for each job, as described for the CDS heuristic.

Experiments were conducted on the comparative performance of heuristics on randomly generated instances of $F m \mid l t d$, perm $\mid C_{\max }$ with up to $n=50$ jobs, $m=2$ or 3 , and a variety of values for buffer capacities $\beta_{1}, \beta_{2}$. For $m=2$, it was found that NEH, LGE, CDS, and $S_{0}^{\star}$ produce the best solution amongst the heuristics tested $87,83,74$ and $73 \%$ of the time respectively. When the metric becomes obtaining a solution within $1 \%$ of the best makespan value, the corresponding percentages become $93,92,76$ and $95 \%$. These figures point to the superiority of NEH, LGE, and $S_{0}^{\star}$ for $F 2 \mid l t d$, perm $\mid C_{\max }$ problems. When $m=3$, NEH significantly outperforms other heuristics producing the best makespan value in $91 \%$ of the instances tested ( $98 \%$ within $1 \%$ ). This
result is surprising as well as impressive because NEH is not specifically designed for $F m \mid$ block, perm $\mid C_{\text {max }}$. It was also found that LGE adaptations to allow for job passing yielded inferior results to the permutation version of the heuristic. This indicates that makespan benefits due to passing are negligible.

As we discuss shortly, heuristics PF and MM exhibit superior performance to LPT and NEH. Heuristic NEH, however, is based on an initial sequence: the LPT sequence. Alternatively, NEH may be executed using initial sequences PF, or MM. Let PFE and MME be the resulting heuristics. Through experimentation, Ronconi (2004) proposed the value $\alpha=0.6$ as best for $M M$ and value $\alpha=0.75$ as best for $M M E$. The following heuristics were examined in the experiment of Ronconi (2004) on the problems in the Taillard (1993) suite: LPT, NEH, PF, MM, PFE and MME. It was found that PFE outperforms all other heuristics especially when $n \geq 200$. Also, the performance of PFE improves as $n / m$ increases. Recall that PFE aims to minimize the sum of blocking plus idle times. The superiority of PFE indicates that these times have a stronger impact on the makespan when the number of jobs per machine increases.

Recall the LAP-based lower bounding scheme presented in Section 7.5.2. Each application of the linear assignment problem yields a permutation; i.e., a heuristic solution for $F m \mid$ block, $($ perm $) \mid C_{\max }$. In the computational experiment presented in Karabati et al. (1992) for randomly generated instances with $n=12$ and the equivalent of $m=10$ machines, a total of 1200 permutations are produced and the best amongst them is evaluated against the LAP-based lower bound. The average relative deviation from this bound is found to be about $4 \%$, and in most instances the resulting heuristic solution is better than the one obtained by PF.

### 7.5.5 Metaheuristics

One cohort of heuristics for $F m \mid l t d$, perm $\mid C_{\max }$ is based on Tabu Search, TS for short. Details on TS are provided in Chapter 4. In brief, we start with an initial job sequence, perform a neighborhood search to find a better permutation, move to the improved schedule, and repeat. A tabu list of recently visited solutions is kept, to avoid cycling and to broaden the search region.

TS implementations that have appeared in the literature are based on the characteristics of the critical path from the start of $T_{11}$ to the end of $T_{m n}$ in a precedence network associated with a permutation sequence $S=$ $(1,2, \ldots, n)$. It turns out to be nontrivial to construct such a network for the flow shop with buffers. Nowicki (1999) argues that to extend the network shown in Figure 7.3, which assumes blocking ( $\beta_{k}=0 \forall k$ ), to the case with buffers ( $\beta_{k} \geq 0$ ) is impractical. Rather than replacing each unit of buffer capacity with a storage machine, he proposes a new precedence graph, as shown in Figure 7.4 for a nine-job four-machine instance with interstage storage capacities $\left(\beta_{2}, \beta_{3}, \beta_{4}\right)=(2,0,1)$. Here, each task is represented by a
node labeled with its processing time, and horizontal and vertical arcs impose the usual precedence ordering between jobs in a permutation schedule and between tasks in a job. However, the precedence arcs which reflect the buffer constraints, shown by the dashed arrows slanting up, add a negative time to the path length, as shown on the figure. Thus, if a path leaves a node via a diagonal arc, the contribution of that node to the path length is cancelled out. To see how this works, consider the selected path (in dark) in Figure 7.4. Between nodes $(3,4)$ and $(1,9)$, the path visits $(3,5)$ and $(2,6)$, but the contributions of those two nodes are negated. Thus on this path, $T_{34}$ has immediate precedence over $T_{19}$. This reflects the possibility that, if $p_{34}$ is very long, $M_{2}$ and its buffer may fill up with waiting jobs, so $J_{8}$ cannot advance from $M_{1}$ (allowing $J_{9}$ to start) until $J_{4}$ vacates $M_{3}$.


Fig. 7.4 Precedence graph of $S=(1, \ldots, 9)$ for $F 4 \mid l t d$, perm $\mid C_{\text {max }}$ with $\underline{\beta}=(2,0,1)$

This figure will be useful in illustrating the TS procedure to be described. We shall be chiefly interested in the way a neighborhood of a given sequence is defined. The neighbors are generated by moving a job to a new position. Since there are far too many such moves at each step, we must narrow down to a few moves most likely to produce improvement.

As usual, we shall need to specify the times at which tasks are processed. In the past, we have spoken of completion times $C_{k j}(S)$, or more recently of departure times $D_{k j}(S)$. It will now be convenient to define the third possibility
$B_{k j}(S)=$ the start time of the task in position $j$ on $M_{k}$ in schedule $S$.
As before, we shall generally suppress the dependence on $S$. Also, let $(k, j)$ denote $T_{k j}$ for short.

Beginning with an arbitrary job sequence $S=(1,2, \ldots, n)$, Nowicki (1999) and other subsequent papers restrict the search to a relatively few neighbors with a procedure outlined in the following steps:

1. Compute start times. Iteratively, for $j=1, \ldots, n$, and for each $j$ for $k=1, \ldots, m$, find

$$
\begin{gathered}
B_{k j}=\max _{i=1,2,3} f_{i}(k, j), \quad \text { where } \\
f_{1}(k, j)=B_{k-1, j}+p_{k-1, j}, \\
f_{2}(k, j)=B_{k, j-1}+p_{k, j-1}, \\
f_{3}(k, j)=B_{k+1, j-\beta_{k+1}-1},
\end{gathered}
$$

with $B_{k 0}=B_{0 j}=B_{m+1, j}=0$ and $p_{i 0}=p_{0 j}=0$. This of course comes from the requirement that $B_{k j} \geq f_{i}(k, j), i=1,2,3$; see the discussion of (7.1), which is the equivalent recursion in terms of departure times. We can also think of a task beginning when the last of its immediate predecessors completes.

For future reference, as we compute the start time $B_{k j}$ of each task $(k, j)$, it will be useful to keep track of its type, where $(k, j)$ is of type $i$ if $B_{k j}=$ $f_{i}(k, j)$.
2. Find the critical path. Step 1 has found us the early start times of all tasks, and in particular the makespan $C_{m}(S)=B_{m n}+p_{m n}$. As usual with dynamic programming, after the forward recursion comes backtracking. We move task by task back along the critical path, starting at $(m, n)$. At any point, if the current task $(k, j)$ is of type 1 , then its immediate predecessor is $(k-1, j)$; if of type 2 , then $(k, j-1)$; if of type 3 , then $\left(k+1, j-\beta_{k+1}-1\right)$. Ties are broken arbitrarily. Of course, we then move to the new task and iterate until we reach $(1,1)$. This procedure should be clear from the way $B_{k j}$ is calculated. In Figure 7.4, assuming the selected path is critical, the type-1 critical tasks are $(2,4),(3,4),(2,9),(3,9)$ and $(4,9)$; type-2 tasks are $(1,2)$, $(1,3),(1,4)$ and $(3,5)$; type-3 tasks are $(2,6)$ and $(1,9)$. Task $(1,1)$, lacking predecessors, has no type.
3. Define blocks of jobs. To introduce the idea of blocks, consider first a maximal set of consecutive critical tasks all on the same machine, say $\{(k, a),(k, a+1), \ldots,(k, b)\}$; i.e., a set of jobs $\left\{J_{a}, J_{a+1}, \ldots, J_{b}\right\}$ processed consecutively on $M_{k}$ without delay. In the figure, such a set is $\left\{J_{1}, J_{2}, J_{3}, J_{4}\right\}$ on $M_{1}$. Let $J_{a}$ and $J_{b}$ be the end jobs and the rest the interior jobs. Note that any rearrangement of the interior jobs will not affect the length of the current critical path. Possibly another path will become critical, but if so the makespan will be longer. Thus, no such rearrangement can produce a better schedule. We call such a set of jobs a block, $\mathcal{B}=\{a, a+1, \ldots, b\}$. It can be identified as the jobs associated with a maximal set of consecutive tasks on the critical path all of type 2 , plus the job that immediately precedes them (which is also on the same machine).

Sets of type-1 tasks (e.g., $(1,4),(2,4)$ and $(3,4)$ in the figure) are all from the same job, so do not form blocks. Sequential critical type-3 tasks, plus the jobs between them, plus one more at the start, do form blocks. For example, the maximal set of type-3 tasks $\left\{(k, a),\left(k-1, a+\beta_{k}+1\right),\left(k-2, a+\beta_{k}+\beta_{k-1}+\right.\right.$ $2), \ldots,(k-r, b)\}$, for some $r \geq 0$, that are consecutive on the critical path,
plus the task preceding $(k, a)$ on the critical path, namely $\left(k+1, a-\beta_{k+1}-1\right)$, produce the block of jobs $\mathcal{B}=\left\{a-\beta_{k+1}-1, a-\beta_{k+1}, \ldots, b\right\}$. In Figure 7.4, the consecutive type-3 tasks $(2,6)$ and $(1,9)$ give rise to the block $\mathcal{B}=\{5,6,7,8,9\}$.
4. The use of blocks. We have now defined blocks of two types; henceforth we will not differentiate between type-2 and type-3 blocks. All jobs are included in at least one block; end jobs are in both adjacent blocks. Assuming there are $h$ blocks, let $\mathcal{B}_{k}=\left\{u_{k-1}, \ldots, u_{k}\right\}$, where $1=u_{0}<u_{1}<\ldots<u_{h}=n$. The three blocks in Figure 7.4 are shown above the graph.

The important characteristic of blocks is that any job movement inside a block will not lower the makespan, so need not be considered. On the other hand, the number of possible insertions across blocks is $O\left(n^{2}\right)$ for a given sequence $S$, thus making it computationally expensive. Nowicki (1999) experimented with the following insertions, each for $k=1, \ldots, h$ :

- move jobs in positions $u_{k-1}, \ldots, u_{k}-1$ to position $u_{k}$;
or
- move interior jobs in $\mathcal{B}_{k}$ to position $u_{k-1}\left[u_{k-1}-1\right]$ if the final machine of $\mathcal{B}_{k-1}$ is different from [the same as] the starting machine in $\mathcal{B}_{k} ;$
or
- move the job in position $u_{k}$ to position $u_{k-1}$.

Nowicki's (1999) TS implementation used a tabu list of size 8, limit of 1,000 iterations without improving the incumbent solution and starts with the permutation obtained by the adaptation of NEH to $F m \mid l t d$, perm $\mid C_{\text {max }}$. This implementation was found to outperform the NEH on problems in the Taillard (1993) suite by $3.5 \%$. No significant dependence is found between algorithmic performance and the number of machines or the size of the buffers. Recall that, according to the experiment in Leisten (1990), NEH outperforms all other constructive heuristics. The caveat is that the TS implementation just described takes significantly more CPU time than NEH.

Using his tabu search metaheuristic, Nowicki (1999) examined the decrease in makespan as the buffer size increases for the problem instances in the Taillard (1993) suite. For instances with $n=20$ and $m=5$, it is found that buffers with $\beta=1$ improve makespan by only $0.5 \%$ compared to blocking, i.e., $\beta=0$. For $\beta>1$ the marginal makespan improvement is $0.04 \%$. For larger instances, e.g. $n=100$ and $m=20$, the size of buffers is of greater significance: a $5.5 \%$ improvement going from $\beta=0$ to $\beta=1$. In all instances, $\beta=4$ provided nearly the same makespan performance as the case of unlimited capacity buffers.

A different neighborhood search was used in the TS implementation presented in Grabowski and Pempera (2007) which we will call the GP implementation. It focuses on the problem without buffers, $F m \mid b l o c k$, perm $\mid C_{\max }$. For a given block $\mathcal{B}_{k}=\left\{u_{k-1}, \ldots, u_{k}\right\}$, for $k=1, \ldots, h$, the following insertion moves are considered:

- move jobs in positions $u_{k-1}, \ldots, u_{k}-1$ to position $u_{k}$;
or
- move jobs in positions $u_{k-1}+1, \ldots, u_{k}$ to position $u_{k-1}$.

The best amongst the above moves, say $v^{\star}$, is selected to produce a new heuristic sequence, and replaces the oldest move in the tabu list.

To improve the CPU performance of GP, the authors developed a variant referred to as GP-M, which periodically applies a multimove. A multimove is comprised of simultaneous insertions in every block. For each $\mathcal{B}_{k}$ :

1. consider moving each interior job to position $u_{k}$ and select the move with the greatest improvement on the incumbent makespan value (if any);
2. building on Step 1, consider moving each interior job to position $u_{k-1}$, again selecting the move giving greatest improvement (if any).

Evidently, there are at most $2 h$ such insertions; two per block. In a multimove, all these insertions are implemented simultaneously, in hopes that this will result to a significant improvement in makespan and a drastically different permutation. Indeed, GP-M is found to converge to better solutions than GP, faster. After 3,000 iterations, GP-M exhibits similar performance to 30,000 iterations of GP. When both variants are run for 30,000 iterations and the initial permutation is obtained by NEH, GP-M delivers on average $0.9 \%$ better schedules (with respect to makespan) than GP for the instances in the Taillard (1993) suite.

Alternate metaheuristic implementations include the genetic algorithm GA of Caraffa et al. (2001). Here, the starting sequence is chosen randomly and the mutation and crossover operations are standard. The algorithm is tested on problems with up to $m=20$ and $n=150$. Grabowski and Pempera (2007) tested algorithm GP on problems in the Taillard (1993) suite against the above GA. When GP is run for 10,000 iterations while GA is run for 100 , algorithm GP provides solutions with $7.9 \%$ lower makespan on average. However, in most cases the CPU time spent by GP is 20 or more times greater than GA. Algorithm GP-M on the other hand is much faster and produces schedules with an average of $6.75 \%$ better makespan than GA when both run for 100 iterations.

### 7.6 Extensions

We present two applications that extend the concept of a flow shop with blocking, where the objective remains makespan minimization.

### 7.6.1 Operating Room Scheduling

The application presented in Dekhici and Belkadi (2010) reflects a hospital operating theater consisting of $k_{1}$ operating rooms (ORs), from which a
patient must be transferred without delay to one of $k_{2}$ revival beds (RBs) in a post-anesthesia care unit. We thus have a two-stage hybrid flow shop $F\left(k_{1}, k_{2}\right) \mid$ block $\mid C_{\text {max }}$. There are additionally a number of constraints to be satisfied: (1) patients with higher priority must be scheduled earlier, (2) two patients should, or should not, be scheduled consecutively, (3) a patient may require surgery before or after a certain date and time, and (4) a patient may have to be assigned a particular OR, or OR-RB pair.

A tabu search algorithm is developed for this problem. During a preprocessing stage of the algorithm, a penalty function is used to identify schedules that minimize constraint violations, using local search. Then the problem is solved with a tabu list of size 7, a candidate list of size 500, and 1000 iterations. Throughout the algorithm, restricted neighborhood search like the one in Belkadi et al. (2006) ensures that the solutions in the candidate list satisfy all constraints as in the pre-processing stage.

### 7.6.2 Flow Shops with Pallets

In another related application, Wang et al. (1997) studied the complexity of problems where each job is loaded on a pallet upon entering the production system, and remains on it until it exits. Hence, at any time, the number of pallets, $p$, dictates the maximum number of jobs in process or waiting between stages. This problem, denoted by $F m|p a l(p)| C_{\max }$ for $m \geq 2$, is related to $F m|l t d| C_{\max }$ with $\sum_{k=2}^{m} \beta_{k}=p-m$. They are not certainly equivalent problems, except for the special case $m=p=2$ when $F 2|p a l(2)| C_{\text {max }}$ is equivalent to $F 2 \mid n w t$ or $b l o c k \mid C_{\max }$ and is, of course, solvable in $O(n \log n)$ time using the GGV algorithm presented in Chapter 6. Theorem 7.2 establishes that $F 2|p a l(3)| C_{\text {max }}$ is strongly NP-hard because in this case one pallet resides on each machine, and a third pallet is located at the storage area between $M_{1}$ and $M_{2}$. Wang et al. (1997) also show that $F 3|p a l(2)| C_{\max }$ is strongly NPhard; however, when the no-passing constraint is imposed on the jobs, the reduction is from 2-Partition thus leaving open the exact complexity status of this variant of the problem.

### 7.7 Fm|block, perm| $\boldsymbol{\Sigma} T_{j}$

An adaptation of the lower bound presented in $\operatorname{Kim}(1995)$ for $F m \mid$ perm $\mid \Sigma T_{j}$, is used in Ronconi and Armentano (2001) to develop a lower bound for $F m \mid$ block, perm $\mid \Sigma T_{j}$ which is then used to develop a branch-and-bound algorithm. As usual, we present the bounds in the context of such an algorithm, where a partial schedule $\sigma$ already occupies each $M_{k}$ up to time $C_{k}(\sigma)$. Let $\mathcal{U}$ denote the set of $u$ jobs (actually, job indices) remaining to be scheduled, and hereafter, when we refer to jobs or tasks, we mean unscheduled ones, and the lower bounds we compute are for those jobs only. For bounds on the
total schedule, the tardiness of the jobs in $\sigma$ are a sunk cost that should be added. Of course, if there is no $\sigma$, let all $C_{k}(\sigma)=0$.

Following the logic of Sect. 4.16.1, in a permutation schedule, the task $T_{k[i]}$ processed in position $i$ of $M_{k}$ (let us call this position $k i$ ) is preceded on $M_{k}$ by $i-1$ tasks, and they in turn must be preceded by at least one task on each of $M_{1}, \ldots, M_{k-1}$, machines which are already occupied by the jobs in $\sigma$. Similarly, $T_{k[i]}$ is followed by at least one task on each of $M_{k+1}, \ldots, M_{m}$. The $i-1$ tasks preceding $T_{k[i]}$ on $M_{k}$ are processed with or without blocking caused by the previous job on $M_{k-1}$. Hence, $T_{k[i]}$ and the preceding $i-1$ tasks, occupy $M_{k}$ for at least $\sum_{r=2}^{i} \max \left\{p_{k[r]}, p_{k-1,[r-1]}\right\}$ units of time. It will be convenient to define, for $i \in \mathcal{U}$ :

$$
\begin{aligned}
& q_{k i}=\text { the } i^{t h} \text { smallest task time on } M_{k}, \text { and } \\
& Q_{k i}=\sum_{r=2}^{i} \max \left\{q_{k r}, q_{k-1, r-1}\right\} .
\end{aligned}
$$

Then, $Q_{k i} \leq \sum_{r=2}^{i} \max \left\{p_{k[r]}, p_{k-1,[r-1]}\right\}$ for any permutation of the unscheduled jobs. Accounting for all possible $k$ 's, the completion time $C_{m[i]}$ of task $T_{m[i]}$ is at least

$$
t_{[i]}=\max _{k=1, \ldots, m}\left\{\max _{j=1, \ldots, k}\left[C_{j}(\sigma)+\sum_{h=j}^{m} q_{h 1}+Q_{k i}\right]\right\}
$$

Reindex jobs in $\nearrow d_{j}$ order. Then, as in Sect. 4.16.1, for any permutation schedule,

$$
\sum_{i=1}^{u} T_{i}=\sum_{i=1}^{u}\left(C_{m[i]}-d_{[i]}\right)^{+} \geq \sum_{i=1}^{u}\left(t_{[i]}-d_{i}\right)^{+}
$$

providing the desired lower bound. Using minor adaptations of this lower bound, Ronconi and Armentano (2001) implemented a branch-and-bound algorithm which was tested against the permutation obtained by the NEH heuristic. On randomly generated problems with up to $n=18$ and $m=10$, it was found that the algorithm can solve problem $F m \mid$ block, perm $\mid \Sigma T_{j}$ within 1 hour of CPU time when the stopping criterion is the optimality gap to be within $0.5 \%$. For larger values of $n$, the algorithm often fails to find nearoptimal solutions.

### 7.8 Cyclic Flow Shops with Blocking

We now consider the cyclic scheduling problem, as introduced in Sect. 1.7.1, in which the $n$ given jobs (in the cyclic literature, this is a minimal part set or MPS) are to be manufactured repeatedly so as to minimize the cycle time (or equivalently to maximize the output rate or throughput). This we denote $F m \mid$ block, cyclic|CT. As earlier noted, the static schedule with minimal makespan $C$, repeated every $C$ periods, yields a feasible but generally suboptimal solution to the cyclic problem. Hence, solving the integer program $\mathbf{B}$ given in Sect. 7.5.1 provides an upper bound on the cycle time.

To solve the cyclic problem directly, we can modify $\mathbf{B}$ to replace the makespan objective with the cycle time $Z$, which is the largest machine
occupancy as we discussed in connection with Fig. 1.6, getting:

$$
\begin{array}{ll}
\operatorname{minimize} & Z \\
\text { subject to } & Z \geq d_{k n}-d_{k 1}+\sum_{j=1}^{n} p_{k j} x_{1 j}, k \in \mathcal{M}  \tag{7.12}\\
& \text { and }(7.5),(7.6),(7.7),(7.8),(7.9),
\end{array}
$$

where (7.12) enforces $Z=\max _{k}\left\{d_{k n}-s_{k 1}\right\}$.

## Graphical Portrayal of $\operatorname{Fm} \mid$ block, cyclic $\mid \boldsymbol{C}_{\text {max }}$

As with static flow shops with blocking, cyclic flow shops with blocking accept an elegant pictorial representation. Recall that the makespan associated with a job permutation for $F m \mid$ block $\mid C_{\max }$ was the length of the longest path from $T_{11}$ to $T_{m n}$ as in Fig. 7.3. For problem Fm|block, cyclic|C $\max$, the value of $Z^{*}$ may be attained by any one of the inequalities in (7.11), that is, by the longest occupancy of any machine. We show in Figure 7.5 one cycle (with parts of the adjacent cycles) in the repetitive manufacture of the same set of jobs $(n=6, m=3)$ as in the earlier figure, and how the adjacent cycles connect to it. The heavy arrows mark a hypothetical critical path associated with $M_{2}$, running from the start of $T_{21}$ in one cycle to the start of the same task in the next cycle (bold nodes). The cycle time is the maximum over all machines of these path lengths.


Fig. 7.5 Precedence graph of one cycle for $F 3 \mid b l o c k$, cyclic $\mid C T$

### 7.8.1 Profile Fitting

Among the algorithms for makespan minimization surveyed in Sect. 7.5.4 was the Profile Fitting (PF) heuristic of McCormick et al. (1989). The authors found that PF also performs quite well for Fm|block, cyclic|CT. Furthermore, even better performance was achieved with the following refinement. Recall, each job's suitability to go next is calculated as the sum of the idle plus blocking times it produces on all machines. The new idea is to give greater weight to time wasted on more heavily loaded machines, where the work load on $M_{k}$ is measured by $\Sigma_{j} p_{k j}$. The precise value of the weights was not discussed.

### 7.8.2 Augmented Task Times

A different way to generate schedules for $F m \mid$ block, $p e r m$, cyclic $\mid C_{\text {max }}$ was exploited in Abadi et al. (1997, 2000). Their idea is, given any blocking schedule $S$, to lengthen each task time as needed to fill in the additional time a task remains on a machine after completion due to blocking. This task augmentation converts a blocking schedule into a no-wait schedule. The advantage of this approach is the use of a vast literature for $F m \mid n w t$, perm| $C_{\text {max }}$ to produce schedules that are then adapted for $F m \mid$ block, perm $\mid C_{\text {max }}$. For a given blocking permutation schedule $S=(1,2, \ldots, n)$, the augmentation is calculated by the following linear program $N F(S)$ (modified from the original in Abadi et al. (1997)), which is shown to be equivalent to a mincost Network Flow. Defining the variables

$$
q_{i j}=\text { the amount of increase in } p_{i j}
$$

we have:
$\mathbf{N F}(\mathbf{S}) \quad$ minimize $\sum_{j=1}^{n} q_{1 j}$

$$
\begin{array}{ll}
\text { subject to } & \sum_{i=1}^{k-1}\left(p_{i j}+q_{i j}\right) \geq \sum_{i=2}^{k}\left(p_{i, j-1}+q_{i, j-1}\right),  \tag{7.13}\\
& q_{i j} \geq 0,
\end{array}
$$

where $q_{i 1}=q_{m j}=0$ (the tasks of $J_{1}$, and the tasks on $M_{m}$, are never blocked), and where (7.13) holds for $k=2, \ldots, m, j=2, \ldots, n$.

Abadi et al. (2000) used GENIUS - a package developed by Gendreau, Hertz and Laporte (1992) - to produce near optimal no-wait schedules. Every permutation $S$ of jobs produced in GENIUS is augmented using $N F(S)$ and the resulting schedule is evaluated for $F m \mid b l o c k$, perm $\mid C_{\text {max }}$. The best makespan value and permutation are then reported.

We refer to the above heuristic as PA because it involves a number of Parallel Augmentations. Indeed, GENIUS uses two subroutines to create new permutations: i) GENI which starts with 3 arbitrary jobs and then inserts unscheduled jobs in the best position of the current subsequence, and ii) US which starts with a complete permutation and reinserts jobs one by one to the best position of the current subsequence. For each job, the best sequence is saved, thus US produces $n$ permutations for every start sequence. Both GENI and US use 4 alternative ways to insert or remove jobs for reinsertion, respectively. Evidently, the number of candidate permutations produced by GENIUS is very large. Evaluating all these permutations within heuristic PA delivers great performance. On randomly generated instances with up to $n=250$ jobs and $m=20$ machines, the percentage deviations from the Lagrangean decomposition bounds reported by Abadi et al. (2000) are on average less than $2 \%$. Moreover, these percentage deviations decrease as $m$ and $n$ increase. The study of Abadi et al. appears to be the only one where heuristic performance is measured against a lower bound. In earlier experiments, heuristic performance has been reported in comparative terms.

### 7.9 Conclusions

The body of literature on blocking and limited buffers allows us to draw some key conclusions. First, it is apparent that blocking and limited buffers appear in many production settings. Second, it is surprising to see that the NEH heuristic is simple yet powerful even compared to sophisticated metaheuristics that take a lot more CPU time to output their best solution. Finally, of interest is the observation that a very small number of interstage storage buffers provides the same benefits as unlimited storage does. On the negative side, we found that the literature is sparse for objectives other than makespan.

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## Chapter 8 FLEXIBLE FLOW SHOPS


#### Abstract

We introduce four types of flexibility encountered in a flow shop: job routing through a hybrid shop, machine assignment, allocation of a scarce resource over the tasks to speed up processing, and the composition of daily production batches to satisfy requirements for a finite set of parts over a finite horizon. Two forms of machine flexibility are considered: multitasking where a machine can do more than one task of a job, and multiprocessing where several machines may combine to process a task. In each case, we present some or all of the following: complexity results, solution algorithms or heuristics with worst case performance, and assessments on the benefits of flexibility.


### 8.1 Preliminaries

Flexibility is an important aspect of manufacturing and operations. Unfortunately, though most managers will agree that flexibility is a desirable trait in manufacturing systems, few are clear as to the nature of the flexibility desired or the value of the flexibility to the system's performance. In this chapter, we survey the flow shop scheduling literature that deals with various kinds of flexibility, including ways of measuring the value of flexibility and of exploiting flexibility to improve system performance. Note that the system sometimes referred to as a "flexible flow shop", where each stage has several parallel processors, is not included in our definition of the term "flexible". We refer to that configuration as a "hybrid flow shop", and it is covered in Chap. 5. We also consider hybrid shops in this chapter, but as a setting for flexibility considerations, not because the hybrid system has any intrinsic flexibility.

In this section, we first describe aspects of flexibility and classification schemes. We then describe the basic flow shop problem with which we are concerned. The rest of the chapter includes one section dedicated to each
major type of flexibility encountered in the flow shop literature and a final section with concluding remarks.

### 8.1.1 Types of Flexibility

Several schemes for classifying flexibility have been suggested in the literature, as summarized by Upton (1994). He first distinguishes between external and internal flexibility. External flexibility is concerned with the output of the system that the consumer observes. A firm that can customize its products to meet diverse customer needs is demonstrating external flexibility. Internal flexibility is concerned with the ability of a firm to manufacture a product in a variety of ways. A machine that can perform different tasks when equipped with different tool magazines is a source of internal flexibility. We will be exclusively concerned with internal flexibility.

Upton (1994) further classifies internal flexibility by the dimension of change (the aspect of the system which requires flexibility), the time horizon of change (how often change is required) and the element of change. Three elements of change are identified: range, mobility and uniformity. For any dimension of change, range is the capacity to produce products with diverse values on that dimension; mobility is the ability to change position on the dimension without incurring significant penalties; and uniformity is the ability to maintain system performance as the manufacturing environment changes.

A common approach, which we shall use to organize this chapter, is to categorize manufacturing flexibility simply by type, which combines the categories of dimension and time horizon, above. Upton lists many types of flexibility. We can have short-term or long-term flexibility, flexibility in product or process, in volume, expansion or design change, and many more.

