

Eletromagnetismo

Capacitores

- Podemos armazenar a energia eletrostática!
- Gastamos energia para montar uma configuração de cargas.
- Isto inclui cargas livres e rearranjo de cargas em meios materiais.
- Esta energia está acoplada à própria distribuição do campo elétrico no espaço.
- Como podemos aumentar a eficiência?
- Como aumentar a capacidade de acumular energia?
- Vamos falar de acumuladores, ou *capacitores*.

Acumulando energia

Distribuyendo cargas

$$E = \frac{1}{2} \int_V V(\vec{r}) \rho(\vec{r}) d\tau$$

$$= \frac{1}{2} \sum V(r_i) q_i$$

Creando campo

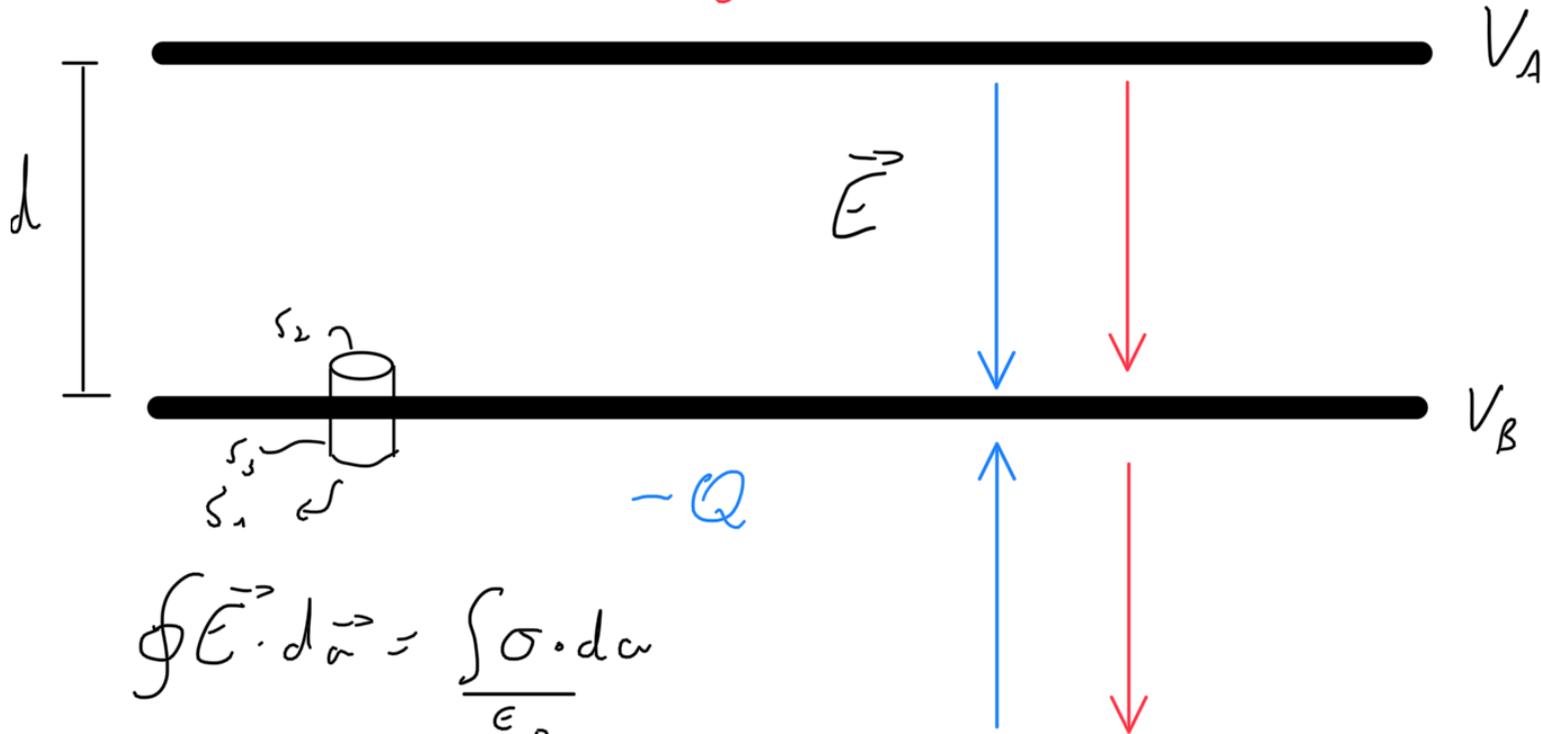
$$E = \frac{1}{2} \int_V \vec{E} \cdot \vec{D} d\tau$$

$$\vec{E} = -\nabla \cdot V$$

$$\nabla \cdot \vec{D} = \rho$$

Energia em um arranjo de cargas

$$Q > 0$$

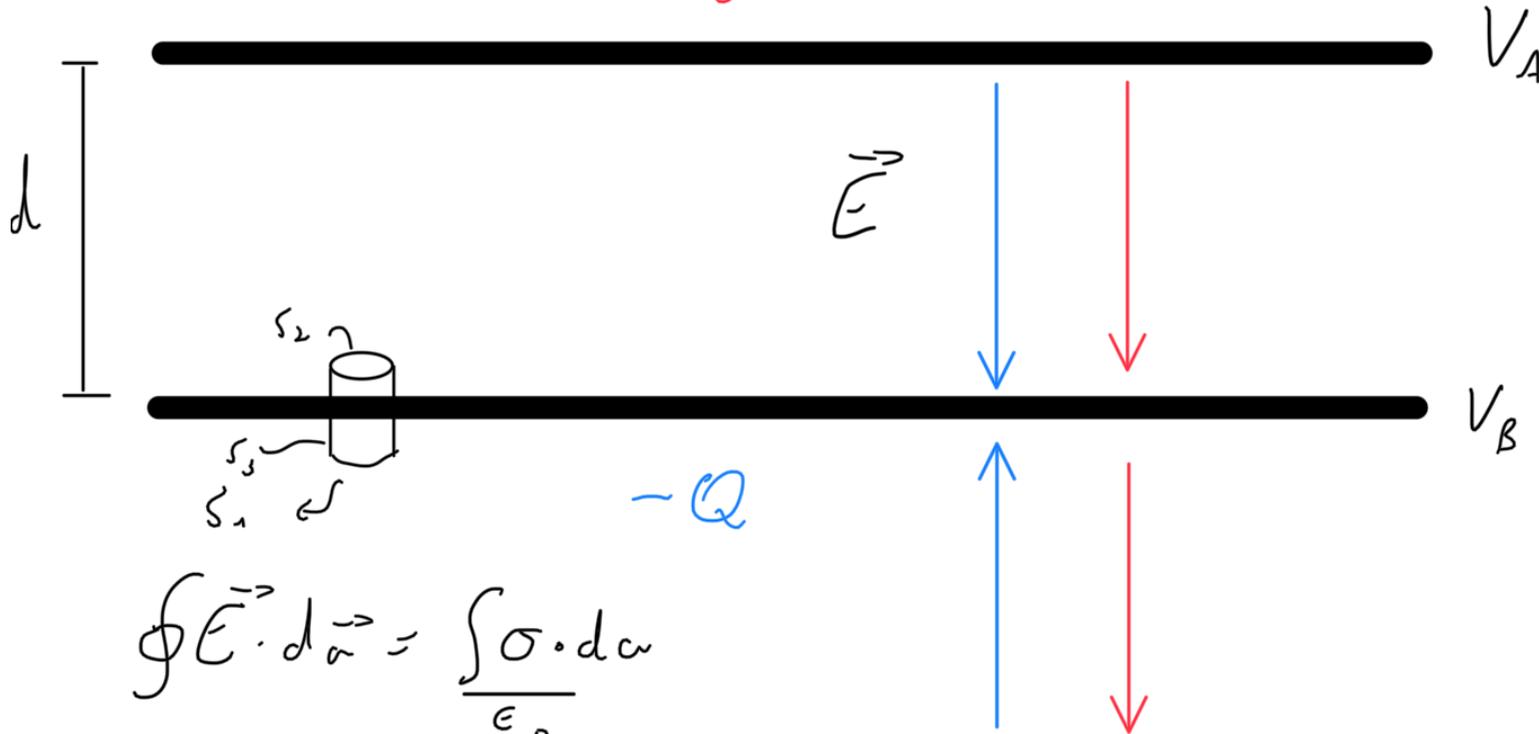


$$\oint \vec{E} \cdot d\vec{a} = \frac{\int \sigma \cdot da}{\epsilon_0}$$

$$\int_{s_1} \vec{E} \cdot d\vec{a} = 0, \quad \int_{s_3} \vec{E} \cdot d\vec{a} = 0$$

Energia em um arranjo de cargas

$$Q > 0$$



$$\oint \vec{E} \cdot d\vec{a} = \frac{\int \sigma \cdot da}{\epsilon_0}$$

$$\oint \vec{E} \cdot d\vec{a} = \int E \cdot da = \int \frac{\sigma}{\epsilon_0} da \Rightarrow E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

$$V_A - V_B = - \int_B^A \vec{E} \cdot d\vec{l} = \frac{Q}{\epsilon_0 A} d$$

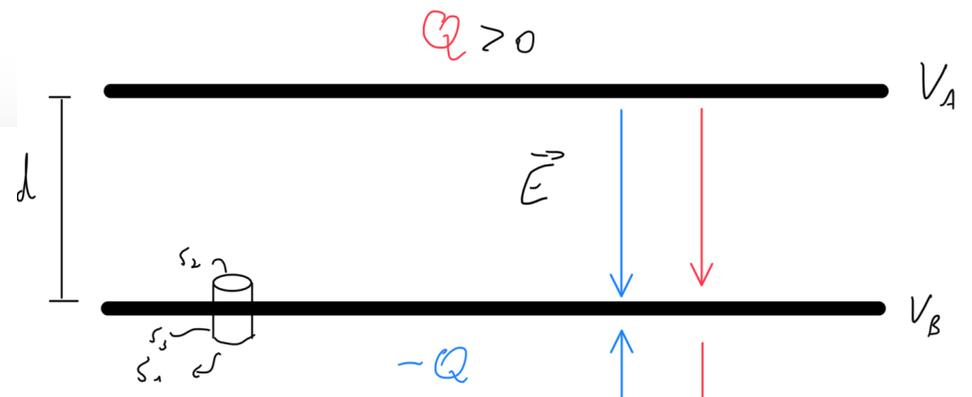
$$V_A - V_B = - \int_B^A \vec{E} \cdot d\vec{l} = \frac{Q}{\epsilon_0 A} d$$

$\Delta V \propto Q \rightarrow$ variações na proporcionalidade
além da geometria

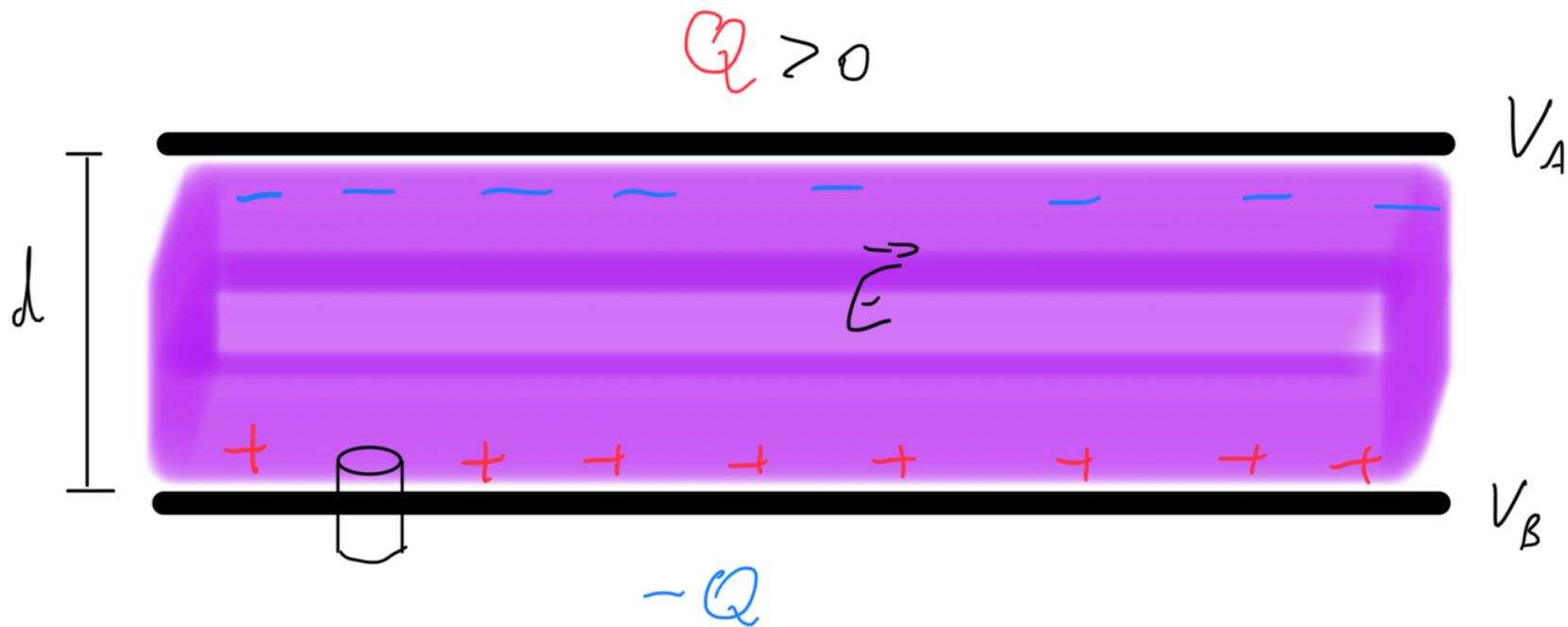
$$\Rightarrow Q = C \Delta V \Rightarrow C = \frac{Q}{\Delta V} \quad [C] = F = \frac{C}{V}$$

Temos um dispositivo de dois contatos (dipolar)
que associa carga acumulada com diferença de potencial!

