Chapter 13 Solutions

13.1 $T = 673$ K, area of each face is $A = 10^{-2}$ m^2, $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}, then $0.97AI_e = 0.97\, A\sigma T^4 = 110$ W.

13.2 $0.97I_e = 0.97\,\sigma(T^4 - T_E^4) = 76.9$ W/m^2 with $T = 306$ K and $T_e = 293$ K is the temperature of the environment. Then $0.97AI_e = 108$ W for the radiated power.

13.3 $I_e = 22.8\times 10^4$ W/m^2, $T = (I_e/\sigma)^{1/4} = 1420$ K.

13.4 $E \sim T^4$, so the energy radiated increases by a factor of 10^4.

13.5 $T = 306$ K, $\lambda_{\max} = 2.8978\times 10^{-3}$ m$K/T = 9.45\times 10^{-6}$ m $= 9.5\mu$ m (in the infrared).

13.6 If the blackbody is at $T = 293$ K, then $\lambda_{\max} = 2.8978\times 10^{-3}$ m K/T $= 9.9\mu$ m (in the IR).

13.7 $T = 4.0 \times 10^4$ K, $\nu_{\max} = c/\lambda_{\max} = cT/2.8978\times 10^{-3}$ m K $= 4.1\times 10^{15}$ Hz (in the UV).

13.8 $T = 2.8978 \times 10^{-3}$ m K$/\lambda_{\max} = 2.8978\times 10^{-3}mK/4.65\times 10^{-7}$ m $= 6230$ K.

13.9 $T = 2.8978 \times 10^{-3}$ m K$/\lambda_{\max} = 4300$ K.

13.10 We have for the total radiated power per unit area of the blackbody

$$P(T) = \int_0^\infty I_\lambda d\lambda = 2\pi hc^2 \int_0^\infty \frac{d\lambda}{\lambda^5(e^{hc/\lambda k_B T} - 1)}$$
$$= 2\pi hc^2 \left(\frac{k_B T}{hc}\right)^4 \int_0^\infty \frac{x^3 dx}{(e^x - 1)},$$

by putting $x = hc/\lambda k_B T$, $d\lambda = -hcdx/k_B T x^2$. The value of the integral over x is $\Gamma(4)\varsigma(4) = 3!\,\pi^4/90 = \pi^4/15$. Therefore the Stefan-Boltzmann law follows,

$$P(T) = \frac{2\pi^5}{15}\frac{(k_B T)^4}{h^3 c^2}.$$

13.11 From Eq (13.4):

$$I_\lambda = \frac{2\pi hc^2}{\lambda^5}\left[\frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}\right]$$
$$2\pi hc^2 = 3.7418\times 10^{-16}$$
$$\frac{hc}{k_B} = 0.00144$$
$$I_\lambda = \frac{3.7418\times 10^{-16}}{\lambda^5}\left[\frac{1}{e^{\frac{0.00144}{\lambda T}} - 1}\right]\text{W/m}^3 = \frac{3.7418\times 10^{-25}}{\lambda^5}\left[\frac{1}{e^{\frac{0.00144}{\lambda T}} - 1}\right]\text{W/m}^2\cdot\text{nm}$$

13.12 $E = hc/\lambda = 1.99\times 10^{-25}$ J m$/\lambda$. Since 1 eV = 1.602×10^{-19} J and 1 nm = 10^{-9} m, this gives $E = 1240$ eV nm$/\lambda$. Therefore the energy of a 600 nm photon is 2.1 eV.

13.13 λ(min) = 300 nm, $E = h\nu = hc/\lambda = 1240$ eV nm$/300$ nm $= 4.14$ eV $= 6.63\times 10^{-19}$ J.

13.14 If the $P = 100$ W light bulb has an efficiency of $\varepsilon = 2.5\%$, then the radiated power is $\varepsilon P = Nh\nu/t$, where N is the number of photons, ν is their frequency, and $t = 1$ ms. In terms of the wavelength $\lambda = 550$ nm, this gives $N = \varepsilon Pt/h\nu = \varepsilon Pt\lambda/hc$. The solid angle subtended by the $d = 3$ cm diameter aperture at distance $r = 100$ s is: aperature area/$r^2 = \pi d^2/4r^2$. Making the assumption that the light bulb emits isotropically, this is the fraction of photon that passes through the aperature, $N\pi d^2/4r^2 = \varepsilon Pt\lambda\pi d^2/4r^2hc = 4.9\times 10^8$ photons.