The four types of manufacturing flexibility encountered in the flow shop literature are: Routing Flexibility, Machine Flexibility, Labor Flexibility and Mix Flexibility. Routing Flexibility is available when jobs may be processed on one of several machines in each stage of a hybrid flow shop. Then a choice of one machine per stage corresponds to a route for a job. When jobs can be processed in the flow shop using different routes, routing flexibility is experienced. Machine Flexibility occurs when the manager has alternative processing choices for tasks. For instance, the choice of tool magazines can render more than one machine capable of processing a particular task. Labor or more generally Resource Flexibility arises when one or more resources can or must be scheduled concurrently with the jobs to be done. The resource may be required by a machine to perform part or all of a task (a worker with a specific skill is needed) or the level of the resource may influence the processing time of a job (assigning more labor to a machine allows it to complete jobs faster). In the first case we say the resource is flexible because it can move between different machines and stages in the flow shop. In the second case we say the resource is flexible because we can allocate different levels during the processing of each job. Finally, Mix Flexibility occurs
when jobs are released to the system in batches. The jobs in each batch are not distinct, rather each job belongs to one of a small number of job types. A job type uniquely determines the processing characteristics of a job; e.g., its processing time. If the manager can control the ratio of job types in a batch then the system exhibits Mix Flexibility. In the rest of this chapter, we consider each of these four main types of manufacturing flexibility in turn.

### 8.2 Routing Flexibility

As previously mentioned, routing flexibility is encountered in the hybrid flow shop setting when a job can be processed on any one of several machines at a stage. Specifically, the $F\left(k_{1}, k_{2}, \ldots, k_{m}\right)$ environment without further restrictions has complete routing flexibility, as each job can be routed through any machine at each stage. In other environments, there may be restrictions on which machine(s) each job may visit in each stage. Such additional routing constraints on the pure $F\left(k_{1}, k_{2}, \ldots, k_{m}\right)$ shop will be indicated in the $\beta$ section of the system's notation. Two versions of $F\left(k_{1}, k_{2}, \ldots, k_{m}\right)|\beta| \gamma$ have appeared in the research literature. The first is the static problem of scheduling $n$ given unrelated jobs through the $F\left(k_{1}, k_{2}, \ldots, k_{m}\right)|\beta| \gamma$ system. The second, dynamic version is concerned with the performance of the system when $n$ different types of jobs arrive randomly and continuously over time. Each job corresponds to processing a fixed number of a certain part type and all jobs are processed on a first-come-first-served basis.

Clearly, the more flexible a particular instance of $F\left(k_{1}, k_{2}, \ldots, k_{m}\right)|\beta| \gamma$ is, the better the performance of the system can be, since all feasible routings for the less flexible instance should be available for the more flexible instance. The literature focuses on measuring the degree of improvement due to increased routing flexibility, as well as on algorithms for better managing this flexibility.

### 8.2.1 Approximating Performance using Merged Machines

We shall be interested in assessing the performance of hybrid flow shops with various degrees of flexibility. These are complex systems and hard to analyze; it will be necessary to find simplifying approximations. An analytical technique that has proven successful is to replace the $k$ parallel identical machines at any stage by a single superserver that works $k$ times as fast. Of course, such a substitution is not actually possible; it is a mathematical device to simplify analysis. To understand the analogy between the two systems, it may be helpful to visualize each job in the original system broken into $k$ equal pieces, which are then processed simultaneously on all $k$ machines. Thus, every job starts and stops synchronously on all machines, which appear to act as one superserver. We will refer to this act of combining several servers into one superserver as merging machines. Due to its higher processing speed,
the superserver does the same work per unit time as the original $k$ machines. The imaginary superserver is a fairly close equivalent to the real system for most objectives under most conditions, and one server is a lot easier to analyze than $k$.

To see how the two systems compare, consider a single stage of $k$ parallel machines. It is not hard to see that the makespan on the superserver is a lower bound, often tight, on the parallel machine makespan. For example, on 2 machines, 3 jobs taking 5, 6, and 7 hours require 11 hours to process without preemption. A single machine doing the jobs sequentially at twice the speed needs only $\frac{5+6+7}{2}=9$ hours. In general, the two makespans will be equal if a perfect packing is possible in the parallel system, or if preemption is allowed. Similar conclusions can be drawn for the total completion time objective. Again, the superserver gives a lower bound. For both objectives, the relative error increases as the number of jobs increases, and as the number of machines increases.

### 8.2.2 Routing Flexibility in $\boldsymbol{F}\left(\boldsymbol{k}_{1}, \boldsymbol{k}_{2}\right)|\boldsymbol{\beta}| C_{\text {max }}$

In Vairaktarakis and Elhafsi (2000) the value of routing flexibility is investigated via two related $F\left(k_{1}, k_{2}\right)|\beta| C_{\text {max }}$ problems. The first and more flexible environment is the standard $F\left(k_{1}, k_{2}\right) \| C_{\text {max }}$. This is compared to a less flexible but related design referred to as the parallel hybrid flow shop or $F\left(k_{1}, k_{2}\right)|\operatorname{par}| C_{\max }$, in which routing is restricted so that each machine at stage $G_{1}$ routes jobs to only $k_{2} / k_{1} G_{2}$-machines. It is assumed that $k_{2}$ is an integer multiple of $k_{1}$. Thus, $F\left(k_{1}, k_{2}\right) \mid$ par $\mid C_{\max }$ comprises $k_{1}$ independent parallel systems or virtual cells, each being an $F\left(1, k_{2} / k_{1}\right)$ hybrid flow shop. Note that the number of routes available for a job in the first environment is $k_{1} k_{2}$ (the largest possible number), while in the parallel system the number of routes is just $k_{2}$ (the smallest possible, since any number smaller than $k_{2}$ would result in unused machines). The unconstrained system is maximally flexible, while the parallel system is minimally flexible.

Hoogeveen et al. (1996) have shown that $F(1,2) \| C_{\max }$ is strongly NPcomplete (see Chap. 5 for a brief description of the proof), and hence both $F\left(k_{1}, k_{2}\right)|\operatorname{par}| C_{\max }$ and $F\left(k_{1}, k_{2}\right) \| C_{\max }$ are NP-complete. Thus, in order to compare the makespan performance of the two designs, the authors use the best known heuristic for $F\left(k_{1}, k_{2}\right) \| C_{\max }$, and develop similar heuristics for $F\left(k_{1}, k_{2}\right)|\operatorname{par}| C_{\max }$. For $F\left(k_{1}, k_{2}\right) \| C_{\max }$, the best theoretical perfomance is exhibited by the heuristic developed in Lee and Vairaktarakis (1994) and Chen (1995), as discussed in Chapt. 5. Since it applies to the unconstrained hybrid flow shop, we call it $\mathrm{H}_{\mathrm{U}}$. We repeat it here for convenience.

Heuristic $\mathbf{H}_{\mathbf{U}}$ for $F\left(k_{1}, k_{2}\right) \| C_{\text {max }}$

1. Let $S=J R\left(\underline{a} / k_{1}, \underline{b} / k_{2}\right)$.
2. Apply the FAM rule to assign the $a$-tasks to $G_{1}$ using $S$.
3. Apply the LBM rule to assign the $b$-tasks to $G_{2}$ using $S$.
4. On each $G_{2}$-machine, resequence the $b$-tasks assigned to it in the order they become available from $G_{1}$, and start each as early as possible.
5. Compute the makespan $C_{U}$ of the resulting schedule $S_{U}$.

In Theorem 5.6, we established the worst case performance of $\mathrm{H}_{\mathrm{U}}$, showing that $C_{U} / C^{\star} \leq 2-1 / \max \left\{k_{1}, k_{2}\right\}$ was a tight bound.

For $F\left(k_{1}, k_{2}\right)|\operatorname{par}| C_{\max }$, Vairaktarakis and Elhafsi (2000) developed a heuristic of comparable performance; we refer to this heuristic for the parallel hybrid as $\mathrm{H}_{\mathrm{P}}$. The algorithm requires that $k_{1}$ be a power of two, and as before that $k_{2}$ be a multiple of $k_{1}$. The authors first develop a dynamic programming algorithm to solve $F(2,2)|\operatorname{par}| C_{\max }$ (i.e., two 2 -machine flow shops in parallel) to optimality. To apply it here, we merge the machines at each stage $i(i=1,2)$ into two superservers, each replacing half the machines in the stage, hence working $k_{i} / 2$ times as fast. By applying the DP algorithm with respect to the processing times $\left\{<2 p_{i 1} / k_{1}, 2 p_{i 2} / k_{2}>: i=1,2, \ldots, n\right\}$ the jobs can be partitioned into two sets, each to be done on half the stage-1 machines and an associated half of the stage- 2 machines.

We now repeat the process, applying the algorithm to each of the two subsets of jobs, and the result is four subsets of jobs, each to be processed on one quarter of the machines at each stage. We continue iterating until we have $k_{1}$ subsets of jobs, each assigned to a single $G_{1}$-machine and $k_{2} / k_{1}$ $G_{2}$-machines. The authors show that this assignment can be achieved even when $k_{1}$ is not a power of two, and when $k_{2}$ is not a multiple of $k_{1}$. This is achieved by introducing dummy work centers and fictitious jobs. Given such an assignment the jobs are then scheduled in the corresponding $F\left(1, k_{2} / k_{1}\right)$ environment using $\mathrm{H}_{\mathrm{U}}$.

## Computational Results

Both $\mathrm{H}_{\mathrm{U}}$ and $\mathrm{H}_{\mathrm{P}}$ are shown in the corresponding papers to achieve nearoptimal makespans, $C_{U}$ and $C_{P}$. For randomly generated problems with uniformly distributed processing times between 1 and $20, \mathrm{H}_{\mathrm{U}}\left[\mathrm{H}_{P}\right]$ is used to solve $F\left(k_{1}, k_{2}\right) \| C_{\max }\left[F\left(k_{1}, k_{2}\right)|\operatorname{par}| C_{\max }\right]$. The value of routing flexibility is evaluated using $\left(C_{P}-C_{U}\right) / C_{U} \times 100 \%$, which measures the (makespan) performance gains of $F\left(k_{1}, k_{2}\right) \| C_{\max }$ over $F\left(k_{1}, k_{2}\right)|p a r| C_{\max }$. For each combination of $n=20,30,40$ or 50 jobs, $k_{1}=2$ or 4 machines and $k_{2}=2,4,8$ or 16 machines, 50 instances were randomly generated.

It was found that, in general, the percentage increase in makespan performance due to increased routing flexibility in $F\left(k_{1}, k_{2}\right) \| C_{\max }$ is negligible. Particularly when $n$ is large ( $n=50$ ), the percent increase was at most $2.2 \%$ and averaged $1.3 \%$ over all combinations considered. For small $n(n=20)$, the percent increase was somewhat higher, averaging $4.1 \%$ over all combinations. In general the percentage gap was seen to decrease with the size of $n$ and increase with the size of $k_{1}+k_{2}$. Overall the results suggest that unless the system handles jobs in small batch sizes, the benefits of the added flexibility of $F\left(k_{1}, k_{2}\right) \| C_{\max }$ are not likely to outweigh the additional investment
and management costs associated with its complex routing structure. Thus when designing plant layouts it will be more cost effective to sacrifice the flexibility of $F\left(k_{1}, k_{2}\right) \| C_{\text {max }}$ for the less complex $F\left(k_{1}, k_{2}\right)|p a r| C_{\max }$ where independent parallel cells are used. This should result in a negligible loss of throughput.

### 8.2.3 Evaluating $\operatorname{Fm}(k, k, \ldots, k)|\beta| \Sigma C_{j}$ as a Queueing Network

In Vakharia et al. (1999) the value of Routing Flexibility is studied when job orders arrive at an $m$-stage $F(k, k, \ldots, k)$ shop as a Poisson process, and receive service in order of arrival. There are $k$ types of jobs (we might call them products or part types). Each arriving order is for one unit of one product, and an order is equally likely to be of any type. The processing times are all equal to $p$, for every job at every stage and of every type. A delay (a setup or switch-over time) $s$ is incurred if a machine is to process a type- $t_{2}$ job after a type- $t_{1}$ job, $t_{1} \neq t_{2}$. When multiple machines in a stage are available to process a job, preference is given to machines which are properly tooled for the job type (i.e., where the delay $s$ can be avoided). Otherwise the machine is selected randomly.

For this system the authors investigate the impact on average flow time of dedicating a single machine in every stage to one job type (thus, each product has its own dedicated $m$-machine flow shop and we have $k$ parallel cells, giving minimal flexibility) versus allowing jobs of all types to be routed through every machine within a stage (the unconstrained system with maximal flexibility). To compute system performance, the $G I|G| m$ queuing approximations from Whitt $(1983,1993)$ were used, after verifying through simulation that errors were less than $10 \%$. In order to use the approximation, a number of parameter values must be designated (mean arrival rate, mean processing time, coefficient of variation of processing times and coefficient of variation of interarrival times); details will be omitted here.

The corresponding average flow times were computed for problems with $m=2,3$ or 4 stages, and $k=2,4$ or 6 machines per stage (and hence job types). The arrival rate for each type of job was fixed at $1 / 6$. The processing times for all tasks took values $p=3$, 4 , or 5 , while the setup times $s$ were set at $.5,1.5$ and 3 . Thus, the ratios of setup time to processing time ranged from 10 to $100 \%$.

It was this ratio of setup times to run times that proved to be the most important factor affecting routing flexibility. Not surprisingly, when the ratio is low (between 10 and $16.67 \%$ ) the unconstrained system outperformed the parallel system by 15 to $22 \%$. In contrast, when the ratio is high (between 70 and $100 \%$ ) the parallel system outperformed the unconstrained one by 25 to $43 \%$. When the ratio is at a medium level it was found that as the total number of machines in the shop and the processing time increased, the performance of the unconstrained shop improved over the parallel shop. For
$p=3$ or $m=2$ stages, the parallel shop dominated. But when $p=4$ and $k=4$ or 6 , the higher level of routing flexibility (provided by the number of machines in each stage) favors the unconstrained shop. It was also seen that an increase in the number of stages $m$ can alleviate the performance loss due to routing inflexibility in the parallel shop. For example, when $p=4$ and $m=4$, the parallel system again performs better.

The above system is, of course, extremely simplified, to provide a tractable model with which to investigate flexibility. For one thing, the number of machines at each stage is the same as the number of job types, so that the minimal flexibility of $k$ parallel cells is attainable. The authors discuss the situation when this is not true. They suggest that, to approximate $k$ parallel cells, machines should be assigned to job types at each stage in proportion to the average processing requirements. However, when the number of machines in a stage is small, it may not be possible to approximate well the desired proportions. To handle this, the authors propose a hybrid system where the problematic stages receive full flexibility as in the unconstrained flow shop and other stages are treated as virtual cells.

### 8.3 Machine Flexibility

Machine or processor flexibility can result from assigning different tool magazines to machines or it may be due to the multiple and/or differing capabilities of machines. The first source is referred to as tooling or equipping. We may be free to retool a machine before each task, in which case the time and cost of such setups necessitate careful job sequencing. If instead we must equip all machines at the beginning of the schedule without the option to change between jobs, efficient scheduling requires appropriate tooling of each machine, followed by a proper assignment of jobs to machines. The second kind of flexibility arises with multipurpose machines, which have fixed capabilities with no equipping possible, but have the versatility to do a variety of task types. A machine may be capable of doing several types of tasks at a stage, or it may be used to process tasks at more than one stage.

In studying this kind of flexibility, our model will be a two-stage hybrid flow shop in which, due either to tooling flexibility or multipurpose equipment, each job can be processed in more than one mode. We consider two possibilities. Briefly, in one case a processor may be capable of doing more than one task of a job (say, a $G_{2}$-machine, $M_{.2}$, can do both tasks $T_{j 1}$ and $T_{j 2}$ of $J_{j}$ in combination), while in the other it may be possible to use more than one processor to do a task. The first we shall call multitask flexibility, and we denote it $\boldsymbol{m t f} \boldsymbol{l} \boldsymbol{x}$ in the $\beta$ field of the problem encoding. With multitask flexibility, each job can either be split between the two stages as usual, or completely processed at either stage (so that the machine at that stage does multiple tasks). We start by considering a simple flow shop (only one machine at each stage), and extend the model to the hybrid flow shop.

The second kind of resource flexibility we shall refer to as multiprocessor flexibility, $\boldsymbol{m p f l x}$ for short. Here, we deal only with the simple flow shop. Each task may be processed at the appropriate stage by one machine as usual, or both machines may combine their capacities and work on a task simultaneously.

### 8.3.1 $F\left(k_{1}, k_{2}\right)|m t f l x| C_{\text {max }}$

We assume now that each $J_{j}$ can be processed in the two-stage shop in any of three ways: it can flow through each stage as usual, with time requirements $a_{j}$ and $b_{j}$, or it can be entirely processed at stage $G_{1}\left[G_{2}\right]$ for a time $p_{j}\left[q_{j}\right]$. Thus, we wish to partition the jobs into three sets:

- $\mathcal{V}_{1}=\left\{j: J_{j}\right.$ is entirely processed at $\left.G_{1}\right\}$
- $\mathcal{V}_{2}=\left\{j: J_{j}\right.$ is entirely processed at $\left.G_{2}\right\}$
- $\mathcal{V}_{3}=\left\{j: J_{j}\right.$ is processed first at $G_{1}$, then at $\left.G_{2}\right\}$
where, for simplicity, we represent each job in $\mathcal{V}_{i}$ by its index. We shall make the obvious assumption that $p_{j}>a_{j}$ and $q_{j}>b_{j}$; otherwise, any job with $p_{j} \leq a_{j}\left[q_{j} \leq b_{j}\right]$ can be automatically placed in $\mathcal{V}_{1}\left[\mathcal{V}_{2}\right]$. We start by considering the case with $k_{1}=k_{2}=1$, and write $\mathrm{F}(1,1)$ as F 2 .


## F2|mtflx| $C_{\text {max }}$

With single machines at each stage, for any given partition, the optimal schedule follows easily, as shown by Jackson (1956): the jobs in $\mathcal{V}_{3}$ should be scheduled first on $M_{1}$, in Johnson's sequence, $J R(\underline{a}, \underline{b})$, followed by the jobs in $\mathcal{V}_{1}$ in any order. On $M_{2}$, start with $\mathcal{V}_{2}$ in any order, then schedule $\mathcal{V}_{3}$ in $J R(\underline{a}, \underline{b})$ order (actually, the work on $M_{2}$ can be sequenced in any order that does not insert idle time). Such a schedule is shown in Fig. 8.1; the extra notation will be introduced shortly. Note that it can be viewed as a permutation schedule:

$$
S^{\star}=\left(\mathcal{V}_{2}, \mathcal{V}_{3}: J R(\underline{a}, \underline{b}), \mathcal{V}_{1}\right)
$$

with the jobs following the same sequence on both processors.


Fig. 8.1
An optimal schedule with $I_{2}=0$

However, the partitioning problem remaining is still difficult. Indeed, the special case with $b_{j}=0$ and $p_{j}>a_{j}=q_{j}$ (note how this cost structure forces
the second tasks of each job to be done on $M_{2}$ where they can be neglected) reduces to makespan minimization on two identical parallel machines: the Partition problem known to be ordinary NP-complete (see Garey and Johnson, 1979).

Using the above property for scheduling a given partition, a pseudopolynomial dynamic program is developed by Kouvelis and Vairaktarakis (1998) to generate optimal schedules for $F 2|m t f l x| C_{\max }$. We present the simpler iteration that solves the special case $p_{j}=q_{j}=a_{j}+b_{j}$. We will schedule the jobs sequentially in the order indicated by $J R(\underline{a}, \underline{b})$; let them be so indexed. At stage $j$, we consider adding $J_{j}$ to schedules involving only $\left\{J_{1}, J_{2}, \ldots, J_{j-1}\right\}$. Define:

$$
\begin{aligned}
f_{j}\left(t, I_{1}, I_{2}\right)= & \text { the makespan of an optimal schedule for }\left\{J_{1}, \ldots, J_{j}\right\} \\
& \text { whose total processing time on } M_{1} \text { of jobs in } \mathcal{V}_{3} \text { is } t, \\
& \text { and whose idle time at the end of job processing on } \\
& M_{1}\left[M_{2}\right] \text { is } I_{1}\left[I_{2}\right] \text { (see Fig. 8.1). }
\end{aligned}
$$

At each stage $j$, the function $f_{j}$ must be evaluated for all possible states $\left(t, I_{1}, I_{2}\right)$, by reaching from all states attained at stage $j-1$. Any state that cannot be achieved gets value infinity. To estimate the number of possible states, note that each component is bounded above by $P_{n}$, where

$$
P_{j}=\sum_{i=1}^{j}\left(a_{i}+b_{i}\right)
$$

is the total processing time of the first $j$ jobs. However, since $I_{i}$ is the time from ending processing on $M_{i}$ to the makespan, only one of $I_{1}$ and $I_{2}$ can be positive. These observations indicate that the state space of the dynamic program is $O\left(n P_{n}^{2}\right)$.

Since each job must be in precisely one of the sets $\mathcal{V}_{1}, \mathcal{V}_{2}, \mathcal{V}_{3}$, and (at least) one of $I_{1}, I_{2}$ is zero, there are six cases. Let

$$
\begin{aligned}
C_{v m}= & \text { the case where } J_{j} \text { goes into } \mathcal{V}_{v}(v=1,2,3), \\
& \text { and the makespan is achieved on } M_{m}(m=1,2) .
\end{aligned}
$$

This motivates the following recurrence relation, which we shall call DP, developed in detail in Kouvelis and Vairaktarakis (1998), where $I_{1}^{\prime}, I_{2}^{\prime}$ denote the values of $I_{1}, I_{2}$ at stage $j-1$ :
$f_{j}\left(t, I_{1}, 0\right)=\min \left\{\begin{array}{l}C_{12}: f_{j-1}\left(t, p_{j}+I_{1}, 0\right) \\ C_{22}:\left\{\begin{array}{l}\left(P_{j}+I_{1}\right) / 2, \quad \text { if } \exists I_{2}^{\prime} \in\left[0, P_{j}+I_{1}\right]: 2 f_{j-1}\left(t, 0, I_{2}^{\prime}\right)=P_{j}-I_{1} \\ \min _{0 \leq I_{1}^{\prime} \leq I_{1}}\left(f_{j-1}\left(t, I_{1}^{\prime}, 0\right)+\left[P_{j}-2 f_{j-1}\left(t, I_{1}^{\prime}, 0\right)+I_{1}^{\prime}\right]^{+}\right), \mathrm{o} / \mathrm{w}\end{array}\right. \\ C_{32}: \min \left\{\begin{array}{l}f_{j-1}\left(\left(t-a_{j},\left(a_{j}+I_{1}-b_{j}\right)^{+},\left(b_{j}-a_{j}-I_{1}\right)^{+}\right)+b_{j}\right. \\ \min _{0 \leq I_{1}<a_{j}+I_{1}-b_{j}}\left\{f_{j-1}\left(t-a_{j}, I_{1}^{\prime}, 0\right)-I_{1}^{\prime}\right\}+a_{j}+I_{1} \\ \min _{I_{2}^{\prime}>b_{j}-a_{j}-I_{1} f_{j-1}\left(t-a_{j}, 0, I_{2}^{\prime}\right)+a_{j}+I_{1} .} .\end{array}\right.\end{array}\right.$
with the initial conditions:

$$
\begin{aligned}
& f_{1}\left(0,0, p_{1}\right)=f_{1}\left(0, p_{1}, 0\right)=f_{1}\left(a_{1}, b_{1}, 0\right)=p_{1} \\
& f_{1}\left(t, I_{1}, I_{2}\right)=\infty, \text { otherwise }
\end{aligned}
$$

and with the optimal solution given by

$$
f_{n}^{\star}=\min _{t, I_{1}, I_{2}} f_{n}\left(t, I_{1}, I_{2}\right) .
$$

Since the computation at each stage is $O\left(P_{n}\right)$ for each of the $O\left(n P_{n}^{2}\right)$ states, the complexity of $\mathbf{D P}$ is $O\left(n P_{n}^{3}\right)$.

As an example, we show how the formula for case $C_{11}$ is derived. Recall, case $C_{11}$ has $J_{j}$ scheduled entirely on $M_{1}$ and the makespan of the resulting schedule is attained on $M_{1}$. There are two possible schedule configurations at stage $j-1$ that could produce this result, as shown in Fig. 8.2. In Fig. 8.2a [8.2b], the makespan at stage $j-1$ is attained on $M_{2}$ [ $M_{1}$ ]; note how this corresponds to having $p_{j}>I_{2}\left[p_{j}<I_{2}\right]$. From Fig. 8.2a, we see that, to reach the state $\left(j, t, 0, I_{2}\right)$ (that is, state $\left(t, 0, I_{2}\right)$ at stage $j$ ), we must have come from state $\left(j-1, t, I_{1}^{\prime}, 0\right)$, with $p_{j}=I_{1}^{\prime}+I_{2}$. Thus we go from $\left(j-1, t, p_{j}-I_{2}, 0\right)$ to $\left(j, t, 0, I_{2}\right)$, with resulting makespan

$$
f_{j-1}\left(t, p_{j}-I_{2}, 0\right)+I_{2}, \text { when } p_{j}>I_{2} .
$$

Similarly, Fig. 8.2b illustrates that, to get to $\left(j, t, 0, I_{2}\right)$, we must come from


Fig. 8.2
Adding $J_{j}$ to a partial schedule in case $C_{11}$ when (a) $I_{2}^{\prime}=0$, and (b) $I_{1}^{\prime}=0$
$\left(j-1, t, 0, I_{2}-p_{j}\right)$, producing a makespan

$$
f_{j-1}\left(t, 0, I_{2}-p_{j}\right)+p_{j}, \text { when } p_{j}<I_{2} .
$$

These two results give us the formula for $C_{11}$ in the above recursion.

We have assumed above that the total processing time for a job remains the same for all modes: $p_{j}=q_{j}=a_{j}+b_{j}$. In the general case, where $p_{j}$ and $q_{j}$ can have any values satisfying $p_{j}>a_{j}$ and $q_{j}>b_{j}$, a similar but more elaborate recursion, $\mathbf{D P}^{\prime}$, is presented in the same paper, having state space $O\left(n P_{n}^{\prime 3}\right)$ and running time $O\left(n P_{n}^{\prime 4}\right)$, where $P_{n}^{\prime}=\sum_{j=1}^{n} \max \left(a_{j}+b_{j}, p_{j}, q_{j}\right)$.

## Heuristic for the Hybrid Flow Shop

With more than one processor at each stage, we approximate the performance of parallel processors by merging machines (see Sect. 2.1), which then permits us to use $\mathbf{D P}$ or $\mathbf{D} \mathbf{P}^{\prime}$ to partition the tasks into the sets $\mathcal{V}_{1}, \mathcal{V}_{2}, \mathcal{V}_{3}$. The following heuristic for $F\left(k_{1}, k_{2}\right)|m t f l x| C_{\max }$ also uses the first available machine ( FAM) and last busy machine ( LBM) rules to schedule the jobs in the first and second stages, respectively (see Sect. 2.2). Since the heuristic handles resource flexibility, we call it $\mathrm{H}_{\mathrm{RF}}$.

Heuristic $\mathbf{H}_{\mathbf{R F}}$ for $F\left(k_{1}, k_{2}\right)|m t f l x| C_{\max }$

1. Let $p_{L}=\max _{j} \min \left\{a_{j}+b_{j}, p_{j}, q_{j}\right\}, p_{H}=\max _{j} \max \left\{a_{j}+b_{j}, p_{j}, q_{j}\right\}$.
2. For every integer $p \in\left[p_{L}, p_{H}\right]$, do
a) Apply DP or $\mathbf{D P}^{\prime}$ to $F(1,1)|r f l x| C_{\text {max }}$ with processing times $\left(\underline{a} / k_{1}, \underline{b} / k_{2}, \underline{p} / k_{1}, \underline{q} / k_{2}\right)$; where $J_{j}$ may be assigned to set $\mathcal{V}_{1}, \mathcal{V}_{2}$ or $\mathcal{V}_{3}$ in the recursion only if $p_{j}<p, q_{j}<p$ or $a_{j}+b_{j}<p$, respectively. Let $S_{p 1}$ and $S_{p 2}$ be the resulting sequences of tasks on merged machines $M_{1}$ and $M_{2}$, respectively.
b) Apply the FAM rule to the $a$-tasks in sequence $S_{p 1}$.
c) Apply the LBM rule to the $b$-tasks in sequence $S_{p 2}$.
d) On each stage-2 machine, schedule the jobs in $\mathcal{V}_{2}$ to start as soon as possible, and the tasks in $\mathcal{V}_{3}$ in FIFO (first come, first served) order. Let $S(p)$ and $C(p)$ be the resulting schedule and makespan, respectively.
3. Let $C_{R F}=\min _{p}\{C(p)\}$, with $S_{R F}$ the associated schedule.

Since $\mathbf{D P}^{\prime}$ requires running time $O\left(n P_{n}^{\prime 4}\right), \mathrm{H}_{\mathrm{RF}}$ runs in $O\left(n P_{n}^{\prime 4} p_{H}\right)$ time. The for-loop in $\mathrm{H}_{\mathrm{RF}}$ is taken over all possible values of $p$, the maximal processing time that any job is allowed to take. For each value of $p$, a schedule is generated. The output of the algorithm, $S_{R F}$, is the best of these.

The for-loop, besides producing a number of schedules from which to choose, also makes possible the worst case performance result below.

Theorem 8.1 For $F\left(k_{1}, k_{2}\right)|m t f l x| C_{\max }$ :

$$
C_{R F} / C^{\star} \leq 2-1 / k, \quad \text { where } k=\max \left\{k_{1}, k_{2}\right\}
$$

and this bound is tight.

## Computational Results

We mention briefly the outcome of experiments run by the authors. They compared $F\left(k_{1}, k_{2}\right)|m t f l x| C_{\max }$ (in the simpler case where $p_{j}=q_{j}=a_{j}+b_{j}$ ) with two special cases: $P\left(k_{1}+k_{2}\right) \| C_{\max }$ and $F\left(k_{1}, k_{2}\right) \| C_{\max }$. In the parallel processor environment, jobs can only be done in the "combined" mode (both tasks together on a single machine), while in the hybrid flow shop only the "separate" mode (each task of a job on a separate machine) is available.