Duas placas paralelas: $C = \epsilon_0 \frac{A}{d}$



Com um meio dielétrico



Quanto vale \vec{E} ? Temos a carga de polarização $\sigma_p = \vec{P} \cdot \hat{n}$
reduzindo a carga efetiva.

$$\oint \vec{E} \cdot d\vec{a} = \frac{\int \sigma \cdot da}{\epsilon_0} - \int \frac{\sigma_p}{\epsilon_0} da = \int \frac{\sigma}{\epsilon_0} da - \int \frac{\vec{P}}{\epsilon_0} \cdot \hat{n} da$$

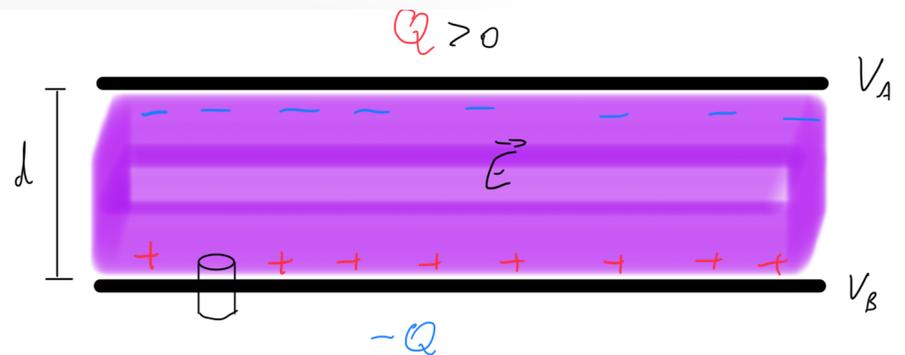
$$\oint \vec{E} \cdot d\vec{a} = \int \frac{\sigma}{\epsilon_0} \cdot d\vec{a} - \int \frac{\sigma_{\text{ext}}}{\epsilon_0} d\vec{a} = \int \frac{\sigma}{\epsilon_0} \cdot d\vec{a} - \int \frac{\vec{P}}{\epsilon_0} \cdot \vec{n} d\vec{a}$$

$$\therefore \int (\epsilon_0 \vec{E} + \vec{P}) d\vec{a} = \int \sigma d\vec{a} \Rightarrow \int \vec{D} d\vec{a} = \int \sigma d\vec{a}$$

$$\Delta V = E \cdot d \quad ; \quad E = \frac{D}{\epsilon} \quad ; \quad D = \frac{Q}{A}$$

$$\Delta V = \frac{d}{\epsilon A} \cdot Q = \frac{Q}{C_\epsilon} \Rightarrow C_\epsilon = \epsilon \frac{A}{d}$$

Para a mesma carga: $\Delta V \propto 1/\epsilon$

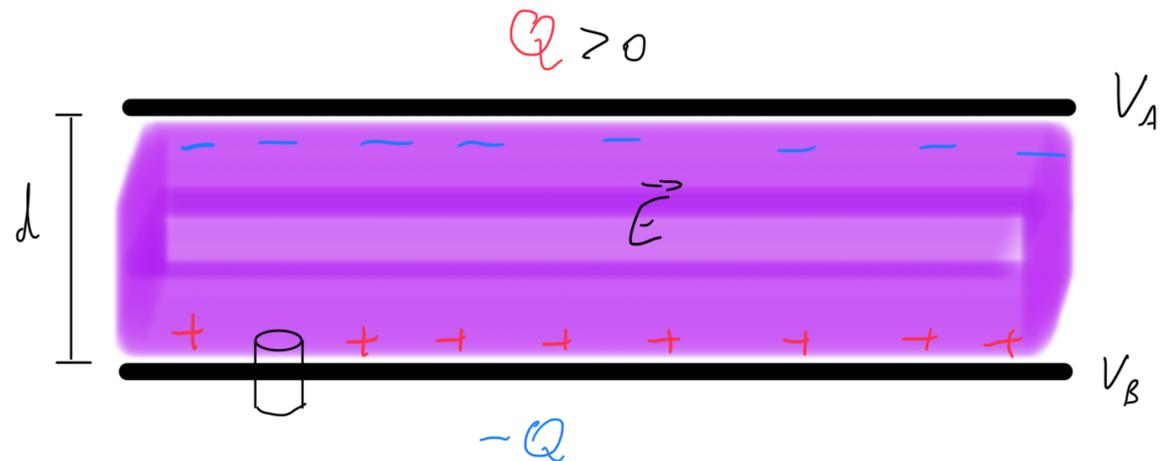


Qual a energia armazenada?

$$\mathcal{E} = \frac{1}{2} \int \vec{E} \cdot \vec{D} \, d\tau = \frac{\epsilon}{2} d \cdot A \cdot E^2 = \frac{\epsilon d \cdot A}{2} \left(\frac{\Delta V}{d} \right)^2$$

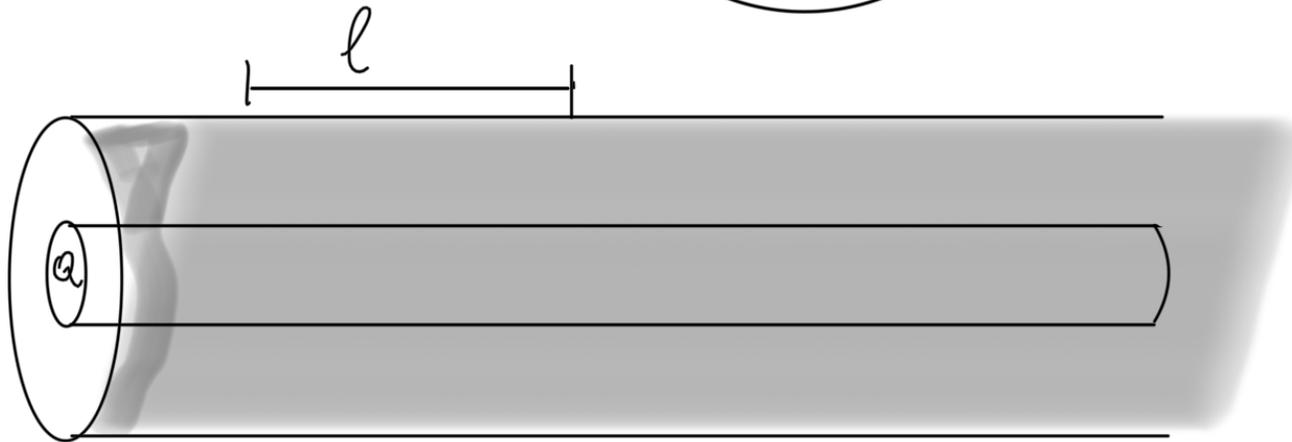
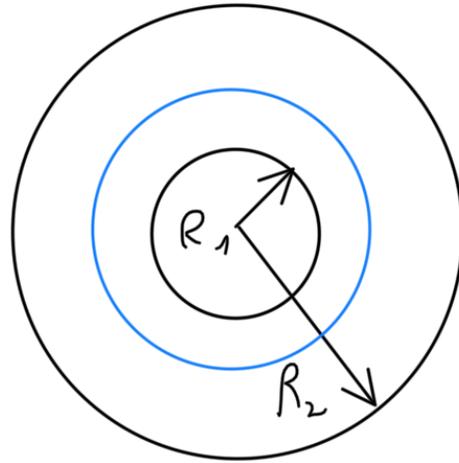
$$\mathcal{E} = \frac{\epsilon A}{d} \frac{\Delta V^2}{2} = \frac{C \Delta V^2}{2} \quad ; \quad C = \frac{Q}{\Delta V}$$
$$= \frac{Q \cdot \Delta V}{2} = \frac{1}{2} \int V \cdot \rho \, d\tau$$

$$\mathcal{E} = \frac{Q^2}{2C}$$



Explorando geometrias

Tubular:



Superfície gaussiana: cilindro de raio r

$$\int \vec{D} \cdot d\vec{a} = \rho \cdot 2\pi r \cdot l = Q$$

$$\int \vec{D} \cdot d\vec{a} = D \cdot 2\pi r \cdot l = Q$$

$$E = \frac{D}{\epsilon} = \frac{Q}{2\pi r l \epsilon} \rightarrow \Delta V = - \int_{R_2}^{R_1} \vec{E} \cdot d\vec{r}$$

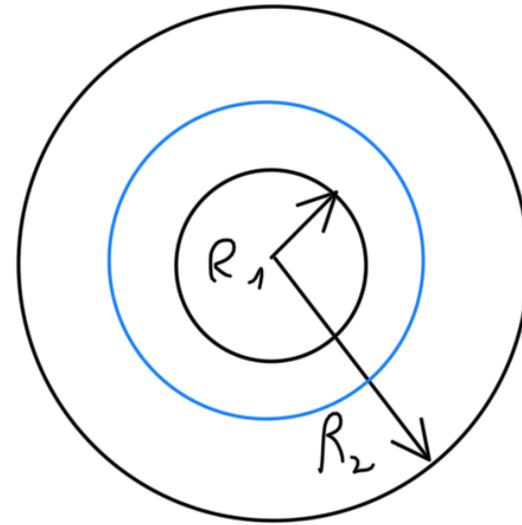
$$\Delta V = - \frac{Q}{2\pi l \epsilon} \int_{R_2}^{R_1} \frac{1}{r} dr = - \frac{Q}{2\pi l \epsilon} \left[\ln r \right]_{R_2}^{R_1}$$

$$= \frac{Q}{2\pi l \epsilon} \left[\ln R_2 - \ln R_1 \right]$$

$$\Delta V = V_{R_1} - V_{R_2} = \frac{Q}{2\pi l \epsilon} \ln \left(\frac{R_2}{R_1} \right) \Rightarrow \frac{C}{l} = \frac{2\pi \epsilon}{\ln \left(\frac{R_2}{R_1} \right)}$$

Capacitor esférico

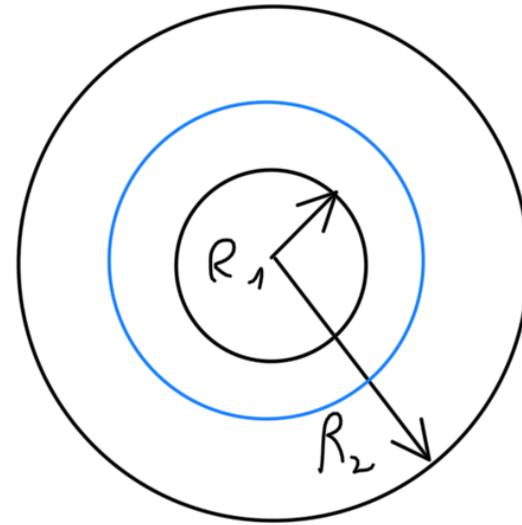
$$\vec{D} = \frac{Q}{4\pi r^2} \hat{r} ; R_1 < r < R_2$$



$$\begin{aligned} \Delta V = V_{R_1} - V_{R_2} &= - \int_{R_2}^{R_1} \vec{E} \cdot d\vec{l} = \int_{R_1}^{R_2} \frac{Q}{4\pi\epsilon} \cdot \frac{1}{r^2} \cdot \hat{r} \cdot \hat{r} dr \\ &= \frac{Q}{4\pi\epsilon} \int_{R_1}^{R_2} \frac{1}{r^2} dr = \frac{-Q}{4\pi\epsilon} \left[\frac{1}{r} \right]_{R_1}^{R_2} \\ &= \frac{Q}{4\pi\epsilon} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \end{aligned}$$

Capacitor esférico

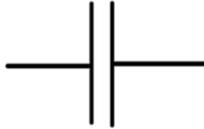
$$\vec{D} = \frac{Q}{4\pi r^2} \hat{r} ; R_1 < r < R_2$$



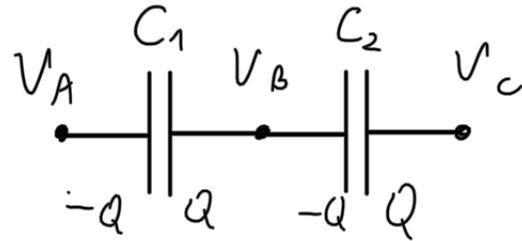
$$C = 4\pi\epsilon \frac{R_1 R_2}{R_2 - R_1}$$

Note: $E = \frac{Q^2}{2C} = \frac{Q^2}{8\pi\epsilon} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$

No vácuo, se $\lim_{R_2 \rightarrow \infty} E = \frac{Q^2}{8\pi\epsilon_0 R_1} \rightarrow$ Energia de uma casca esférica carregada!