13.15 $Nh\nu = Nhc/\lambda = (1.4\times 10^3\ W/m^2)(1\ m^2)(1\ s)$ gives $N = 49\times 10^{20}$.

13.16 The number of Ar atoms present in the chamber is $N = pV/k_BT = 2.69\times 10^{17}$. Taking 1% of this number and using the given excited-state lifetime yields $0.01\,N/1.4\times 10^{-8}\,\text{s} = 1.9\times 10^{23}$ transitions per second.

13.17 With energy density

$$\rho(\nu) = \frac{A_{21}/B_{21}}{(B_{12}/B_{21})e^{h\nu/k_BT} - 1}$$

and $B_{12} = B_{21}$, the ratio of interest is $B_{21}\rho(\nu)/A_{21} = 1/(e^{h\nu/k_BT} - 1)$.

13.18 $k_BT = 4.14\times 10^{-21}\,\text{J}$, $E = 3.2\times 10^{-19}\,\text{J}$, $E/k_BT = 77.3$. Since $\exp(E/k_BT) \gg 1$, the ratio

$$1/(\exp(E/k_BT) - 1) \simeq \exp(-E/k_BT) \approx 2.7\times 10^{-34},$$

extremely low.

13.19 $k_BT = 4.14\times 10^{-19}\,\text{J}$, $E = 3.2\times 10^{-19}\,\text{J}$, $E/k_BT = 0.773$. Then $1/(\exp(E/k_BT) - 1) = 0.86$, much higher than in the previous problem. Stimulated emission is quite likely at this high temperature (as at the surface of a hotter star).

13.20 This statement gives the time rate of change of the population in from level 2 as it drops down to level 1. In thermal equilibrium, the rate of excitation to level 2 must equal the rate of downward transitions to level 1:

$$\frac{dN_2}{dt} = \frac{dN_1}{dt}$$

Using:

$$\frac{dN_2}{dt} = B_{12}u_\nu N_1 - B_{21}u_\nu N_2 - A_{21}N_2$$
$$\frac{dN_1}{dt} = -B_{12}u_\nu N_1 + B_{21}u_\nu N_2 + A_{21}N_2$$
$$-2B_{21}u_\nu N_2 - 2A_{21}N_2 = -2B_{12}u_\nu N_1$$
$$A_{21}N_2 + B_{21}u_\nu N_2 = B_{12}u_\nu N_1$$

13.21 Each photon may be said to be the result of a stimulated emission event.

$$E = hc = \frac{hc}{\lambda} = 4.499\times 10^{-19}\,\text{J}$$

$$\#\ of\ events = \frac{100\times 10^{-3}\ \text{J/s}}{4.499\times 10^{-19}\ \text{J}} = 2.22\times 10^{17}\ \text{photons/sec.}$$

13.22 $N_j/N_i = e^{-(E_j-E_i)/k_BT} \approx 1-(E_j-E_i)/k_BT$ for $|E_j-E_i| \ll k_BT$.

Therefore as $T\to\infty$, N_j/N_i, tends to 1.

13.23 $T = 3.0$ K, $k_BT = 4.14\times10^{-23}$ J,

$E = hc/\lambda = 1.99\times10^{-25}\,\text{J m}/\lambda = 9.51\times10^{-25}$ J,

$E/k_BT = 2.15\times10^{-2}$

and the ratio $1/(\exp(E/k_BT)-1)\simeq 46$, so stimulated emission is very likely, if not dominant. (The number of significant figures is important in such a case.)

13.24 From the example, $E_T = 2.59\times10^7\,\text{J/m}^3$.

$$P = \frac{E}{t} = \frac{2.59\times10^7\,\text{J/m}^3}{230\times10^{-6}\,\text{s}} = 1.13\times10^{11}\,\text{W/m}^3$$

13.25 $\tau = \dfrac{1}{\sum A_{ji}} = \dfrac{1}{7.8\times10^5+1.6\times10^5} = 1.06\ \mu\text{s}$

13.26 $\tau = \dfrac{1}{\sum A_{ji}} = \dfrac{10^{-5}}{259+2.83+2.00+2.26+6.09+6.39+33.9+3.45+13.9+2.55} = 1.63\times10^{-8}\,\text{s}$

13.27 $\Phi \approx 2.44\lambda/D = 5.15\times10^{-4}$ rad, $s = r\Phi = 5.15\times10^{-2}$ m or the diameter of the spot on the wall is 5.1 cm.