They considered instances with 20 to 40 jobs and 5 or 6 total machines. They found that resource flexibility added little to parallel machines: as an overall average, the makespans were only $1 \%$ to $2 \%$ greater when only the combined mode was available. On the other hand, the hybrid with resource flexibility significantly outperformed the simple hybrid: makespans were over $10 \%$ greater when only the separate mode could be used. Both performance gaps increased with the total number of machines, and decreased with the number of jobs.

We conclude that, while the parallel machine environment would not be significantly helped by adding a "job splitting" facility, the efficiency of a flow shop would be much enhanced by adding the capability to combine the tasks of a job, when possible.

### 8.3.2 Fm|mtflx| $C_{\text {max }}$

In this subsection we present results on a design similar to the one presented for $F 2|m t f l x| C_{\max }$ in Subsect. 8.3.1, this time for the $m$-machine flow shop. Specifically, let
$\mathcal{A}_{k}=$ the subset of processors after $M_{k}$ able to process $k$-tasks,
$\mathcal{B}_{k}=$ the subset of processors before $M_{k}$ able to process $k$-tasks,
for $k=1,2, \ldots, m$, where $k$-tasks are the tasks initially assigned to $M_{k}$. It is assumed that, if $M_{l} \in \mathcal{A}_{k} \cup \mathcal{B}_{k}$, then the processing time requirement of $T_{k j}$ on $M_{l}$ is $p_{l j}$. For the makespan objective, Liao et al. (1995) presented a mixed integer programming formulation. Due to a large number of constraints, the LINDO solver could only handle randomly generated problems with $m=4$ and $n \leq 7$ with integer processing times drawn uniformly from $[1,10]$. For larger problems they experimented with the following heuristic.

Heuristic for $F m|m t f l x| C_{\text {max }}$

1. Apply heuristic NEH for $F m \mid$ perm $\mid C_{\max }$; reindex jobs so that $(1,2, \ldots, n)$ is the resulting sequence.
2. Define list $L=\left(T_{11}, \ldots, T_{m 1}, T_{12}, \ldots, T_{m 2}, \ldots, T_{1 n}, \ldots, T_{m n}\right)$, and let
(a) $L^{\prime}$ be the sublist of $L$ that includes all tasks $T_{k j}$ on every $M_{k}: \mathcal{B}_{k}=\phi$, (b) $L^{\prime \prime}=L-L^{\prime}$ be the sublist of tasks on machines with $\mathcal{B}_{k} \neq \phi$.
3. Apply the FAM rule to the concatenation $L_{H}=\left(L^{\prime}, L^{\prime \prime}\right)$.

Heuristic NEH is presented in Sect. 4.9 and is shown to be one of the best performing polynomial time heuristics for problem $\operatorname{Fm} \mid$ perm $\mid C_{\text {max }}$. In Step 2 of the above heuristic, tasks $T_{k j}$ that can be processed by a higher index machine $M_{l}$ with $l>k$, are listed later in $L_{H}$ in hopes of utilizing the idle time inserted in $M_{l}$ by tasks in $L^{\prime}$. In step 3, the FAM rule schedules the next task $T_{k j}$ in $L_{H}$ at the earliest possible time, taking into account all machines in $\mathcal{A}_{k}$.

Example 8.1: With $m=n=4$, suppose that $\mathcal{B}_{4}=\left\{M_{2}\right\}$ and $\mathcal{A}_{1}=$ $\left\{M_{3}\right\}$, and all other $\mathcal{A}_{k}, \mathcal{B}_{k}$ subsets are null. Suppose that NEH yields the permutation (3, 4, 2, 1). Then, $L^{\prime}=\left\{T_{13}, T_{23}, T_{33}, T_{14}, T_{24}, T_{34}, T_{12}, T_{22}, T_{32}\right.$, $\left.T_{11}, T_{21}, T_{31}\right\}$ and $L^{\prime \prime}=\left\{T_{43}, T_{44}, T_{42}, T_{41}\right\}$.

Liao et al. (1995) tested this heuristic on randomly generated problems with $m=2,3,4,5$ and $n=3,5,7,9$ and found that the average relative deviation from optimality is $0.843 \%$ while in the majority of instances it provides an optimal solution. While these findings are quite promising, they do not address instances of large size.

### 8.3.3 F2|mpflx| $C_{\text {max }}$

A set of jobs $\left.\left\{J_{j}=<a_{j}, b_{j}\right\rangle, j=1, \ldots, n\right\}$ is to be scheduled in a twomachine flow shop. It is possible to process the $a$-task of $J_{j}$ in time $p_{j} \leq a_{j}$ using both processors simultaneously. Similarly, the second task of any job can be done in time $q_{j} \leq b_{j}$ using the combined capacities of both processors. All processing times are assumed to be integers. While multiprocessing a task, the two machines are not available for other work. This problem is addressed in Vairaktarakis and Lee (2004), where all the results of this section are presented with additional details.

Industrial applications of such multiprocessor flexibility (we abbreviate this mpflx), such as the production of electronic circuits, often involve flexible machines that can switch quickly between tasks, with detachable tool magazines allowing for off-line setups. In such environments, a task may occupy two processors simultaneously. In another application, if the critical resource is workforce rather than machine capacity, each of the two work stations may be manned by its own group of workers, who can join forces to expedite a task when needed.

As in the previous section, we first need to partition the jobs into sets according to their processing modes:

- $\mathcal{V}_{1}=\left\{j: T_{j 1}\right.$ is done entirely by $M_{1}, T_{j 2}$ entirely by $\left.M_{2}\right\}$
- $\mathcal{V}_{2}=\left\{j: T_{j 1}\right.$ is done jointly by $M_{1}$ and $M_{2}, T_{j 2}$ entirely by $\left.M_{2}\right\}$
- $\mathcal{V}_{3}=\left\{j: T_{j 1}\right.$ is done entirely by $M_{1}, T_{j 2}$ jointly by $M_{1}$ and $\left.M_{2}\right\}$
- $\mathcal{V}_{4}=\left\{j\right.$ : both tasks are done jointly by $M_{1}$ and $\left.M_{2}\right\}$

By job interchange arguments, and by noting that JR minimizes makespan for $\mathcal{V}_{1}$, we can show that there always exists an optimal permutation schedule

$$
S^{\star}=\left(\mathcal{V}_{2}, \mathcal{V}_{1}: J R(\underline{a}, \underline{b}), \mathcal{V}_{3}, \mathcal{V}_{4}\right)
$$

where the job order within each of $\mathcal{V}_{2}, \mathcal{V}_{3}$, and $\mathcal{V}_{4}$ is arbitrary. We show such a schedule in Fig. 8.3, where each set is given a superscript to show its processing mode. Note that there may be idle time (denoted $I_{1}$ and $I_{2}$ in the figure) on either or both processors. We have consolidated the idle time on each machine by giving each task on $M_{1}$ its earliest start time, and starting all work on $M_{2}$ as late as possible (subject to minimizing makespan).


Fig. 8.3 Form of the optimal schedule for $F 2|m p f l e x| C_{\text {max }}$

## The Complexity of $\operatorname{F2}|m p f l x| C_{\text {max }}$

While the schedule is easy to find once the jobs have been partitioned, the hard part, again, is partitioning the jobs into the four sets. This is indeed NP-complete, as we now show.

Theorem 8.2 $F 2|m p f l x| C_{\max }$ is ordinary $N P$-complete
Proof Outline: The reduction is from the NP-complete problem:

## PARTITION

INSTANCE: An integer $V$, and $k$ positive integers $v_{i}: i \in \mathcal{T}=\{1,2, \ldots, k\}$ such that $\Sigma_{i \in \mathcal{T}} v_{i}=2 V$.
QUESTION: Is there subset $\mathcal{T}^{\prime} \subset \mathcal{T}$ such that $\Sigma_{i \in \mathcal{T}^{\prime}} k_{i}=\Sigma_{i \in \mathcal{T}-\mathcal{T}^{\prime}} k_{i}=K$ ?
to the decision version of our problem:

## $F 2|m p f l x| C_{\text {max }} \leq B ?$

INSTANCE: A real $B>0$, and $n$ jobs each with parameters $<a_{j}, b_{j}, p_{j}, q_{j}>$ to be scheduled in a two-machine flow shop, where $a_{j}\left[b_{j}\right]$ is the time of $J_{j}$ on $M_{1}\left[M_{2}\right]$, and $p_{j}\left[q_{j}\right]$ is the time of task 1 [2] if multiprocessed. QUESTION: Does there exist a schedule with $C_{\max } \leq B$ ?

Given an instance of Partition, construct an instance of $F 2|m p f l x| C_{\max }$ as follows:

- $n:=k+1$;
- $<a_{j}, b_{j}, p_{j}, q_{j}>:=<0, v_{j}, 0, v_{j} / 2>, j=1,2, \ldots, k$;
- $\left.\left\langle a_{j}, b_{j}, p_{j}, q_{j}\right\rangle:=<V, 0, V, 0\right\rangle, j=k+1$;
- $B:=3 V / 2$.

Since $J_{t+1}$ must occupy $M_{1}$ for a time $V$, we can only achieve a perfect packing if a subset $\mathcal{T}^{\prime}$ of the other jobs fills the same time on $M_{2}$ (see Fig. 8.4), with the rest being multiprocessed.


Fig. 8.4 The $F 2|m p f l e x| C_{\text {max }}$ instance produced by reduction from Partition

## Dynamic Program for $\operatorname{F2}|m p f l e x| C_{\text {max }}$

The following pseudopolynomial algorithm solves $F 2|m p f l x| C_{\text {max }}$ optimally. Incidentally, its existence shows that our problem is not strongly NP-complete.
Let the jobs be indexed according to $J R(\underline{a}, \underline{b})$, and let

$$
\begin{aligned}
f_{j}\left(t_{1}, t_{2}, t_{3}, I_{1}, I_{2}\right)= & \text { the makespan of an optimal schedule for the } \\
& \text { subset of jobs }\left\{J_{1}, J_{2}, \ldots, J_{j}\right\}
\end{aligned}
$$

where (see Fig. 8.3):

$$
\begin{aligned}
t_{1}\left[t_{2}, t_{3}\right]= & \text { the total processing time of jobs in } \mathcal{V}_{1}\left[\mathcal{V}_{2}, \mathcal{V}_{3}\right] \text { on } \\
& M_{1}\left[M_{2}, M_{1}\right], \text { respectively; }
\end{aligned}
$$

$$
I_{1}\left[I_{2}\right]=\text { the idle time on } M_{1}\left[M_{2}\right] .
$$

There is no need to add the variable $t_{4}$; it is determined by the other five.
The recurrence below is in the spirit of the one for $F 2|m t f l x| C_{\max }$ in the previous section, and the reader is referred to Vairaktarakis and Lee (2004) for derivation and discussion. Since, given a partial schedule for the jobs $\left\{J_{1}, J_{2}, \ldots, J_{j-1}\right\}$, the next job $J_{j}$ can be assigned to any one of the four sets $\mathcal{V}_{1}$ through $\mathcal{V}_{4}$, we can go from a given state $\left(t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime}, I_{1}^{\prime}, I_{2}^{\prime}\right)$ at stage $j-1$ to four states $\left(t_{1}, t_{2}, t_{3}, I_{1}, I_{2}\right)$ at the next stage. We get:

$$
\begin{aligned}
& f_{i}\left(t_{1}, t_{2}, t_{3}, I_{1}, I_{2}\right)= \\
& \min \left\{\begin{array}{l}
J_{i} \in \mathcal{V}_{1}:\left\{\begin{array}{l}
f_{i-1}\left(t_{1}, t_{2}, t_{3}, I_{1}+a_{i}-b_{i}, I_{2}\right)+b_{i}, \quad \text { if } I_{1}>b_{i} \\
\min _{I_{1}^{\prime} \leq a_{i}} f_{i-1}\left(t_{1}, t_{2}, t_{3}, I_{1}^{\prime}, I_{2}-a_{i}+I_{1}^{\prime}+b_{i}\right)+a_{i}-I_{1}^{\prime}, \\
f_{i-1}\left(t_{1}, t_{2}, t_{3}, I_{1}-b_{i}+a_{i}, I_{2}\right)+b_{i}, \quad \text { if } b_{i}<I_{i} \leq I_{1} \leq I_{1}^{\prime}, b_{i} \leq t_{3}, I_{1}=0
\end{array}\right. \\
J_{i} \in \mathcal{V}_{2}:\left\{\begin{array}{l}
\text { if } I_{1}, \\
\min _{I_{2}^{\prime} \leq b_{i} f_{i-1}\left(t_{1}, t_{2}-b_{i}, t_{3}, I_{1}-b_{i}+I_{2}^{\prime}, I_{2}^{\prime}\right)+p_{i}+b_{i}-I_{2}^{\prime},} \text { if } I_{2}=0 \\
f_{i-1}\left(t_{1}, t_{2}-b_{i}, t_{3}, I_{1}, I_{2}+b_{i}\right)+p_{i}, \\
\text { in } I_{2}>0
\end{array}\right. \\
J_{i} \in \mathcal{V}_{3}: \begin{cases}\min _{I_{1}^{\prime} \leq a_{i} f_{i-1}\left(t_{1}, t_{2}-a_{i}, t_{3}, I_{1}^{\prime}, I_{2}-a_{i}+I_{1}^{\prime}\right)+q_{i}+a_{i}-I_{1}^{\prime},}, \text { if } I_{1}=0 \\
f_{i-1}\left(t_{1}, t_{2}, t_{3}-a_{i}, I_{1}+a_{i}, I_{2}\right)+q_{i},\end{cases} \\
J_{i} \in \mathcal{V}_{4}: f_{i-1}\left(t_{1}, t_{2}, t_{3}, I_{1}, I_{2}\right)+p_{i}+q_{i}
\end{array}\right.
\end{aligned}
$$

with boundary conditions:
$f_{1}\left(t_{1}, t_{2}, t_{3}, I_{1}, I_{2}\right)= \begin{cases}J_{1} \in \mathcal{V}_{1}: I_{2}+b_{1}, & \text { if } a_{1}+I_{1}=I_{2}+b_{1}, t_{1}=a_{1}, t_{2}=b_{1}, t_{3}=0 \\ J_{1} \in \mathcal{V}_{2}: p_{1}+I_{2}+b_{1}, & \text { if } I_{1}=I_{2}+b_{1}, t_{1}=t_{3}=0, t_{2}=b_{1} \\ J_{1} \in \mathcal{V}_{3}: I_{2}+q_{1}, & \text { if } I_{1}=I_{2}-a_{1}, t_{1}=t_{2}=0, t_{3}=a_{1} \\ J_{1} \in \mathcal{V}_{4}: I_{1}+p_{1}+q_{1}, & \text { if } I_{1}=I_{2}, t_{1}=t_{2}=t_{3}=0 \\ \infty, & \text { otherwise }\end{cases}$
The optimal solution is:
$f_{n}^{\star}=\min \left\{f_{n}\left(t_{1}, t_{2}, t_{3}, I_{1}, I_{2}\right): t_{1}+t_{2} \leq \sum_{i} a_{i}, t_{3} \leq \sum_{i} b_{i}, I_{1} \leq \max _{i} b_{i}, I_{2} \leq \max _{i} a_{i}\right\}$
To determine complexity, we first note from Fig. 8.3 that $I_{1} \leq t_{2}+t_{4} \leq$ $\Sigma_{j} b_{j}$, and $I_{2} \leq t_{1}+t_{3} \leq \Sigma_{j} a_{j}$. Thus, each of the five arguments is bounded by $T=\max \left\{\Sigma_{j} a_{j}, \Sigma_{j} b_{j}\right\}$, so that the state space for our dynamic program is $O\left(n T^{5}\right)$. Since the effort required per iteration is of order $O\left(p_{\max }\right)$, where $p_{\max }=\max \left\{\max _{i} a_{i}, \max _{i} b_{i}\right\}$, the complexity of the dynamic program is $O\left(n T^{5} p_{\max }\right)$.

## Heuristics

The above algorithm is viable for small problems, but its complexity is prohibitive for medium and large ones. The same authors offer several heuristics, of which we present two. The first is based on the relaxation where all tasks are multiprocessed.

## Heuristic R: Relaxation where $\mathcal{V}_{1}=\mathcal{V}_{2}=\mathcal{V}_{3}=\phi$

With all jobs in $\mathcal{V}_{4}$, all tasks are multiprocessed, so the problem reduces to a trivial single-machine sequencing problem. Each job $J_{j}$ has processing time $p_{j}+q_{j}$, and any sequence is optimal. Though seemingly a crude approximation, R is optimal when $p_{j} \leq a_{j} / 2$ and $q_{j} \leq b_{j} / 2$ for all $j=1,2, \ldots, n$, as we show in the following theorem. This case reflects production environments where capacity is additive and/or multiprocessing offers operational synergies that reduce task times.
Theorem 8.3 Let $C_{R}(\theta)$ be the makespan of the schedule $S_{R}(\theta)$ produced by $R$ for an instance of $F 2|m p f l x| C_{\max }$ with parameter $\theta=\max _{j}\left\{p_{j} / a_{j}, q_{j} / b_{j}\right\}, \theta \in[0,1]$. Then:

$$
C_{R}(\theta) / C^{\star} \leq \begin{cases}1 & \text { if } \theta \in\left[0, \frac{1}{2}\right] \\ 2 \theta & \text { if } \theta \in\left[\frac{1}{2}, 1\right]\end{cases}
$$

and these bounds are tight.
Proof: (a) $\boldsymbol{\theta} \in\left[0, \frac{1}{2}\right]$. Let $\mathcal{V}_{1}^{\star}, \mathcal{V}_{2}^{\star}, \ldots, \mathcal{V}_{4}^{\star}$ be an optimal partition, giving a schedule $S^{\star}$ with makespan $C^{\star}$. Suppose that $S^{\star}$ has the form depicted in Fig. 8.3, and let $z$ be the time that the machines are not multiprocessing: $z=t_{1}+t_{3}+I_{1}$. Then:

$$
C^{\star}=\sum_{j \in \mathcal{V}_{2}^{\star} \cup \mathcal{V}_{4}^{\star}} p_{i}+\sum_{j \in \mathcal{V}_{3}^{\star} \cup \mathcal{V}_{4}^{\star}} q_{i}+z
$$

On the other hand, R multiprocesses all tasks. Of the tasks that are not multiprocessed in $S^{\star}$, there are at most $z$ time units on each machine which
are sequentially processed in $S_{R}$, and these tasks take at most $\theta$ times as long to process. This gives:

$$
C_{R}(\theta) \leq \sum_{j \in \mathcal{V}_{2}^{\star} \cup \mathcal{V}_{4}^{\star}} p_{i}+\sum_{j \in \mathcal{V}_{3}^{\star} \cup \mathcal{V}_{4}^{\star}} q_{i}+2 \theta z \leq C^{\star}
$$

Since $C^{\star}$ is the minimal makespan, we must have $C^{\star}=C_{R}(\theta)$.
(b) $\theta \in\left[\frac{1}{2}, \mathbf{1}\right]$. Now, with $2 \theta>1$, we get in place of the last inequality:

$$
C_{R}(\theta) \leq \sum_{j \in \mathcal{V}_{2}^{\star} \cup \mathcal{V}_{4}^{\star}} p_{i}+\sum_{j \in \mathcal{V}_{3}^{\star} \cup \mathcal{V}_{4}^{\star}} q_{i}+2 \theta z \leq 2 \theta C^{\star}
$$

The two-job instance where the job parameters $\left.<a_{j}, b_{j}, p_{j}, q_{j}\right\rangle$ are $<K, 0, \theta K, 0>$ and $<0, K, 0, \theta K>$ clearly has $C^{\star}=K$ and $C_{R}(\theta)=2 \theta K$, which shows that this bound is tight.

## Heuristic H: Golden Section Threshold

The basic idea is simply that the first [second] task of $J_{j}$ should be multiprocessed whenever the multiprocessed time $p_{j}\left[q_{j}\right]$ is sufficiently smaller than the singly-processed time $a_{j}\left[b_{j}\right]$, as determined by a threshold value, $\lambda$, for the ratio, $p_{j} / a_{j}\left[q_{j} / b_{j}\right]$. It can be shown that the optimal threshold is the "golden section" parameter

$$
\lambda=(\sqrt{5}-1) / 2 \approx 0.618
$$

Specifically, the steps of heuristic H are:

1. For every $j=1,2, \ldots, n$ :
(a) if $p_{j}<\lambda a_{j}$, multiprocess the stage- 1 task of $J_{j}$; otherwise process it on $M_{1}$.
(b) if $q_{j}<\lambda b_{j}$, multiprocess the stage- 2 task of $J_{j}$; otherwise process it on $M_{2}$.
2. Given the partition $\mathcal{V}_{1}, \mathcal{V}_{2}, \mathcal{V}_{3}, \mathcal{V}_{4}$ resulting from step 1 , construct an optimal schedule as in Fig. 8.3.

The complexity of H is $O(n \log n)$. It gives a schedule that is at most $62 \%$ longer than optimal. We state this formally, omitting the proof:

Theorem 8.4 Let $C_{H}$ be the makespan of the schedule produced by heuristic $H$ for an instance of $F 2|m p f l x| C_{\max }$. Then:

$$
C_{H} / C^{\star} \leq(\sqrt{5}+1) / 2 \approx 1.618
$$

and the bound is tight.
Finally, we make the commonplace observation that, when we have several easily-implemented heuristics, it makes sense to try them all and use the best result. For worst case analysis when using both heuristics R and H , we apply each algorithm in that range of $\theta$ where it outperforms the other. This gives:

Corollary 8.1 Let $C_{B}$ be the makespan of the schedule produced by applying both heuristics $R$ and $H$ to an instance of $F 2|m p f l x| C_{\max }$. Then:

$$
C_{B} / C^{\star} \leq \begin{cases}1 & \text { if } \theta \in\left[0, \frac{1}{2}\right] \\ 2 \theta & \text { if } \theta \in\left[\frac{1}{2}, \frac{\sqrt{5}+1}{4}\right] \\ (\sqrt{5}+1) / 2 & \text { if } \theta \in\left[\frac{\sqrt{5}+1}{4}, 1\right]\end{cases}
$$

and these bounds are tight.

### 8.4 Resource Flexibility

We described resource flexibility in Subsect. 8.1.1 as arising when one or more limited resources can or must be allocated across jobs and stages. This scarce commodity (we will generally assume there is only one) may be a consumable or material resource of which a fixed amount is available, or a reusable or work resource of which a fixed number may be assigned in each period. In this section we will focus on the latter type, of which labor provides a common example. Labor flexibility can be achieved by cross-training workers in the skills needed for various tasks associated with multiple processing centers.

If each task requires a fixed and predetermined amount of the resource, there are no flexibility issues, though the scheduling problem is already complex (see the complexity discussion in Sect. 8.4.1 below), since tasks must not be scheduled to overlap if their total resource requirements exceed the supply. Here, we are concerned with the still more difficult situation where we can allocate various amounts of the resource to each task, to reduce its processing time: the more of the resource we allocate to a task, the quicker it can be processed. We will call this resource flexibility, denoted rflx.

### 8.4.1 Fm|perm, $\mathrm{rflx} \mid \mathrm{C}_{\text {max }}$

The simple permutation flow shop with work resource flexibility is discussed by Daniels and Mazzola $(1993,1994)$. They assume that $R$ units of a single resource are available in each period. Each task $T_{i j}$ (task $i$ of job $j$ ) has a finite number of processing modes, such that $u_{i j k}$ units of resource allocated to $T_{i j}$ in mode $k$ produces a processing time $p_{i j k}$. As the language above implies, there is one machine at each of $m$ stations, every machine processes jobs in the same order, and no preemption is allowed. We wish to determine a sequence of jobs, the processing mode for each task, and the start time for each task, so as to minimize the makespan.

## The Complexity of $\operatorname{Fm} \mid$ perm, $r f l x \mid C_{\text {max }}$

Since the special case $F m \mid$ perm $\mid C_{\text {max }}$ is strongly NP-complete for $m>2$, the same is true for the present problem. Daniels and Mazzola (1994) show that the following two special cases of $F m \mid$ perm, $r f l x \mid C_{\max }$ are just as hard.

- The resource-constrained flow shop, in which each task has a single mode of processing, so that the resource requirement $u_{i j}$ and processing time $p_{i j}$ of a task $T_{i j}$ are given constants, is strongly NP-complete even for $m=2$.
- The fixed-sequence resource-constrained flow shop, which in addition prespecifies the order in which the jobs must be processed through each machine, is strongly NP-complete.


## Solution Approaches to $\operatorname{Fm} \mid$ perm, $r f l x \mid C_{\text {max }}$

In Daniels and Mazzola (1994) (which, despite the publication date, is evidently earlier work than Daniels and Mazzola (1993)), an integer program and a branch-and-bound formulation are presented, but neither is capable of solving problems of realistic size. A heuristic based on tabu search is given, but we will discuss the improved version given in Daniels and Mazzola (1993).

Our problem requires the simultaneous optimization of three interrelated subproblems: job sequencing, resource allocation, and the specification of task start times. Let $\pi$ be a job permutation, and $\mathcal{P}$ be the set of all such permutations. Let $\rho$ be a vector whose $n m$ components are the resource allocations to each task, and $\mathcal{R}$ the set of all such feasible allocations (i.e., allocations where no more than $R$ units of the resource are used in each period). Finally, let $\sigma$ be an $n m$-vector containing the start times of each task on each machine, with $\mathcal{S}(\pi, \rho)$ the set of all feasible start times $\sigma$ (that is, start times that are achievable using sequence $\pi$ and resource allocation $\rho)$. Our goal is to find

$$
C_{\max }^{\star}=\min _{\pi \in \mathcal{P}}\left\{\min _{\rho \in \mathcal{R}}\left\{\min _{\sigma \in \mathcal{S}(\pi, \rho)}\left\{C_{\max }(\pi, \rho, \sigma)\right\}\right\}\right\}
$$

where $C_{\max }(\pi, \rho, \sigma)$ is the makespan resulting from the solution $(\pi, \rho, \sigma)$.
We see from this formulation how the problem can be decomposed into three nested subproblems. The heuristic of Daniels and Mazzola (1993) exploits this using a nested search strategy. It uses two tabu searches, one (STABU) to select the best sequence and the other (RTABU) to select the corresponding best resource allocation. For the innermost subproblem, given a sequence and an allocation, subroutine START assigns start times sequentially as early as possible. When so many tasks are available for simultaneous processing that the resource limit is exceeded, preference is given to those producing the greatest time reduction per unit resource.

Thus, STABU starts by picking a job sequence using an initializing rule. A neighborhood of the initial sequence (a neighboring sequence is generated by moving the job in position $i$ to position $j$ for any $i, j$ ) is searched for a schedule with smaller value (i.e., makespan), and we move to the improved solution, define a new neighborhood, and repeat. Each sequence considered is evaluated by calling up RTABU and START. We will not discuss further the many other details and complexities of the algorithm.

The authors tested their procedure on over 1600 randomly generated instances. Where optimal solutions could be identified (four machines or less),
the algorithm produced them over $70 \%$ of the time. On larger problems, it showed that the flow shop with resource flexibility performed significantly better than the fixed-resource flow shop: an average decrease in makespan of about $25 \%$.

### 8.5 Mix Flexibility

Consider now a shop where each job belongs to one of several job types or part types, and jobs are batched for processing. Batches may consist of a mixture of job types. Jobs of the same type require the same processing time, and no setup is required between them. Mix flexibility occurs when the mix of each batch (the proportion of parts of each type in a batch) can be controlled.

Wittrock (1985) uses linear programming to batch jobs over a finite horizon. Here, we adapt the original more general formulation to the simple job shop. We wish to produce $d_{j}$ units of part type $j(j=1,2, \ldots, n)$ in an $m$ machine flow shop over $H$ days, where a part of type $j$ requires time $p_{i j}$ at stage $i(i=1,2, \ldots, m)$. We assume that a single batch will be processed in its entirety each day. The primary goal is to keep the makespans low each day (thus maximizing the time available to react to breakdowns or other crises), and secondarily to keep the number of part types in each batch small (this is a proxy for minimizing setup costs, which are not explicitly handled). Wittrock proposes that, as a surrogate for daily makespan minimization, the workload for each machine be leveled over the planning horizon. This appears in the linear program below as a constraint requiring the workload at each stage on each day to be a constant, $w_{i}$. The second objective, too, is replaced by a roughly equivalent objective. Each part type is assigned a hypothetical unit cost of production on each day. On most days the cost is high, and different parts are given different low-cost days. Then cost minimization tends to concentrate production of each part in its few low-cost days. Let

$$
\begin{aligned}
& c_{j t}=\text { the imputed unit cost of producing a type }-j \text { part on day } t ; \\
& u_{j t}=\text { the number of type- } j \text { parts produced on day } t .
\end{aligned}
$$

A possible cost function would be $c_{j t}=(t-j) \bmod n$, giving each part type its own least-cost period every $n$ periods, with other low-cost opportunities adjacent. The linear program is:

$$
\begin{array}{lll}
\operatorname{minimize} & \sum_{t=1}^{H} \sum_{j=1}^{n} c_{j t} u_{j t} \\
\text { subject to } & \sum_{j=1}^{n} p_{i j} u_{j t}=w_{i}, \quad i=1,2, \ldots, m, \quad t=1,2, \ldots, H \\
& \sum_{t=1}^{H} u_{j t}=d_{j}, & j=1,2, \ldots, n, \\
& u_{j t} \geq 0 & j=1,2, \ldots, n, \quad t=1,2, \ldots, H
\end{array}
$$

Though extensive testing of the performance of this heuristic was not conducted, the author reported positive results for a real production line.

### 8.6 Conclusions

In this chapter, we presented various modes of internal flexibility in a flow shop, including routing, resource, mix, multitask and multiprocessor flexibilities. The intricacy of the results presented makes it apparent that theoretical analyses of these environments are extremely difficult. This is probably the reason why existing literature lags behind practice, where a myriad of flexible modes are utilized every day in factories across the world. The importance of flexibility in manufacturing creates the need for further analyses of flexible systems whose mode of flexibility has not yet appeared in the literature. This is a major research agenda that needs to be expanded further. Whenever theoretical models of manufacturing flexibility are found to be intractable, empirical research should be done to lay the groundwork for future study.