Associações: Capacitor 

Série:



$$V_C - V_B = \frac{Q}{C_2} = V_2 \quad V_B - V_A = \frac{Q}{C_1} = V_1$$

$$\Delta V = V_C - V_A = V_C - V_B + V_B - V_A = Q \left(\frac{1}{C_2} + \frac{1}{C_1} \right) = \frac{Q}{C_{eq}}$$

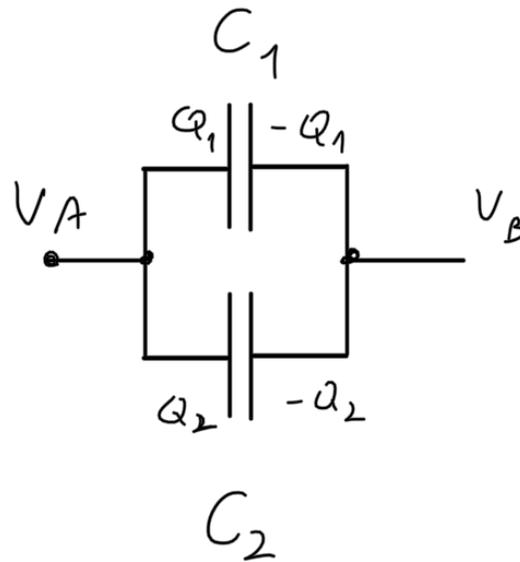
$$C_{eq} = \frac{C_2 C_1}{C_1 + C_2}$$

$$V_1 = \frac{Q}{C_1} \quad V_2 = \frac{Q}{C_2} \quad \Rightarrow \quad V_1 \cdot C_1 = V_2 \cdot C_2$$

Paralelo:

$$V_A - V_B = \frac{Q_1}{C_1} = \Delta V$$

$$V_A - V_B = \frac{Q_2}{C_2} = \Delta V$$



$$Q = Q_1 + Q_2 = C_1 \cdot \Delta V + C_2 \Delta V = \Delta V \cdot (C_1 + C_2)$$

$$C_{eq} = C_1 + C_2$$

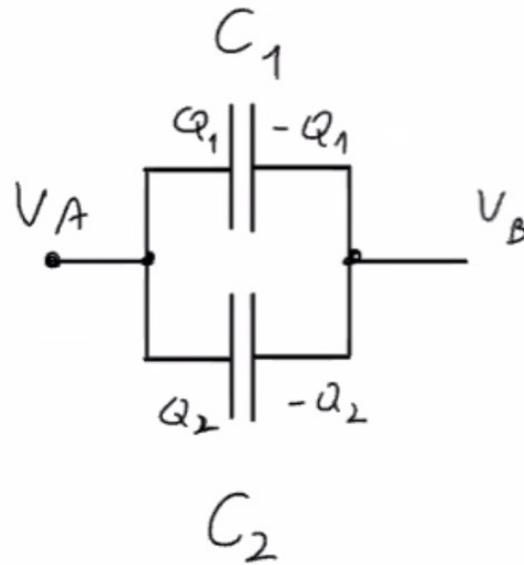
$$V_1 = V_2$$

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

Paralelo:

$$V_A - V_B = \frac{Q_1}{C_1} = \Delta V$$

$$V_A - V_B = \frac{Q_2}{C_2} = \Delta V$$



$$Q = Q_1 + Q_2 = C_1 \cdot \Delta V + C_2 \Delta V = \Delta V \cdot (C_1 + C_2)$$

$$C_{eq} = C_1 + C_2$$

$$V_1 = V_2$$

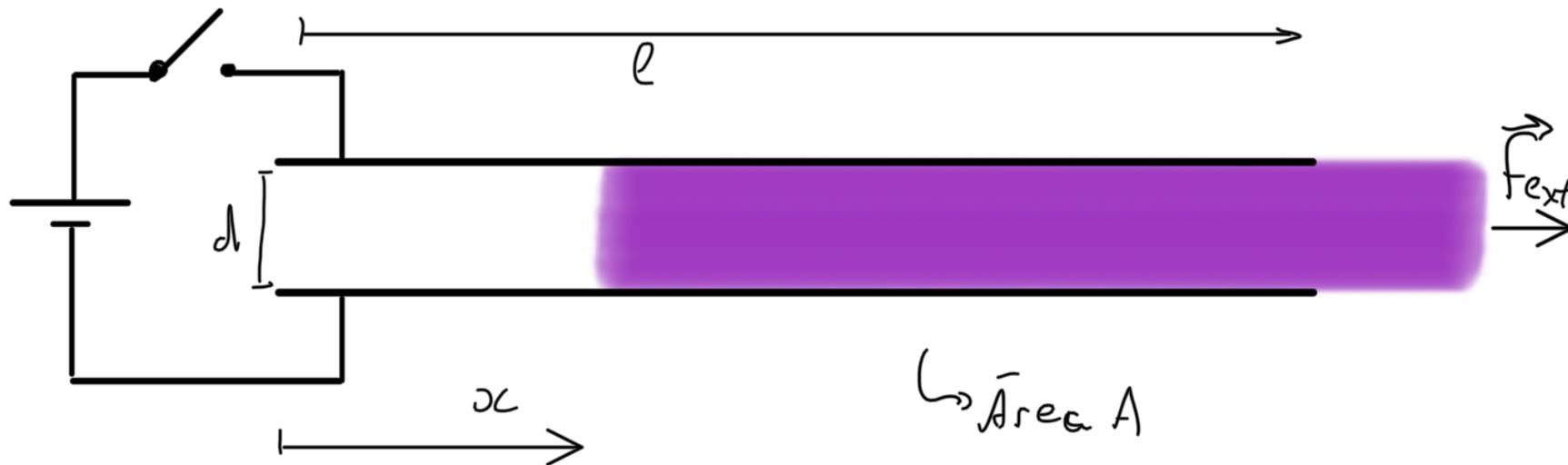
$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

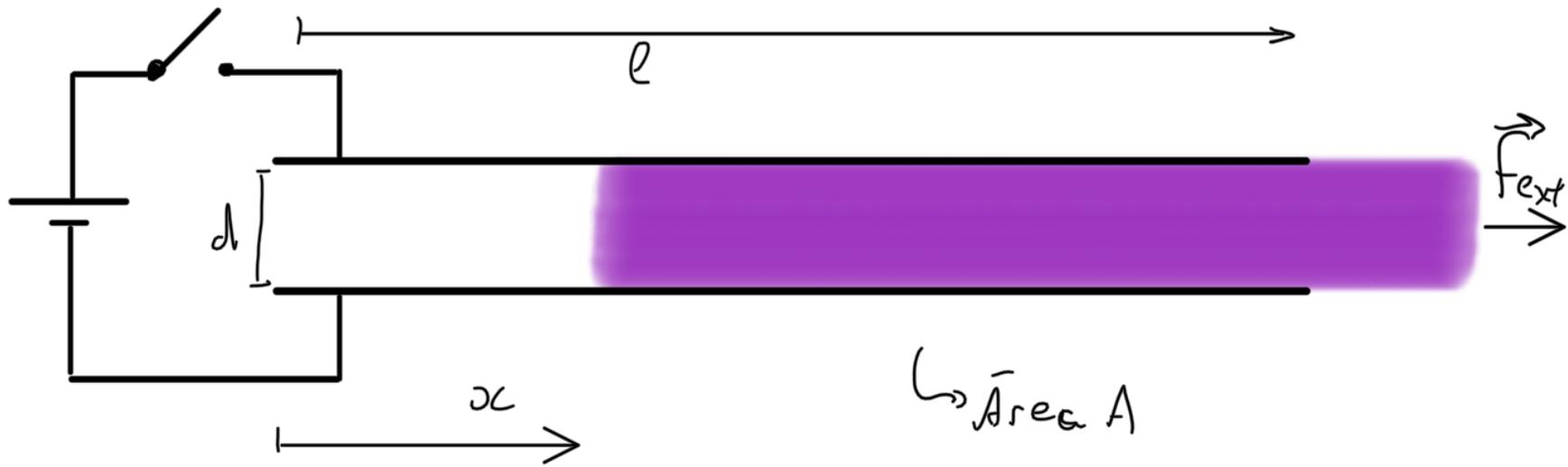
$$V_1 = V_A - V_B = V_2$$

$\hookrightarrow C_1$

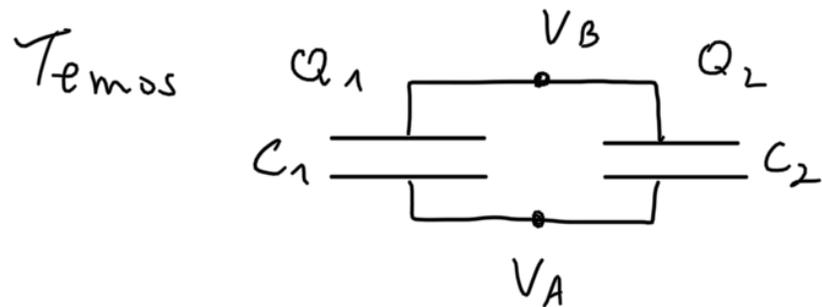
$\hookrightarrow C_2$

Qual o trabalho da inserção de um dielétrico?





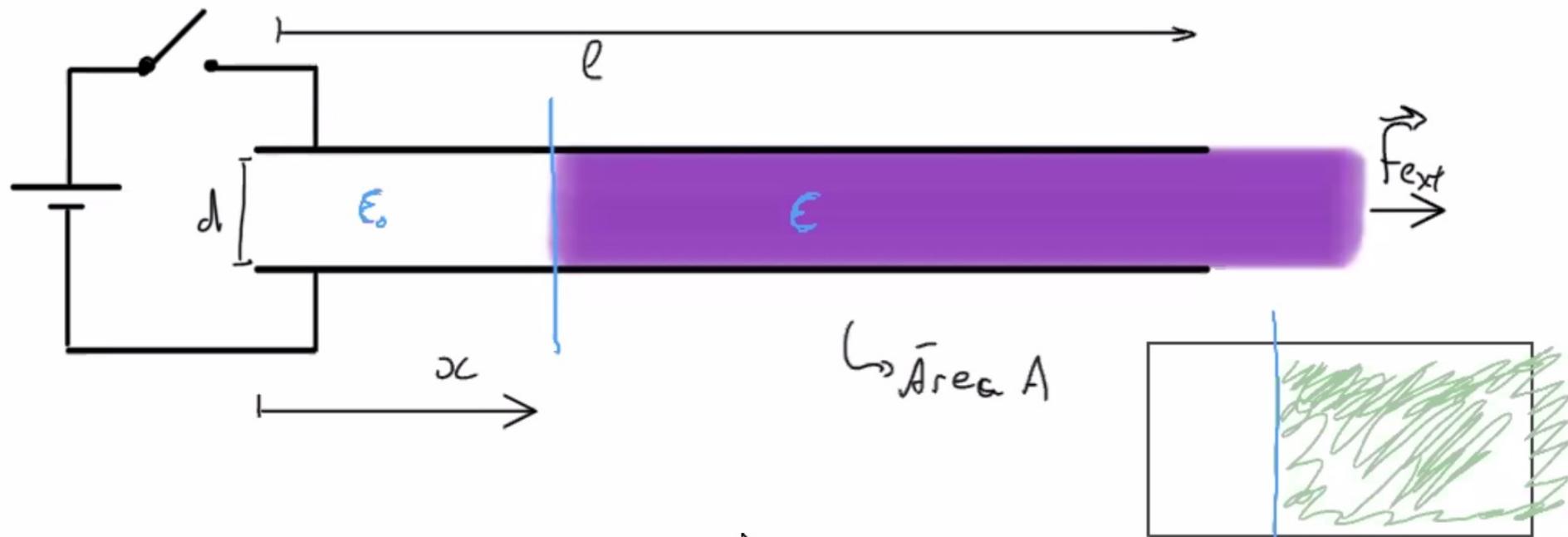
Capacitor inicialmente carregado: Q constante



