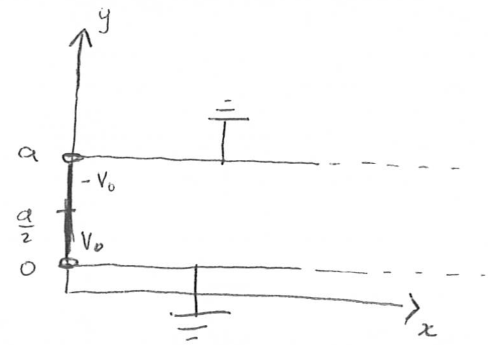
Exercicio fixação aula 4



$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$V(0,y) = V_0 \qquad 0 \leq y \leq a/2$

$V(0,y) = -V_0 \qquad a/2 < y \leq a$

$V(x,a) = V(x,0) = 0 \quad \forall x$

$V(x \to \infty, y) = 0 \quad \forall y$

Condições de contorno

⇒ Separação de variáveis

$$V(x,y) = X(x)\,Y(y)$$

$$Y(y)\,\frac{\partial^2 X(x)}{\partial x^2} + X(x)\,\frac{\partial^2 Y(y)}{\partial y^2} = 0$$

$$\frac{1}{X(x)}\,\frac{\partial^2 X(x)}{\partial x^2} = \frac{-1}{Y(y)}\,\frac{\partial^2 Y(y)}{\partial y^2} = \alpha$$

$$\frac{\partial^2 X(x)}{\partial x^2} = \alpha X(x) \quad ; \quad \frac{\partial^2 Y(y)}{\partial y^2} = -\alpha Y(y)$$

$X(x) = e^{-rx}$

$r^2 = \alpha$

$r = \pm\sqrt{\alpha}$

$X(x) = A e^{\sqrt{\alpha}x} + B e^{-\sqrt{\alpha}x}$

$Y(y) = e^{-r}$

$r^2 = -\alpha$

$r = \pm i\sqrt{\alpha}$

$Y(y) = D\cos(\sqrt{\alpha}\,y) + C\sin(\sqrt{\alpha}\,y)$

$$V(x,y) = X(x)\,Y(y)$$

$$= \left(A e^{\sqrt{\alpha}\,x} + B e^{-\sqrt{\alpha}\,x}\right)\left(D\cos(\sqrt{\alpha}\,y) + C\sin\sqrt{\alpha}\,y\right)$$

$$\Rightarrow V(x,0) = \left(A e^{\sqrt{\alpha}x} + B e^{-\sqrt{\alpha}\,x}\right) D = 0$$

$$D = 0$$

$A e^{\sqrt{\alpha}x} = -B e^{-\sqrt{\alpha}x}$ mas isso leva à solução trivial

$$\rightarrow V(x,a) = \left(A e^{\sqrt{\alpha}x} + B e^{-\sqrt{\alpha}x}\right) C\sin(\sqrt{\alpha}\,a) = 0$$

$$\sin(\sqrt{\alpha}\,a) = 0$$

$$\sqrt{\alpha}\,a = n\pi \qquad n \in \mathbb{N}$$

$$\sqrt{\alpha} = \frac{n\pi}{a}$$

$$\rightarrow \lim_{x\to\infty} V(x,y) = \left(A e^{\sqrt{\alpha}x} + B e^{-\sqrt{\alpha}x}\right) C \sin\left(\frac{n\pi y}{a}\right) = 0$$

$A \cdot C = 0$ ∞ não converge

$$\longrightarrow \quad V(x,y) = \sum_n F_n \sin\left(\frac{n\pi}{a}y\right) e^{-\frac{n\pi}{a}x}$$

$$V(0,y) = \sum_n F_n \sin\left(\frac{n\pi y}{a}\right) = V_0 \quad , 0 < y < a/2$$

$$V(0,y) = \sum_n F_n \sin\left(\frac{n\pi y}{a}\right) = -V_0 \quad , \frac{a}{2} < y < a$$

Falta aplicar as condições de contorno para $x=0$.

Para isso vamos aproveitar a ortogonalidade do sin

$$V(x,y) = \sum_n F_n \sin\left(\frac{n\pi}{a} y\right) e^{-\frac{n\pi}{a}x}$$

- Se $n<0$; sabemos que $\sin(-\alpha) = -\sin(\alpha)$

então $$\sum_{n\in\mathbb{Z}} \sin(n\alpha) = \sum_{n\in\mathbb{N}} [\sin(n\alpha) - \sin(n\alpha)] = 0$$

então $n \geq 0$

- Se $n=0$;

$$V(x,y) = 0$$ o que não aporta nada pro potencial

então podemos deixar $n>0$

$$\longrightarrow V(x,y) = \sum_{n=1}^{N} F_n \sin\left(\frac{n\pi}{a} y\right) e^{-\frac{n\pi}{a}x}$$

$$\int_0^a V(x,y) \sin\left(\frac{m\pi}{a} y\right) dy = \sum_{n=1} F_n \int \sin\left(\frac{m\pi}{a} y\right) \sin\left(\frac{n\pi}{a} y\right) e^{-\frac{n\pi}{a}x} dy$$

$x=0$, $0<y<a/2$

$$V_0 \int_0^{a/2} \sin\left(\frac{m\pi}{a} y\right) dy = \sum_n F_n \int_0^{a/2} \sin\left(\frac{m\pi}{a} y\right) \sin\left(\frac{n\pi}{a} y\right) dy$$

$$V_0 \left(-\frac{a}{m\pi}\cos\left(\frac{m\pi}{a} y\right)\right)\Bigg|_0^{a/2} = \sum_n F_n \frac{a}{2} \delta_{n,m}$$

$$-\frac{a V_0}{m\pi}\left[\cos\left(\frac{m\pi}{2}\right) - 1\right] = \sum_n F_n \frac{a}{2} \delta_{n,m}$$

- Se $m = 2p+1$

$$\cos\left(\frac{m\pi}{2}\right) - 1 = -1$$

- Se $m = 2p$

$$\cos\left(p\pi\right) - 1 = 0$$

Por tanto, para m impar

$$\frac{2V_0}{m\pi} = F_m$$

$$\longrightarrow V^{(1)}(x,y) = \frac{2V_0}{\pi} \sum_{p=0}^{\infty} \frac{1}{2p+1} e^{-\frac{(2p+1)\pi}{a}} \sin\left(\frac{(2p+1)\pi}{a} y\right) \qquad 0 < y < a/2$$

A solução para $\frac{a}{2} < y < a$ val ser igual com a diferença do potencia $V(0,y) = -V_0$

$$\longrightarrow V^{(2)}(x,y) = -\frac{2V_0}{\pi} \sum_{p=0}^{\infty} \frac{1}{2p+1} e^{-\frac{(2p+1)\pi}{a}} \sin\left(\frac{(2p+1)\pi}{a} y\right)$$

Como o potencial cumpre com superposição

$$V_T(x,y) = V^{(1)}(x,y) + V^{(2)}(x,y) = 0$$