Even for the flexible modes presented in this chapter, most of the analyses are limited to 2-stage production. Extending to more than 2 stages is extremely difficult but worthy of attention due to its practical relevance. Further, the bulk of the existing literature focuses on the makespan objective, which by itself is hard enough. However, total cost objectives where cost may be captured by weighted flow time, earliness/tardiness penalties, etc. find applications in great many production settings and to this date have not received much attention. Again, empirical analysis may be the way to start such studies.

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## Chapter 9 <br> REENTRANT FLOW SHOPS


#### Abstract

We introduce flow shops that revisit certain processors, and define the common patterns of flow: cyclic, chain, hub, and V-shaped. We show that even the simplest case, the ( $1,2,1$ )-reentrant shop, is NPhard, establish properties that facilitate a branch-and-bound algorithm, and present two simple but very effective heuristics. With m machines, we give for chain-reentrance simplifying properties, for hub-reentrance a DP based on simplifying assumptions that yet performs well, for Vreentrance a solvable special case. For cyclic production of a single product in the general m-machine reentrant shop, we give an algorithm for finding the efficient frontier between cycle time and flow time, and a heuristic for larger instances. For the hybrid reentrant system, if all jobs require the same time for each production step but have different due dates, dispatching rules are recommended and compared.


### 9.1 Preliminaries

Many manufacturing layouts take the form of job shops or flow shops in which jobs progress from station to station without ever visiting the same stage twice. However, in some industries, as in semiconductor fabrication, product design may call for jobs to recirculate or revisit a stage in the manufacturing process. In general, a reentrant flow shop is distinguished from a simple flow shop by the requirement that one or more jobs may need to be processed repeatedly at one or more stations. Each of the visits of a job to the same station is called a pass.

Consider a flow shop with $m$ stations. Let $\phi_{i}$ be the $i$ th stage visited by each job, where $\phi_{i} \in\{1,2, \ldots, m\}$. Then the flow vector $\bar{\phi}=\left(\phi_{1}, \phi_{2}, \ldots, \phi_{l}\right)$ is the sequence in which jobs visit the stages of the manufacturing system. The reentrant property occurs when at two steps in the sequence, $\phi_{i}=\phi_{j}$ for $i \neq j$. In maximal generality, the station sequence determined by the
manufacturing process may be arbitrary, provided all jobs follow the same sequence (i.e., we have a flow shop) and at least one stage is revisited.

Throughout this monograph, we have dealt largely with deterministic flow shops in which a fixed set of distinct jobs with known characteristics is to be scheduled. However, in much of the literature on reentrant manufacturing systems, jobs are of one or a few types, each type having identical processing times, and we are not scheduling a given predetermined set of them. Instead, work proceeds continuously, with jobs arriving either at random intervals as a stochastic process, or according to an input release policy that decides when a fresh job should be started. Such research is part of queuing network theory and largely beyond our scope, but we will touch on some of the highlights.

Although we use the language of hybrid flow shops in these preliminary remarks, it must be admitted that most of the published research to be discussed deals with simple flow shops (one machine at each stage). Also, almost always it is concerned with makespan minimization.

### 9.1.1 Common Flow Patterns

While the station sequence may often be arbitrary, there are certain specific patterns of flow that arise frequently in practice and are reported in the literature. They include:

- cyclic-reentrant: In a cyclic-reentrant flow shop, or simply a cyclic shop, all jobs make $r$ passes $(r>1)$ through all the stations in order. Thus, for $r=3$, the route is

$$
\bar{\phi}=(1,2, \ldots, m, 1,2, \ldots, m, 1,2, \ldots, m)
$$

- chain-reentrant: In a chain-reentrant flow shop, jobs start their processing at the primary station $G_{1}$, then visit the secondary stages $G_{2}, G_{3}, \ldots, G_{m}$ in order, and finally return to $G_{1}$ for the finishing operation. Thus:

$$
\bar{\phi}=(1,2, \ldots, m, 1) .
$$

- hub-reentrant: In a reentrant shop with a hub every job visits alternately the hub $G_{1}$ for the primary operations, and stations $G_{2}, \ldots, G_{m}$ for the secondary operations, in the sequence

$$
\bar{\phi}=(1,2,1,3, \ldots, 1, m, 1)
$$

- V-reentrant: In a V-reentrant flow shop, or simply V-shop, the primary operation at $G_{m}$ is preceded by a sequence of preliminary tasks, and followed by the same tasks in reverse order. This gives a flow vector:

$$
\bar{\phi}=(1,2, \ldots, m-1, m, m-1, \ldots, 2,1) .
$$

- (1, 2, 1)-reentrant: This simple reentrant flow shop has two stations, with flow

$$
\bar{\phi}=(1,2,1)
$$

In fact, chain-, hub- and V-reentrant shops all reduce to $(1,2,1)$ when $m=2$. Simplest of all is the two-machine version, which will be the first model we discuss.

### 9.1.2 Examples

Many instances of reentrant flows are encountered in manufacturing. Here are some examples, from semiconductor and other production systems.

- Photolithography is one of the most complex steps in the wafer fabrication process of semiconductor production. It is an optical process used to etch multiple layers of circuit patterns on the silicon wafer. Every layer is etched by visiting the photolithography station several times. Between successive visits to that station, it has to be processed on other machines. As we shall see, this facility has the form of a hub-reentrant shop.
- In manufacturing printed circuit boards (PCB's), surface-mounted and pin-through-hole devices must be applied. The bare PCB is first fed into a machine (say, $M_{1}$ ) where surface-mounted devices are attached to the upper side. Then the board is transferred to another machine $\left(M_{2}\right)$ where pin-through-hole devices are inserted. It then returns to $M_{1}$ where other surface-mounted devices are applied to the underside. The flow pattern, therefore, is $(1,2,1)$.
- Consider the assembly and testing of electronic circuits (or dies) stacked on top of each other. Every time a new die is attached, a set of machines is revisited. If the same machines are visited in the same order each time, we have a cyclic-reentrant shop.
- Two-machine cyclic shops arise wherever two processes alternate. For example, in a painting shop, parts have to move back and forth between the painting and baking departments for successive coats of paint.


### 9.2 F2| 1 (1, 2, 1)-reentrant $\mid C_{\text {max }}$

The results of this section come mainly from Wang et al. (1997). The notation used includes the usual $<a_{j}, b_{j}, c_{j}>$ for the task times of $J_{j}$, and in addition:

- $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}: \boldsymbol{a}=\Sigma_{j} a_{j}, \boldsymbol{b}=\Sigma_{j} b_{j}, \boldsymbol{c}=\Sigma_{j} c_{j}$.
- $x \prec[\succ] y: T_{x}$ precedes [follows] $T_{y}$ (or, in a permutation schedule, $J_{x}$ precedes [follows] $J_{y}$ ).
- $x \prec\left[\succ \succ\right.$ ] $y: T_{x}$ (or $J_{x}$ ) immediately precedes [follows] $T_{y}$ (or $J_{y}$ ).

Note that the above ordering is not by precedence or preference; it simply means that the tasks or jobs happen to be ordered that way in the schedule being considered.

### 9.2.1 Complexity

Theorem 9.1 (Lev and Adiri, 1984)
$F 2 \mid(1,2,1)$-reentrant $\mid C_{\max }$ is (at least ordinary) NP-complete.
Proof: The reduction is from the NP-complete problem:

## PARTITION

INSTANCE: An integer $V$, and $k$ positive integers $v_{i}: i \in \mathcal{T}=\{1,2, \ldots, k\}$ such that $\Sigma_{i \in \mathcal{T}} v_{i}=2 V$.
QUESTION: Is there subset $\mathcal{T}^{\prime} \subset \mathcal{T}$ such that $\Sigma_{i \in \mathcal{T}^{\prime}} v_{i}=\Sigma_{i \in \mathcal{T}-\mathcal{T}^{\prime}} v_{i}=V$ ?
Given an instance of Partition, construct an instance of the decision problem $F 2 \mid(1,2,1)$-reentrant $\mid C_{\max } \leq B$ ? as follows:

- $n:=k+1$;
- $\left\langle a_{j}, b_{j}, c_{j}>:=<0, v_{j}, 0>, j=1,2, \ldots, k\right.$;
- $\left.<a_{j}, b_{j}, c_{j}\right\rangle:=<V, V, V>, j=k+1$;
- $B:=3 V$.

To achieve the makespan $3 V, J_{k+1}$ must be scheduled without delay. This leaves two intervals on $M_{2}$, each of length $V$, into which the other jobs must fit. Thus, only if the partition exists can the makespan equal $3 V$.

### 9.2.2 Dominance of a Simple Class of Schedules

Let the set of all the $i^{\text {th }}$ tasks constitute entry $i$, abbreviated $\mathcal{E}_{i},(i=1,2,3)$. For example, the $a$-tasks make up $\mathcal{E}_{1}$. We shall say that $\mathcal{E}_{i}$ is compact in a schedule if all the tasks of $\mathcal{E}_{i}$ are scheduled contiguously.

Theorem 9.2 (Wang et al., 1997)
For $F 2 \mid(1,2,1)$-reentrant $\mid C_{\max }$, there exists an optimal schedule where:
(a) all three entries are compact;
(b) all three entries maintain the same job order.

Proof: In any optimal schedule where the $a$-tasks are not all scheduled before the $c$-tasks, there must be a $c$-task that immediately precedes some $a$-task. They can be interchanged without loss of optimality (that is, the schedule remains feasible and the makespan is no greater). Repeat this as often as necessary, then slide all $a$-tasks as far left as possible, and all $c$-tasks to the right as far as possible, keeping the last $c$-task fixed. We now have made $\mathcal{E}_{1}$ and $\mathcal{E}_{3}$ compact without loss of optimality.

To see that $\mathcal{E}_{2}$ can also be made compact, let

$$
\begin{aligned}
b_{j^{\star}}= & \text { the task on } M_{2} \text { that bridges (or ends during) the transition } \\
& \text { from } \mathcal{E}_{1} \text { to } \mathcal{E}_{3} \text { on } M_{1} \\
= & \text { the first } b \text {-task that finishes later than } \boldsymbol{a} .
\end{aligned}
$$

Call $J_{j^{\star}}$ the partition job. Now, all tasks in $\mathcal{E}_{2}$ to the left of $b_{j \star}$ can be moved to the right as needed, and tasks to the right can be moved left, to compactify $\mathcal{E}_{2}$.

Next, suppose the job order of $\mathcal{E}_{1}$ differs from that of $\mathcal{E}_{2}$ in an optimal schedule, in violation of Theorem $9.2(\mathrm{~b})$. Let $J_{i}$ and $J_{j}$ be the first jobs such that $a_{i} \prec \prec a_{j}$ but $b_{i} \succ b_{j}$. Again, we can interchange $a_{i}$ and $a_{j}$ without loss of compactness or optimality. This can be repeated until the job order of $\mathcal{E}_{1}$ matches that of $\mathcal{E}_{2}$. A similar argument reorders $\mathcal{E}_{3}$ to conform with $\mathcal{E}_{2}$.

We now have a schedule of the type illustrated in Fig. 9.1 for an instance with five jobs, arbitrarily sequenced in order of index. Note how the three blocks of tasks that make up the entries are each scheduled as early as possible, subject to precedence constraints (see the up and down arrows in Fig. 9.1), and how $\mathcal{E}_{1}$ and $\mathcal{E}_{3}$ are separated by an idle interval $I \geq 0$. The partition job has $j^{\star}=3$.

Incidentally, it often happens that $I=0$, in which case $C_{\max }=\boldsymbol{a}+\boldsymbol{c}$. Such a schedule is clearly optimal. Unless the $b$-tasks tend to be significantly longer than the $a$-tasks and $c$-tasks ( as they are in Fig. 9.1), an instance may have many such optimal schedules.


Fig. 9.1 A compact permutation schedule for $F 2 \mid(1,2,1)$-reentrant $\mid C_{\max }$

The authors establish one other important property of an optimal schedule. Given any compact permutation schedule, the partition job $J_{j \star}$ separates the other jobs into two sets, $\mathcal{J}_{1}=\left\{J_{j}: J_{j} \prec J_{j^{\star}}\right\}$, and $\mathcal{J}_{2}=\left\{J_{j}: J_{j} \succ J_{j^{\star}}\right\}$. A partition $\left\{\mathcal{J}_{1}, J_{j^{\star}}, \mathcal{J}_{2}\right\}$ is admissible if it can be generated from a schedule in this way.

Theorem 9.3 For $F 2 \mid(1,2,1)$-reentrant $\mid C_{\max }$, there exists an optimal schedule:

$$
S^{\star}=\left(\mathcal{J}_{1}: J R(\underline{a}, \underline{b}), J_{j^{\star}}, \mathcal{J}_{2}: J R(\underline{b}, \underline{c})\right)
$$

where $\left\{\mathcal{J}_{1}, J_{j^{\star}}, \mathcal{J}_{2}\right\}$ is an admissible partition of the jobs.
That is, start with the jobs in $\mathcal{J}_{1}$, sequenced according to Johnson's Rule using task times from $\mathcal{E}_{1}$ and $\mathcal{E}_{2}$; put $J_{j^{\star}}$ next; then add $\mathcal{J}_{2}$ ordered by Johnson' Rule applied to the $b$-tasks and $c$-tasks. The theorem is somewhat appealing intuitively; the proof is a fairly straightforward but tedious consideration of cases; we shall omit it.

## Solution Algorithms: Optimization

A branch-and-bound algorithm is outlined, based on Theorem 9.3. In what follows, it will be convenient to define:

$$
\begin{aligned}
C\left(\mathcal{J}_{1}\right)\left[C_{2}\left(\mathcal{J}_{1}\right)\right]= & \text { completion time on } M_{1}\left[M_{2}\right] \text { of a job set } \mathcal{J}_{1} \\
& \text { ordered by } J R(\underline{a}, \underline{b}) .
\end{aligned}
$$

At the first level of branching, each of the $n$ nodes represents a choice of a partition job. At lower levels, binary branching places a job either in $\mathcal{J}_{1}$ or $\mathcal{J}_{2}$, which start out as empty sets and gradually fill as branching proceeds. A node can be fathomed if the partial schedule becomes inadmissible; e.g., if $C_{2}\left(\mathcal{J}_{1}\right)>\boldsymbol{a}$, which would violate the definition of $J_{j^{\star}}$.

If we define $\overline{\mathcal{J}}_{1}$ as the set of all jobs not in $\mathcal{J}_{1}$, then two lower bounds that can be used at a node are:

$$
\begin{aligned}
& C_{2}\left(\mathcal{J}_{1}\right)+\sum_{J_{j} \in \overline{\mathcal{J}}_{1}} b_{j}+\min _{J_{j} \in \overline{\mathcal{J}}_{1}} c_{j}, \text { and } \\
& C\left(\mathcal{J}_{1}\right)+\sum_{J_{j} \in \overline{\mathcal{J}}_{1}} c_{j}
\end{aligned}
$$

For further details, consult Wang et al. (1997).

## Solution Algorithms: Approximation

The authors propose three simple heuristics. We omit one of them; it is too simplistic and dominated by the other two. Heuristics $\mathrm{H}_{\mathrm{F}}$ (the forward heuristic) and $\mathrm{H}_{\mathrm{B}}$ (the backward one) generate schedules $S_{F}$ and $S_{B}$ with makespans $C_{F}$ and $C_{B}$, respectively. Both make use of the partition property.

Heuristic $\mathbf{H}_{\mathbf{F}}$ for $F 2 \mid(1,2,1)$-reentrant $\mid C_{\text {max }}$

1. Let $S_{a b}=J R(\underline{a}, \underline{b})$, with makespan $C_{a b}$. If $C_{a b}=\boldsymbol{a}+\boldsymbol{c}$, stop; $S_{a b}$ is optimal.
2. Find the partition $\left\{\mathcal{J}_{1}, J_{j^{\star}}, \mathcal{J}_{2}\right\}$ for $S_{a b}$.
3. Let $S_{F}=\left(\mathcal{J}_{1}: J R(\underline{a}, \underline{b}), J_{j \star} \cup \mathcal{J}_{2}: J R(\underline{b}, \underline{c})\right)$, with makespan $C_{F}$.

The other heuristic is essentially $\mathrm{H}_{\mathrm{F}}$ with time reversed. Clearly, with $\mathcal{E}_{1}$ and $\mathcal{E}_{3}$ interchanged, we have an equivalent problem. However, in the heuristic, the schedule in Step 1 and the partition in Step 2 will be different.

Heuristic $\mathbf{H}_{\mathbf{B}}$ for $F 2 \mid(1,2,1)$-reentrant $\mid C_{\text {max }}$

1. Interchange $a_{j}$ and $c_{j}, j=1, \ldots, n$, and repeat $\mathrm{H}_{\mathrm{F}}$.

Finally, the makespan found heuristically is $C_{H}=\min \left\{C_{F}, C_{B}\right\}$.
The authors state without proof the following worst case performance guarantee, which is probably not tight. Of course, it holds for $C_{B}$, too.

Theorem 9.4 For $F 2 \mid(1,2,1)$-reentrant $\mid C_{\text {max }}$,

$$
C_{F} / C^{\star} \leq 3 / 2
$$

## Computational Experiments

The performance of the combined heuristics was tested in extensive experiments; we present here a small part of the reported results. Processing times $a_{j}, b_{j}$, and $c_{j}$ were generated from uniform distributions with means $\mu_{a}, \mu_{b}$, and $\mu_{c}$, respectively. Table 9.1 shows two sets of values for these parameters. Each row represents the average of 200 experiments.

|  |  |  | \% of instances where |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mu_{a}$ | $\mu_{b}$ | $\mu_{c}$ | $n$ | $C^{\star}=\boldsymbol{a}+\boldsymbol{c}$ | $C_{H}=C^{\star}$ |
| 75 | 100 |  | 50 | 10 | $78 \%$ |
|  |  |  | 30 | $95 \%$ | $96 \%$ |
|  |  |  | 100 | $100 \%$ | $99 \%$ |
| 50 | 200 | 75 | 20 | $100 \%$ |  |
|  |  |  | 50 | $0 \%$ | $87 \%$ |
|  |  |  | 000 | $0 \%$ | $100 \%$ |

Table 9.1 Performance of heuristics for $F 2 \mid(1,2,1)$-reentrant $\mid C_{\max }$

As we remarked earlier, unless the $b$-tasks are "large" relative to the $a$ tasks and $c$-tasks, we are likely to get the trivial solution $C_{\max }=\boldsymbol{a}+\boldsymbol{c}$. "Large", as it turns out, means roughly $\mu_{b} \geq \mu_{a}+\mu_{c}$. In the first case of Table 9.1, where $\mu_{b}<\mu_{a}+\mu_{c}, C_{\max }=\boldsymbol{a}+\boldsymbol{c}$ almost always; in the second case, with $\mu_{b}$ large, almost never. In any case, however, the heuristic gives the optimal solution most of the time. When it does not, the heuristic solution is very close to the optimum, so that in all the experiments run the average relative error was tiny (less than 0.0001).

### 9.3 Fm|chain-reentrant $\mid C_{\text {max }}$

In a chain-reentrant flow shop, where $\bar{\phi}=(1,2, \ldots, m, 1)$, Wang et al. (1997) establish some properties of the general problem, but do not propose a solution algorithm (except for the case of $m=2$, reported in Sect. 9.2).

### 9.3.1 Complexity Results

For $m=2$, we showed in Theorem 9.1 of Sect. 9.2 that the problem was ordinary NP-complete. For larger $m$ we have

Theorem 9.5 (Wang et al., 1997)
For $m \geq 3$, Fm|chain-reentrant $\mid C_{\max }$ is strongly NP-complete.
Proof: It is sufficient to show this for $m=3$. This time, the simple reduction is from $F 3 \| C_{\max }$. Given an instance of the latter with processing times $<a_{j}, b_{j}, c_{j}>$, construct an instance of $F 3 \mid$ chain-reentrant $\mid C_{\max }$ with times $<a_{j}, b_{j}, c_{j}, 0>$. The equivalence of the two problems should be obvious.

### 9.3.2 Dominance of a Simple Class of Schedules

The following theorem, an obvious extension of Theorem 9.1, can help to narrow down the search for an optimum. As before, we call the set of all the $k^{t h}$ tasks, $\left\{T_{k 1}, T_{k 2}, \ldots, T_{k n}\right\}$, entry $k$, abbreviated $\mathcal{E}_{k},(k=1, \ldots, m+1)$, and we say $\mathcal{E}_{k}$ is compact if all its tasks are scheduled contiguously.

Theorem 9.6 (Wang et al., 1997)
For $F m \mid$ chain-reentrant $\mid C_{\max }$, there exists an optimal schedule where:
(a) $\mathcal{E}_{1}$ is compact, and $\mathcal{E}_{m+1}$ is compact;
(b) job order is unchanged from $M_{1}$ to $M_{2}$, and from $M_{m}$ to $M_{1}$.

The proof is essentially the same as that of Theorem 9.1, and will be omitted.

### 9.4 Fm|hub-reentrant $\mid \Sigma C_{j}$

The problem of minimizing mean flow time in a simple hub-reentrant flow shop has been considered by Kubiak et al. (1996). Recall, a hubreentrant shop, as shown in Fig. 9.2a (where $m=3$ ), has flow vector $\bar{\phi}=(1,2,1,3, \ldots, 1, m, 1)$. Each of the $m$ visits of a job to the hub, $M_{1}$, is called a pass. Thus, the ith primary operation of $J_{j}$ is performed on pass $i$ through $M_{1}$.

Having observed that the problem is strongly NP-complete $\left(F 2|-| \Sigma C_{j}\right.$ can be reduced to the two-machine hub-reentrant case with $\left.c_{j}=0 \forall j\right)$, they define a heavily constrained version of the problem that is more tractable.

## Notation and Assumptions

Let us define
$P_{i j}=$ primary operation $i$ of $J_{j}$, requiring time $p_{i j}(i=1, \ldots, m)$;
$T_{i j}=$ secondary operation $i$ of $J_{j}$, with time $t_{i j}(i=1, \ldots, m-1)$. Note that $T_{i j}$ is performed on $M_{i+1}$.

The following simplifying assumptions are made:
Assumption 1 No setups are required for secondary operations. However, $M_{1}$ must be set up when successive tasks are on different passes. The setup time is pass-dependent, but not job-dependent. Let

$$
\begin{aligned}
s_{i i^{\prime}}= & \text { the setup time on } M_{1} \text { for } P_{i^{\prime} j^{\prime}} \text { if it follows } P_{i j} \text {, regardless } \\
& \text { of which jobs } J_{j} \text { and } J_{j^{\prime}} \text { are involved. Note that } s_{i i}=0 .
\end{aligned}
$$

Assumption 2 On each secondary machine, all jobs have the same processing time, so we can write $t_{i j}=t_{i}$. The same results hold for jobdependent $t_{i j}$, provided that $t_{i j} \leq s_{i-1, i}$; that is, the times to process secondary operations are shorter than the ensuing setups. But this turns the
definition of $s_{(i j)\left(i^{\prime} j^{\prime}\right)}$ (see below) into a triviality.
Figure 9.2a is labeled with the processing times and setup times for an arbitrary $J_{j^{\prime}}$, under Assumptions 1 and 2.


Fig. 9.2
(a) A three-machine hub reentrant shop, and (b) the one-machine equivalent

## Assumption 3 (Bottleneck Assumption)

$$
\min _{i, j} p_{i j} \geq \max _{i, j} t_{i j} .
$$

That is, roughly speaking, jobs spend most of their time on $M_{1}$.
Assumption 4 (No-passing Assumption) Given the order in which the first operations of the jobs are sequenced, that order is maintained among the second primary operations of all jobs, and the third, etc. That is, if pass 1 of $J_{j}$ is processed before pass 1 of $J_{j^{\prime}}$, then all other passes of those two jobs are similarly ordered. We can now speak of scheduling the jobs in a particular order. Although this does not fully specify the schedule, it does restrict the selection to a small subset of schedules. For example, For $m=n=2$, the job sequence $(1,2)$ is maintained if the schedule on $M_{1}$ is $\left(P_{11}, P_{21}, P_{12}, P_{22}\right)$ (all of $J_{1}$ done before $J_{2}$ ), or ( $P_{11}, P_{12}, P_{21}, P_{22}$ ) (the first primary operations of both jobs are done before the second), but not by $\left(P_{11}, P_{12}, P_{22}, P_{21}\right)$.

Assumption 5 (Hereditary Order) If we define $p_{j}=\Sigma_{i} p_{i j}$, then:

$$
p_{j} \leq p_{j^{\prime}} \Rightarrow p_{i j} \leq p_{i j^{\prime}}, \text { for all } i
$$

That is, if the total primary processing time of one job is bigger than another's, then so is the time of each primary operation individually.

## The One-Machine Equivalent System

These assumptions allow us to collapse the original problem shown in Fig. 9.2a into the equivalent single machine reentrant system shown in Fig. 9.2b; where equivalent means that, for any given schedule, every task has the same completion time in both systems. Each job still has $m$ primary operations with the same processing times $p_{i j}$, but with new setup times.

The modified setup time (i.e., the enforced delay between finishing one task and starting the next) is now job-dependent as well as pass-dependent, and is defined:

$$
\begin{aligned}
s_{(i j)\left(i^{\prime} j^{\prime}\right)} & =\text { setup time when } P_{i j} \text { is followed by } P_{i^{\prime} j^{\prime}} \\
& = \begin{cases}s_{i i^{\prime}} & \text { if } j \neq j^{\prime} \\
\max \left\{s_{i i^{\prime}}, t_{j}\right\} & \text { if } j=j^{\prime}\end{cases}
\end{aligned}
$$

In the first case, $J_{j^{\prime}}$ is either on its first pass or was processed earlier on $M_{1}$ and, by Assumption 3, its secondary operation is negligible compared to $p_{i j}$. In the second case, the same job immediately returns to $M_{1}$ (so $i^{\prime}$ must equal $i+1$ ), and it cannot start until its secondary operation is complete and $M_{1}$ has been set up for it, both of which are proceeding simultaneously.

## The SPT Rule and Cluster Schedules

Hereafter we deal only with the single machine model, so the only tasks being scheduled are primary operations. Given the assumptions above, the authors show that (at least) one of the SPT schedules is always optimal, where $S=(1,2, \ldots, n)$ is an SPT schedule if $p_{1} \leq p_{2} \leq \ldots \leq p_{n}$, or by Assumption 5, $p_{i 1} \leq p_{i 2} \leq \ldots \leq p_{i n}$ for any $i$. Lets assume jobs are indexed that way.

Using this observation to limit their search, the authors develop a dynamic program to find the optimal sequence of primary operations among the SPT schedules. The algorithm runs in time $O\left(m n^{m}\right)$, which is prohibitive for problems with more than five machines. They then further restrict the search to "cluster schedules".

In a cluster schedule, the jobs are partitioned into $k$ subsets or clusters, $k=1, \ldots, n$. We then require that all of the tasks of a cluster be scheduled contiguously, with the first pass tasks coming first (in SPT order) then the second pass, etc. Thus, each cluster results in a block of processing, with setup times between each pass (note there is no setup between the tasks of the same pass since $i=i^{\prime}$ and $j \neq j^{\prime}$.) The SPT requirement also means that the first cluster to be scheduled contains the shortest jobs, the second cluster the next shortest, etc. Between clusters there is a setup time of $s_{m 1}$.

It is now a straightforward if tedious calculation to compute the contribution of any cluster to the total flow time (including the flow times of all following jobs) of any clustered schedule in which it appears, noting that this contribution is independent of the rest of the schedule. Let $c_{u v}$ be the contribution of cluster $\left\{J_{u}, J_{u+1}, \ldots, J_{v}\right\}$. The search for the optimal clustering can now be formulated as a Shortest Path problem, in which a typical node, labeled $(i, j)$, denotes all the partial schedules in which the first $i$ clusters contain $j$ jobs. Thus, a start node is connected to $n$ nodes $\{(1, j), j=1, \ldots, n\}$ representing all the possible sizes of the first cluster, by arcs costing $c_{1 j}$. Each outgoing arc adds another cluster, with the arc from $(i, j)$ to $\left(i+1, j^{\prime}\right)$ costing $c_{j+1, j^{\prime}}$. Hopefully this gives the idea; further details will be omitted. The algorithm to find this shortest path has time complexity $O\left(n^{3}\right)$.

Although scheduling jobs in clusters is not necessarily optimal (except, it turns out, for $m=2$ ), it does tend to give good schedules since it keeps down the number of setups. Simulations by Wang and Lou (1991) have shown that clustered SPT scheduling performs well. The authors also give a tight worst case bound (too complicated to give here) for the ratio of the total flow time of an optimal clustered schedule to the total flow time of an optimal schedule. They argue that, if the total processing times $p_{j}$ are roughly equal, and for large values of $n$, the bound is approximately 2 .

### 9.5 Fm|V-reentrant $\mid C_{\text {max }}$ or $\Sigma_{j} C_{j}$

The V-reentrant shop, or V-shop, with flow $\bar{\phi}=(1,2, \ldots, m-1, m, m-$ $1, \ldots, 2,1$ ), is discussed in Lev and Adiri (1984). For $F 2 \mid V$-reentrant $\mid C_{\max }$, which reduces to $\bar{\phi}=(1,2,1)$, see Sect. 9.2.

### 9.5.1 Complexity

Theorem 9.7 (Lev and Adiri, 1984)
(a) $F m \mid V$-reentrant $\mid C_{\max }$ is ordinary $N P$-complete, for $m=2$.
(b) $F m \mid V$-reentrant $\mid C_{\max }$ is strongly $N P$-complete, for $m \geq 3$.
(b) $F m \mid V$-reentrant $\mid \Sigma_{j} C_{j}$ is strongly $N P$-complete, for $m \geq 2$.