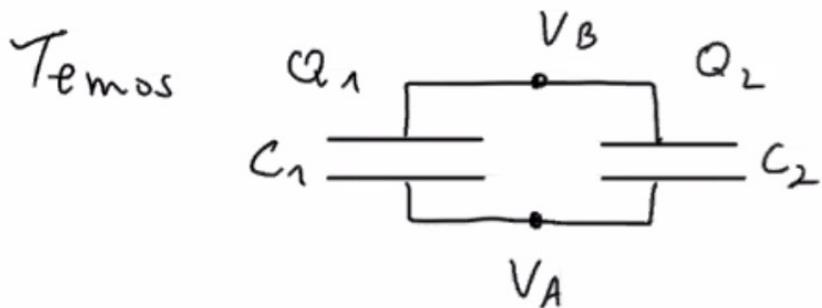
$$V = V_B - V_A = \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

$$C_1 = \epsilon_0 \frac{A \cdot x}{d \cdot l}$$

$$C_2 = \epsilon \frac{A \cdot (l-x)}{d \cdot l}$$



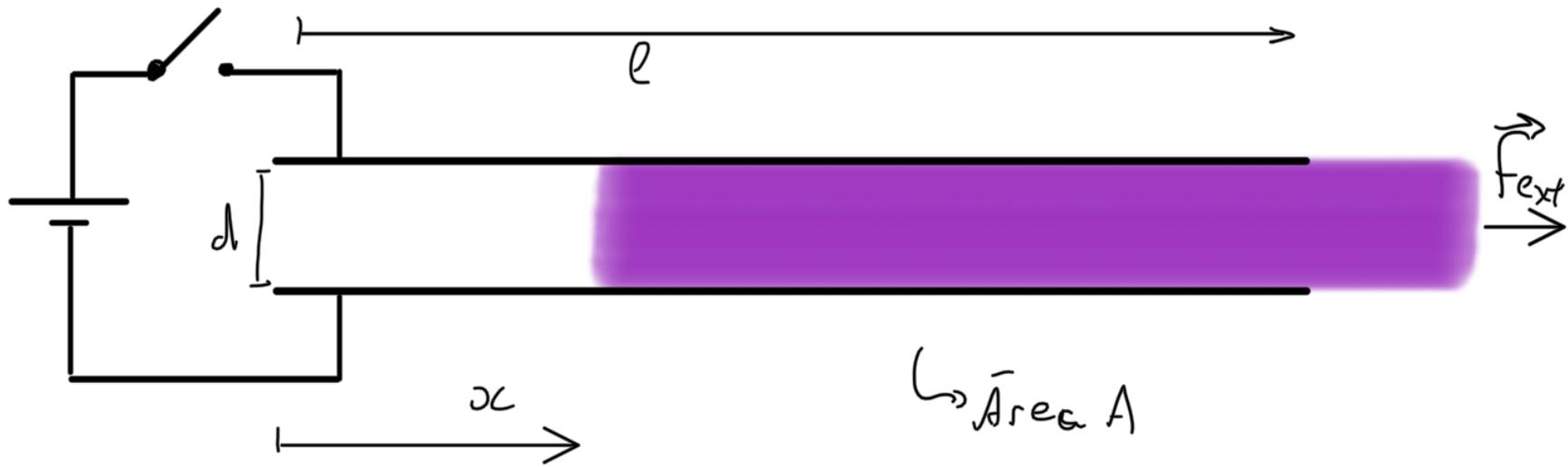
Capacitor inicialmente carregado: Q constante



$$V = V_B - V_A = \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

$$C_1 = \epsilon_0 \frac{A \cdot x}{d \cdot l}$$

$$C_2 = \epsilon \frac{A \cdot (l-x)}{d \cdot l}$$



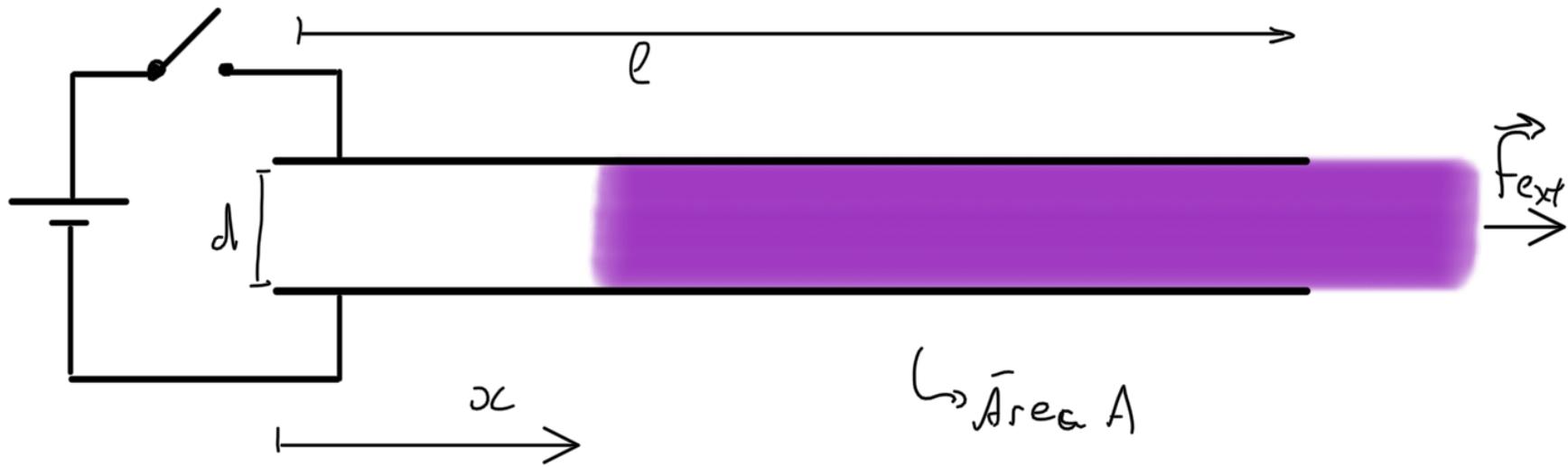
$$C_1 = \epsilon_0 \frac{A \cdot x}{d \cdot l}$$

$$C_2 = \epsilon \frac{A \cdot (l-x)}{d \cdot l}$$

Energia armazenada: $E = \frac{Q^2}{2C} = \frac{C V^2}{2} = \frac{Q \cdot V}{2}$

Qual usar?

Energia em $C_1 \rightarrow E_1 = \frac{Q_1^2}{2C_1}$ / em $C_2 \rightarrow E_2 = \frac{Q_2^2}{2C_2}$



$$\text{Energia em } C_1 \rightarrow E_1 = \frac{Q_1^2}{2C_1} \quad / \quad \text{em } C_2 \rightarrow E_2 = \frac{Q_2^2}{2C_2}$$

$$E = E_1 + E_2 = \left(\frac{Q_1^2}{C_1} + \frac{Q_2^2}{C_2} \right) \frac{1}{2} = \left(\frac{Q_1}{C_1} \cdot Q_1 + \frac{Q_2}{C_2} \cdot Q_2 \right) \frac{1}{2}$$

$$= (V \cdot Q_1 + V \cdot Q_2) \frac{1}{2} = \frac{V \cdot Q}{2}$$

Trabalho da Força externa $dW = \vec{F}_{ext} \cdot d\vec{l} = F_{ext} dx$

$dW = dE \rightarrow$ variação de energia no sistema

$$\frac{dE}{dx} = \frac{d}{dx} \left(\frac{Q^2}{2C} \right) = \frac{\partial}{\partial C} E \cdot \frac{dC}{dx}$$

$$\frac{\partial E}{\partial C} = -\frac{Q^2}{2C^2} = -\frac{V^2}{2}$$

$$C(x) = \epsilon_0 \frac{A}{d} \cdot \frac{x}{l} + \epsilon_0 \frac{A}{d} \frac{l-x}{l}$$

$$C(x) = \epsilon_0 \frac{A}{d} \cdot \frac{x}{l} + \epsilon \frac{A}{d} \frac{l-x}{l}$$

$$\frac{dC}{dx} = \epsilon_0 \frac{A}{d} \frac{1}{l} - \frac{\epsilon A}{dl} = -(\epsilon - \epsilon_0) \frac{A}{dl}$$

$$\Rightarrow \frac{dE}{dx} = -\frac{V^2}{2} \cdot -(\epsilon - \epsilon_0) \frac{A}{dl} = \frac{V^2}{2} \frac{A}{dl} (\epsilon - \epsilon_0) > 0$$

\Rightarrow Energia E mínima com $x=0$

(como vimos na última aula)

$$\Rightarrow \frac{dE}{dx} = -\frac{V^2}{2} \cdot \frac{-(\epsilon - \epsilon_0)A}{dl} = \frac{V^2}{2} \frac{A}{dl} (\epsilon - \epsilon_0) > 0$$

\Rightarrow Energia é mínima com $x=0$

(como vimos na última aula)

\Rightarrow Realizamos trabalho para remover o dielétrico $dW > 0$

\Rightarrow Força externa compensa a força atrativa que o capacitor exerce no meio

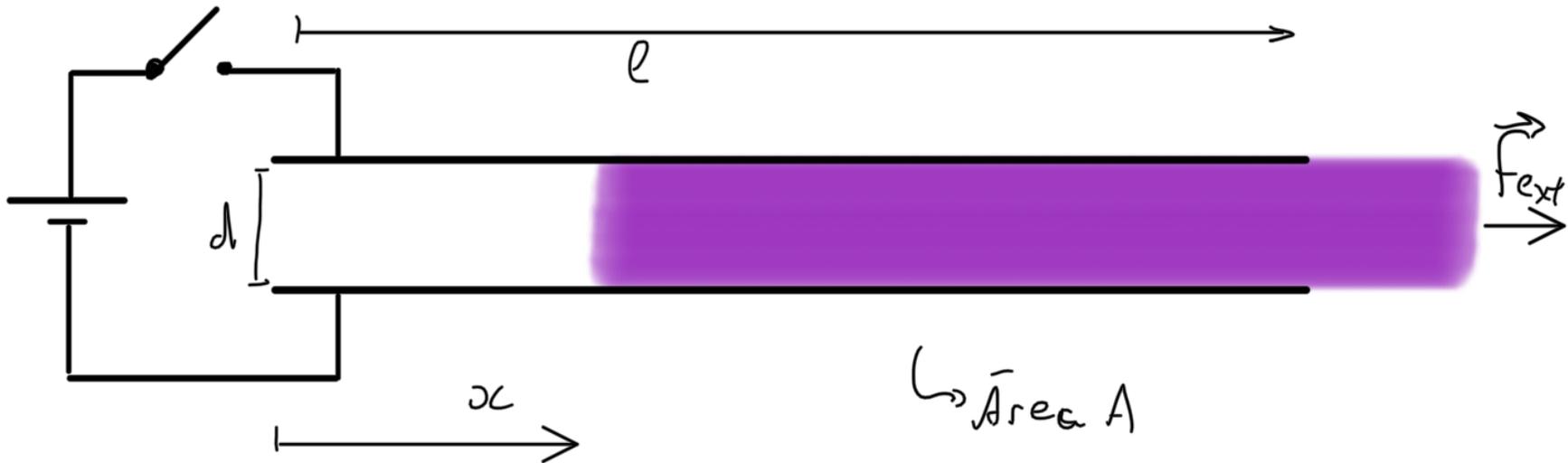
$$\Rightarrow F_e = -F_{\text{int}} = -\frac{dE}{dx} \rightarrow \text{-variação da energia potencial}$$

Trabalho realizado: $W = \int_0^l \frac{V^2}{2} \frac{A}{dl} (\epsilon - \epsilon_0) dx$

Carga constante:

Se a carga inicial for $Q \Rightarrow Q = C(x) \cdot V(x) = \epsilon \frac{A}{d} V_0$

$$V(x) = \epsilon \frac{A}{d} V_0 \cdot \frac{1}{C(x)}$$



Carga constante:

Se a carga inicial for $Q \Rightarrow Q = C(x) \cdot V(x) = \epsilon \frac{A}{d} V_0$

$$V(x) = \epsilon \frac{A}{d} V_0 \cdot \frac{1}{C(x)}$$

$$C(x) = \epsilon_0 \frac{A}{d} \cdot \frac{x}{l} + \epsilon \frac{A}{d} \frac{l-x}{l} = \epsilon \frac{A}{d} \left[1 - \frac{x}{l} \left(\frac{\epsilon - \epsilon_0}{\epsilon} \right) \right]$$

$$V(x) = V_0 \cdot \frac{1}{1 - x \cdot \alpha}$$

$$\alpha = \frac{\epsilon - \epsilon_0}{\epsilon l}$$

$$W_Q = \frac{A V_0^2 (\epsilon - \epsilon_0)}{2 d l} \int_0^l \frac{1}{(1 - x \alpha)^2} dx$$

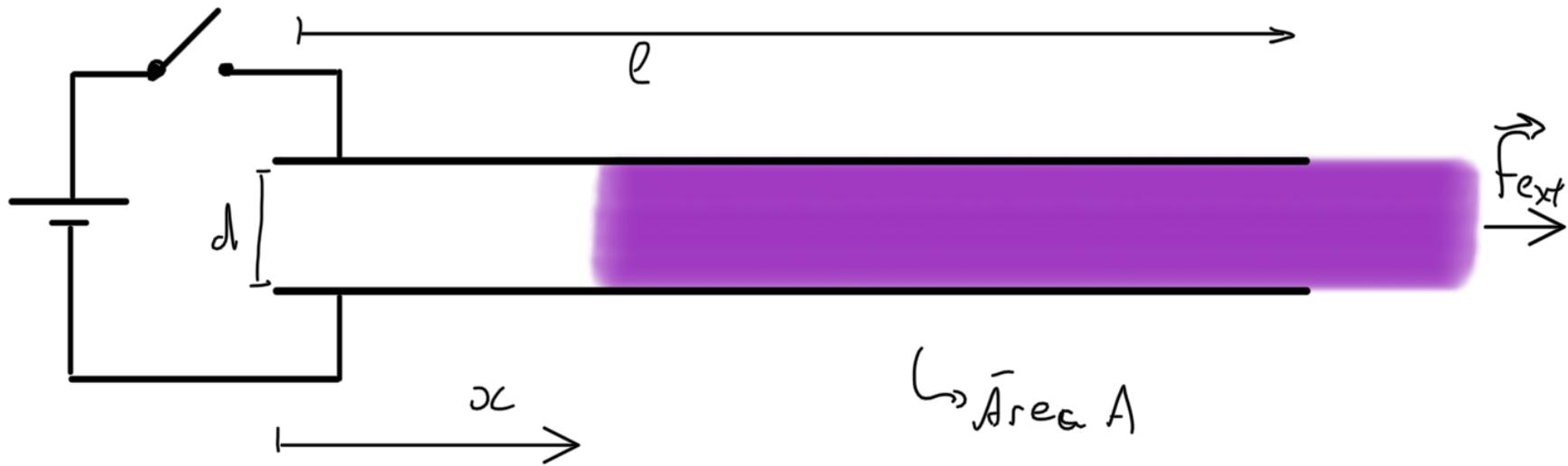
$$W_Q = \frac{A V_0^2 (\epsilon - \epsilon_0)}{2d l} \int_0^l \frac{1}{(1 - x\alpha)^2} dx$$

$$1 - x\alpha = y \quad dy = -\alpha dx$$

$$\int_0^l \frac{1}{(1 - x\alpha)^2} dx = -\frac{1}{\alpha} \int_1^{1-l\alpha} \frac{1}{y^2} dy = \frac{1}{\alpha} \left[\frac{1}{y} \right]_1^{1-l\alpha} = \frac{1}{\alpha} \left[\frac{1}{1-l\alpha} - 1 \right] =$$

$$= \frac{1}{\alpha} \left[\frac{1 - 1 + l\alpha}{1 - l\alpha} \right] = \frac{l}{1 - l\alpha} = l \frac{1}{1 - \frac{(\epsilon - \epsilon_0)}{\epsilon}} = l \frac{\epsilon}{\epsilon_0}$$

$$W_Q = \frac{A V_0^2}{2d} (\epsilon - \epsilon_0) \cdot \frac{\epsilon}{\epsilon_0} = \frac{A V_0^2}{2d} \left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon} \right) \epsilon^2$$



$$W_Q = \frac{A V_0^2}{2d} (\epsilon - \epsilon_0) \cdot \frac{\epsilon}{\epsilon_0} = \frac{A V_0^2}{2d} \left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon} \right) \epsilon^2$$