13.28 The volume of the crystal is $V = (\pi/4)D^2L = 9.8\times10^{-7}\,\text{m}^3$. Therefore the mass of Cr_2O_3 present is

$$0.05\times10^{-2}(3.7\times10^3)9.8\times10^{-7} = 1.81\times10^{-6}\ \text{Kg}.$$

The mass of one Cr_2O_3 molecule is 152 amu or 2.52×10^{-25} Kg. Therefore approximately 7.17×10^{18} Cr_2O_3 molecules. are present. Assuming that each contributes two Cr^{3+} ions to lasing, $N_{\text{ions}} = 1.4\times10^{19}$ ions participate in the lasing action, at $\Delta E = 2.87\times10^{-19}$ J. Then $E_{\text{tot}} = N_{\text{ions}}\,\Delta E = 4.0$ J; the corresponding power is $E_{\text{tot}}/t = 4.0\ \text{J}/5.0\times10^{-6}\text{s} = 8\times10^6\ \text{W} = 800$ kW.

13.29 $N/t = P/\Delta E = 1.0\times10^{-3}\,\text{J s}^{-1}/(1.96)(1.602\times10^{-19}\,\text{J}) = 3.2\times10^{15}$ transitions per second.

13.30 $V = \pi r^2 h = \pi(5\times10^{-3})^2(0.2)\ \text{m}^3 = 1.5708\times10^{-5}\ \text{m}^3$

# of photons = (*# of ions*) (*V*) (*efficiency*) = $(4.0\times10^{19}\ \text{ions/cm}^3)(15.708\ \text{cm}^3)(0.02)$

$= 1.2566\times10^{19}$ photons

$$E = hc = \frac{hc}{\lambda} = 2.834\times10^{19}\,\text{J}$$

$(1.2566\times10^{19}\ \text{photons})(2.834\times10^{-19}\ \text{J/photon}) = 3.56$ J

13.31 $\Delta\lambda_0 = \bar{\lambda}_0^2\Delta\nu/c = 8.0\times10^{-5}$ nm.

13.32 $\Delta\nu = c/2L = 6\times10^8$ Hz, using $\nu = c$ when $n = 1$.

13.33 $\Delta\nu = \dfrac{c}{2L} = \dfrac{3\times10^8 \text{ m/s}}{(2)(1 \text{ m})} = 1.5\times10^8 \text{Hz}$

$\dfrac{2.9\times10^9 \text{ Hz}}{1.5\times10^8 \text{ Hz}} = 18 \text{ intervals}$

19 modes

13.34 $\nu_m = \dfrac{m\upsilon}{2L} = \dfrac{mc}{2nL} = \dfrac{c}{\lambda}$

$m = \dfrac{2nL}{\lambda} = \dfrac{2(0.04 \text{ m})(1)}{6.00\times10^{-7} \text{m}} = 1.33\times10^6$

13.35 The condition $\Delta\nu = 1.4\times10^9 = c/2L$ for $n = 1$ gives $L = c/2\Delta\nu = 11$ cm.

13.36 $g_{th} = \alpha + \left(\dfrac{1}{2L}\right)\ln\left(\dfrac{1}{R_1R_2}\right) = 10 \text{ cm}^{-1} + \left(\dfrac{1}{2(0.03 \text{ cm})}\right)\ln(6.25) = 40.5 \text{ cm}^{-1}$

13.37 $I = (\upsilon/2)\varepsilon E_0^2 = (n/2)(\varepsilon_0/\mu_0)^{1/2} E_0^2$, where $\mu \approx \mu_0$, $E_0^2 = 2(\mu_0/\varepsilon_0)^{1/2} I/n$, $(\mu_0/\varepsilon_0)^{1/2} = 376.730\Omega$, so $E_0 = 27.4(I/n)^{1/2}$.

13.38 $\Phi \approx 2.44\lambda/D = 2.6\times10^{-3}$ rad.

13.39 The three crossed gratings form a type of triangular lattice. The diffraction spots will appear along the directions of the dual lattice, which are directions connecting the centroids of the original lattice. As usual, there will be a central spot of highest irradiance (intensity) and the irradiance decreases with distance from this central spot. Reciprocals of multiples of lattice constants of the original lattice are proportional to the spatial frequencies present in the diffraction pattern. (Strictly speaking, the lattice should have infinite extent in order to consider it a mathematically periodic structure.)