Proof: (a) See Sect. 9.2
(b, c) Garey et al. (1976) showed that $F 3 \| C_{\max }$ and $F 2 \| \Sigma_{j} C_{j}$ are strongly NP-complete. But, clearly, the simple flow shop is a special case of the V-shop where the last $m-1$ task times of each job are zero.

The authors present a number of very special cases in which the problem is polynomially solvable. We will mention just one, where all processing times are equal. The following algorithm simultaneously minimizes both schedule length and sum of completion times.

Algorithm for $F m \mid V$-reentrant, $p_{i j}=1 \mid C_{\max }$ or $\Sigma_{j} C_{j}$

1. $i:=1$.
2. Schedule $J_{i}$ as early as possible on each machine over the complete route $\bar{\phi}=(1,2, \ldots, m-1, m, m-1, \ldots, 2,1)$.
3. If $i=n$, stop. Else $i:=i+1$ and return to Step 2.

We omit the proof of this algorithm, which has time complexity $O(m n)$, but illustrate it for the instance with $m=3$ and $n=6$ in Fig. 9.3. The jobs are shaded to help in grasping how they fit together. Note especially how, although scheduled singly, they pair up; a peculiarity that remains true for all values of $m$ and $n$.

The minimal job completion times are

$$
C_{j}= \begin{cases}2(m-2)+2 j, & j=2,4,6, \ldots \\ 2(m-2)+2 j+1, & j=1,3,5, \ldots\end{cases}
$$



Fig. 9.3 Optimal schedule for a V-shop with all task times equal

### 9.6 Cyclic Scheduling of a Single Product

First, note that the word "cyclic" has multiple meanings. In a cyclic shop, there may be a variety of jobs (different setup and processing times, etc.), but they all follow the same route, visiting the same sequence of work stations several times. If instead, as here, we have a cyclic schedule, we are generally manufacturing one product (or a few products), making multiple copies of the same thing. The pattern of visiting the stations is generally not cyclic, but repetition produces a cycle of production.

Aldakhilallah and Ramesh (2001) model a simple (not hybrid) reentrant shop in which a single product is produced repetitively, in batches, in a predetermined sequence of operations; call this batch for short. They seek the batch size and schedule of operations that will minimize both flow time and cycle time. A cyclic schedule is repeated every $Z$ time units (the cycle time), producing one batch per cycle. While cycle time is the time between successive batch completions, flow time is the time spent in the system by one unit or batch. We illustrate these concepts with the following example.

Example 9.1: The manufacture of a certain product entails a sequence of five operations or tasks on three machines. The machine assignment $\left(m_{i}\right)$, setup time $\left(s_{i}\right)$, and processing time $\left(p_{i}\right)$ of the tasks is as follows:

| $i$ | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{i}$ | 1 | 2 | 1 | 3 | 1 |
| $s_{i}$ | 1 | 1 | 1 | 2 | 2 |
| $p_{i}$ | 3 | 11 | 4 | 7 | 5 |

The setups are assumed to be detached (often called separable): they can be done in advance of the task's arrival from the previous stage. Attached setups can simply be incorporated into the processing time. Incidentally, there might be other time-consuming preparations or follow-ups associated
with a task: material handling, transportation, etc. We assume they have been incorporated into the processing time if attached, or into the setup if detached. Of course, for a batch of $Q$ units, any times that are incurred once per batch should simply be added, while per-unit times should be multiplied by $Q$. A task is considered to start on a machine when its setup begins.

One possible schedule is shown in Fig. 9.4, assuming a batch size of one. Note that the cycle time, $Z=16$, is as small as possible ( $M_{1}$ is fully loaded), while the flow time $F$, as determined by precedence constraints and shown by the sequence of darkly shaded tasks, is 48 .


Fig. 9.4 Three cycles of a schedule for the sample instance
Each of the two objectives has its merit, cycle time being a measure of throughput while flow time is correlated with work-in-process inventory. Unfortunately, they tend to be in conflict, so actually minimizing both simultaneously is usually impossible. Even finding an undominated cyclic schedule (i.e., a schedule such that no other schedule is better in both measures) is strongly NP-complete (Roundy 1992).

The authors build on the work of Roundy (1992), who gives a search algorithm similar to branch-and-bound to solve a somewhat simpler formulation. They seek to minimize flow time $(F)$ for given cycle time $(Z)$ and lot size $(Q)$, then searching over $Z$ and $Q$ for the overall optimum. After formulating the problem, which we might summarize $F m \mid$ batch, reentrant, $s_{i} \mid(F \mid Z, Q)$, as a mathematical program, they give a heuristic solution.

### 9.6.1 A Mixed Integer Program

The program MIP uses the following notation (some already introduced above but repeated here):

- $Z, F$ : cycle time, product flow time.
- $Q$ : batch size.
- $T_{i}$ : task $i$.
- $m_{i}, s_{i}, p_{i}$ : machine assignment, setup time, processing time of $T_{i}$.
- $X_{i}$ : start time of $T_{i}$ in the batch flow across cycles (see Fig. 9.4).
- $Y_{i}$ : start time of $T_{i}$ within the cycle (see Fig. 9.4).
- $\sigma(i)$ : immediate successor of $T_{i}$ on $m_{i}$.
- $O_{i}$ : cycle offset between $T_{i}$ and $T_{i+1}$ :

$$
O_{i}=\left\{\begin{array}{l}
0, \text { if } T_{i+1} \text { begins in the same cycle as } T_{i}, \\
k, \text { if } T_{i+1} \text { starts } k \text { cycles after } T_{i}
\end{array}\right.
$$

The following mixed integer program minimizes flow time $(F)$ for given cycle time $(Z)$ and lot size $(Q)$ :

$$
\begin{array}{rrl}
\text { MIP minimize } & F=X_{n}+s_{n}+Q p_{n} & \\
\text { subject to } & X_{1}=Y_{1}=0 & \\
X_{i} \geq X_{i-1}+s_{i-1}+Q p_{i-1}-s_{i} & i=2, \ldots, n \\
Y_{i}=X_{i}-Z \sum_{k=1}^{i-1} O_{k} & i=2, \ldots, n \\
Y_{\sigma(i)} \geq Y_{i}+s_{i}+p_{i} Q & i=1, \ldots, n \\
O_{i} \in\{0,1,2\} ; & Z>Y_{i} \geq 0 ; \quad Q>0, \text { integer } & i=1, \ldots, n \tag{9.4}
\end{array}
$$

Constraints (9.1) enforce precedence on the successive tasks of the product. Constraints (9.2) amount to $Y_{i}=X_{i} \bmod Z$, and determine the position of each task within the cycle. Constraints (9.3) prevent tasks overlapping on any machine. Constraints (9.4) on $O_{j}$ reflect the authors' observation that the offset is never more than two (recall, this is the offset from the previous task, not from $T_{1}$ ).

### 9.6.2 A Heuristic

We now outline a heuristic given by Aldakhilallah and Ramesh (2001) that also searches over $Q$ and $Z$ for the best solution, using the criterion $R=$ $Z / Z_{\min }+F / F_{\min }$, where $Z_{\min }\left[F_{\min }\right]$ are the smallest possible cycle [flow] times. The heuristic proceeds in the following steps:

1. For $Q=1$ to $Q_{\max }$, repeat steps $2-8$.
2. (a) Compute $Z_{\text {min }}=\max _{k=1}^{m} \Sigma_{i: m_{i}=k}\left(s_{i}+Q p_{i}\right)$.

In Fig. 9.4, $Z_{\min }$ is determined by the total load on $M_{1}$.
(b) Compute $F_{\min }=s_{1}+\sum_{i=1}^{n} Q p_{i}$.
3. For $Z=Z_{\text {min }}$ to $Z_{\text {min }}+m$, repeat steps $4-7$
4. Determine product start times for each $T_{i}$ :
$X_{1}=0, \quad X_{i}=X_{i-1}+s_{i-1}+Q p_{i-1}-s_{i}$, for $i=2, \ldots, n$.
5. Determine preliminary cycle start times:
$Y_{1}=0, \quad Y_{i}=X_{i} \bmod Z$, for $i=2, \ldots, n$.
6. In increasing order of $Y_{i}$, adjust each $Y_{i}$ as follows:
(a) If $T_{i-1}$ is already scheduled, put $T_{i}$ immediately after it.
(b) If not, put $T_{i}$ as early as possible.
(c) If $T_{i}$ cannot be scheduled, delay other tasks to make room.
7. If possible, reduce $Z$ without increasing $F$.
8. Given the schedule for each cycle, find the flow time $F$, and compute the criterion $R$.
9. Output the solution with smallest $R$, over all $Q, Z$ values.

### 9.6.3 Performance of the Heuristic

The authors considered how well their heuristic did from several points of view.

## Worst Case Analysis

The worst case for this problem is defined as the cyclic schedule with the longest flow time for a given cycle time. Since the flow time for a given cycle time is $F=Y_{n}+s_{n}+Q p_{n}+Z \sum_{j=1}^{n-1} O_{j}$, minimizing the total cycle offsets minimizes total flow time. This, however, turns out to be an NP-complete problem (Rao and Jackson 1993). We can give attainable bounds for the cycle offsets: $0 \leq \Sigma_{j=1}^{n-1} O_{j} \leq n-1$.

## Trade-off between Cycle Time and Flow Time

For any instance, the plot of $Z$ versus $F$ is not a smooth curve; clearly, the particular values of processing and other times may fit one cycle time better than slightly shorter or longer cycle times. Of course, when the cycle time is so long that all the tasks can be accommodated end-to-end (i.e., $\left.Z \geq F_{\min }=s_{1}+\sum_{j=1}^{n} Q p_{j}\right)$, the flow time is minimized.

## Computational Results

Twenty-four experiments were conducted, each characterized by the number of machines $(m=3,4,5$, or 6$)$ and the number of tasks $(n=10,11, \ldots$, 15). Each experiment consisted of 25 randomly generated instances. For each instance, the flow and cycle times generated by the heuristic were compared to their smallest possible values by computing their relative deviations, ( $F-$ $\left.F_{\min }\right) / F_{\min }$ and $\left(Z-Z_{\min }\right) / Z_{\min }$. As an overall average, they found a flow time deviation of only $0.7 \%$ from the smallest possible, but a cycle time deviation of $53 \%$ from the (admittedly unrealistically small) minimum.

### 9.7 Dispatching in a Reentrant Hybrid Flow Shop

Lu and Kumar (1991) consider a problem encountered in semiconductor and some flexible manufacturing where jobs of wafers, each job with a due date, follow roughly the same sequence of work stations, each station having one or more identical machines, often revisiting the same sequence of stations repeatedly for the application of several layers. Again, a problem far too complex to admit optimization. They use the approach often employed in studying large job shops: propose one or more criteria for evaluating schedules, propose several dispatching rules for sequencing the jobs at each work station, and evaluate the scheduling rules against the criteria using simulation. A dispatching rule is applied independently at each work station, selecting the next task to process from among the jobs waiting at the station, using a criterion based solely on the characteristics of those jobs. This is sometimes
called distributed scheduling. Since there is no coordination between stations, this is clearly a suboptimal approach, but it is simple to grasp and implement and often gives satisfactory results.

We note that, rather than simulation, Narahari and Khan (1996) have applied mean value analysis to analyze the performance of re-entrant manufacturing systems. This technique, often used to analyze queueing networks, is beyond the scope of this monograph.

### 9.7.1 Single Part Model

Consider a reentrant flow shop with $m$ work stations (also called machine groups or service centers) producing large numbers of a single part or product. Thus, every job or item is identical except that each may have its own due date. Machine group $G_{i}(i=1, \ldots, m)$ has $k_{i}$ identical machines in parallel. All jobs follow the same arbitrary route $\bar{\phi}$. For example, Fig. 9.5 shows a cyclic-reentrant shop with three stations and two cycles. At step $r$ of the production sequence, the item is first stored in a buffer $b_{r}$ (with effectively unlimited capacity) prior to processing at station $\phi_{r}$. No setup times are considered. The job will then require $p_{r}$ time units of processing by one of the machines $M_{h, \phi_{r}}\left(h=1, \ldots, k_{\phi_{r}} ; \phi_{r}=1, \ldots, m\right)$.


Fig. 9.5 A cyclic reentrant hybrid shop with three machine groups

We use the following additional notation for this model:

- $u(t)$ : number of items released in time interval $[0, t]$;
- $\lambda$ : arrival rate of items;
- $r_{j}$ : time that an item $J_{j}$ is released into the system;
- $C_{j}$ : time that $J_{j}$ exits the system;
- $D_{j}$ : delay or cycle time of $J_{j} . D_{j}=C_{j}-r_{j}$;
- $d_{j}$ : due date of $J_{j}$;
- $B_{i}$ : set of buffers serving machine group $G_{i}$;
and we make the following assumptions:

1. Bursty Arrivals Arrivals are called "bursty" when they satisfy the following regularity constraint used by Cruz (1991):

$$
u(t)-u(s) \leq \lambda(t-s)+\gamma, \text { for all } 0 \leq s \leq t
$$

and for some constant $\gamma \geq 0$. The parameter $\gamma$ allows for the burstiness in the arrivals.
2. Feasible Load on the System For the system to be able to meet the demand, the arrival rate $\lambda$ satisfies: $\rho \equiv \max _{i} \lambda w_{i}<1$, where $\rho$ is the load on the system and $w_{i}$ is the time units of work per machine that a job brings to $G_{i}$, given by the expression: $w_{i}=\Sigma_{r: b_{r} \in B_{i}} p_{r} / k_{i}$.
3. Non-Idling All scheduling policies that will be considered are nonidling. A policy is non-idling if machine $M_{h i}$ is allowed to remain idle only if all buffers in $B_{i}$ are empty.
4. All scheduling policies are nonpreemptive.

### 9.7.2 Scheduling Policies

Two types of scheduling policies are considered for this model, namely: buffer priority policies and due date policies. A buffer priority policy selects the job that is first in line at a specified buffer, while a due date policy chooses on the basis of the jobs' due dates.

The specific policies chosen for evaluation are listed below.

| Buffer Priority Policy | Description of Priorities |
| :--- | :--- |
| First Buffer First Serve (FBFS) | An idle machine at $G_{i}$ processes the <br> first job in buffer $b_{r} \in B_{i}$ only if all <br> other buffers $b_{r^{\prime}} \in B_{i}, r^{\prime}<r$, are <br> empty. |
| Last Buffer First Serve (LBFS) | An idle machine at $G_{i}$ processes the <br> first job in buffer $b_{r} \in B_{i}$ only if all <br> other buffers $b_{r^{\prime}} \in B_{i}, r^{\prime}>r$, are <br> empty. |
| First Come First Serve (FCFS) | An idle machine at $G_{i}$ processes the <br> waiting job that arrived earliest, re- <br> gardless of which buffer $b_{r} \in B_{i}$ it is <br> in. |


| Due Date Policy | Description of Priorities |
| :--- | :--- |
| Earliest Due Date (EDD) | An idle machine at $G_{i}$ processes a job with <br> earliest due date among jobs in any $b_{r} \in B_{i}$. |
| Least Slack (LS) | For every buffer $b_{r} \in B_{i}$, let there be a $\delta_{r} \geq 0$ <br> that represents an estimate of the remaining |
| delay for a job $J_{j}$ commencing service in $b_{r}$. |  |
| Slack for such a unit is defined as $d_{j}-\delta_{r}$. An |  |
| idle machine at $G_{i}$ processes a job with the |  |
| smallest slack among the jobs in any $b_{r} \in B_{i}$. |  |

These policies are called distributed scheduling policies or dispatching rules because they can be implemented independently of each other at each station without any need of coordination. From the structure of the policies, one gets the insight that the LBFS attempts to "greedily" empty the system, hurrying along the jobs closest to completion. Thus, it could be expected to minimize the number of units in the system which is also equivalent to minimizing mean delay. The LS policy on the other hand takes account of due dates, and tries to make all units equally late or early, thereby making it tend to minimize the variance of the delay. These insights will be verified through simulation experiments.

### 9.7.3 Stability of the Scheduling Policies

An important property of a scheduling policy is its stability. A policy is stable if the delay, or time-in-system, of each part is bounded, whenever the arrival rate is within the system's capacity. That is, for any $\lambda$ that gives $\rho<1$ (see Assumption 2),

$$
D_{j}=C_{j}-r_{j} \leq B \quad \text { for all } J_{j}, \quad \text { for some } B \geq 0
$$

By Little's theorem, stability also guarantees that the maximum work in process is bounded.

You might think that satisfying Assumption 2 would always guarantee the stability of a scheduling policy, but this is not true as shown by a counterexample in the paper. Proofs of the stability of the FBFS, LBFS, EDD and LS policies are given in the paper. However, the stability of FCFS is still an open issue.

### 9.7.4 Generalizations

The stability results above continue to hold under the following generalizations:

- Jobs may have different processing times. If $p_{i j}$ is the time for $J_{j}$ at $b_{i}$, then define $p_{i}$ as $\max _{j} p_{i j}$.
- The $k_{i}$ machines at station $G_{i}$ may differ in speed. When we say that $s_{h i}$ is the speed of $M_{h i}$, we mean that a job requiring time $p$ on a "standard" machine takes $p / s_{h i}$ on $M_{h i}$. In this case, we simply replace the formula for $w_{i}$ in Assumption 2 by $w_{i}=\Sigma_{r: b_{r} \in B_{i}} p_{r} / \sum_{h=1}^{k_{i}} s_{h i}$.
- A state-dependent transportation delay may be incurred between centers, as long as such times are uniformly bounded.

The authors also briefly discuss the extension to multiple parts, with part $v$ having its own arrival rate $\lambda_{v}$, processing times $p_{v r}$, buffers $b_{v r}$, and route $\left(\phi_{v 1}, \phi_{v 2}, \ldots, \phi_{v, l_{v}}\right)$. Of course, since routes differ, this is no longer a flow shop. They extend the definitions and assumptions, and define two new policies, Interleaved First Buffer First Serve and Uniform Last Buffer First Serve,
as tabulated below. These two policies are proven to be stable. However, the stability of the interleaved versions of LBFS and EDD has not yet been established.

| Buffer Priority Policy | Description of Priorities |
| :--- | :--- |
| Interleaved First Buffer |  |
| First Serve (IFBFS) | Order the buffers $b_{v r} \in B_{i}$ in a list $L_{i}$, in in- <br> creasing order of pass number $r$. For given $r$, <br> buffers of different part types may be ordered <br> arbitrarily. Then, an idle machine at $G_{i}$ pro- <br> cesses the first job in the first nonempty buffer <br> in $L_{i}$. |
| Uniform Last Buffer First | Order the buffers $b_{v r} \in B_{i}$ in a list $L_{i}$, with <br> Serve (ULBFS) <br> all the buffers for product $v$ contiguous in de- <br> creasing order of pass number $r$. The products <br> may be ordered arbitrarily. Then, an idle ma- <br> chine at $G_{i}$ processes the first job in the first <br> nonempty buffer in $L_{i}$. |

### 9.7.5 Simulation Results

A simulation experiment was conducted on a single product reentrant flow shop with five service centers having one machine each and each machine being visited five times. Thirty simulations for the FBFS, FCFS, LBFS and LS policies were conducted for three different loads on the system for a total of 90 simulation runs. Results showed that the LS policy had the smallest variance in delay in 81 of the 90 simulation runs. LBFS had the smallest mean delay and the smallest mean + three standard deviation values in the simulation runs. The FBFS on the other hand consistently performed the worst among the four policies. From these 90 runs, the results support the insights mentioned earlier that the LBFS performs well for mean delay and the LS policy for the variance of the delay. We cannot however say that these results are conclusive, but merely indicative for a particular set of instances. More simulations are needed to validate these results for larger and more general systems.

### 9.8 A Two-Machine Reentrant Job Shop

The problem of minimizing makespan in a two-machine side frame press shop of a truck manufacturing company is discussed by Hwang and Sun (1997). Although the jobs do not all flow through the machines in the same sequence, we mention the paper briefly because the problem, though not originally a flow shop, is reformulated as one.

Jobs move from stage to stage in one of three patterns: $\bar{\phi}_{1}=(1,1,2,1)$, $\bar{\phi}_{2}=(1,2,1)$, and $\bar{\phi}_{3}=(1,1,1)$. Successive operations of a job on the same
processor cannot be combined because of setup times that are sequence dependent; details will be omitted. The device used by the authors is to redefine each job as a sequence of shorter jobs interrelated by precedence. Each new job either has flow pattern $(1,2)$ or is a single operation on $M_{1}$, in which case we add a dummy task of length zero on $M_{2}$ so it too follows the path $(1,2)$. For example, a job $J_{j}$ that follows route $\bar{\phi}_{1}$ with processing times $\left.<p_{1}, p_{2}, p_{3}, p_{4}\right\rangle$ is replaced by three jobs $\left.J_{j 1}=\left\langle p_{1}, 0\right\rangle, J_{j 2}=<p_{2}, p_{3}\right\rangle$, and $J_{j 3}=<p_{4}, 0>$, with $J_{j 1} \rightarrow J_{j 2} \rightarrow J_{j 3}$. Thus, we end up with a simple two-machine flow shop, without reentry but with precedence constraints.

A dynamic program is developed to solve $F 2 \mid$ prec, $s_{i j} \mid C_{\text {max }}$. Computation time is further improved by exploiting sequence dominance properties of feasible schedules. For further discussion of this algorithm, see Hwang and Sun (1997).

### 9.9 Concluding Remarks

This chapter exposes the scarcity of research for the reentrant flow shop. One explanation may be that this system is closer to a job shop than any of the flow shop systems studied in this book. Available research shows that the reentrant shop is widely found in the industry. It also shows that even the simplest imaginable reentrant shops of practical interest are NP-hard. Research on the reentrant flow shop is limited both on the objectives considered, as well as on the number of machines involved. In fact, the only systems considered are those that can be approximated by two-machine flow shops. The corresponding systems are shown computationally to provide near-optimal solutions. The greater question still remains: can the special reentrant flow structure be exploited to yield satisfactory algorithms for general systems?

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## Chapter 10 THE ROBUST FLOW SHOP


#### Abstract

In this chapter, job processing requirements are considered to be uncertain. They are no longer assumed to be deterministically known. One modeling approach would be to consider processing time probability distributions, and indeed this is done in a later chapter. Here, we provide a standard minimax regret approach, which however has been exploited very little in the context of scheduling. The limited research that has been conducted on this topic demonstrates that the minimax regret approach is particularly useful in scheduling problems because the corresponding solutions are quite robust on processing time variations. On a negative note, the available research indicates that this technique is at least a level of magnitude harder to deal with than its deterministic counterpart.


### 10.1 Preliminaries

So far, the processing time of each task has been assumed to be a known constant, at least to a good approximation. When planning production in the face of uncertain task durations, perhaps due to unpredictable conditions or incomplete/unreliable information, there are two approaches we might consider. The first, surveyed in Chap. 11, requires that we assign a probability distribution to the quantities that cannot be precisely predicted. This, the stochastic approach, treats each processing time as a random variable. Most often, we then seek to minimize the expected value of the objective function.

If we find it too difficult to accurately estimate the necessary distributions, or too difficult to deal with the complex stochastic processes that arise, we may fall back on the second approach. We now seek schedules that will perform well in all eventualities. This robust performance is the focus of the present chapter.

Our objectives have always been to make some measure of cost as small as possible. Now, instead of minimizing the value of our criterion (deterministic), or minimizing its mean value (stochastic), our goal will be to minimize the maximal value of our cost measure. Without sufficient information to find the best solution, we shall at least avoid the worst.

There are two ways in which we can specify what limited information we have about the processing times.

## Processing Time Scenarios

In this paradigm, we are given a finite set of processing time scenarios, $\Lambda$. Each scenario $\lambda \in \Lambda$ represents a unique assignment of a processing time to each task, which can be realized with some positive (unknown) probability. Let
$p_{k j}^{\lambda}=$ the processing time of $J_{j}$ on $M_{k}$ in scenario $\lambda$.

## Processing Time Intervals

Another type of uncertain task times can be described by specifying for each task the range or interval of processing times that can occur. For example, the processing time of $J_{j}$ on $M_{k}$ may fall within the range bounded by $\underline{p}_{k j}$ and $\bar{p}_{k j}$. In most cases, $\underline{p}_{k j}$ and $\bar{p}_{k j}$ can be more easily and confidently specified than the entire probability distribution. Note that structuring processing time uncertainty in this way implies an infinite number of scenarios.

We will refer to "using scenarios" or "using intervals", and will sometimes use one, sometimes the other, to express task time uncertainty. To begin with, we use scenarios.

In our literature survey we were not able to identify real-world applications of the robust flow shop. It appears that robustness is presented more as a modeling approach rather than a way to solve specific problems. Or, the information collection process is too time-consuming for busy managers. Nevertheless, as we will see shortly, the robust approach teaches us a lot on the special structure of the corresponding solution. Also, robustness offers a viable alternative when compared to the stochastic version of the problem where distributional information has to be developed. Or it can be used as a benchmark to the stochastic solution.

### 10.2 The Minimax Regret Makespan Problem

Unfortunately, the only objective considered in the literature for the robust flow shop is makespan. Even then, only the case $m=2$ has been considered, so henceforth we use $\left\langle a_{j}, b_{j}\right\rangle$ instead of $\left\langle p_{1 j}, p_{2 j}\right\rangle$. The results in this chapter are mostly drawn from Kouvelis et al. (2000).

Let $S$ denote a given schedule, and $S_{\lambda}^{\star}$ the optimal schedule given processing time scenario $\lambda \in \Lambda$. We can easily construct $S_{\lambda}^{\star}$ using Johnson's
$O(n \log n)$ algorithm. If $C(S, \lambda)$ denotes the makespan of schedule $S$ given processing time scenario $\lambda$, then $C\left(S_{\lambda}^{\star}, \lambda\right)=\min _{S} C(S, \lambda)$.

Not knowing in advance which scenario will actually be realized, we want a robust solution, one that performs well no matter what the outcome. One may imagine several notions of robustness. The following approach is known in the decision theory literature as minimax regret. The performance of any schedule $S$ differs, depending on which scenario is realized. For a given scenario, the difference between the realized value (in our case, the length of the makespan) using $S$, and the value that could have been achieved had we known the scenario and chosen the schedule accordingly, is called the regret. Since the scenario cannot be predicted, we propose to choose the schedule whose maximal regret is smallest. This idea is the basis for the following definition.

Definition 10.1. If schedule $S$ is chosen and scenario $\lambda$ is then realized, the regret is $C(S, \lambda)-C\left(S_{\lambda}^{\star}, \lambda\right)$. Let

$$
R(S)=\max _{\lambda \in \Lambda}\left\{C(S, \lambda)-C\left(S_{\lambda}^{\star}, \lambda\right)\right\}
$$

be the maximal regret. The maximizing $\lambda$ is the maximal regret scenario, $\lambda_{S}$. Then the minimax regret schedule, $S^{\star}$, satisfies

$$
R\left(S^{\star}\right)=\min _{S}\{R(S)\}
$$

The objective function value when using intervals is quite similar, hence we limit our discussion to the problem using scenarios.

The minimax regret makespan (MRM) scheduling problem will be the focus of our attention throughout this chapter. As is usual with two-machine flow shops, we will limit our attention to permutation schedules. We do this without loss of optimality, as we now show.

Theorem 10.1 In the two-machine flow shop with makespan objective, there always exists a permutation schedule that achieves minimax regret.

Proof: Suppose a minimax regret schedule $S_{N P}$ is not permutation. Its regret for any given $\lambda$ is $C\left(S_{N P}, \lambda\right)-C\left(S_{\lambda}^{\star}, \lambda\right)$. But with $\lambda$ fixed we have an ordinary deterministic flow shop, where $S_{N P}$ can be replaced by a permutation schedule $S_{P}$ in which no task is moved on $M_{2}$, so that $C\left(S_{P}, \lambda\right)=C\left(S_{N P}, \lambda\right)$. Since the regret is the same for each $\lambda$, the maximal regret is unchanged: $R\left(S_{P}\right)=R\left(S_{N P}\right)$. Thus, if $S_{N P}$ minimizes $R(\cdot), S_{P}$ is an alternative optimum.

We can now call the problem of interest $F 2 \mid($ perm $) \mid M R M$.

### 10.2.1 An Integer Program for $F 2 \mid($ perm $) \mid M R M$

The following integer program determines $S^{\star}$. Letting [i] denote the index of the job scheduled in position $i$, define the following decision variables for $i, j=1, \ldots, n$ and $\lambda \in \Lambda$ :

$$
\begin{aligned}
& x_{i j}= \begin{cases}1, & \text { if } \left.[i]=j \text { (i.e., if } J_{j} \text { takes position } i\right), \\
0, & \text { otherwise, }\end{cases} \\
& B_{i}^{\lambda}=\text { start time of } J_{[i]} \text { on } M_{2} \text { given scenario } \lambda .
\end{aligned}
$$

The binary variables, of course, assign jobs to positions. For example, using them and given scenario $\lambda$, the makespan is $B_{n}^{\lambda}+\sum_{j=1}^{n} b_{j}^{\lambda} x_{n j}$. We can now formulate $F 2 \mid($ perm $) \mid M R M$ as:

MRM minimize $R$
subject to

$$
\begin{align*}
B_{n}^{\lambda}+\sum_{j=1}^{n} b_{j}^{\lambda} x_{n j}-C\left(S_{\lambda}^{\star}, \lambda\right) \leq R, & \lambda \in \Lambda  \tag{10.1}\\
\sum_{\ell=1}^{i} \sum_{j=1}^{n} a_{j}^{\lambda} x_{\ell j} \leq B_{i}^{\lambda}, & i=1, \ldots, n ; \lambda \in \Lambda  \tag{10.2}\\
B_{i}^{\lambda}+\sum_{j=1}^{n} b_{j}^{\lambda} x_{i j} \leq B_{i+1}^{\lambda}, & i=1, \ldots, n-1 ; \lambda \in \Lambda  \tag{10.3}\\
\sum_{i=1}^{n} x_{i j}=1, & j=1, \ldots, n  \tag{10.4}\\
\sum_{j=1}^{n} x_{i j}=1, & i=1, \ldots, n  \tag{10.5}\\
x_{i j} \in\{0,1\}, & i, j=1, \ldots, n \tag{10.6}
\end{align*}
$$

### 10.2.2 Complexity of $\operatorname{F2} \mid($ perm $) \mid M R M$

The complexity of the minimax regret makespan scheduling problem is established by the following result.