Colocando em termos da carga $Q = \epsilon \frac{A}{d} V_0 \Rightarrow V_0^2 = \frac{Q^2}{(\epsilon A/d)^2}$

$$W_Q = \frac{Q^2}{\epsilon^2 A^2} \cdot d^2 \cdot \frac{A \epsilon^2}{2d} \left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon} \right) = \frac{Q^2 d}{2A} \left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon} \right)$$

Diretamente do capacitor: $E = Q^2 / 2C \Rightarrow$

$$E_i = \frac{Q^2}{2C_i} = \frac{Q^2 \cdot d}{2\epsilon A} \quad ; \quad E_f = \frac{Q^2}{2C_f} = \frac{Q^2 \cdot d}{2\epsilon_0 A}$$

$$E_f - E_i = Q^2 \cdot \frac{d}{2A} \left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon} \right)$$

Potencial final: $V_f \cdot C_f = V_0 \cdot C_0$

$$V_f = V_0 \cdot \frac{C_0}{C_f} = V_0 \cdot \frac{\epsilon A}{d} \cdot \frac{1}{\epsilon_0 A} = \frac{\epsilon}{\epsilon_0} V_0$$

Energia armazenada:

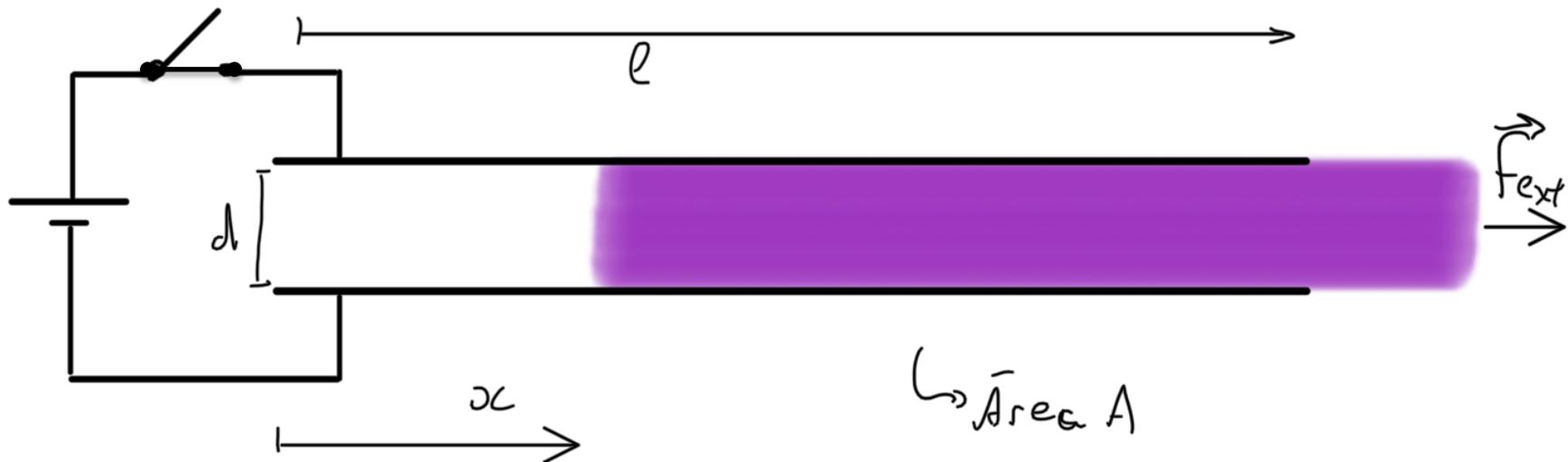
$$E_f = \frac{Q \cdot V}{2} = \frac{\epsilon}{\epsilon_0} \frac{Q V_0}{2} = \frac{\epsilon}{\epsilon_0} E_i$$

$$\frac{E_f - E_i}{E_i} = \left(\frac{\epsilon - \epsilon_0}{\epsilon_0} \right)$$

Bateria ligada $V = \text{cte}$

Trabalho realizado:
$$W_V = \int_0^l \frac{V_0^2}{2} \frac{A}{d \cdot l} (\epsilon - \epsilon_0) dx = \frac{V_0^2 A}{2d} (\epsilon - \epsilon_0)$$

Energia final armazenada:
$$E_{fV} = \frac{Q \cdot V}{2}$$



Energia final armazenada: $E_{fv} = \frac{Q_f \cdot V}{2}$

Como: $Q(x) = C(x) \cdot V \Rightarrow \frac{Q_0}{C_0} = \frac{Q_f}{C_f}$

$$Q_f = Q_0 \cdot \frac{C_f}{C_0} = Q_0 \frac{d}{\epsilon A} \cdot \frac{\epsilon_0 A}{d} = Q_0 \cdot \frac{\epsilon_0}{\epsilon}$$

$$E_{fv} = \frac{Q_0 \epsilon_0}{2 \epsilon} \cdot V_0 = \frac{\epsilon_0}{\epsilon} E_{iv}$$

$$E_{fv} - E_{iv} = \left(\frac{\epsilon_0 - \epsilon}{\epsilon} \right) E_{iv} < 0$$

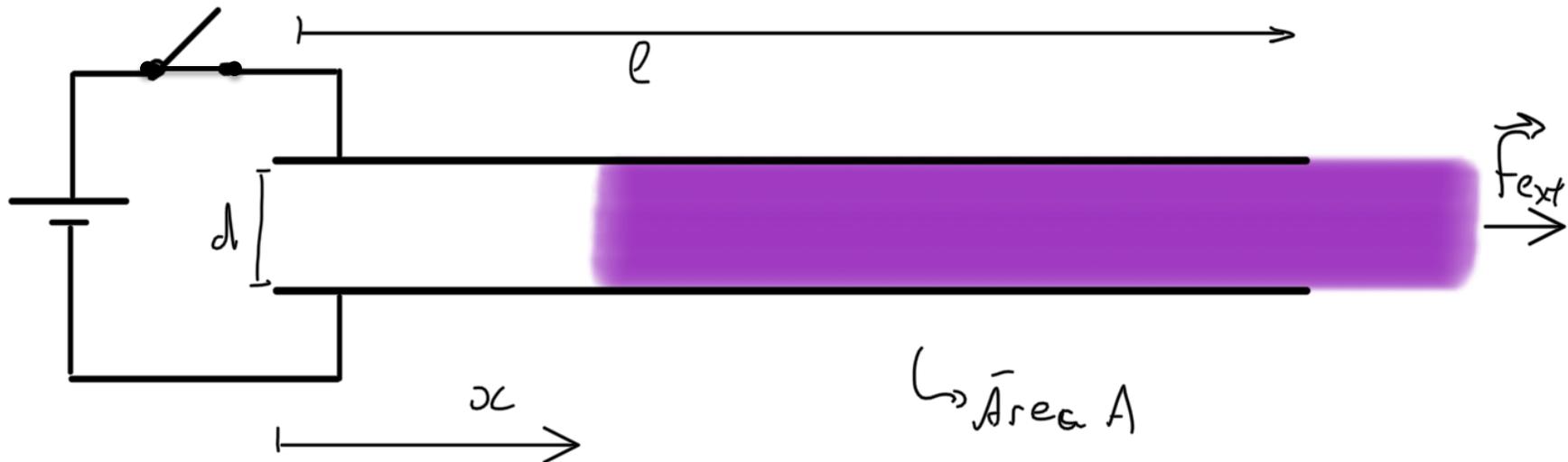
$$W_V = \frac{V_0}{2} \cdot \underbrace{V_0 \left(\frac{A \epsilon}{d} \right)}_{Q_0} \left(\frac{\epsilon - \epsilon_0}{\epsilon} \right) = E_{iv} \cdot \frac{\epsilon - \epsilon_0}{\epsilon}$$

Comparando: $W_Q = \frac{v_0^2 A}{2d} \cdot (\epsilon - \epsilon_0) \cdot \frac{\epsilon}{\epsilon_0} = W_V \cdot \frac{\epsilon}{\epsilon_0} > W_V$

Energia: $(\epsilon_{fv} - \epsilon_{iv}) - W_Q = \left(\frac{\epsilon - \epsilon_0}{\epsilon_0}\right) \epsilon_{iv} - \left(\frac{\epsilon - \epsilon_0}{\epsilon_0}\right) \epsilon_{iv} = 0$

$(\epsilon_{fv} - \epsilon_{iv}) - W_V = -2 W_V$

Para onde foi esta energia?

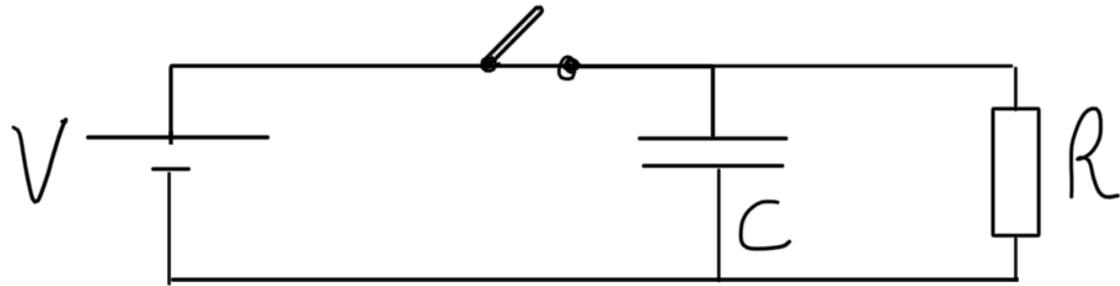


Eletromagnetismo

Complementos

- Construindo um capacitor.
- Comparando uma esfera dielétrica e uma esfera metálica em um campo eletrostático.