13.40 In this case, the four crossed gratings form a sort of rectangular lattice, whose dual lattice is again rectangular. Therefore the diffraction spots will be located along horizontal and vertical lines. The central spot has the highest irradiance and the irradiance of the others decreases with distance from the center. A horizontal slit filter will keep the vertical lined grating the same in the altered image, but blur the horizontal lines.

13.41 The horizontal grating gives a row of diffraction spots, with the central spot of highest irradiance. The details in the picture image are contained in many high spatial frequency components. The picture can be enhanced by using a filter which blocks out the diffraction pattern of the horizontal grating.

13.42 The circular grating present will generate a central spot of highest irradiance, together with successive rings. In order to enhance the picture, a spatial filter which blocks out these contributions should be used.

13.43 The filter is a long slit, perpendicular to the observed image.

13.44 From the geometry, $f_t\theta = f_i\Phi : k_O = k\sin\theta$ and $k_1 = k\sin\Phi$, hence $\sin\theta \approx \theta \approx k_O\lambda/2\pi$ and $\sin\Phi \approx \Phi \approx k_I\lambda/2\pi$, therefore $\theta/\Phi = k_O/k_I$ and $k_I = k_O(\Phi/\theta) = k_O(f_t/f_i)$. When $f_i > f_t$ the image will be larger than the object, the spatial periods in the image will also be larger, and the spatial frequencies in the image will be smaller than in the object.

13.45 $a = (1/50)$ cm: $a \sin\theta = m\lambda$, $\sin\theta \approx \theta$, hence $\theta = 5000\, m\lambda$ and the distance between orders on the transform plane is $f\theta = 5000\lambda f = 2.7$ mm.

13.46 (a) As in Figure 11.10, the transform of the cosine function will be a pair of δ-functions, at $f_x = \pm 1/\mathrm{d}$, where d is the spatial period of the cosine plus a zero-order term at $x = 0$. To pass only the first order terms, we need a filter with holes at these positions, for the specific wavelength, as given by $x/f \simeq \sin\theta_1 = \lambda(1/d)$; $x = f\lambda/d = [(2.0\text{ m})(5\times10^{-7}\text{ m})/(1\times10^{-5}\text{ m})] =$ 0.1 m, above and below center. (b) Any "DC" components, and all higher order components, are removed. A smoothly varying cosine function should be seen in the image. (c) A filter with a hole in the center would pass only the "DC" term, resulting in a lower intensity, uniform image.

13.47 Each point on the diffraction pattern corresponds to a single spatial frequency, and if we consider the diffracted wave to be made up of plane waves, it also corresponds to a single-plane wave direction. Such waves, by themselves, carry no information about the periodicity of the object and produce a more or less uniform image. The periodicity of the source arises in the image when the component plane waves interfere.

13.49 The relative field amplitudes are 1.00, 0.60, and 0.60; hence $E \propto 1 + 0.60\cos(+ky') + 0.60\cos(-ky') = 1 + 1.2\cos ky'$. This is a cosine oscillating about a line equal to 1.0. It varies from +2.2 to –0.2. The square of this will correspond to the irradiance, arid it will be a series of tall peaks with a relative height of $(2.2)^2$, between each pair of which there will be a short peak proportional to $(0.2)^2$; notice the similarity with Fig. 11.32.

13.50 a Sin $\theta = \lambda$, here $f\theta = 50\lambda f = 0.20$ cm; hence $\lambda = 0.20/50(100) = 400$ nm. The magnification is 1.0 when tile focal lengths are equal, hence the spacing is again 50 wires/cm.

13.51 The random (dots will add considerable "noise" to the pattern. The spatial frequency is 1/(0.1 mm) = 10 mm^{-1}. A filter that is the transform of the regular pattern will remove the random dots.

13.52 The array of top hats corresponds to the pixels, so that each "selects" the amplitude (density) of the picture within its radius. The transform will look like a regular array of dots of varying amplitude. As in Figure 13.39, filtering out the higher frequency components will yield a continuous image.

13.53 The pinhole blocks the high-frequency components, which correspond to the rapid spatial variations in the beam.

13.54 The randomly, but more or less uniformly, distributed particles in the milk will tend to block the "regular" part of the beam, and thus enhance the relative intensity of the speckle.