Theorem 10.2 (Kouvelis et al., 2000) The $F 2 \mid($ perm $) \mid M R M$ problem using the scenario paradigm is NP-complete.

Outline of Proof: We reduce the NP-complete problem

## PARTITION

INSTANCE: Finite set of positive integers $v_{i}, i \in \mathcal{T}$.
QUESTION:Does there exist a subset $\mathcal{T}_{0} \subset \mathcal{T}$ such that
$\Sigma_{i \in \mathcal{T}_{0}} v_{i}=\Sigma_{i \in \mathcal{T}-\mathcal{T}_{0}} v_{i}$ ?
to the decision version of the $F 2 \mid($ perm $) \mid M R M$.
For a given set $\left\{v_{i}, i \in \mathcal{T}\right\}$, suppose $|\mathcal{T}|=k$. Construct an instance of $F 2 \mid($ perm $) \mid M R M$ with two processing time scenarios, $k+1$ jobs ( $J_{0}$ and $\left.J_{j}, j \in \mathcal{T}=\{1,2, \ldots, k\}\right)$. The following define the processing times of $J_{j}$ on $M_{1}$ and $M_{2}$ in each of the two scenarios.

$$
\begin{aligned}
& <a_{j}^{1}, b_{j}^{1}>= \begin{cases}<\frac{1}{2} \sum_{i=1}^{k} v_{i}, \frac{1}{2} \sum_{i=1}^{k} v_{i}>, & \text { for } j=0, \\
<0, v_{j}>, & \text { for } j \in \mathcal{T}\end{cases} \\
& <a_{j}^{2}, b_{j}^{2}>= \begin{cases}<\frac{1}{2} \sum_{i=1}^{k} v_{i}, \frac{1}{2} \sum_{i=1}^{k} v_{i}>, & \text { for } j=0, \\
<v_{j}, 0>, & \text { for } j \in \mathcal{T}\end{cases}
\end{aligned}
$$

Observe that the Johnson order has makespan $3 \Sigma_{i=1}^{k} v_{i} / 2$ for both scenarios. The authors show that there exists a partition with $\Sigma_{i \in \mathcal{T}_{0}} v_{i}=\Sigma_{i \in \mathcal{T}-\mathcal{T}_{0}} v_{i}$ if and only if $F 2 \mid($ perm $) \mid M R M$ has an optimal solution with $\Delta^{*}=0$.

Assume that a set $\mathcal{T}_{0} \subset \mathcal{T}$ of jobs is scheduled before $J_{0}$, and the remaining jobs are scheduled after $J_{0}$. The above claim is easily verified for each of the following three cases:
(1) $\Sigma_{i \in \mathcal{T}_{0}} v_{i}>\Sigma_{i \in \mathcal{T}} v_{i} / 2>\Sigma_{i \in \mathcal{T}-\mathcal{T}_{0}} v_{i}$,
(2) $\Sigma_{i \in \mathcal{T}_{0}} v_{i}<\Sigma_{i \in \mathcal{T}} v_{i} / 2<\Sigma_{i \in \mathcal{T}-\mathcal{T}_{0}} v_{i}$,
(3) $\Sigma_{i \in \mathcal{T}_{0}} v_{i}=\Sigma_{i \in \mathcal{T}} v_{i} / 2=\Sigma_{i \in \mathcal{T}-\mathcal{T}_{0}} v_{i}$.

### 10.3 The Two-Machine MRM Problem using Scenarios

In the following two sections we present results drawn from Kouvelis et al. (2000) for minimax regret makespan schedules when processing times of jobs are specified in terms of scenarios or intervals respectively. We start with processing time scenarios.

The worst case scenario for any given sequence (i.e., the scenario that maximizes regret) can be determined using a process referred to as WorstCase, that requires $|\Lambda|$ iterations. For each scenario $\lambda \in \Lambda$, the makespan of the given sequence is computed in $O(n)$ time, and the corresponding optimal makespan is determined in $O(n \log n)$ time using JR. If the difference between the two makespans exceeds the largest difference encountered in iterations 1 through $\lambda-1$, the new value and associated scenario are retained. After $|\Lambda|$ iterations, the stored value represents the worst case absolute deviation from optimality for the given sequence, and the associated set of processing times represents the worst case scenario for that sequence. By determining the set of optimal makespans exactly once, the complexity of Worst-Case is $O(|\Lambda| n)$.

### 10.3.1 Dominance Properties

The following dominance properties are also employed to enhance the computational efficiency of the branch-and-bound procedure described below.

Property 10.1 For $F 2 \mid($ perm $) \mid M R M$ using scenarios, if for two jobs $J_{i}$ and $J_{j}, a_{i}^{\lambda} \leq a_{j}^{\lambda}$ and $b_{i}^{\lambda} \geq b_{j}^{\lambda}$ for all $\lambda \in \Lambda$, then $i \stackrel{\text { global }}{\longrightarrow} j$.

Property 10.2 For $F 2 \mid($ perm $) \mid M R M$ using scenarios, if for two adjacent jobs $J_{i}$ and $J_{j}, \min \left\{a_{i}^{\lambda}, b_{j}^{\lambda}\right\} \leq \min \left\{a_{j}^{\lambda}, b_{i}^{\lambda}\right\}$ for all $\lambda \in \Lambda$, then $i \leadsto j$.

These properties are presented without proof. Similar results using intervals will be presented later in more detail.

### 10.3.2 Branch-and-Bound Algorithm

A depth first search branch-and-bound algorithm is presented, where at each branching a job is added to an initial partial schedule $\sigma$. Let $\mathcal{U}$ denote the unscheduled jobs at a node $\sigma$, with $|\mathcal{U}|=u$. To compute a bound for this node, we must consider each of the scenarios. At iteration $\lambda$ of this process, jobs are assigned their processing times in scenario $\lambda$. The $u$ unassigned jobs are then arranged according to $J R\left(\underline{a}^{\lambda}, \underline{b}^{\lambda}\right)$, and appended to $\sigma$. The difference between the makespan of this schedule and the corresponding optimal makespan (both computed with respect to $\left\langle a_{j}^{\lambda}, b_{j}^{\lambda}\right\rangle$ ) represents a lower bound on the deviation from optimality (the regret) that could result given $\sigma$ and $\lambda$. The maximum obtained by repeating this process for each $\lambda \in \Lambda$ is a lower bound on the minimax regret that could result from any schedule starting with the given partial sequence. Again, if the Johnson sequence associated with each processing time scenario is determined external to the bounding process, then the complexity of the procedure is $O(|\Lambda| n)$.

### 10.3.3 Heuristic

The heuristic procedure in Kouvelis et al. (2000) requires $|\Lambda|$ iterations. It is a straightforward neighborhood search, at each iteration updating the current overall best sequence, $S^{o}$, along with its worst case deviation from optimality, $d\left(S^{o}\right)$, initially set to infinity. For every $\lambda \in \Lambda$, with processing times set to $\left\langle a_{j}^{\lambda}, b_{j}^{\lambda}\right\rangle$, we first find $J R\left(\underline{a}^{\lambda}, \underline{b}^{\lambda}\right)$ and record this as the current best sequence for scenario $\lambda, S^{\lambda}$, along with its worst case deviation from optimality, $d\left(S^{\lambda}\right)$, found using procedure Worst-Case. Defining the neighborhood of a sequence as all $n(n-1)$ possible insertions and $n(n-1) / 2$ pairwise interchanges, we next search the neighborhood of $S^{\lambda}$, and the performance of each neighbor is evaluated and compared with $d\left(S^{\lambda}\right)$. If no improvement is found, iteration $\lambda$ of the process is completed, and the procedure advances to scenario $\lambda+1$. Otherwise, the neighbor yielding the lowest worst case deviation from optimality becomes $S^{\lambda}$, and its neighborhood is searched, the process being repeated until no further improvement results and we advance to the next scenario. Each time $\lambda$ is incremented, we update the overall best-yet solution: if $d\left(S^{\lambda}\right)<d\left(S^{o}\right)$, set $S^{o}:=S^{\lambda}$. After $|\Lambda|$ iterations, we output $S^{o}$.

The above branch-and-bound and heuristic algorithms were tested on problems with $n=9,12$, and 15 jobs and $|\Lambda|=4,8$, and 12 processing time scenarios drawn from a uniform distribution of integers $\left[10 \beta_{j},(10+40 \alpha) \beta_{j}\right]$, where $\alpha$ is a parameter that allows the variability of processing times across jobs in a given problem instance to be controlled, and $\beta_{j}$ represents the relative processing requirement on $M_{j}$, thus allowing the location of processing bottlenecks to be controlled. Three values of $\alpha(\alpha=0.2,0.6,1.0)$ and three vectors representing the relative processing requirements on $M_{1}$ and $M_{2}\left(\left[\beta_{1}, \beta_{2}\right]=[1.0,1.0],[1.2,1.0],[1.0,1.2]\right)$ were included in the experimental design.

It was found that the $\beta$ values do not have an appreciable effect on the quality of the heuristic solution. Also, the computational effort required to run the branch-and-bound algorithm increases moderately (not factorially) as $n$ increases. Optimal solutions are increasingly difficult to obtain as the number of processing time scenarios is decreased, suggesting that the additional computational effort required to identify the worst case scenario for each sequence is more than offset by the enhanced effectiveness of the lower bounding process as the number of processing time scenarios increases. The heuristic required only modest computational effort to generate approximate solutions. The heuristic produced the optimal solution in over $95 \%$ of the problems tested, while the error incurred in approximating the optimal solution averaged $0.5 \%$, with a maximum of $15.4 \%$.

In addition, assuming that each processing time scenario is equally likely, the sequence that optimizes expected makespan performance was determined for each test problem using a branch-and-bound process. It was found that the resulting schedule was an average of $21.3 \%$ off the performance of the optimal robust schedule with a corresponding maximum error of $190.4 \%$. In contrast, the expected makespan performance of the robust schedule closely approximated the optimal expected makespan, with an average percentage deviation of $0.4 \%$, and a maximum of $3.3 \%$. Evidently, the robust schedule optimally hedges against the worst case realization of job processing times while maintaining excellent expected makespan performance.

### 10.4 The Two-Machine MRM Problem using Intervals

We now consider the case where processing time uncertainty is captured by means of time intervals: the times for the tasks of $J_{j}$ on $M_{1}$ and $M_{2}$ must be chosen from the intervals $\left[\underline{a}_{j}, \bar{a}_{j}\right]$ and $\left[\underline{b}_{j}, \bar{b}_{j}\right]$, respectively. Though we are using intervals rather than scenarios, it will be convenient to continue to use the term "scenario" to refer to an assignment of a feasible time to every task. The set of all scenarios, $\Lambda$, is now uncountably infinite:

$$
\Lambda=\{\lambda\}, \text { where } \lambda=\left(a_{1}^{\lambda}, b_{1}^{\lambda}, \ldots, a_{n}^{\lambda}, b_{n}^{\lambda}\right): \underline{a}_{j} \leq a_{j}^{\lambda} \leq \bar{a}_{j}, \underline{b}_{j} \leq b_{j}^{\lambda} \leq \bar{b}_{j} .
$$

The following theorem helps identify the maximal regret scenario for any $S$. Recall the portrayal of a two-machine flow shop as a precedence network (see Fig. 2.1), in which the makespan of $S=([1], \ldots,[n])$ is the length of the critical (i.e., longest) path, $\Sigma_{i=1}^{c} a_{[i]}+\sum_{i=c}^{n} b_{[i]}$, where $J_{[c]}$ is the critical job, with $b_{[c]}$ starting just as $a_{[c]}$ ends. In case there is more than one critical path, choose the latest critical job in what follows.

Theorem 10.3 For $F 2 \mid$ perm $\mid M R M$ using intervals, for any schedule $S$, there exists a maximal regret scenario $\lambda_{S}$ such that
(i) The time for every task on the critical path is at its upper bound;
(ii) The time for every other task is at its lower bound.

Proof Outline: Consider sequence $S$ and scenario $\lambda$ that is supposed to be maximal regret, with critical job $J_{[c]}$. Note that the tasks on the critical path are $\left\{a_{[i]}^{\lambda}, i=1, \ldots, c\right\}$ and $\left\{b_{[i]}^{\lambda}, i=c, \ldots, n\right\}$. Suppose, contrary to the theorem, that any of the following four cases holds, for the processing time requirements $<a_{[i]}^{\lambda}, b_{[i]}^{\lambda}>$ of job $J_{[i]}$ in scenario $\lambda$ :
a) $a_{[i]}^{\lambda}<\bar{a}_{[i]}$ for some $i=1, \ldots, c$,
b) $b_{[i]}^{\lambda}<\bar{b}_{[i]}$ for some $i=c, \ldots, n$,
c) $a_{[i]}^{\lambda}>\underline{a}_{[i]}$ for some $i=c+1, \ldots, n$,
d) $b_{[i]}^{\lambda}>\underline{b}_{[i]}$ for some $i=1 \ldots, c-1$.

In cases a) and b), increasing $a_{[i]}^{\lambda}$ or $b_{[i]}^{\lambda}$ by $\epsilon$ results in an increase in the makespan of schedule $S, C(S, \lambda)$, of $\epsilon$, while increasing the optimal system makespan, $C\left(S_{\lambda}^{\star}, \lambda\right)$ by no more than $\epsilon$. In cases c) and d), decreasing $a_{[i]}^{\lambda}$ or $b_{[i]}^{\lambda}$ by $\epsilon$ results in no change in $C(S, \lambda)$, and does not increase $C\left(S_{\lambda}^{\star}, \lambda\right)$. Thus, if any of the four cases holds, the $\epsilon$ adjustment can only increase the regret. With $\epsilon$ chosen as large as feasible, we get $\lambda_{S}$.

The above result motivates exhaustive enumeration of the scenarios described in Theorem 10.3 to find the maximal regret scenario associated with a given permutation $S$ of jobs. In iteration $k$ of the procedure, $J_{[k]}$ is assumed to be critical and the processing times of all jobs are determined using Theorem 10.3. Subsequently, the makespan of $S$ is computed for the resulting scenario and compared to the corresponding optimal makespan determined by JR. If the deviation between the two makespan values is greater than the largest deviation encountered in iterations 1 through $k-1$, the new value and associated scenario are retained. The maximal regret scenario is found after $n$ iterations. The complexity of the resulting procedure is $O\left(n^{2} \log n\right)$.

### 10.4.1 Dominance Properties

The following two dominance properties can be used to establish the relative position of pairs of jobs in an MRM schedule.

Property 10.3 For $F 2 \mid$ perm $\mid M R M$ using intervals, if for two jobs $J_{i}$ and $J_{j}, \bar{a}_{i} \leq \underline{a}_{j}$ and $\underline{b}_{i} \geq \bar{b}_{j}$, then $i \stackrel{\text { global }}{\sim} j$.

Proof: Let $\left(a_{i}, b_{i}\right)$ and $\left(a_{j}, b_{j}\right)$ be the processing times in an optimal (minimax regret) sequence. Then, $a_{i} \leq \bar{a}_{i} \leq \underline{a}_{j} \leq a_{j}$ and similarly $b_{i} \geq \underline{b}_{i} \geq$ $\bar{b}_{j} \geq b_{j}$. A simple interchange argument shows that the makespan is no worse when $J_{i}$ precedes $J_{j}$.

Property 10.4 For $F 2 \mid$ perm $\mid M R M$ using intervals, if for two adjacent jobs $J_{i}$ and $J_{j}, \min \left\{\bar{a}_{i}, \bar{b}_{j}\right\} \leq \min \left\{\underline{a}_{j}, \underline{b}_{i}\right\}$, then $i \sim j$.

Proof: Let $\left(a_{i}, b_{i}\right)$ and $\left(a_{j}, b_{j}\right)$ be the processing times in an optimal (minimax regret) sequence. Then, $\min \left\{a_{i}, b_{j}\right\} \leq \min \left\{\bar{a}_{i}, \bar{b}_{j}\right\} \leq \min \left\{\underline{a}_{j}, \underline{b}_{i}\right\} \leq$
$\min \left\{a_{j}, b_{i}\right\}$. For these $a_{i}, b_{i}, a_{j}, b_{j}$ values, a simple interchange argument shows that the makespan is no worse when $J_{i}$ immediately precedes $J_{j}$.

### 10.4.2 Lower Bounds and Branch-and-Bound Algorithm

The above dominance properties are incorporated into a branch-and-bound algorithm for determining the minimax regret schedule. The branching procedure is similar to the one used with the scenario paradigm. At each node of the search tree a lower bound is computed on the deviation of the partial sequence corresponding to this node. Assume a partial sequence $\sigma$, of $s$ jobs, with the scheduling of the remaining $u=n-s$ jobs as yet unknown. At iteration $k \leq s, J_{k}$ is assumed to be critical and the processing times of the jobs are assigned using Theorem 10.3. A lower bound can then be computed on the makespan associated with the given partial sequence by

$$
L B=\sum_{i=1}^{k} \bar{a}_{i}+\sum_{i=k}^{n} \bar{b}_{i}
$$

and compared to the corresponding Johnson makespan for this scenario. If the difference between the two makespan values is greater than the incumbent, the new value is retained. After $s$ iterations, the procedure yields a lower bound on the maximal regret for all completions of the given partial sequence. The complexity of this stage of the process is $O\left(n^{2} \log n\right)$.

To strengthen the lower bound associated with a node, one can evaluate the impact of each of the unassigned jobs being the last critical job while occupying either position $s+1$ or position $n$ in the final schedule. Suppose unassigned job $J_{k}$ is the last critical job and occupies position $s+1$. Then:

- $J_{k}$ should have $a_{k}=\bar{a}_{k}$ and $b_{k}=\bar{b}_{k}$,
- $J_{i}$ should have $a_{i}=\bar{a}_{i}$ and $b_{i}=\underline{\underline{b}}_{i}$ for $i=[1], \ldots,[s]$,
- $J_{i}$ should have $a_{i}=\underline{a}_{i}$ and $b_{i}=\bar{b}_{i}$ for $i=[s+2], \ldots,[n]$.

Computing a lower bound on the difference between the makespan associated with the given partial sequence, $L B^{\prime}=\Sigma_{i=1}^{s+1} \bar{a}_{[i]}+\sum_{i=s+1}^{n} \bar{b}_{[i]}$, and the corresponding optimal makespan using JR, we obtain an upper bound on the worst case performance, since job $k$ need not necessarily be either the last critical job or occupy position $s+1$ in sequence. However, if we repeat the process for each of the $n-s$ unassigned jobs, the minimal difference represents a lower bound on the maximal regret associated with the given partial sequence, since one unassigned job must ultimately occupy position $s+1$.

Similarly, if $J_{k}$ is the last critical job and occupies position $n$ in sequence, then $J_{k}$ should again have $a_{k}=\bar{a}_{k}$ and $b_{k}=\bar{b}_{k}$, with all $n-1$ remaining jobs having $a_{i}=\bar{a}_{i}$ and $b_{i}=\underline{b}_{i}$ for $i=[1], \ldots,[n-1]$. Computing a lower bound on the makespan associated with the given partial sequence, $L B^{\prime \prime}=\sum_{i=1}^{n} \bar{a}_{[i]}+\bar{b}_{[n]}$, and the corresponding optimal makespan using the Johnson algorithm, we again obtain an upper bound on the worst case performance, since job $k$ need not necessarily be either the last critical job or
occupy position $n$ in sequence. However, if we repeat the process for each of the $n-s$ unassigned jobs, the minimal difference represents a lower bound on the worst case absolute deviation from optimality associated with the given partial sequence, since one unassigned job must ultimately occupy position $n$. The complexity of each of these two stages of the process is $O\left(n^{2} \log n\right)$.

The dominance properties and bounding procedure are easily incorporated into a branch-and-bound solution framework to construct robust schedules.

### 10.4.3 Heuristic

Suppose $J_{c}$ is critical in a permutation sequence, and set(s) $\mathcal{L}[\mathcal{R}]$ of jobs preceding [following] $J_{c}$ are given. Then, Theorem 10.3 is used to determine the maximal regret scenario for the bipartition $\mathcal{L}$ and $\mathcal{R}$. A promising sequence may be constructed by arranging the jobs $J_{i}, i \in \mathcal{L}$ according to JR with respect to $\left(\bar{a}_{i}, \underline{b}_{i}\right)$, followed by $J_{c}$, followed by $J_{i}, i \in \mathcal{R}$ arranged in Johnson order with respect to ( $\underline{a}_{i}, \bar{b}_{i}$ ). This property is not necessarily characteristic of the optimal solution and in fact $J_{c}$ may no longer remain critical when the task times are given the extreme values prescribed in Theorem 10.3.

According to Theorem 10.3 , each $J_{i}$ will contribute at least $\min \left\{\bar{a}_{i}, \bar{b}_{i}\right\}$ to the makespan of the robust schedule. Hence, we can compute $\min \left\{\bar{a}_{i}, \bar{b}_{i}\right\}$ for each job, and assign $J_{i}$ to $\mathcal{L}$ if $\bar{a}_{i} \leq \bar{b}_{i}$ and to $\mathcal{R}$ if $\bar{a}_{i}>\bar{b}_{i}$. The hope with this strategy is that the resulting permutation provides a good starting solution for the minimax schedule. Performing all $n(n-1)$ possible insertions and $n(n-1) / 2$ possible pairwise interchanges can further improve the initial solution.

An experiment similar to that for the scenario paradigm was performed for the interval paradigm. The lower end of the processing time range for each task was drawn from a uniform distribution of integers over $\left[10 \beta_{k},(10+\right.$ $\left.40 \alpha_{1}\right) \beta_{k}$ ], where $\alpha_{1}$ is a parameter that allows the variability of processing times across jobs in a given problem instance to be controlled, and $\beta_{k}$ again represents the relative processing requirement on $M_{k}$. The upper end of the range for each task time was next randomly drawn from a uniform distribution of integers on the interval $\bar{a}_{i} \in\left[\underline{a}_{i}, \underline{a}_{i}\left(1+\alpha_{2}\right)\right], \bar{b}_{i} \in\left[\underline{b}_{i}, \underline{b}_{i}\left(1+\alpha_{2}\right)\right]$, where $\alpha_{2}$ controls the variability of a given job's processing time. Three values of $\alpha_{1}(0.2,0.6,1.0)$, three values of $\alpha_{2}(0.2,0.6,1.0)$, and three vectors representing the relative processing requirements on $M_{1}$ and $M_{2}\left(\left[\beta_{1}, \beta_{2}\right]=[1.0,1.0]\right.$, $[1.2,1.0],[1.0,1.2])$ were included in the experimental design.

In addition, the sequence that optimizes expected makespan performance was directly determined by assuming independent, exponential processing time distributions fit to the minimal and maximal processing times specified for each job on each machine, and using the results of Cunningham and Dutta (1973) (see also Pinedo, 1984 and Ku and Niu, 1986). As before, the performance of the heuristic is not affected by the values $\left[\beta_{1}, \beta_{2}\right]$ ). Also. the computational effort required to obtain optimal solutions using the branch-and-bound procedure grew rapidly with problem size but not factorially. Also,
problem difficulty was significantly affected by the variability of individual job processing times, as measured by the interval ranges.

As before, the heuristic produced an optimal solution in over $95 \%$ of the 810 test problems, while the average error was $0.2 \%$ with a maximum of $16.7 \%$. In addition, the sequence that optimizes expected makespan performance was determined for each test problem using a branch-and-bound process that assumes that each processing time scenario is equally likely. The solution quality associated with the resulting schedule was considerably lower, with an average approximation error of $22.1 \%$, and a maximum error of $289.1 \%$. In contrast, the expected makespan performance of the robust schedule closely approximated the optimal expected makespan, with an average percentage deviation of $0.2 \%$, and a maximum of $3.9 \%$.

### 10.5 Conclusions

Using the interval paradigm, robust scheduling requires a decision maker to specify processing time ranges for uncertain task durations. The robust approach yields schedules that minimize the worst case deviation from optimal makespan performance, i.e., find the minimax regret, for the processing time ranges specified. Experiments indicate that small to moderate increases in the minimal deviation from optimality are realized when processing time ranges are widened by $50 \%$ or so, and that the identity of the robust schedule remains relatively constant as processing time ranges are enlarged. Thus, a decision maker can have considerable confidence in the minimax regret schedule even when processing time ranges are relatively inaccurate.

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## Chapter 11 STOCHASTIC FLOW SHOPS


#### Abstract

When job parameters are uncertain or unpredictable, new types of policies become possible. Besides static policies, we now should consider dynamic policies, with or without preemption. Objectives too have more variety. The makespan, for example, is now random; we usually choose to minimize its expectation.

We present new conditions under which permutation schedule are optimal for two or three machines. If these do not hold, then to minimize expected makespan on two machines: - A simple optimum exists when task times are exponential; - for arbitrary distributions, Johnson's Rule is asymptotically optimal; - three simple heuristics have been tested using log normal distributions, and give good results in combination. For more than two machines, we report the very limited results that have been published.


### 11.1 Preliminaries

We now present the somewhat limited results that are known concerning the flow shop when any parameters of the problem, especially processing times, are not given fixed values known in advance, but are specified as random variables with some probability distribution. In practice, such uncertainty is frequently encountered (processing times are not often precisely predictable), but a good guess often suffices. However, when experience shows that some parameter varies a lot from case to case, and there is enough historical data to estimate a distribution, a stochastic model may be appropriate. In any case, such analysis often provides valuable insights, as we hope to show.

The reader is expected to have a basic knowledge of probability theory and stochastic processes. He should recognize the common continuous and discrete distributions, and understand the uses of the functions that describe
them: the density and mass functions, cumulative distribution function, mean and variance. We adopt the usual conventions, denoting random variables by capital letters, with the corresponding lower case letter often used for a particular value. Thus, $X$ is a random variable with cumulative distribution function (CDF) $F_{X}(x)=\mathrm{P}(X \leq x)$. A continuous $X$ has density function $f_{X}(x)=d F_{X}(x) / d x$, while $f_{X}(x)=\mathrm{P}(X=x)$ for discrete $X$. In particular, we capitalize the lower case letters we have been using for known fixed parameters, to emphasize their randomness. For example:

- $P_{i j}:$ Random processing time of $T_{i j}$.
- $A_{j}, B_{j}$ : Random task times of $T_{1 j}, T_{2 j}$ respectively, when $m=2$.
- $a_{j}, b_{j}$ : Average task times: $a_{j}=E\left(A_{j}\right), b_{j}=E\left(B_{j}\right)$.

Random variables are assumed to be independent unless otherwise noted. We will enter $P_{i j}$ in the second field of the problem descriptor to indicate random task times.

### 11.1.1 The Exponential Distribution

We give here some of the basic properties of the exponential variate, on which many of the results in this chapter are based. An exponential random variable, $X$, with rate $\lambda$, has density function $f(x)=\lambda e^{-\lambda x}(x>0)$, distribution $F(x)=1-e^{-\lambda x}$, and mean value $E(X)=1 / \lambda$. An important characteristic of this distribution is

Theorem 11.1 If $X$ is exponential with rate $\lambda$, then

$$
P(X>x+t \mid X>t)=P(X>x)
$$

This is called the memoryless property.
Proof: $P(X>x+t \mid X>t)=P(X>x+t) / P(X>t)=e^{-\lambda x}$
Thus, if the time to an anticipated event is exponential, then, no matter how much time has passed, if the event has not yet occurred, the time remaining is just as long, probabilistically, as it was at the start. It shares this property with the discrete geometric distribution, which has PMF $P(X=x)=(1-$ p) $p^{x},(x=0,1,2, \ldots, 0<p<1)$ and $E(X)=p /(1-p)$.

Other results we shall find useful are
Theorem 11.2 If $X$ and $Y$ are exponential with rates $\lambda$ and $\mu$, then $\min \{X, Y\}$ is exponential with rate $\lambda+\mu$.
Proof: $P(\min \{X, Y\} \leq t)=1-P(X>t) \cdot P(Y>t)=1-e^{-(\lambda+\mu) t}$.
Theorem 11.3 If $X$ and $Y$ are exponential with rates $\lambda$ and $\mu$, then

$$
P(X<Y)=\lambda /(\lambda+\mu) .
$$

Proof: Integrating over the joint probability space:

$$
P(X<Y)=\lambda \mu \int_{0}^{\infty} e^{-\mu y} \int_{0}^{y} e^{-\lambda x} d x d y=\mu \int_{0}^{\infty} e^{-\mu y}\left(1-e^{-\lambda y}\right) d y=\frac{\lambda}{\lambda+\mu}
$$

Theorem 11.4 If $X$ and $Y$ are exponential with rates $\lambda$ and $\mu$, then $X+Y$ has the distribution

$$
P(X+Y>t)=\frac{\mu}{\mu-\lambda} e^{-\lambda t}+\frac{\lambda}{\lambda-\mu} e^{-\mu t}
$$

Proof: Simply compute the convolution:

$$
P(X+Y>t)=\int_{0}^{t} f_{X}(x)\left(1-F_{Y}(t-x)\right) d x+1-F_{X}(t)
$$

### 11.1.2 Types of Policies

In the study of flow shops, randomness introduces some new considerations. For example, consider a schedule as it evolves over time. In the deterministic world, having chosen a schedule, we know exactly what will happen from start to finish. But as we watch an unpredictable sequence of events unfold, as random variables are realized, we gain new information that might suggest a change of plans for the remaining work. Thus, two types of scheduling policies emerge:

1. static policies specify schedules that are fixed at the start, and can never thereafter be changed. Research on stochastic shops, up to now, has been limited to the class of static list policies, giving rise to permutation schedules, where a predetermined job ordering is given at time zero, that must be followed on all machines. Thus, whenever a machine becomes idle, the next job on the list is processed as soon as it is available.

As an example of a static policy that is not permutation, we might specify different job sequences to be followed on different machines. As we know, for deterministic shops, such schedules may be optimal, at least for more than three machines.