Aula 9: Descarga



Carga inicial $Q = C \cdot V$

Descarga: $I(t) = V/R = -\frac{d}{dt} Q(t)$

$$\Rightarrow \frac{d}{dt} Q(t) = -\frac{Q}{CR} \Rightarrow Q(t) = A e^{-t/\tau}$$

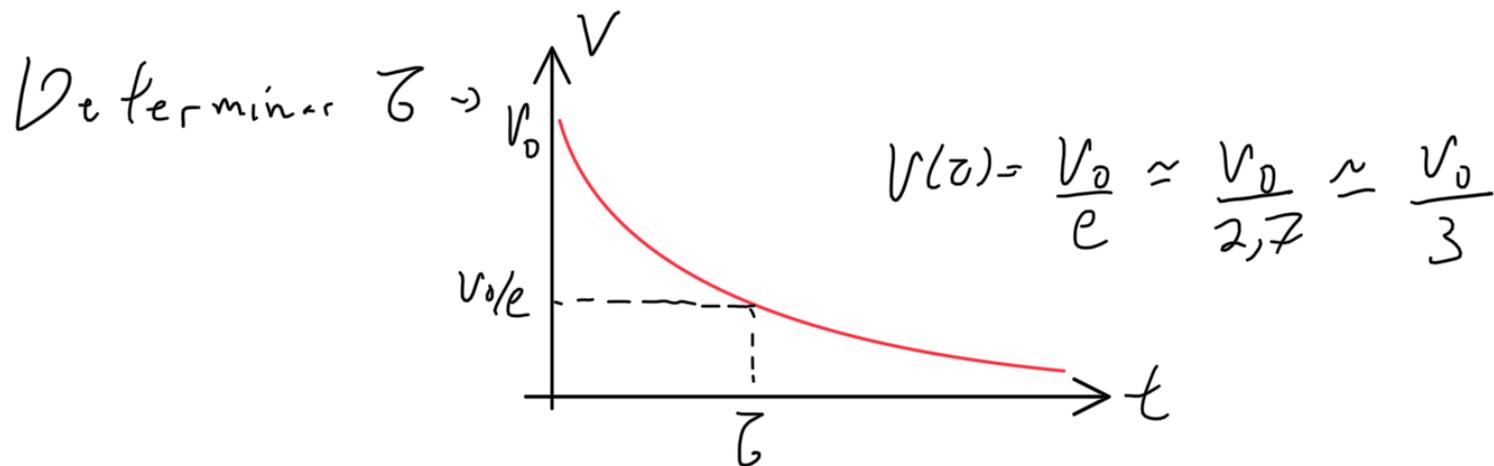
$$V(t) = \frac{Q}{C} = \frac{A}{C} e^{-t/\tau} = V_0 e^{-t/\tau} \quad \tau = R \cdot C$$

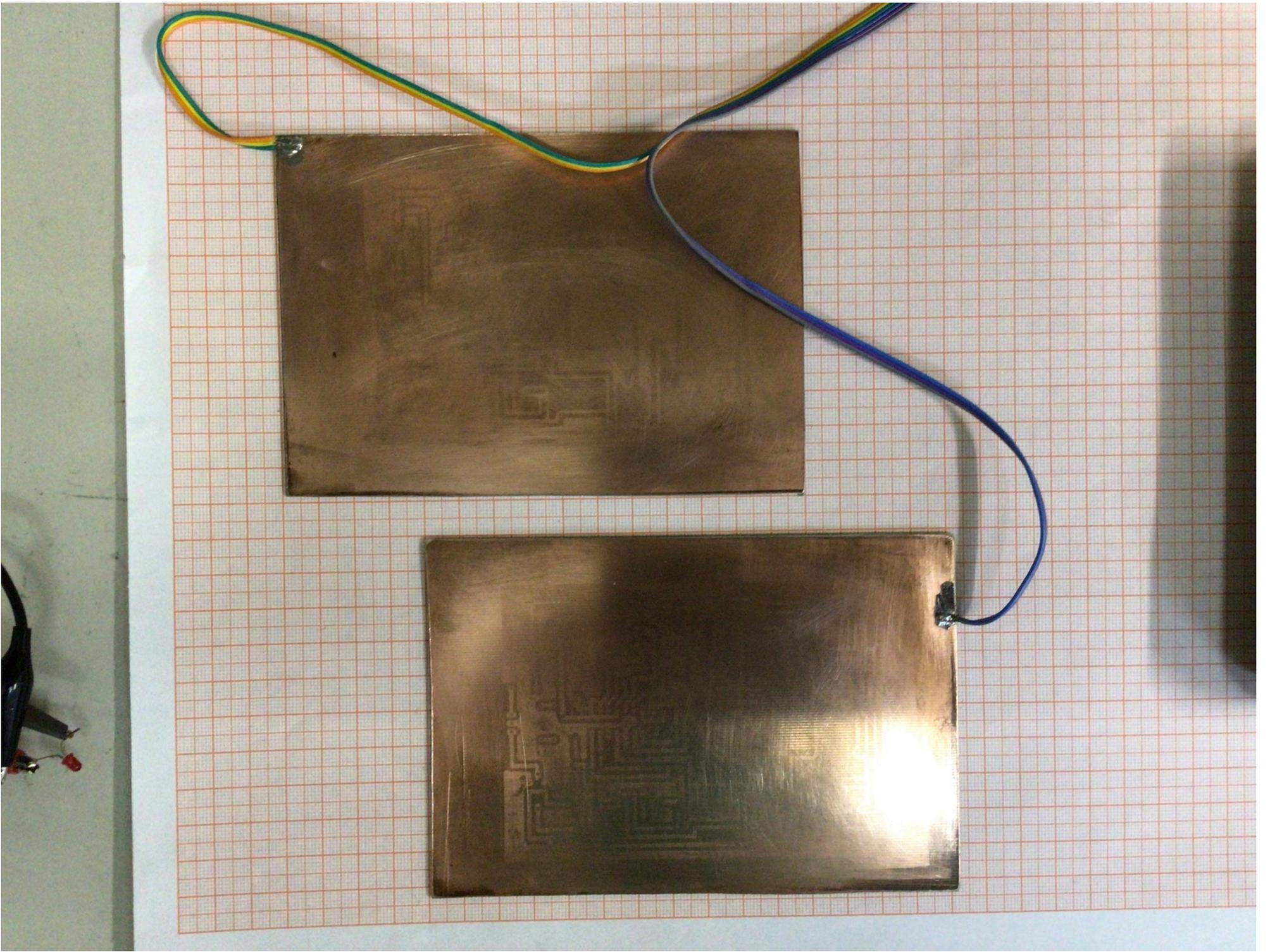
$$V(t) = \frac{Q}{C} = \frac{A}{C} e^{-t/\tau} = V_0 e^{-t/\tau} \quad \tau = R \cdot C$$

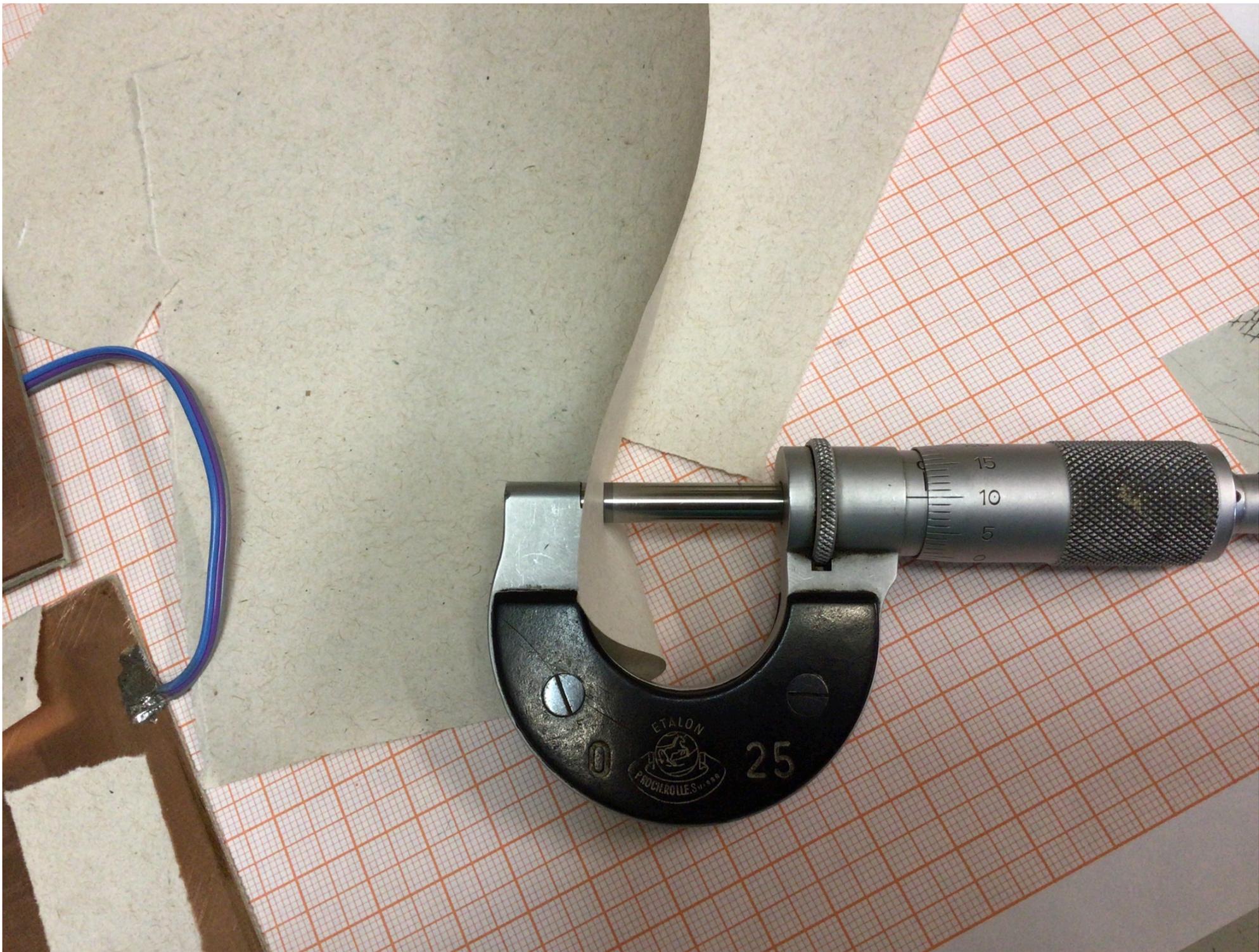
Energia dissipata: $P = \frac{d}{dt} \mathcal{E} = V \cdot I = \frac{V^2}{R} = \frac{V_0^2}{R} e^{-2t/\tau}$

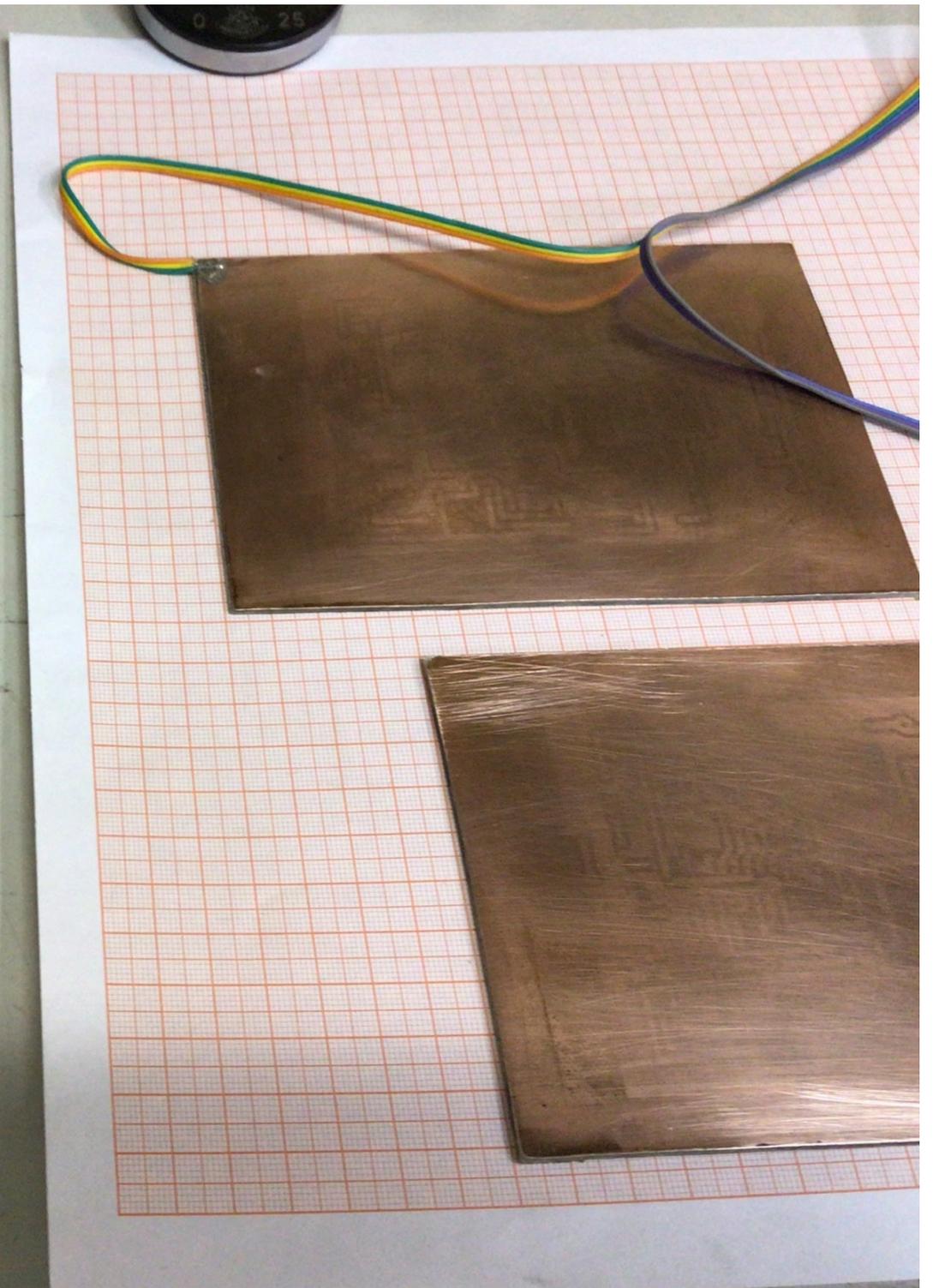
$$\mathcal{E}(t) = \int_0^t P dt = \frac{V_0^2}{R} \left[e^{-2t/\tau} \right]_0^t \cdot \frac{-RC}{2}$$

$$= -\frac{V_0^2 C}{2} \lim_{t \rightarrow \infty} \left[e^{-2t/\tau} - 1 \right] = \frac{V_0^2 C}{2}$$