Evidently, the above definition precludes preemption: once a task is started, it will process to completion. Pinedo (2008) makes an exception for the case where jobs have different release times. If $J_{1}$ is released earlier than $J_{2}$ but has lower priority, it should be preempted, provided of course preemption is allowed and $J_{1}$ is in process at $r_{2}$. This may be called static, in that a predetermined job list is followed, but has the flavor of dynamic scheduling in that the actual order of the tasks (or parts of tasks) is not known at the start. We will not include this case among static policies, and so for us static policies are always nonpreemptive.
2. dynamic policies give schedules that are determined over time:
(a) without preemption: when any task is completed, we can reassess the situation and make any feasible task assignment to the freed-up processor.
(b) with preemption: at any moment, we can reassess the situation and make any feasible change of task assignments.

Dynamic policies may be very complex and hard to specify, involving many "If this, then do that" statements. We can, however, fully spell them out at time zero. What we cannot do is promise to always work on a job at the top of a predetermined priority list.

A final remark. An assumption that is universally made but rarely stated explicitly is that, subject to precedence requirements and to the policy being followed, each task should be processed as early as possible. This is fine for regular measures (see Chap. 1 for definition of regularity), but if, for example, there are costs for completing a job early, we may want to delay a task. Criteria involving inserted idle time have not been considered for stochastic flow shops; their analysis would be very complicated.

## Preemption in the Stochastic Setting

In the context of random task times, the term "preemption" deserves more careful definition. Given a fixed task length, if the work is interrupted part way through, it is clear enough how much is left to be completed later on. But if a random variable is interrupted, what is the length of the residual task time? We give two possible answers to this question.
(a) Task length fixed at start. It may be that the task time is determined at the instant processing begins. Thereafter, of course, it behaves like a deterministic time, and preemption is well understood.
(b) Task length revealed at completion. If the duration remains uncertain until processing ends, then when a task is preempted at time $t$, its residual time is a random variable. Let the complete task time, $X$, have distribution $F(x)=P(X \leq x)$. Then the time remaining after preemption, $Y=X-t$, has distribution:

$$
P(Y \leq y \mid X>t)=P(X \leq y+t \mid X>t)=\frac{F(y+t)-F(t)}{1-F(t)}
$$

We do not propose to make use of this formula, but include it to make precise the underlying assumptions that allow us to say: the total time to process a job, deterministically or in distribution, is the same, whether it is preempted or not.

These results can easily be extended to the case of multiple preemptions of a task, and to a random preemption time, as when a higher-priority job is simultaneously in process on the preceding machine.

### 11.1.3 Types of Objectives and Stochastic Ordering

With random elements introducing uncertainty into schedule times, the best policy is no longer as clear. The makespan, or other measure, is now a random variable. Thus, in comparing two schedules $S_{1}$ and $S_{2}$, we may find that, with some probability $p, S_{1}$ has the shorter makespan, while $S_{2}$ wins out with probability $1-p$. In either case, the difference in makespan may be a little or a lot. It is no longer obvious which schedule is preferred. We are led to consider how to rank random variates, a topic called stochastic ordering or stochastic dominance.

As usual, let $X$ be a random variable with density (or mass) function $f_{X}(x)$, and CDF $F_{X}(x)$; and similarly for $Y$. Three ways in which they might be compared are as follows:

- Expectation Ordering The simplest, most practical and most commonly used basis of comparison:
$X$ is smaller than $Y$ in expectation $\Longleftrightarrow E(X) \leq E(Y)$.


## - Stochastic Ordering

$X$ is stochastically smaller than $Y \Longleftrightarrow P(X>t) \leq P(Y>t) \quad \forall t$.
This of course implies that $F_{X}(t) \geq F_{Y}(t) \forall t$. We denote this $X \leq_{s t} Y$. It means roughly that $X$ has more of its probability mass at low values, so that the CDF builds up faster. It is a very strong result to show that a certain solution is stochastically minimal. It may not be possible: CDF's often intersect each other, in which case no single solution dominates all the rest.

## - Almost Sure Ordering

$X$ is almost surely smaller than $Y \Longleftrightarrow P(X \leq Y)=1$
In such a case, we write $X \leq_{a s} Y$. If we could establish this relationship, say between two makespans (we will not be able to), it would be an incredibly strong result. When we assume it, say between processing times, the situation will not be very different from the deterministic case, and we might expect to get similar results.

Lemma $11.1 \quad X \leq_{s t} Y \Rightarrow E(X) \leq E(Y)$.
Proof: If $X$ and $Y$ are continuous:

$$
E(X)=\int_{0}^{\infty}\left[1-F_{X}(t)\right] d t \leq \int_{0}^{\infty}\left[1-F_{Y}(t)\right] d t=E(Y)
$$

### 11.1.4 Permutation Schedules

Only rather limited results exist for stochastic flow shop scheduling, and most of them find optimal schedules in the class of static list policies; that is, they
seek the best permutation schedule. The question arises: when are permutation schedules dominant over all static schedules? That is, when can we be sure that there exists a permutation schedule that is optimal in the class of static schedules? For two-machine deterministic flow shops, we have seen (in Theorem 1.1) that this is the case for a wide range of objective functions. A similar result (which appears to be widely known but not previously published) holds for the stochastic case.

Theorem 11.5 In a simple flow shop with random processing times, to minimize the expected value of any regular measure over all static policies, there exists an optimal schedule with the same job order on the first two processors.

Proof: When optimizing over static policies, as in the deterministic proof, if we are given an optimal schedule in which the job order on the first two machines is not identical, we find the first two consecutive jobs on $M_{1}$ whose order is reversed on $M_{2}$, and interchange them on $M_{1}$. For every sample path (that is, every set of possible values of the random variables, producing a realization of the entire schedule), the completion times of all tasks at all stations (except for one task on $M_{1}$ ) are made no later (some may move earlier). Since this is true in particular for all tasks on $M_{m}$, i.e., all job completion times, a regular measure cannot be made larger in any realization, hence its expected value does not increase. We can, of course, make as many such interchanges as are needed.

With fixed processing times, as we saw in Chap. 1, this result holds for a larger class of objectives: all measures that are functions of job completion times. But here we cannot hold completion times fixed when we reorder tasks on earlier machines. With random times, we cannot, for example, insert idle time to avoid earliness penalties, because we cannot predict how much will be needed. All we can do is process tasks as early as possible, and this is all we need for regular objectives.

Another similar result carries over from deterministic scheduling:
Theorem 11.6 In a simple flow shop with random processing times, to minimize the expected value of the makespan
(a) over all static policies, or over all nonpreemptive dynamic policies, there exists an optimal schedule with the same job order on the last two processors;
(b) over all preemptive dynamic policies, preemption is never needed on the last machine, where tasks may be processed in the order they become available.

Proof: First, without preemption, the argument is simply this: as long as $M_{m}$ keeps working on whatever task is available, the sequence does not matter: the total time to complete all work will have the same distribution. Thus, whatever sequences are optimal on the first $m-1$ machines, the order on $M_{m-1}$ may as well be followed on $M_{m}$. Similarly, if preemption is allowed,
it gives no advantage on $M_{m}$, as the total remaining processing time is unchanged by interrupting the in-process task.

We may note that the last two theorems extend from expected value minimization to stochastic minimization, but only in the following very limited sense. Since we now have only a partial ordering of schedules, there may not exist one with stochastically minimal makespan (or other objective), and indeed there hardly ever will. All we can say is that, if such a schedule can be found in the class of static policies, then we can construct an equally good schedule that satisfies the theorems.

## $11.2 \boldsymbol{F} 2|(p m t n)| E\left(C_{\text {max }}\right)$

Even for the two-machine case, there are no analytic results without stringent conditions. We first present the one rather well-known case in which a simple list schedule is optimal, namely expected makespan minimization when all task times have exponential distribution. We will then briefly discuss heuristic approaches for general distributions.

### 11.2.1 $F 2\left|(p m t n), \exp P_{i j}\right| E\left(C_{\max }\right)$

The simplest result in stochastic flow shop theory is the extension of the classic two-machine makespan problem to the case where the processing times, $A_{j}\left[B_{j}\right]$, of $J_{j}$ on $M_{1}\left[M_{2}\right]$ are exponentially distributed, and we seek to minimize the expected makespan. Let the task times have means $a_{j}=E\left(A_{j}\right), b_{j}=E\left(B_{j}\right)$, and rates $\alpha_{j}=1 / a_{j}, \beta_{j}=1 / b_{j}$. For convenience, we will let $A_{j}\left[B_{j}\right]$ represent the tasks of $J_{j}$, as well as their (random) processing times.

We might start by asking, in this two-machine makespan setting, if the task times are random, can we simply use $E\left(A_{j}\right)$ and $E\left(B_{j}\right)$ in JR? The answer is "no". We still have $C_{\max }=\max _{j=1, \ldots, n} R_{j}$ (see (2.1)), but $R_{j}$ is now a random variable: $R_{j}=\Sigma_{i=1}^{j} A_{i}+\sum_{i=j}^{n} B_{i}$. Unfortunately, $E\left(C_{\max }\right)=$ $E\left(\max _{j} R_{j}\right) \neq \max _{j} E\left(R_{j}\right)$.

Nevertheless, if the task times are exponentially distributed, a simple solution still exists. It is sometimes called Talwar's Theorem. The proof below is due to Weiss (1982), as presented in Pinedo (2008).

Theorem 11.7 (Talwar 1967; Bagga 1970; Cunningham and Dutta 1973) For $F 2\left|(p m t n), \exp P_{i j}\right| E\left(C_{\max }\right)$, using static or dynamic scheduling, with or without preemption:

$$
S^{\star}=\searrow\left(\alpha_{j}-\beta_{j}\right) .
$$

Proof: First, we will show that $S^{\star}$ is optimal over all static policies, by contradiction. We have by Theorem 11.5 that only permutation schedules need be considered. Consider any schedule $S \neq S^{\star}$. There must be adjacent
jobs, $J_{j}$ and $J_{k}$, in $S$, with $J_{j}$ preceding $J_{k}$ but $\alpha_{j}-\beta_{j}<\alpha_{k}-\beta_{k}$. We will show that interchanging these jobs reduces the expected makespan.

As shown in Fig. 11.1, the jobs preceding $J_{j}$ in $S$ occupy $M_{1}\left[M_{2}\right]$ up to time $C_{1 i}\left[C_{2 i}\right]$, where $J_{i}$ is the last job before $J_{j}$. Let $D_{i}=C_{2 i}-C_{1 i}$ be the excess occupancy or stagger of $M_{2}$ over $M_{1}$. Similarly, following $J_{k}$, the stagger is $D_{k}=C_{2 k}-C_{1 k}$, and let the next job be $J_{l}$.


Fig. 11.1
A realization of schedule $S$

Now interchange $J_{j}$ and $J_{k}$, leaving the rest of the schedule unchanged. Call the new schedule $S^{\prime}$, and distinguish the new completion times by primes. The occupancy of $M_{1}$ preceding $J_{l}$ is unchanged, since $C_{1 j}^{\prime}=C_{1 k}=C_{1 i}+$ $A_{j}+A_{k}$. That is, $C_{1 j}^{\prime}$ and $C_{1 k}$ are stochastically identical: they have the same distribution. What about the occupancy of $M_{2}$ ? How does $C_{2 k}$ compare to $C_{2 j}^{\prime}$ ? Since things are unchanged on $M_{1}$, this is equivalent to asking how $D_{k}$ compares with $D_{j}^{\prime}$. We seek to show that the interchange improves the schedule, or specifically that the stagger after the interchange is stochastically smaller than it was before: $D_{j}^{\prime}<_{s t} D_{k}$. There are two cases.

- If $D_{i} \geq A_{j}+A_{k}$, or $C_{2 i} \geq C_{1 k}$ (see Fig. 11.1), then, as on $M_{1}$, interchange makes no difference: $D_{j}^{\prime}=D_{k}=D_{i}+B_{j}+B_{k}-A_{j}-A_{k}$.
- If $D_{i} \leq A_{j}+A_{k}$, then before the interchange $D_{k}$ has the distribution

$$
P\left(D_{k}>t \mid D_{i} \leq A_{j}+A_{k}\right)=\frac{\beta_{j}}{\alpha_{k}+\beta_{j}} e^{-\beta_{k} t}+\frac{\alpha_{k}}{\alpha_{k}+\beta_{j}}\left(\frac{\beta_{k}}{\beta_{k}-\beta_{j}} e^{-\beta_{j} t}+\frac{\beta_{j}}{\beta_{j}-\beta_{k}} e^{-\beta_{k} t}\right)
$$

To see this, first note that, with $D_{i} \leq A_{j}+A_{k}, A_{k}$ and $B_{j}$ start at the same time (actually, if $D_{i}>A_{j}, A_{k}$ starts earlier than $B_{j}$, but when $D_{i}$ ends, the memoryless property allows us to assume that $A_{k}$ starts afresh). The first term on the right results when, with probability $\beta_{j} /\left(\alpha_{k}+\beta_{j}\right), B_{j}$ ends first (see Theorem 11.3), in which case $D_{k}=B_{k}$. In the second case, when $B_{j}>A_{k}$ as illustrated in Fig. 11.1, $D_{k}$ equals the residual (which, memoryless, equals total) $B_{j}$ plus $B_{k}$, and this sum has the distribution given in Theorem 11.4.

After the job interchange, $P\left(D_{j}^{\prime}>t \mid D_{i} \leq A_{j}+A_{k}\right)$ has the same formula, with subscripts $j$ and $k$ interchanged. We can now show that this interchange has made the stagger, $D_{k}$, stochastically smaller. Thus, after much algebra, we find that, always conditional on $D_{i} \leq A_{j}+A_{k}$ :

$$
P\left(D_{k}>t\right)-P\left(D_{j}^{\prime}>t\right)=\frac{\beta_{j} \beta_{k}}{\left(\alpha_{j}+\beta_{k}\right)\left(\alpha_{k}+\beta_{j}\right)} \frac{e^{-\beta_{k} t}-e^{-\beta_{j} t}}{\beta_{j}-\beta_{k}}\left(\alpha_{k}-\beta_{k}-\left[\alpha_{j}-\beta_{j}\right]\right) \geq 0
$$

Using a sample path argument, it follows that, for each realization of the schedule before and after $J_{j}$ and $J_{k}$, if $C_{1 k}$ is unchanged and $C_{2 k}$ is reduced, the makespan can only be smaller. Thus, with $D_{j}^{\prime}$ stochastically smaller than $D_{k}$, the expected makespan is reduced. This ends the proof for static policies.

As for dynamic nonpreemptive policies, we first note that on $M_{2}$ the order of jobs is immaterial. Given the order on $M_{1}$, the work coming to $M_{2}$ must simply be done as soon as possible; rearrangement will not affect makespan. On $M_{1}$, suppose a task has just been completed. What new information could we now observe, to cause us to change our initial ordering? The only relevant information is the amount of work left to be done on $M_{2}$, the stagger, which is now a random variable yet to be determined. But for any realization, the decision is the same, as shown above in the static case, and so it must remain the best choice.

Preemption is even more easily disposed of, using the memoryless property. On $M_{2}$, the residual time of the in-process task is always the same; there can be no reason to change an earlier decision. On $M_{1}$, we might wish to preempt a task because of its own duration ("its taking too long"), but memorylessness makes this pointless; or because of the evolving situation on $M_{2}$, but the argument in the last paragraph applies here again, leaving the static list schedule intact.

It was later shown by Ku and Niu (1986) that this simple list schedule stochastically minimizes the makespan. It looks very different from Johnson's Rule; for one thing, a job's position depends on both its task times, not just on the smaller one. However, it turns out to have a close relationship. Recall, from Theorem 2.14, the basic preference ordering result in the deterministic case, which led to JR:

$$
j \leadsto k \Longleftrightarrow \min \left(a_{j}, b_{k}\right) \leq \min \left(a_{k}, b_{j}\right)
$$

It turns out that our new preference order $\left(j \leadsto k \Longleftrightarrow \alpha_{j}-\beta_{j} \geq \alpha_{k}-\beta_{k}\right)$ is nothing but

$$
\begin{equation*}
j \leadsto k \Longleftrightarrow E\left(\min \left(A_{j}, B_{k}\right)\right) \leq E\left(\min \left(A_{k}, B_{j}\right)\right) \tag{11.1}
\end{equation*}
$$

as is quickly seen using Theorem 11.2.

### 11.2.2 Asymptotic Optimality of Johnson's Rule

While the optimal schedule cannot be specified when task distributions are arbitrary, Portougal and Trietsch (2006) have shown that, under reasonable conditions, the following heuristic is asymptotically optimal, as the number of jobs increases.

Johnson's Heuristic Replace task times by their mean values and solve the deterministic equivalent problem using JR. One additional rule is added: if two tasks have the same mean time, then to break ties the one with smaller
variance is considered shorter. Let the resulting schedule be $S_{J}=J R(\underline{a}, \underline{b})$, with expected makespan $C_{J}$.

If $S^{\star}$ is the optimal schedule, with expected makespan $C^{\star}$, the precise result is as follows.

Theorem 11.8 If, for all $n=1,2, \ldots$ and for finite positive $\gamma$ and $\delta$ :
(a) $(1 / n) \sum_{j=1}^{n}\left(E\left(A_{j}\right)+E\left(B_{j}\right)\right) \geq \delta$,
(b) $(1 / n) \sum_{j=1}^{n=1}\left(V\left(A_{j}\right)+V\left(B_{j}\right)\right) \leq \gamma^{2}$,
then, for all $\epsilon>0$, there exists a number $n_{\epsilon}$ such that, for every instance of $F 2 \mid($ pmtn $) \mid E\left(C_{\max }\right)$ with $n>n_{\epsilon}$ :

$$
\left(C_{J}-C^{\star}\right) / C^{\star} \leq_{a s} \epsilon
$$

The authors show that a value $n_{\epsilon}=(4 \gamma / \delta \epsilon)^{2}$ satisfies the theorem, but find experimentally that about half that value is sufficient.

### 11.2.3 Heuristics

No optimal algorithms exist for distributions other than exponential. Baker and Trietsch (2009) propose the following heuristics for $F 2|(p m t n)| E\left(C_{\max }\right)$.

- Talwar's Heuristic Use $\searrow\left(\alpha_{j}-\beta_{j}\right)$.

This, of course, treats each task time as though it were exponential.

- Johnson's Heuristic Use $J R(\underline{a}, \underline{b})$.

This is the same stochastic extension of JR specified in the last section.

- API (Adjacent Pair Interchange) Heuristic Starting with any sequence, use an API neighborhood search to iteratively improve it. Even to evaluate a schedule can be prohibitive, so we use a heuristic to assess the difference between two neighboring solutions. This is based on Johnson's Relation, for which the stochastic equivalent given in (11.1) is valid, as we have just seen, when processing times are exponential. Otherwise, it only holds when the two jobs are being sequenced in isolation; it is not strictly true for two adjacent jobs in a longer schedule. Nevertheless, the API heuristic uses it as the criterion for comparing two neighboring schedules in searching for an improved solution. As usual, adjacent jobs are compared, and interchanged when indicated, repeatedly from left to right until no further improvement is possible.

It should be pointed out that even this rather simple test criterion can be intractable for some distributions: there may be no analytic formula for $E(\min (A, B))$. The authors suggest approximating the task times with the normal distribution. In this case, it can be shown that

Lemma 11.2 If $X$ and $Y$ are independent normal random variables with means $\mu_{x}$ and $\mu_{y}$ with $\mu_{x} \geq \mu_{y}$, and variances $\sigma_{x}^{2}$ and $\sigma_{y}^{2}$ with $\sigma_{t}^{2}=\sigma_{x}^{2}+\sigma_{y}^{2}$, and letting $\rho=\left(\mu_{x}-\mu_{y}\right) / \sigma_{t}$, we have:

$$
E(\min (X, Y))=\mu_{x}-\sigma_{t}[\varphi(\rho)+\rho \Phi(\rho)]
$$

where $\varphi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density and distribution functions.

We have required that $\mu_{x} \geq \mu_{y}$ simply to make $\rho \geq 0$; this is not necessary.
These heuristics are easy to use; no simulation or implicit search is needed. Moreover, Baker and Trietsch (2011) report in a later paper that computational tests show good performance of all three. Briefly, the methodology employed was as follows. First, a family of processing time distributions was selected. After some initial experimentation with exponential and uniform distributions, the authors settled on the log normal for all task time distributions. To specify a problem instance now requires the choice of the number of jobs, and the mean and variance of two task times of each job, although we may note that the first two heuristics only utilize mean values. Means and variances were drawn randomly from given intervals; for example, the means of ten jobs might be 20 samples uniformly drawn from [10, 20].

Given the job parameters, the heuristics quickly produced three candidate job sequences. For each, 100,000 simulations were run using log normal distributions, to estimate the expected value of the makespan. Additionally, a proprietary genetic algorithm, the Evolutionary solver, was used to find a near-optimal benchmark solution.

Most of the testing was for 10 -job problems (the relative merits of the heuristics seemed insensitive to problem size). A variety of ranges were tested for means and variances. The authors found that:

- None of the heuristics was dominated; each performed best under certain conditions.
- While it did not dominate the others, The API Heuristic was a clear favorite, coming in first or a close behind under almost all conditions. It did worst when task time variance was high.
- Johnson's Heuristic excelled when the probability distributions had little or no overlap, as when variances are small and/or means are widely separated. Under these conditions, the sizes of task times retain their rank ordering under (almost) all realizations, so it is not surprising that the problem behaves like its deterministic counterpart.
- On the other hand, Talwar's Heuristic tended to do best when task time distributions overlap a great deal. Of course, it is unequivocally best when the distributions are exponential.
- All the heuristics can be used to dispatch jobs; that is, to select the next job to process from a set of jobs that is constantly changing over time. Both Talwar's and Johnson's Rules are of course list schedules, so the job order remains the same when a job is added or removed. The API Heuristic is not a list schedule (we cannot compute a number for each job that determines it place in the order), but a new arrival can be inserted into the earliest stable position without rearranging the others.


### 11.3 Fm $\mid$ perm, $P_{i j}=P_{j} \mid E\left(C_{\max }\right)$ or $E\left(\Sigma C_{j}\right)$

Henceforth, we will have to make rather restrictive assumptions in order to get any results. Throughout this chapter, unless otherwise indicated, we limit consideration to nonpreemptive static policies; that is, to permutation schedules. In this section, we assume that the task times of a job are the same on every machine. That is, the processing times of $J_{j}$ on the $m$ machines are independent random variables with the same distribution $F_{j}$ and mean value $p_{j}$. Our shorthand for this will be $P_{i j}=P_{j}$. This notation should not be misinterpreted to mean that all $P_{i j}$ for a given $j$ are the same random variable. Think of $P_{j}$ as symbolizing any one of the $m$ independent, identically distributed task times of $J_{j}$.

### 11.3.1 $\mathrm{Fm} \mid$ perm, $P_{i j}=P_{j} \mid E\left(C_{\max }\right)$

We start by giving the rather trivial deterministic result.
Lemma 11.3 For $F m \mid($ perm $),(p m t n), p_{i j}=p_{j} \mid C_{\max }$, any permutation schedule is optimal, with

$$
C_{\max }=\sum_{j=1}^{n} p_{j}+(m-1) p_{\max }, \quad \text { where } p_{\max }=\max _{j=1, \ldots, n} p_{j}
$$

Proof: First, limit consideration to permutation schedules. Note that, in any partial schedule, the staggers (i.e., the differences between the occupancies of successive machines) are all the same, and equal to the largest $p_{j}$ so far scheduled. Now, let $S=(1,2, \ldots, n)$ be an arbitrary schedule, and suppose $p_{k}=p_{\max }$. The above observation implies that the stagger remains the same for all jobs after $J_{k}$, which in turn implies that there is no further idle time on any machine; in particular, on $M_{m}$. Of course, the jobs preceding $J_{k}$ are contiguous on $M_{1}$ (all jobs on $M_{1}$ are contiguous). Thus, the makespan is determined by the total time on $M_{1}$ preceding $J_{k}\left(\sum_{j=1}^{k-1} p_{j}\right)$, plus the time required to process $J_{k}\left(m p_{k}\right)$, plus the total time on $M_{m}$ following $J_{k}\left(\sum_{j=k+1}^{n} p_{j}\right)$. This gives the desired result.

But this is clearly a lower bound on the makespan, so it must be optimal over all schedules, with or without preemption.

For the stochastic analogue, we have:
Lemma 11.4 For Fm|perm, $P_{i j}=P_{j} \mid E\left(C_{\max }\right)$, any schedule has

$$
E\left(C_{\max }\right) \geq \sum_{j=1}^{n} p_{j}+(m-1) p_{\max }
$$

where $p_{j}=E\left(P_{j}\right), \quad p_{\max }=\max _{j=1, \ldots, n} p_{j}$.
Proof: By the logic of the deterministic counterpart, and again letting $J_{k}$ be the job with largest processing time in an arbitrary $S=\{1,2, \ldots, n\}, C_{\max }$ is the sum of three independent time intervals:

1. the time on $M_{1}$ until $J_{k}$ begins, whose mean is $\sum_{j=1}^{k-1} p_{j}$;
11.3 Fm $\mid$ perm, $P_{i j}=P_{j} \mid E\left(C_{\max }\right)$ or $E\left(\Sigma C_{j}\right)$
2. the time to process $J_{k}$ through all stages, with mean at least $m p_{\max }$ (the $m$ tasks of $J_{k}$ must be done sequentially, but some may be delayed due to random variation of other task times); and
3. the time on $M_{m}$ after $J_{k}$ is finished, with mean at least $\sum_{j=k+1}^{n} p_{j}$.

We now extend the concept of pyramidal schedules to incorporate randomness.

Definition 11.1 If $J_{j}$ has task times $P_{j}$ with the same distribution $F_{j}$ with mean $p_{j}$ on all $m$ machines, for $j=1,2, \ldots, n$, then an arbitrary permutation schedule $S=(1,2, \ldots, n)$ is pyramidal if there is a job $J_{k}$ such that $p_{1} \leq$ $p_{2} \leq \ldots \leq p_{k}$ and $p_{k} \geq p_{k+1} \geq \ldots \geq p_{n}$. Of course, $p_{k}=p_{\max }$.

Pyramidal schedules are also called SEPT-LEPT (see Pinedo 2008). $S E P T$ and LEPT schedules are special cases, when $k=n$ and $k=1$, respectively.

Theorem 11.9 (Pinedo 1982) For $F m \mid$ perm, $P_{i j}=P_{j} \mid E\left(C_{\max }\right)$, if

$$
P_{1} \leq_{a s} P_{2} \leq_{a s} \ldots \leq_{a s} P_{n}
$$

then any pyramidal schedule is optimal, with

$$
E\left(C_{\max }\right)=\sum_{j=1}^{n} p_{j}+(m-1) p_{\max }
$$

Proof: By hypothesis, $p_{\max }=p_{n}$. In any pyramidal schedule, let $\mathcal{B}[\mathcal{A}]$ be the set of jobs scheduled before [after] $J_{n}$. The jobs in $\mathcal{B} \cup\left\{J_{n}\right\}$, in SEPT order and with nonoverlapping distributions, are never delayed at any machine, so their task times are independent and the expected time until the end of $J_{n}$ is $\Sigma_{j \in \mathcal{B}} p_{j}+m p_{n}$. After $J_{n}$, decreasing task times and nonoverlapping distributions imply that no machine has any idle time, so the expected time remaining on $M_{m}$ is $\Sigma_{j \in \mathcal{A}} p_{j}$. Optimality follows by Lemma 11.4.

### 11.3.2 $\operatorname{Fm} \mid$ perm, $P_{i j}=P_{j} \mid E\left(\Sigma C_{j}\right)$

Based on the above results, we can extend the analysis to cover the objective of minimizing total expected completion times.

Theorem 11.10 (Pinedo 2008) For Fm $\mid$ perm, $P_{i j}=P_{j} \mid E\left(\Sigma C_{j}\right)$, if

$$
P_{1} \leq_{a s} P_{2} \leq_{a s} \ldots \leq_{a s} P_{n}
$$

then SEPT is optimal, with

$$
E\left(\sum C_{j}\right)=\sum_{j=1}^{n}(m+n-j) p_{j} .
$$

Proof: By Theorem 11.9, to minimize the expected completion time in any position $k$, the $k$ smallest jobs should be scheduled first, in any pyramidal sequence. Thus, $\mathrm{SEPT}=(1,2, \ldots, n)$ simultaneously minimizes $E\left(C_{k}\right)$ for all $k$, so is optimal for $E\left(\Sigma C_{j}\right)$. The formula for $E\left(\Sigma C_{j}\right)$ follows easily.

### 11.4 F2|perm, no-wait or block, $P_{i j} \mid E\left(C_{\max }\right)$

As in the deterministic case, it remains true with random task times that with two machines, nondelay scheduling is equivalent to scheduling with blocking. As usual, for $J_{j}$, we let $A_{j}\left[B_{j}\right]$ denote both its tasks and their processing times on $M_{1}\left[M_{2}\right]$, with distribution $F_{A j}(\cdot)\left[F_{B j}(\cdot)\right]$ and mean value $a_{j}\left[b_{j}\right]$. Also, recall that permutation schedules must be static and nonpreemptive.

### 11.4.1 A Traveling Salesman Formulation

As we saw repeatedly in the chapter on no-wait flow shops, the no-wait feature locks a job's operations together so that, if $J_{j}$ follows $J_{i}$, a fixed amount is added to the makespan regardless of the rest of the schedule. As noted in Pinedo (1982), this permits a TSP formulation, with $n+1$ cities $J_{0}, J_{1}, \ldots, J_{n}$, where $J_{0}$ is the dummy city where the tour starts and ends, having $A_{0}=B_{0}=0$.

Suppose $J_{j}$ immediately follows $J_{i}$ in any schedule. The expected time added, from the start of $B_{i}$ to the start of $B_{j}$, is

$$
d_{i j}=E\left(\max \left\{A_{j}, B_{i}\right\}\right)
$$

Noting that, if $Z=\max \{X, Y\}$, then $F_{Z}(t)=F_{X}(t) F_{Y}(t)$, we have:

$$
d_{i j}=\int_{0}^{\infty}\left[1-F_{A j}(t) F_{B i}(t)\right] d t
$$

This is a deterministic TSP, but it does not have the special structure that lends itself to polynomial solution.

### 11.4.2 F2|perm, no-wait or block, $P_{i j}=P_{j} \mid E\left(C_{\max }\right)$

We now again make the assumption (denoted $P_{i j}=P_{j}$ ) that the task times of a job on all machines are independent variates with the same distribution $F_{j}$.
Theorem 11.11 (Pinedo 1982)
For $F 2 \mid$ perm, no-wait or block, $P_{i j}=P_{j} \mid E\left(C_{\max }\right)$, if

$$
P_{1} \leq_{s t} P_{2} \leq_{s t} \ldots \leq_{s t} P_{n}
$$

then

$$
S^{\star}=(1,3,5, \ldots, n, \ldots, 6,4,2), \text { or its reverse. }
$$

We omit the proof; it is found in Pinedo $(1982,2008)$.

### 11.5 Conclusions

While some results, both practical and theoretical, exist for two machines, almost nothing is known about the stochastic problem with $m>2$. We need
at least some testing of simple heuristics, so that satisfactory schedules can be provided for practical use.

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## Appendix A THE COMPLEXITY OF PROBLEMS

## A. 1 Preliminaries

Many scheduling problems are combinatorial in nature: problems where we seek the optimum from a very large but finite number of solutions. Sometimes such problems can be solved quickly and efficiently, but often the best solution procedures available are slow and tedious. It therefore becomes important to assess how well a proposed procedure will perform.

The theory of computational complexity addresses this issue. The seminal papers of complexity theory date from the early 70's (e.g., Cook, 1971 and Karp, 1972). Today, it is a wide field encompassing many sub-fields. For a formal treatment, the interested reader may wish to consult Papadimitriou (1994). As we shall see, the theory partitions all realistic problems into two groups: the "easy" and the "hard" to solve, depending on how complex (hence how fast or slow) the computational procedure for that problem is. The theory defines still other classes, but all except the most artificial mathematical constructs fall into these two.

It should be noted that "easy" or "hard" does not simply mean quickly or slowly solved. Sometimes, for small problem instances, "hard" problems may be more quickly solved than "easy" ones. As we shall see, the difficulty of a problem is measured not by the absolute time needed to solve it, but by the rate at which the time grows as the problem size increases.

To this point, we have not used the accepted terminology; we introduce it now. A problem is a well-defined question to which an unambiguous answer exists. Solving the problem means answering the question. The problem is stated in terms of several parameters, numerical quantities which are left unspecified but are understood to be predetermined. They make up the data of the problem. An instance of a problem gives specified values to each parameter. A combinatorial optimization problem, whether maximization or minimization, has for each instance a finite number of candidates from which the answer, or optimal solution, is selected. The choice is based on a realvalued objective function which assigns a value to each candidate solution. A
decision problem or recognition problem has only two possible answers, "yes" or "no".

An example of an optimization problem is a linear program, which asks "what is the greatest value of $\mathbf{c x}$ subject to $\mathbf{A x} \leq \mathbf{b}$ ?", where bold characters denote $n$-dimensional vectors (lower case) or $n \times n$ matrices (upper case). To make this a combinatorial optimization problem, we might make the variable x bounded and integer-valued so that the number of candidate solutions is finite. A decision problem is "does there exist a solution to the linear program with $\mathbf{c x} \geq k$ ?"

To develop complexity theory, it is convenient to state all problems as decision problems. An optimization (say, maximization) problem can always be replaced by a sequence of problems of determining the existence of solutions with values exceeding $k_{1}, k_{2}, \ldots$. An algorithm is a step-by-step procedure which provides a solution to a given problem; that is, to all instances of the problem. We are interested in how fast an algorithm is. We now introduce a measure of algorithmic speed: the time complexity function.

## A. 2 Polynomial versus Exponential Algorithms

Note that we always think of solving problems using a computer. Thus, an algorithm is a piece of computer code. Similarly, the size of a problem instance is technically the number of characters needed to specify the data, or the length of the input needed by the program. For a decision problem, an algorithm receives as input any string of characters, and produces as output either "yes" or "no" or "this string is not a problem instance." An algorithm solves the instance or string in time $k$ if it requires $k$ basic operations (e.g., add, subtract, delete, compare, etc.) to reach one of the three conclusions and stop.

It is customary to use as a surrogate for instance size, any number that is roughly proportional to the true value. We shall use the positive integer $n$ to represent the size of a problem instance. In scheduling, this usually represents the number of jobs to be scheduled. In summary, for a decision problem $\Pi$ :

Definition A. 1 The Time Complexity Function (TCF) of algorithm $A$ is: $T_{A}(n)=$ maximal time for $A$ to solve any string of length $n$.
In what follows, the big oh notation introduced by Hardy and Wright (1979) will be used when expressing the time complexity function. We say that, for two real-valued functions $f$ and $g, f(n)$ is $O(g(n)$ ), or $f(n)$ is of the same order as $g(n)$ if $|f(n)| \leq k \cdot|g(n)|$ for all $n \geq 0$ and some $k>0$.

An efficient, polynomially bounded, polynomial time, or simply polynomial algorithm is one which solves a problem instance in time bounded by a power of the instance size. Formally:

Definition A. 2 An algorithm $A$ is polynomial time if there exists a polynomial $p$ such that

$$
T_{A}(n) \leq p(n), \quad \forall n \in Z^{+} \equiv\{1,2, \ldots\} .
$$

More specifically, an algorithm is polynomial of degree $c$, or has complexity $O\left(n^{c}\right)$, or runs in $O\left(n^{c}\right)$ time if, for some $k>0$, the algorithm never takes longer than $k n^{c}$ (the TCF) to solve an instance of size $n$.

Definition A. 3 The collection $\mathcal{P}$ comprises all problems for which a polynomial time algorithm exists.

Problems which belong to $\mathcal{P}$ are the ones we referred to earlier as "easy". All other algorithms are called exponential time or just exponential, and problems for which nothing quicker exists are "hard". Although not all algorithms in this class have TCF's that are technically exponential functions, we may think of a typical one as running in $O\left(c^{p(n)}\right)$ for some polynomial $p(n)$. Other examples of exponential rates of growth are $n^{n}$ and $n!$.

We can now see how, as suggested earlier, the terms "hard" and "easy" are somewhat misleading, even though exponential TCFs clearly lead to far more rapid growth in solution times. Suppose an "easy" problem has an algorithm with running time bounded by, say $k n^{5}$. Such a TCF may not be exponential, but it may well be considered pretty rapidly growing. Furthermore, some algorithms take a long time to solve even small problems (large $k$ ), and hence are unsatisfactory in practice even if the time grows slowly. On the other hand, an algorithm for which the TCF is exponential is not always useless in practice. The concept of the TCF is a worst case estimate, so complexity is only an upper bound on the amount of time required by an algorithm. This is a conservative measure and usually useful, but it is too pessimistic for some popular algorithms. The simplex algorithm for linear programming, for example, has a TCF that is $O\left(2^{m}\right)$ where $m$ is the number of constraints, but it has been shown (see Nemhauser et al., 1989) that for the average case the complexity is only $O(n m)$ where $n$ is the number of variables. Thus, the algorithm is actually very fast for most problems encountered.

Despite these caveats, exponential algorithms generally have running times that tend to increase at an exponential rate and often seem to "explode" when a certain problem size is exceeded. Polynomial time algorithms usually turn out to be of low degree ( $O\left(n^{3}\right)$ or better), run pretty efficiently, and are considered desirable.

## A. 3 Reducibility

A problem can be placed in $\mathcal{P}$ as soon as a polynomial time algorithm is found for it. Sometimes, rather than finding such an algorithm, we may place it in $\mathcal{P}$ by showing that it is "equivalent" to another problem which is already known to be in $\mathcal{P}$. We explain what we mean by equivalence between problems with the following definitions.

Definition A. 4 A problem $\Pi^{\prime}$ is polynomially reducible, or simply reducible to a problem $\Pi\left(\Pi^{\prime} \propto \Pi\right)$ if, for any instance $I^{\prime}$ of $\Pi^{\prime}$, an instance $I$ of $\Pi$ can be constructed in polynomially bounded time, such that, given the solution $S_{I}$ to $I$, the solution $S_{I^{\prime}}$ to $I^{\prime}$ can be found in polynomial time.

We call the construction of the $I$ that corresponds to $I^{\prime}$ a polynomial transformation of $I^{\prime}$ into $I$. Later, we will briefly mention a more general type of reducibility, in which the polynomial time requirements for constructing $I$ and finding $S_{I^{\prime}}$ are relaxed. Until then, reduction will mean polynomial reduction.

Definition A. 5 Two problems are equivalent if each is reducible (or simply reduces) to the other.

Since reduction, and hence equivalence, are clearly transitive properties, we can define equivalence classes of problems, where all problems in the same equivalence class are reducible (or equivalent) to each other. Consider polynomial problems. Clearly, for two equivalent problems, if one is known to be polynomial, the other must be, too. Also, if two problems are each known to be polynomial, they are equivalent. This is because any problem $\Pi^{\prime} \in \mathcal{P}$ is reducible to any other problem $\Pi \in \mathcal{P}$ in the following trivial sense. For any instance $I^{\prime}$ of $\Pi^{\prime}$, we can pick any instance of $\Pi$, ignore its solution, and find the solution to $I^{\prime}$ directly. We conclude that $\mathcal{P}$ is an equivalence class.

We state a third simple result for polynomial problems as a theorem.
Theorem A. 1 If $\Pi \in \mathcal{P}$, then $\Pi^{\prime} \propto \Pi \Rightarrow \Pi^{\prime} \in \mathcal{P}$.
Proof: Given any instance $I^{\prime}$ of $\Pi^{\prime}$, one can find an instance $I$ of $\Pi$ by applying a polynomial time transformation to $I^{\prime}$. Since $\Pi \in \mathcal{P}$, there is a polynomial time algorithm that solves $I$. Hence, using the transformation followed by the algorithm, $I^{\prime}$ can be solved in polynomial time.

Normally, to "reduce" means to "make simpler". Not so here. Keep in mind that if $\Pi^{\prime}$ reduces to $\Pi\left(\Pi^{\prime} \propto \Pi\right)$ then, unless they are equivalent, $\Pi$ is the more difficult problem. We can say that $\Pi^{\prime}$ is a special case of $\Pi$.

## A. 4 Classification of Hard Problems

In practice, we do not usually use reduction to show a problem is polynomial. We are more likely to start optimistically looking for an efficient algorithm directly, which may be easier than seeking another problem known to be polynomial, for which we can find an appropriate transformation. But suppose we cannot find either an efficient algorithm or a suitable transformation. We begin to suspect that our problem is not "easy" (i.e., is not a member of $\mathcal{P})$. How can we establish that it is in fact "hard"? We start by defining a larger class of problems, which includes $\mathcal{P}$ and also all the difficult problems we may ever encounter. To describe it, consider any combinatorial decision
problem. For a typical instance, there may be a very large number of possible solutions which may have to be searched. Picture a candidate solution as a set of values assigned to the variables $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$. The question may be "for a given vector $\mathbf{c}$ is there a feasible solution $\mathbf{x}$ such that $\mathbf{c x} \leq B$ ?" and the algorithm may search the solutions until it finds one satisfying the inequality (whereupon it stops with the answer "yes") or exhausts all solutions (and stops at "no").

This may well be a big job. But suppose we are told "the answer is 'yes', and here is a solution $\mathbf{x}$ that satisfies the inequality". We feel we must at least verify this, but that is trivial. Intuitively, even for the hardest problems, the amount of work to check that a given candidate solution confirms the answer "yes" should be small, even for very large instances. We will now define our "hard" problems as those which, though hard to solve, are easy to verify, where as usual "easy" means taking a time which grows only polynomially with instance size. To formalize this, let:

$$
\begin{aligned}
V_{A}(n)= & \text { maximal time for } A \text { to verify that a given solution } \\
& \text { establishes the answer "yes" for any instance of length } n .
\end{aligned}
$$

Definition A. 6 An algorithm $\tilde{A}$ is nondeterministic polynomial time if there exists a polynomial $p$ such that for every input of length $n$ with answer "yes", $V_{\tilde{A}}(n) \leq p(n)$.

Definition A. 7 The collection $\mathcal{N} \mathcal{P}$ comprises all problems for which a nondeterministic polynomial algorithm exists.

It may be noted that a problem in $\mathcal{N P}$ is solvable by searching a decision tree of polynomially bounded depth, since verifying a solution is equivalent to tracing one path through the tree. From this, it is easy to see that $\mathcal{P} \subseteq \mathcal{N} \mathcal{P}$. Strangely, complexity theorists have been unable to show that $\mathcal{P} \subset \mathcal{N} \mathcal{P}$; it remains possible that all the problems in $\mathcal{N} \mathcal{P}$ could actually be solved by polynomial algorithms, so that $\mathcal{P}=\mathcal{N} \mathcal{P}$. However, since so many brilliant researchers have worked on so many difficult problems in $\mathcal{N P}$ for so many years without success, this is regarded as being very unlikely. Assuming $\mathcal{P} \neq \mathcal{N} \mathcal{P}$, as we shall hereafter, it can be shown that the problems in $\mathcal{N} \mathcal{P}$ include an infinite number of equivalence classes, which can be ranked in order of increasing difficulty; where an equivalence class $\mathcal{C}$ is more difficult than another class $\mathcal{C}^{\prime}$ if, for every problem $\Pi \in \mathcal{C}$ and every $\Pi^{\prime} \in \mathcal{C}^{\prime}, \Pi^{\prime} \propto \Pi$ but $\Pi \not \propto \Pi^{\prime}$. There also exist problems that cannot be compared: neither $\Pi \propto \Pi^{\prime}$ nor $\Pi^{\prime} \propto \Pi$.

Fortunately, however, all problems that arise naturally have always been found to lie in one of two equivalence classes: the "easy" problems in $\mathcal{P}$, and the "hard" ones, which we now define.

The class of NP-hard problems $(\mathcal{N P \mathcal { H }})$ is a collection of problems with the property that every problem in $\mathcal{N \mathcal { P }}$ can be reduced to the problems in this class. More formally,

Definition A. $8 \mathcal{N} \mathcal{P} \mathcal{H}=\left\{\Pi: \forall \Pi^{\prime} \in \mathcal{N} \mathcal{P}, \Pi^{\prime} \propto \Pi\right\}$
Thus each problem in $\mathcal{N} \mathcal{P H}$ is at least as hard as any problem in $\mathcal{N} \mathcal{P}$. We know that some problems in $\mathcal{N \mathcal { P H }}$ are themselves in $\mathcal{N P}$, though some are not. Those that are include the toughest problems in $\mathcal{N P}$, and form the class of NP-complete problems ( $\mathcal{N P \mathcal { P } ) \text { . That is, }}$

Definition A. $9 \mathcal{N} \mathcal{P C}=\left\{\Pi:(\Pi \in \mathcal{N} \mathcal{P})\right.$ and $\left.\left(\forall \Pi^{\prime} \in \mathcal{N} \mathcal{P}, \Pi^{\prime} \propto \Pi\right)\right\}$
The problems in $\mathcal{N P C}$ form an equivalence class. This is so because all problems in $\mathcal{N} \mathcal{P}$ reduce to them, hence, since they are all in $\mathcal{N} \mathcal{P}$, they reduce to each other. The class $\mathcal{N} \mathcal{P C}$ includes the most difficult problems in $\mathcal{N} \mathcal{P}$. As we mentioned earlier, by a surprising but happy chance, all the problems we ever encounter outside the most abstract mathematical artifacts turn out to belong to either $\mathcal{P}$ or $\mathcal{N} \mathcal{P C}$.

When tackling a new problem $\Pi$, we naturally wonder whether it belongs to $\mathcal{P}$ or $\mathcal{N P C}$ : is it "easy" or "hard"? As we said, to show that the problem belongs to $\mathcal{P}$, we usually try to find a polynomial time algorithm, though we could seek to reduce it to a problem known to be polynomial. If we are unable to show that the problem is in $\mathcal{P}$, the next step generally is to attempt to show that it lies in $\mathcal{N P} \mathcal{C}$; if we can do so, we are justified in not developing an efficient algorithm.

To show that our problem $\Pi$ is hard, we look for a problem, call it $\Pi^{\prime}$ that has already been proven hard, and can be reduced to our problem. That is, for any instance of the hard problem, we can efficiently construct an instance of our problem such that knowing the answer to our problem will immediately tell us the answer to the hard problem. Effectively, the hard problem $\Pi^{\prime}$ is a special case of our problem $\Pi$. Now, if our problem is easy, the hard problem would be easy. But it is not. So our problem must be hard, too.

This logic is summarized in the following theorem, which should be clear enough to require no proof.

Theorem A. $2 \forall \Pi, \Pi^{\prime} \in \mathcal{N} \mathcal{P},\left(\Pi^{\prime} \in \mathcal{N} \mathcal{P C}\right)$ and $\left(\Pi^{\prime} \propto \Pi\right) \Rightarrow \Pi \in \mathcal{N P \mathcal { C }}$
Thus, we need to find a problem $\Pi^{\prime} \in \mathcal{N P \mathcal { P }}$ and show $\Pi^{\prime} \propto \Pi$, thereby demonstrating that $\Pi$ is at least as hard as any problem in $\mathcal{N P C}$. To facilitate this, we need a list of problems known to be in $\mathcal{N P C}$. Several hundred are listed in Garey and Johnson (1979) in a dozen categories such as Graph Theory, Mathematical Programming, Sequencing and Scheduling, Number Theory, etc., and more are being added all the time. Even given an ample selection, a good deal of tenacity and ingenuity are usually needed to pick one with appropriate similarities to ours and to fill in the precise details of the transformation.

In the next section, we describe the basic technique for theorem proving in complexity theory, and conclude with an illustrative example.

## A. 5 Strong NP-Completeness

We now introduce one of the various ways NP-complete problems can be classified into smaller subclasses, the only one we will use in this monograph: the partitioning of the class $\mathcal{N P C}$ into the two sets, ordinary and strongly NPcomplete problems. For a detailed description of these classes see Garey and Johnson (1979). In practical terms, an ordinary NP-complete problem can be solved using implicit enumeration algorithms like dynamic programming. In this case, the time complexity of the algorithm is not polynomial in the length of input data, but it is polynomial in the size of these data. For instance, Partition is an NP-complete problem (to be defined shortly, in Sect. A.7), for which the input data are $k$ positive integers $v_{i}(i=1,2, \ldots, k)$. Let $V$ be the size of this data: $V=\Sigma_{i} v_{i}$. Partition is solvable by dynamic programming in $O(n V)$ time (see Martello and Toth, 1990). Evidently, this complexity is polynomial in $V$. To see that this complexity bound is not polynomial in the length of the data, consider the binary encoding scheme. In this scheme each $v_{i}$ can be represented by a string of length $O\left(\log v_{i}\right)$, and hence $v_{1}, \ldots, v_{n}$ can be described by a string of length $O\left(\Sigma_{i} \log v_{i}\right)$ which is no greater than $O(n \log V)$. We see that the time complexity $O(n V)$ of the dynamic program (DP) is polynomial in the size $V$ of the data, but not polynomial in the length of the input data, $O(n \log V)$. When the complexity of an algorithm is polynomial in the size of the data, but not the length of the input, we refer to it as a pseudo-polynomial algorithm. A NP-complete problem solvable by a pseudo-polynomial algorithm is called ordinary NP-complete. Else, the problem is strongly NP-complete.

## A.5.1 Pseudo-Polynomial Reduction

As we know, to show ordinary NP-completeness of $\Pi$, we start with an ordinary NP-complete $\Pi^{\prime}$ and provide a polynomial reduction to $\Pi$. That is, for any instance $I^{\prime}$ of $\Pi^{\prime}$ we produce an instance $I$ of $\Pi$ in polynomial time, and given the solution $S_{I}$ of $I$, we produce a solution $S_{I^{\prime}}$ of $I^{\prime}$, also in polynomial time. Now, if we could solve $\Pi$ in polynomial time, we would have a sequence of three polynomial steps that would solve $\Pi^{\prime}$. But we know $\Pi^{\prime}$ is not polynomially solvable, and so $\Pi$ cannot be, either.

The same logic applies if we start with a strongly NP-complete $\Pi^{\prime}$. Given a polynomial reduction, $\Pi$ must also be strongly NP-complete: if $\Pi$ were anything less (polynomial or ordinary NP-complete), $\Pi^{\prime}$ would be, too. But now note: if either or both the steps in the reduction were pseudo-polynomial, and if $\Pi$ could be solved polynomially or pseudo-polynomially, we would still have an overall pseudo-polynomial solution to $\Pi^{\prime}$, giving us the contradiction we need. This should provide the motivation for the following analogue of Definition A.4:

Definition A. 10 A problem $\Pi^{\prime}$ is pseudo-polynomially reducible to a problem $\Pi\left(\Pi^{\prime} \bar{\propto} \Pi\right)$ if, for any instance $I^{\prime}$ of $\Pi^{\prime}$, an instance $I$ of $\Pi$ can be
constructed in pseudo-polynomially bounded time, such that, given the solution $S_{I}$ to $I$, the solution $S_{I^{\prime}}$ to $I^{\prime}$ can be found in pseudo-polynomial time.

This definition leads to the following extension of Theorem A.2:
Theorem A. $3 \forall \Pi, \Pi^{\prime} \in \mathcal{N} \mathcal{P}$, if $\Pi^{\prime}$ is strongly $N P$-complete, and $\Pi^{\prime} \bar{\propto} \Pi$, then $\Pi$ is strongly NP-complete.

This is a stronger result than Theorem A.2. However, it is not to our knowledge ever used, partly because Theorem A. 2 seems to be sufficient, partly because pseudo-polynomial transformations are much harder to find than polynomial ones, and finally because Theorem A. 3 does not seem to be widely known.

## A. 6 How to show a Problem is NP-Complete

We now summarize the process of actually proving the NP-completeness, whether ordinary or strong, of a new Problem $\Pi$ of interest. Recall, we are dealing only with decision problems.

## 1. Show that $\Pi \in \mathcal{N} \mathcal{P}$.

That is, given a solution $S_{\Pi}$ of $\Pi$ we must be able to check whether $S_{\Pi}$ provides a "yes" or "no" answer for $\Pi$ in polynomial time. This is a technical requirement. After all, as we said earlier, "all the problems we ever encounter outside the most abstract mathematics turn out to belong to either $\mathcal{P}$ or $\mathcal{N} \mathcal{P C}$ " and hence to $\mathcal{N} \mathcal{P}$. Thus, in practice, this step is commonly assumed without mention.

## 2. Find a problem $\Pi^{\prime} \in \mathcal{N} \mathcal{P C}$ that reduces to $\Pi$.

This, of course, is the crux of the matter. It is not easy to do, requiring technical skills born of insight and experience. If a candidate problem $\Pi^{\prime}$ is to serve our purposes, then by the definition of reduction in Sect. A.3, the following must be true and verifiable:

- For any instance $I^{\prime}$ of $\Pi^{\prime}$, we must be able to construct an instance $I$ of $\Pi$ such that $I$ has the solution $S_{I}=$ yes if and only if $I^{\prime}$ has the solution $S_{I^{\prime}}=$ yes.
- The times required to construct $I$ from $I^{\prime}$, and to construct $S_{I^{\prime}}$ from $S_{I}$, must be polynomial [may be polynomial or pseudo-polynomial] in the size of (i.e., the length of input data required to specify) $I^{\prime}$, when $I^{\prime}$ is ordinary [strongly] NP-complete.


## 3. Determining whether $\Pi$ is ordinary or strongly NP-complete

The precise complexity of $\Pi$ depends largely on the complexity status of the known NP-complete problem $\Pi^{\prime}$.

- If $\Pi^{\prime}$ is strongly NP-complete, then $\Pi$ is strongly NP-complete.
- If $\Pi^{\prime}$ is ordinary NP-complete, then $\Pi$ is at least ordinary NP-complete. If in addition a pseudo-polynomial algorithm exists for $\Pi$, it is confirmed to be ordinary NP-complete.

Finally we summarize, in the decision tree of Fig. A.1, the sequence of logical steps required to show the complexity of a new problem by reduction of a known problem. We have presented the steps as they are usually given, leaving out the complication that in some cases the reduction may be pseudopolynomial.


Fig. A. 1 Establishing the complexity status of a problem $\Pi$

## A. 7 A Sample Proof

Here is a very simple application of the reduction process outlined previously. More ingenious reductions will be found scattered through this monograph.

## A.7.1 Partition $\propto P 2 \| C_{m a x}$

We wish to show that the following problem is NP-complete:

## $\boldsymbol{P 2} \| \boldsymbol{C}_{\text {max }} \leq \boldsymbol{B}$ ?

INSTANCE: Two parallel identical processors, a set $\mathcal{J}=\left\{J_{1}, J_{2}, \ldots, J_{n}\right\}$ of jobs with a processing time $p_{j}$ for each $J_{j}$, and threshold value $B$.
QUESTION: Is there a nonpreemptive assignment of the $n$ jobs to the two processors so that at any time each machine processes at most one job, and the completion time of $J_{j}$ is $C_{j} \leq B$ for every $j=1,2, \ldots, n$ ?

This is the decision version of the problem $P 2 \| C_{\text {max }}$, replacing a minimization problem with a yes-or-no question to answer. It can be solved repeatedly for different values of $B$ in order to find the minimal makespan. To prove it "hard", we will show that the following problem, known to be NP-complete, can be reduced to it:

## PARTITION

INSTANCE: A set of $k$ positive integers $v_{i}: i \in \mathcal{T}=\{1,2, \ldots, k\}$.
QUESTION: Is there a subset $\mathcal{T}^{\prime} \subset \mathcal{T}$ such that

$$
\sum_{i \in \mathcal{T}}, v_{i}=\sum_{i \in \mathcal{T}-\mathcal{T}^{\prime}} v_{i} ?
$$

We must first show that $P 2 \| C_{\max } \in \mathcal{N} \mathcal{P}$. That is, given a schedule $S$ of the $n$ jobs, we must be able to check in polynomial time whether the associated makespan $C_{\max }(S) \leq B$. To perform the check, we need to find the completion time of the last job processed by each of the processors. This requires no more than $n$ additions involving the processing times of the jobs in $J$. Thus, $C_{\max }(S)$ can be computed in $O(n)$ time, and subsequently, whether $C_{\max }(S) \leq B$ or not can be established in $O(1)$ time. Hence, $P 2 \| C_{\max } \in \mathcal{N} \mathcal{P}$.

Next, we must construct an instance $I$ of $P 2 \| C_{\max }$ corresponding to an instance $I^{\prime}$ of Partition. Let $v_{1}, \ldots, v_{k}$ be the integers in $I^{\prime}$. Then $I$ is simply defined by letting $n=k, p_{i}=v_{i}(i=1, \ldots, n)$, and $B=(1 / 2) \Sigma_{i} p_{i}$. The construction of $I$ requires $n+2$ assignments and $n+1$ basic operations to compute $B$, so the total amount of effort is $O(n)$.

To confirm that this is indeed a reduction, we need to show that the answer is "yes" for the instance $I^{\prime}$ of Partition if and only if the answer is "yes" for the instance $I$ of $P 2 \| C_{\max }$. Indeed, given any $I$ ' with answer "yes", let $\mathcal{T}^{\prime}$ be a subset of $\mathcal{T}$ giving $\Sigma_{i \in \mathcal{T}} v_{i}=\Sigma_{i \in \mathcal{T}-\mathcal{T}}, v_{i}$. We can now construct a solution for $I$ by assigning all jobs $J_{j}: j \in \mathcal{T}^{\prime}$ to be processed (in any order) by $M_{1}$, and all jobs $J_{j}: j \in \mathcal{T}-\mathcal{T}^{\prime}$ to be processed (in any order) by $M_{2}$. Let $S$ be the resulting schedule for $P 2 \| C_{\max }$. By definition of $\mathcal{T}^{\prime}$,

$$
\sum_{i \in \mathcal{T}}, p_{i}=\sum_{i \in \mathcal{T}-\mathcal{T}^{\prime}} p_{i}=\frac{1}{2} \sum_{i} p_{i}=B
$$

Clearly, given $\mathcal{T}^{\prime}$, the schedule $S$ is constructed in $O(n)$ time. Similarly, given a schedule $S$ that solves $P 2 \| C_{\text {max }}$, we can construct the partition $\mathcal{T}^{\prime}, \mathcal{T}-\mathcal{T}^{\prime}$ in $O(n)$ time as well.

Since Partition is an ordinary NP-complete problem, to completely determine the status of $P 2 \| C_{\max }$, we will have to develop a pseudo-polynomial algorithm for it. Such an algorithm can in fact be developed (see Cheng and Sin, 1990) which means that $P 2 \| C_{\max }$ belongs to the class of ordinary NP-complete problems.

## A. 8 Clarification of Terminology

The language of complexity theory can be a bit confusing, with several terms being used in different branches of the literature to refer to the same thing. We have introduced the sets $\mathcal{N \mathcal { P H }}$ and $\mathcal{N \mathcal { P C }}$. All NP-complete problems are NP-hard, and in practice the only NP-hard problems we ever encounter are NP-complete. Though the terms are not synonymous, they have come to be used interchangeably. We have chosen to use "NP-complete" throughout this monograph.

We say a problem is "strongly NP-complete", but we could also say it is "NP-complete in the strong sense", or "unary NP-complete" (a term used in computer science which we will not further motivate). An ordinary NPcomplete problem can be simply called NP-complete, without qualification. It is also acceptable to say "NP-complete in the ordinary sense" or "binary NP-complete".

## A. 9 Conclusion

In this appendix we have presented an introduction to the foundations of computational complexity together with some basic techniques used in proving NP-completeness results. Following Cook's seminal paper (Cook, 1971), the first list of reductions for combinatorial problems was compiled in Karp (1972). The example described in this article can be found in Garey and Johnson (1979).

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