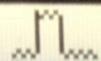








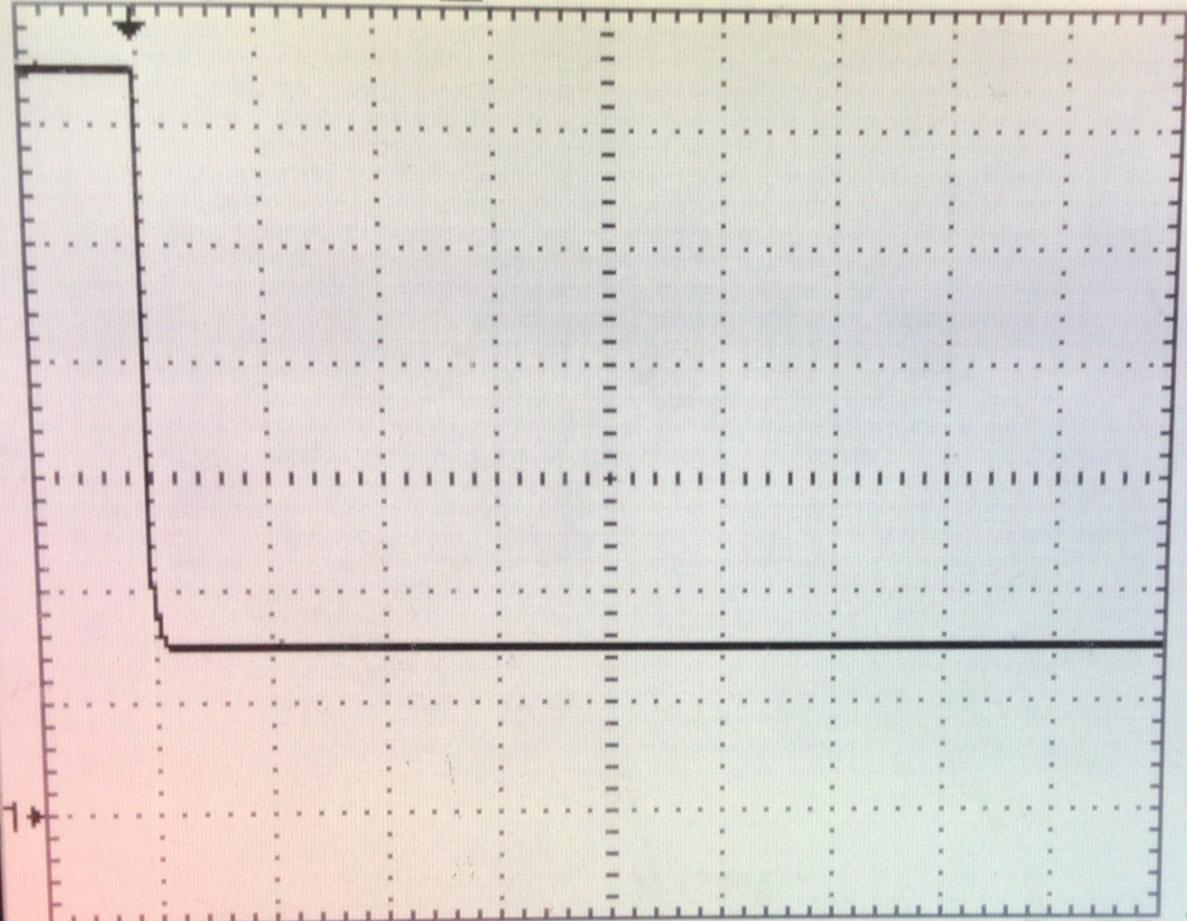
Tek



T Trig'd

M Pos: 10.10ms

SAVE/REC



Action
Save All

PRINT
Button
Saves All
To Files

Select
Folder

About
Save All

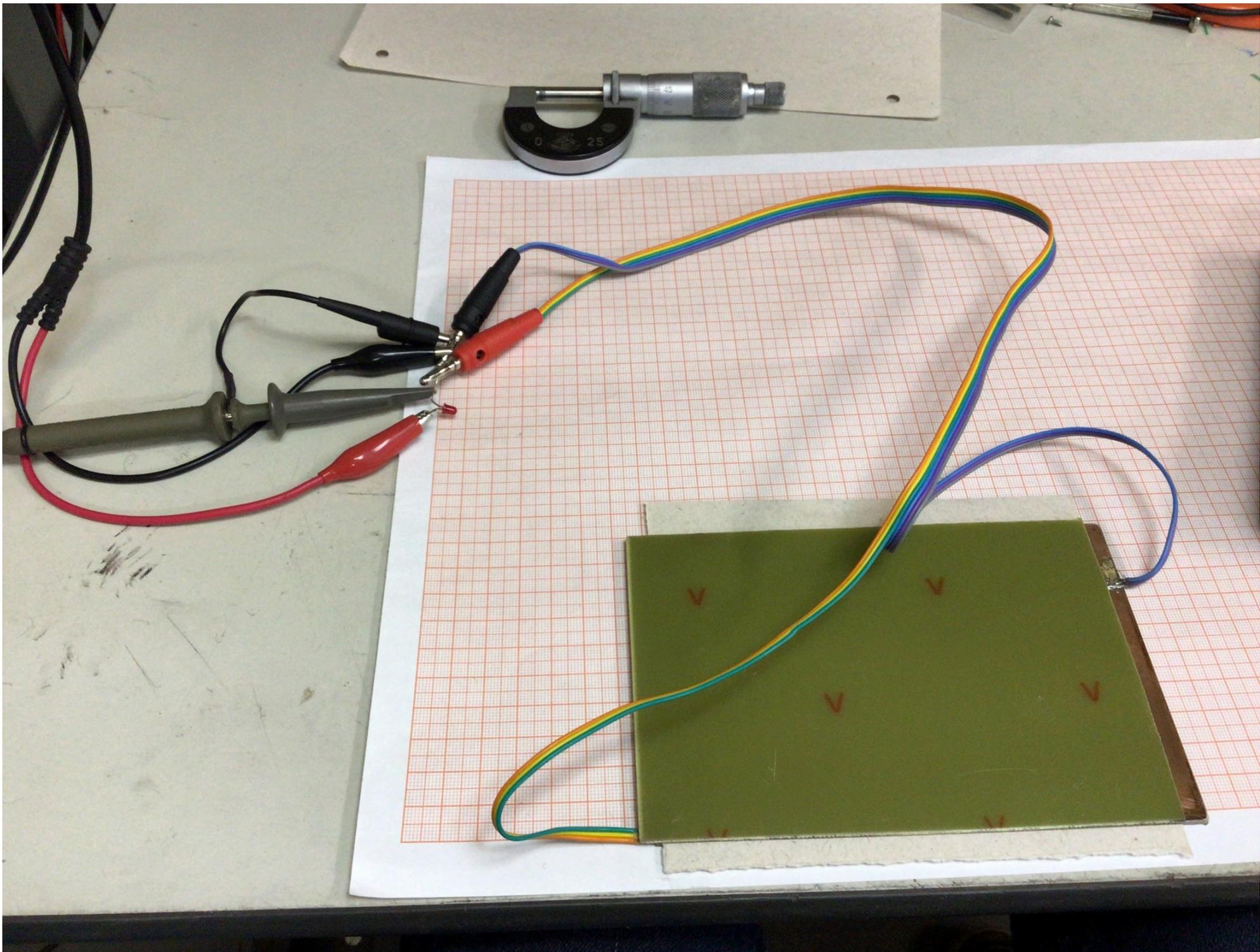
CH1 1.00V

M 2.50ms

Ext/5 1.88V

14-Sep-21 02:16

10.0000Hz



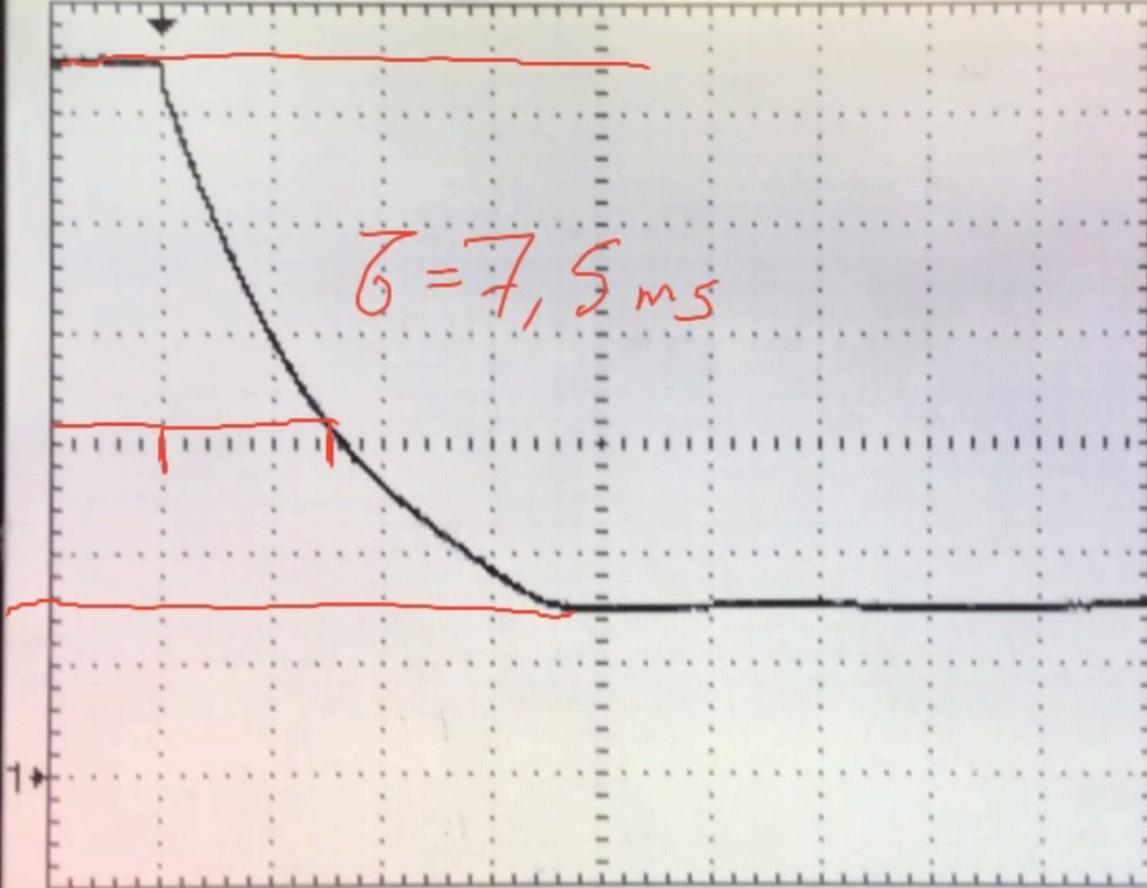
Tek



T Trig'd

M Pos: 20.00ms

SAVE/REC



Action
Save All

PRINT
Button
Saves All
To Files

Select
Folder

About
Save All

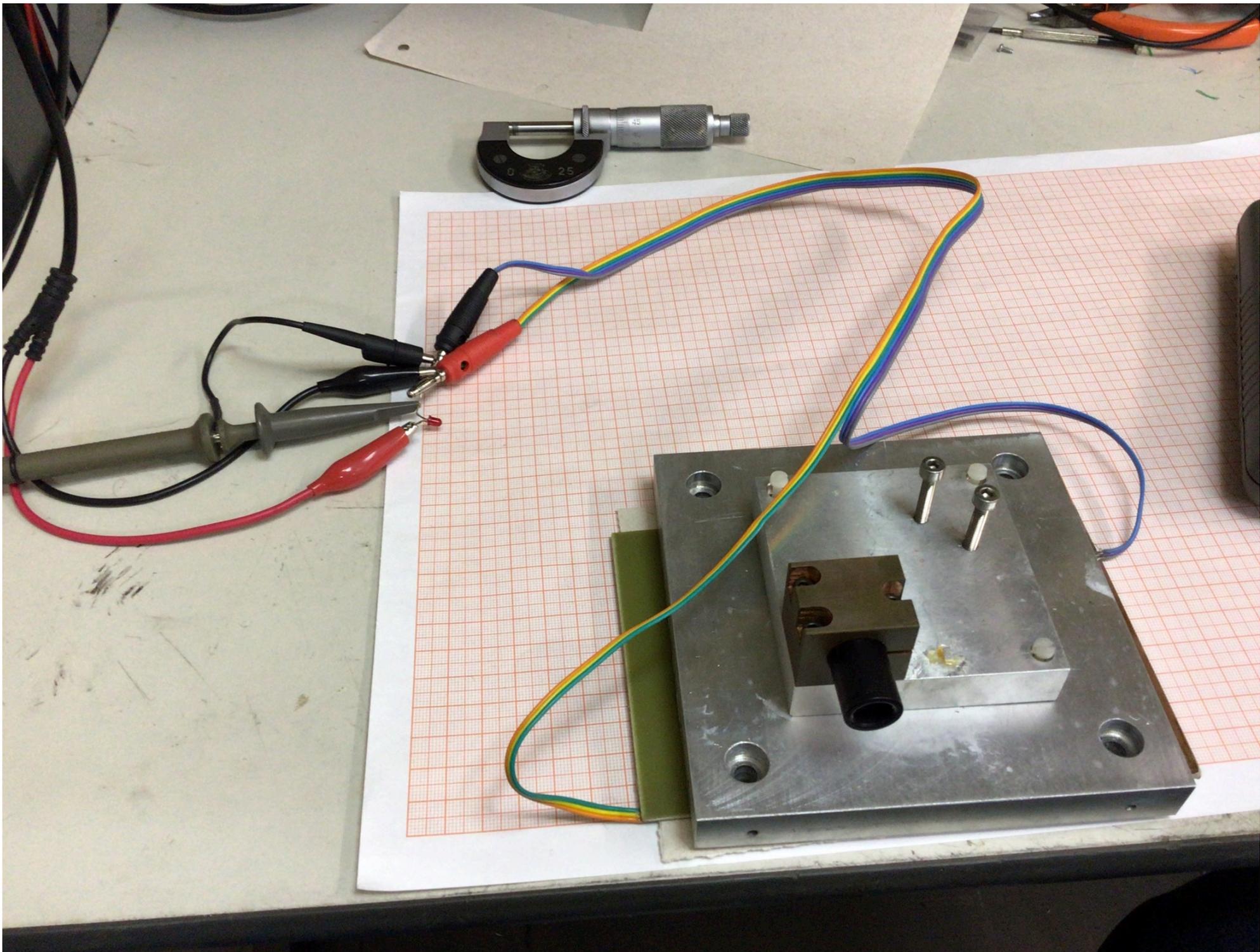
CH1 1.00V

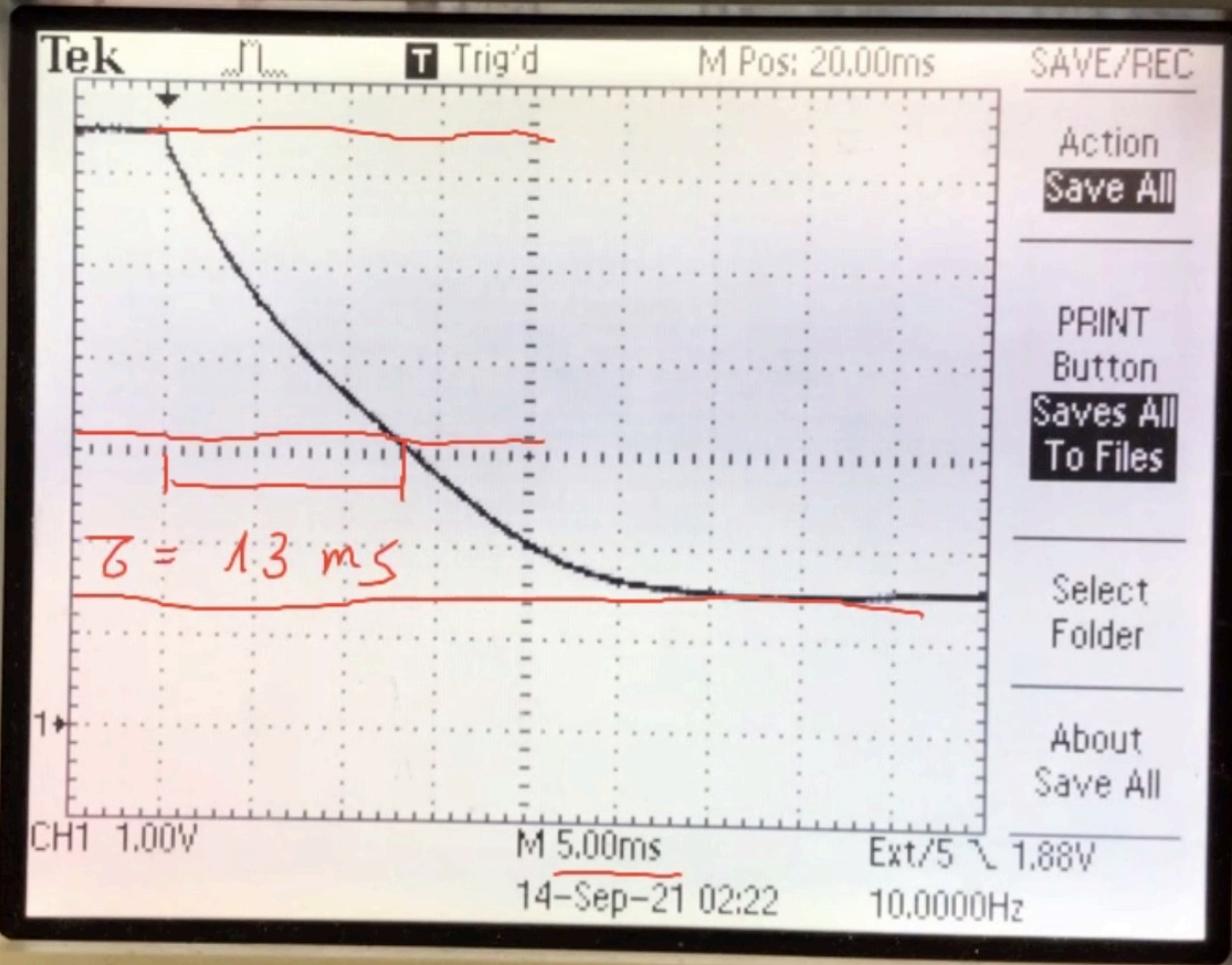
M 5.00ms

Ext/5 1.88V

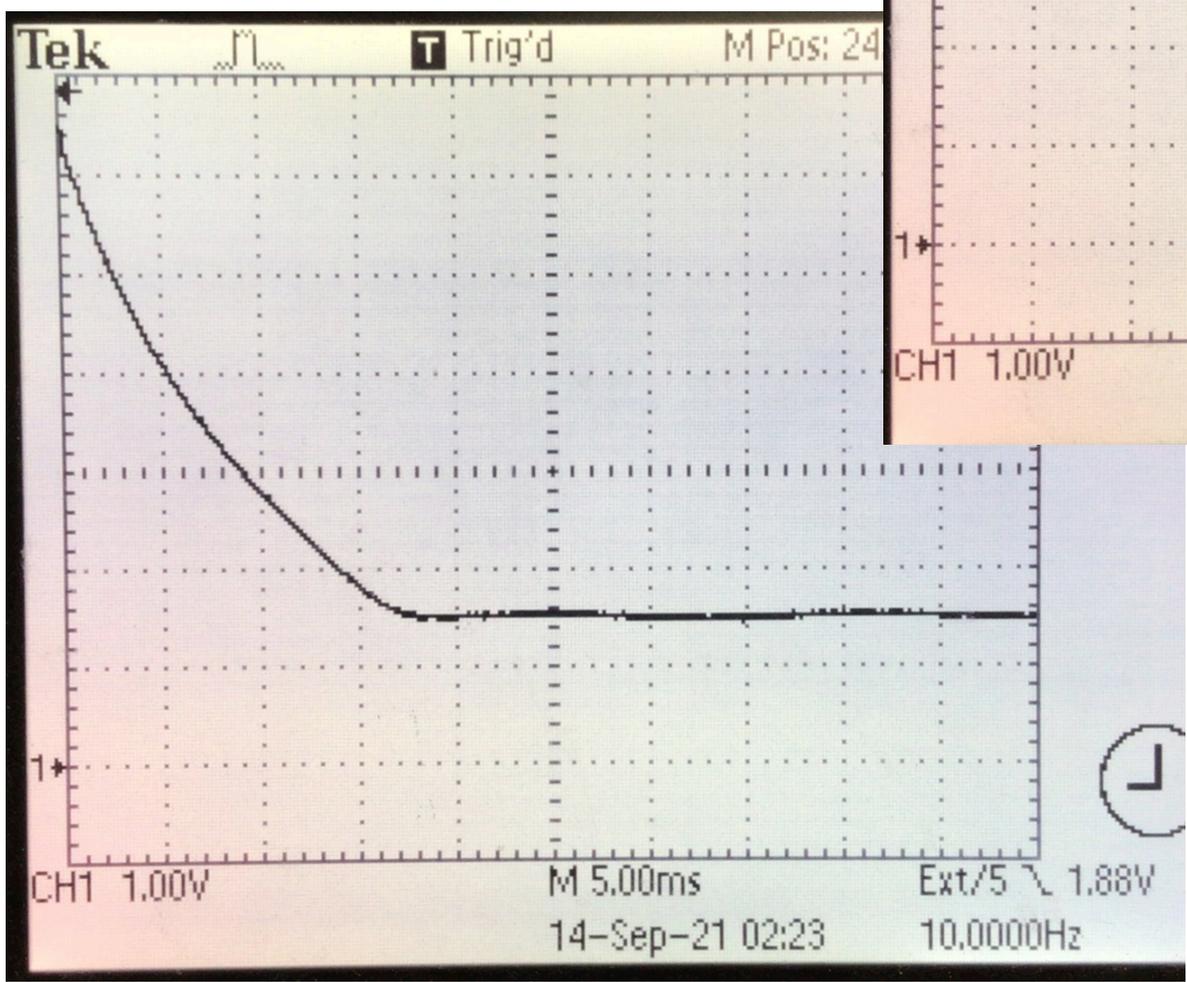
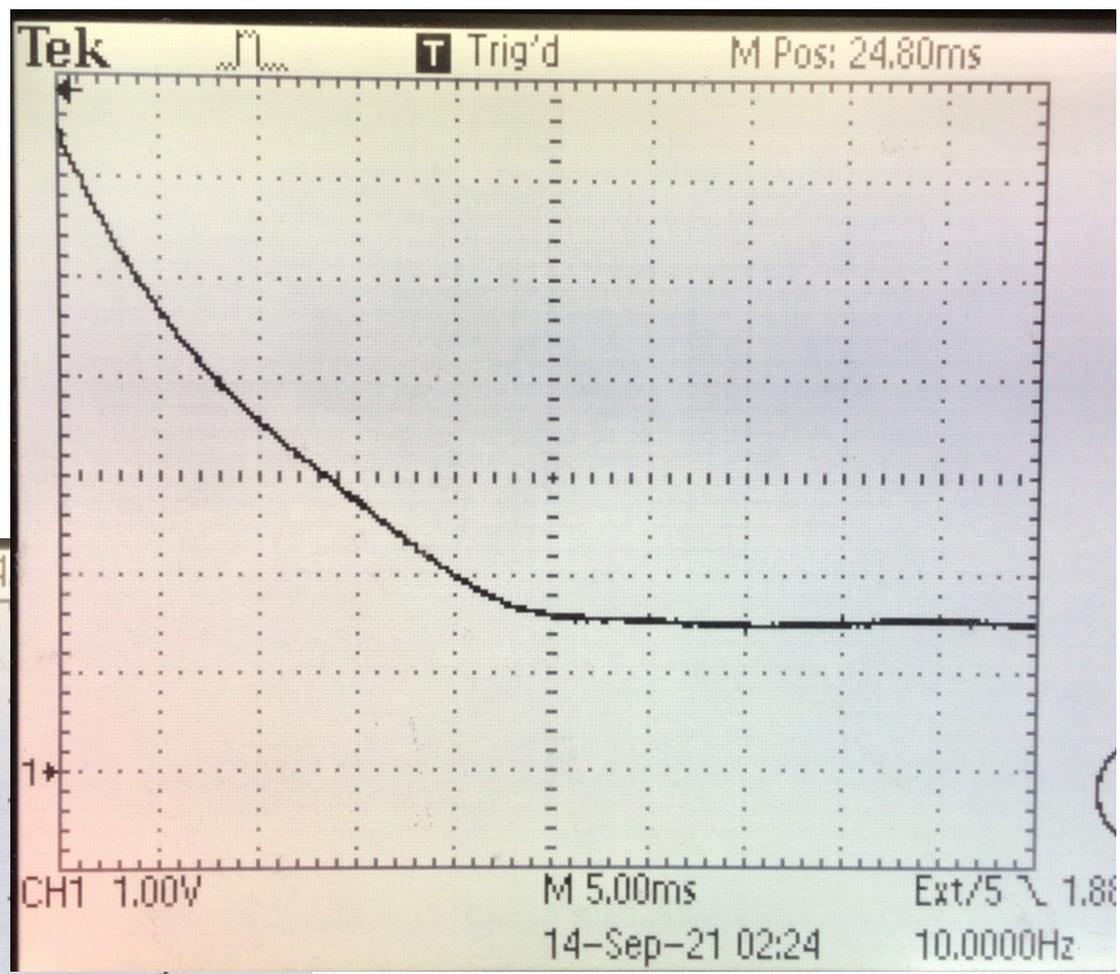
14-Sep-21 02:22

10.0000Hz





USB



$$\text{sem peso: } \tau_u \rightarrow S_{2,7} = 1,85 \quad / \quad S_{1,3} = 1,7$$

$$\sim 3 \times 2,5 \text{ ms} \quad 7,5 \text{ ms}$$

$$R = 10 \text{ M}\Omega \quad C = \frac{7,5 \cdot 10^{-3}}{10^7}$$

$$= 7,5 \cdot 10^{-10} \text{ F} = 750 \text{ pF}$$

$$A = 0,1 \text{ m} \times 0,14 \text{ m}$$

$$\epsilon_0 = 8,8 \cdot 10^{-12} \frac{\text{F}}{\text{m}}$$

$$C = \frac{8,8 \cdot 10^{-12} \text{ F} \cdot 1,4 \cdot 10^{-2} \text{ m}^2}{\text{m} \cdot 10^{-4} \text{ m}}$$

$$d = 0,1 \cdot 10^{-3} \text{ m}$$

$$= 12,32 \cdot 10^2 \text{ pF} = 1,2 \text{ nF}$$

$$\tau = 1,2 \text{ ms}$$

Tipos de capacitores:

