Chapter 11 Solutions

11.1 $E_0 \sin k_p x = E_0(\exp(ik_p x) - \exp(-ik_p x))/2i$;

$$F(k) = \frac{E_0}{2i}\left[\int_{-L}^{L} \exp(i(k+k_p)x)\,dx - \int_{-L}^{L} \exp(i(k-k_p)x)\,dx\right]$$
$$= -\frac{iE_0 \sin(k+k_p)L}{(k+k_p)} + \frac{iE_0 \sin(k-k_p)L}{(k-k_p)}$$
$$F(k) = iE_0 L[\mathrm{sinc}(k-k_p)L - \mathrm{sinc}(k+k_p)L].$$

11.2 (11.5) $F(k) = \int_{-\infty}^{\infty} f(x)\exp(ikx)dx = \int_{-L}^{L} \sin^2 k_p \exp(ik_p x)\,dx$

$$= \int_{-L}^{L} \sin^2 k_p x \cos k_p x dx + i\int_{-L}^{L} \sin^3 k_p x\,dx$$
$$= \frac{1}{3k_p}\sin^3 k_p x\Big|_{-L}^{L} + 0 = (2/3k_p)(\sin^3 k_p L).$$

11.3 $\cos^2 \omega_p t = 1/2 + (1/2)\cos 2\omega_p t = 1/2 + (1/4)(\exp(2i\omega_p t) + \exp(-2i\omega_p t)).$

$$F(\omega) = \frac{1}{2}\int_{-T}^{T} \exp(i\omega t)\,dt + \frac{1}{4}\int_{-T}^{T} \exp(i(\omega+2\omega_p t))\,dt + \frac{1}{4}\int_{-T}^{T} \exp(i(\omega-2\omega_p)t)\,dt$$
$$= \frac{1}{\omega}\sin \omega T + \frac{1}{2(\omega+2\omega_p)}\sin(\omega+2\omega_p)T + \frac{1}{2(\omega-2\omega_p)}\sin(\omega-2\omega_p)T$$
$$F(\omega) = T\ \mathrm{sinc}\,\omega T + (T/2)\mathrm{sinc}(\omega+2\omega_p)T + (T/2)\mathrm{sinc}(\omega-2\omega_p)T.$$

11.4 Show that $F^{-1}\{F(K)\} = f(x)$, where

$$f(x) = 1, \quad F(K) = 2\pi\delta(K).$$

(11.4) $$f(x) = (1/2\pi)\int_{-\infty}^{\infty} F(K)\exp(-iKx)\,dK$$
$$= (1/2\pi)\int_{-\infty}^{\infty} 2\pi\delta(K)\exp(-iKx)\,dK = \exp(0) = 1.$$

11.5 $f(x) = A\cos K_0 x = \dfrac{A}{2}(\exp(iK_0 x) + \exp(-iK_0 x))$

$$F(K) = \frac{A}{2}\int_{-\infty}^{\infty} (\exp(iK_0 x) + \exp(-iK_0 x))\exp(iKx)\,dx = \frac{A}{2}[\delta(K-K_0) + \delta(K+K_0)]$$
$$= \frac{A}{2}\left[\delta\left(f_x - \frac{K_0}{2\pi}\right) + \delta\left(f_x + \frac{K_0}{2\pi}\right)\right]$$

11.6 $E(\omega) = \int (E_0 \exp(-t^2/2\tau^2)\exp(-i\omega_0 t))\exp(i\omega t)\,dt$

$$E(\omega) = E_0\int \exp(-t^2/2\tau^2 - i(\omega-\omega_o)t)\,dt$$
$$E(\omega) = E_0(2\tau^2\pi)^{\frac{1}{2}}\exp(-\tfrac{1}{4}[(i(\omega-\omega_0))^2 2\tau^2])$$
$$E(\omega) = E_0(2\tau^2\pi)^{\frac{1}{2}}\exp(-\tfrac{1}{4}(\omega-\omega_0)^2 2\tau^2)$$
$$E(\omega) = \sqrt{2\pi}E_0\tau\exp(-\tau^2(\omega-\omega_0)^2/2)$$

11.7 $E(t)=\dfrac{\sqrt{2\pi}}{2\pi}\int_{-\infty}^{\infty}(E_0\tau\exp(-\tau^2(\omega-\omega_0)^2/2))\exp(-i\omega t)\,d\omega$

$$E(t)=\frac{E_0\tau}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-\tfrac{\tau^2}{2}\,\omega^2+(\tau^2\omega_0-it)\omega-\tfrac{-\tau^2}{2}\omega_0^2)\,d\omega$$

Using the integral identity from problem 11.6:

$$\int_{-\infty}^{\infty}\exp(-ax^2+bx+c)\,dx=\left(\frac{\pi}{a}\right)^{1/2}\exp(\tfrac{1}{4}(b^2/a)+c)$$

$$E(t)=\frac{E_0\tau}{\sqrt{2\pi}}\left(\frac{2\pi}{\tau^2}\right)^{1/2}\exp(\tfrac{1}{4}(\tau^2\omega_0-it)/(\tau^2/2)^2-\tfrac{-\tau^2}{2}\omega_0^2)$$

$$E(t)=\frac{E_0\tau}{\sqrt{2\pi}}\left(\frac{2\pi}{\tau^2}\right)^{1/2}\exp(\tfrac{1\tau^4\omega_0^2}{2\tau^2}-\tfrac{t^2}{2\tau^2}-\tfrac{i\tau^2\omega_0 t}{\tau^2}-\tfrac{-\tau^2}{2}\omega_0^2)$$

$$E(t)=\frac{E_0\tau}{\sqrt{2\pi}}\left(\frac{2\pi}{\tau^2}\right)^{1/2}\exp(-\tfrac{t^2}{2\tau^2}-i\omega_0 t)$$

11.8 $F(k)=\int f(x)e^{-ikx}dx=\int f(x)\cos kx\,dx+\int f(x)i\sin kx\,dx$

$$=\int f_R(x)\cos kx\,dx+\int f_I(x)\sin kx\,dx+\int f_I(x)\cos kx\,dx+\int f_R(x)\sin kx\,dx$$

If $f(x)$ is real, then $f_I(x)\to 0$:

$$=\int f_R(x)\cos kx\,dx+\int f_R(x)\sin kx\,dx$$

If $f(x)$ is even:

$$F(k)=\int f_R(x)\cos kx\,dx$$

11.9 $F[af(x)+bh(x)]=aF(k)+bH(k)$.

11.11 $F(K)=L\,\mathrm{sinc}^2 kL/2.\ F(0)=L, F(\pm 2\pi/L)=0.$
or, $F(f_x)=L\,\mathrm{sinc}^2(\pi f_x L)$, where $k=2\pi f_x$.

11.12 $F(K)=\int_{-\infty}^{\infty}f(x)\exp(ikx)\,dx$, (11.5). Let $x\to x/a$;

$$F(K')=\int_{-\infty}^{\infty}f(x/a)\exp(ik'(x/a))\ d(x/a).$$

So, $K'\to Ka$, and $F\{f(x/a)\}=F(Ka)$. If, $a=-1$, $F\{f(-x)\}=F(-K)$.

11.13 $F(K)=F\{f(x)\}=\int_{-\infty}^{\infty}f(x)\exp(ikx)\,dx$, (11.5) (a function of K).

$$F\{F\{f(x)\}\}=F\{F(K)\}=\int_{-\infty}^{\infty}F(K)\exp(ikx)\,dK.$$

(11.4) $f(x)=(1/2\pi)\int_{-\infty}^{\infty}F(K)\exp(-iKx)\,dK$,

so, $2\pi f(-x)=\int_{-\infty}^{\infty}F(K)\exp(iKx)\,dK=F\{F\{f(x)\}\}\neq f(x)$.

11.14 $F\{\text{rect}|(x-x_0)/a|\} = \int_{-\infty}^{\infty}\{\text{rect}|(x-x_0)/a|\}\exp(iKx)\,dx = \int_{-1/2}^{1/2}\exp(iKx)\,dx$

$$= \frac{1}{iK}\exp(iKx)\Big|_{-1/2}^{1/2} = \frac{1}{iK}(\exp(iK/2)-\exp(-iK/2)) = \frac{2}{K}\sin(K/2) = \text{sinc}(K/2)$$

$$F\left\{\text{rect}\left|\frac{x}{a}\right|\right\} = a\,\text{sinc}(\pi f_x a)$$

Now apply the shift theorem by x_0:

$$F\{\text{rect}|(x-x_0)/a|\} = a\,\text{sinc}\,(\pi f_x a)\exp(-i2\pi f_x x_0) = a\,\text{sinc}\,(Ka/2)\exp(-iKx_0)$$

where, $K = 2\pi f_x$.

11.15 $F\{\text{rect}\,|x|\} = \text{sinc}\,(K/2)$, from Problem 11.10, $F\{F\{f(x)\}\} = 2\pi f(-x)$.

So, $F\{(1/2\pi)F\{f(-x)\} = f(x)$, let

$$f(x) = \text{rect}\,|x|, \quad F\{\text{rect}\,|x|\} = \text{sinc}\,(K/2);$$
$$F\{(1/2\pi)F\{\text{rect}\,|x|\}\} = F\{(1/2\pi)\text{sinc}\,(K/2)\} = f(x) = \text{rect}\,|x|,$$

since sinc $(-x)$ = sinc (x).

11.16 $F^{-1}\{F\{f(x)\}\} = (1/2\pi)\int_{-\infty}^{\infty}\exp(-iKx)\,dK\int_{-\infty}^{\infty}f(x')\,\exp(iKx')\,dx'$

$$= \int_{-\infty}^{\infty}dx'(\int_{-\infty}^{\infty}\exp(iK(x'-x))\,dK)\,f(x')$$
$$= \int_{-\infty}^{\infty}\delta(x-x')\,f(x')\,dx' = f(x),$$

since the integral is zero except at $x = x'$.

11.17 $F\{f(x-x_0)\} = \int_{-\infty}^{\infty}f(x-x_o)\,\exp(iKx)\,dx$. Change variables, $x' \equiv x - x_o$,

$dx' = dx$. $F\{f(x')\} = \int_{-\infty}^{\infty}f(x')\exp(iK(x'+x_o))\,dx' = \exp(iKx_o)\int_{-\infty}^{\infty}f(x')\exp(iKx')\,dx'$. So

that $F\{f(x-x_o)\}$ differs from $F\{f(x)\}$ by only the phase factor $\exp(iKx_o)$.

11.18 $\int_{-\infty}^{\infty}f(x)h(X-x)\,dx = -\int_{\infty}^{-\infty}f(X-x')h(x')\,dx' = \int_{-\infty}^{\infty}h(x')f(X-x')\,dx'$

where $x' = X - x$, $dx = -dx'$. $f*h = h*f$ or

$F(f*h) = F(f)\cdot F(h) = F(h)\cdot F(f) = F(h*f)$.

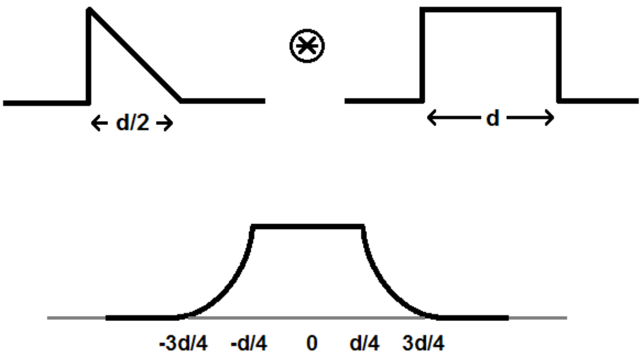
11.19 $\int(f*h)dt = \int\left[\int(u)h(u-t)\,du\right]dt$

$$= \int f(u)\left[\int h(u-t)\,dt\right]du = \left[\int f(u)\,du\right]\left[\int h(t)\,dt\right]$$

11.20 $g(X)$ is the area under the product function $f(x)h(X-x)$. The shape of $g(X)$ reflects the overlap between f and h: It is non-zero in regions where the two functions overlap; it is maximum at the coordinate X where the two functions have their largest overlap. Since f and h are symmetrical about $X = 0$, their product (and thus g) will be symmetrical and have its maximum at $X = 0$. The width of $g(X)$ is the sum of the widths of the functions $f(x)$ and $h(x)$: Thus the width is 3. The peak value is 0.75.

11.22 A point on the edge of $f(x, y)$, for example, at $(x = d, y = 0)$, is spread out into a square $2l$ on a side centered on $X = d$. Thus it extends no farther than $X = d + l$, and so the convolution must be zero at $X = d + l$ and beyond.

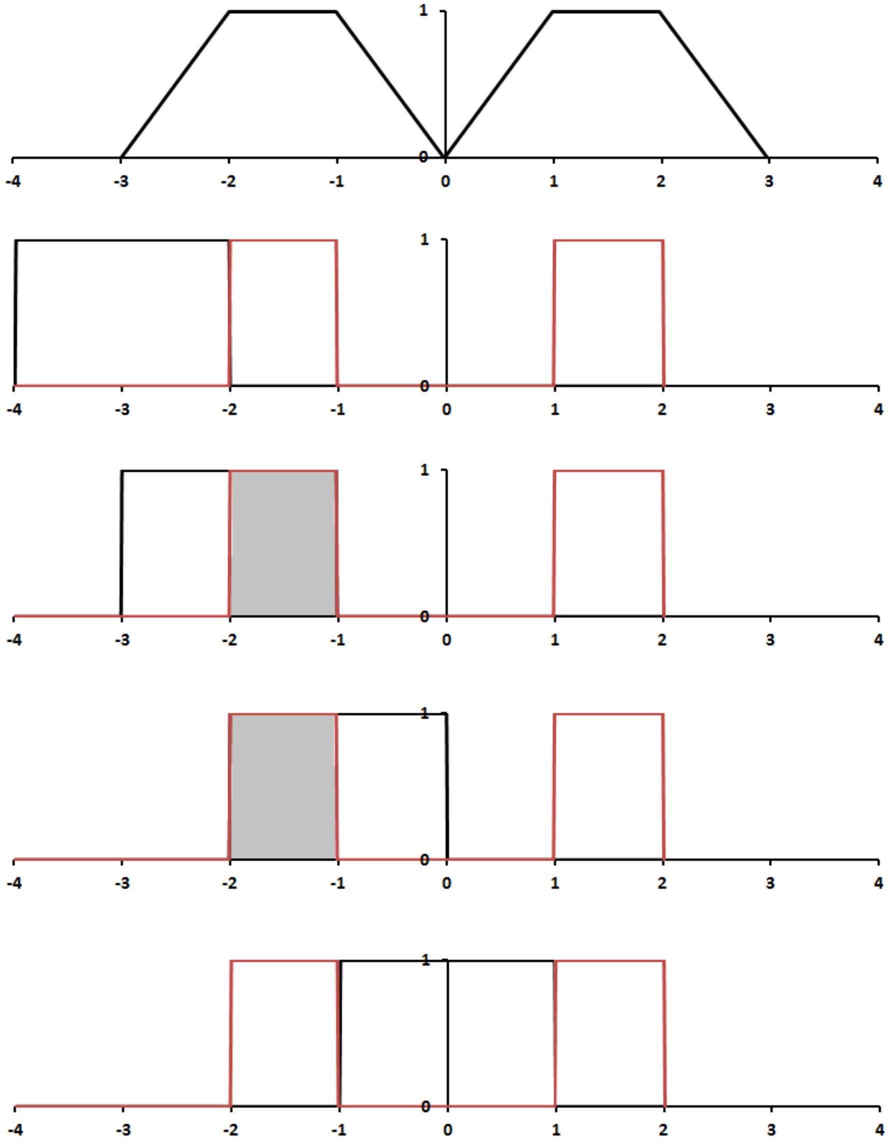
11.23:



11.24 $f(x - x_0) * h(x) = \int_{-\infty}^{\infty} f(x - x_0)\, h(X - x)\, dx,$ and setting $x - X_0 = \alpha$, this becomes $\int_{-\infty}^{\infty} f(\alpha)\, h(X - \alpha - x_0)\, d\alpha = g(X - x_0)$.

11.25 The convolution is asymmetric because the sawtooth function itself is asymmetric (*i.e.*, the shape of the convolution reflects the overlap between the two sawtooth functions). The convolution begins at zero because there is no overlap at positions $x < 0$. The width of the convolution is the sum of the widths of the functions; it is twice the width of the sawtooth function and is thus 2.

11.26 The width of $g(X)$ is the sum of the widths of the functions $f(x)$ whose width is 2, and $h(x)$ whose width is 4: Thus the width is 6.



11.27 $g(X) = \int_{-\infty}^{\infty} f(x)h\left(X-x\right)dx,$ (11.52)

$$= \int_{-\infty}^{\infty} \delta(x)h(X-x)\,dx = h(X-0)\int_{-\infty}^{\infty} \delta(x)\,dx, \text{ (see Section 11.2.3)},$$

$$= h(X), \qquad \text{since } \int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

11.28 For the solution to this problem, please refer to the textbook.

11.29 $F\{f(x)\cos K_o x\} = F\{f(x)(1/2)(\exp(iK_o x)+\exp(-iK_o x))\}$

$$= (1/2)\left[\int_{-\infty}^{\infty} f(x)\exp(i(K+K_o)x)\,dx + \int_{-\infty}^{\infty} f(x)\exp(i(K-K_o)x)\,dx\right]$$

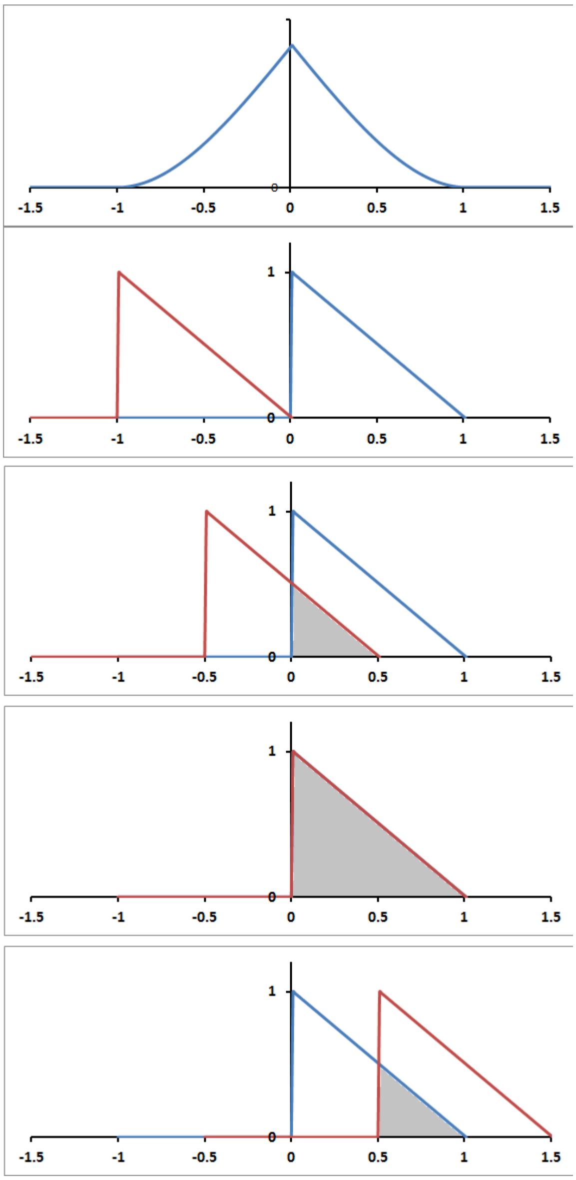
$$= (1/2)[F(K+K_o)+F(K-K_o)].$$

$F\{f(x)\sin K_o x\} = F\{f(x)(1/2i)(\exp(iK_o x)-\exp(-iK_o x))\}$

$$= (1/2i)\left[\int_{-\infty}^{\infty} f(x)\exp(i(K-K_o)x)\,dx - \int_{-\infty}^{\infty} f(x)\exp(i(K+K_o)x)\,dx\right]$$

$$= (1/2i)[F(K-K_o)-F(K+K_o)].$$

11.31 The solution is symmetrical because $f(x) = h(-x)$. The peak occurs when there is maximum overlap of $f(x)$ and $h(x)$ (*i.e.*, at 0). The width of $g(X)$ is the sum of the widths of the functions $f(x)$ and $h(x)$ whose widths are 1: Thus the width is 2. The graphical convolution is seen below.

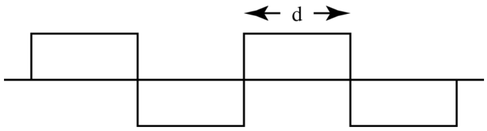


11.32 We see that $f(x)$ is the convolution of a rect-function with two δ-functions, and from the convolution theorem,

$$\begin{aligned} F(k) &= F\{(\text{rect}(x) * [\delta(x-a)+\delta(x+a)])\} \\ &= F[\text{rect}(x)] \cdot F\{[\delta(x-a)+\delta(x+a)]\} \\ &= a \text{ sinc } (ka/2) \cdot (\exp(ikx)+\exp(-ika)) = a \text{ sinc } (ka/2) \cdot 2\cos ka. \end{aligned}$$

11.33 $f(x) * h(x) = [\delta(x+3)+\delta(x-2)+\delta(x-5)] * h(x) = h(x+3)+h(x-2)+h(x-5).$

11.34



11.35 $F\left\{\text{rect}\left|x/(d/2)\right|\right\} = \dfrac{d}{2}\text{sinc}\left(\dfrac{kd}{4}\right) = F(k)$

$F\left\{\sum_{n=-\infty}^{\infty} \delta(x-nd)\right\} = \sum_{n=-\infty}^{\infty} \exp(iKnd) = H(K);\ G\left(K\right) = H(K)\,F(K);$

$F(K) = F\left\{f(x)\right\}$ is zero at $knd = n\pi$ or $kd = \pi.$

11.36 For the solution to this problem, please refer to the textbook.

11.37 $A(y,z) = A(-y,-z).$

$$E\left(Y, Z, t\right) \propto \iint A(y,z)\exp(i(k_Y y + k_Z z))\, dy\, dz.$$

Change Y to $-Y$, Z to $-Z$, y to $-y$, z to $-z$, then k_Y goes to $-k_Y$ and k_Z to $-k_Z$.

$$E\left(Y, Z, t\right) \propto \iint A(-y,-z)\exp(i(k_Y y + k_Z z))\, dy\, dz.$$

Therefore $E(-Y, -Z) = E(Y, Z)$.

11.38 From Eq. (11.63),

$$E\left(Y, Z\right) = \iint A(y,\, z)\exp(ik(Yy+Zz)/R)\, dydz.$$

$$E'\left(Y, Z\right) = \iint A(\alpha y,\, \beta z)\exp(ik(Yy+Zz)/R)\, dydz;$$

now let $y' = \alpha y, \quad z' = \beta z$:

$$E'(Y,Z) = \frac{1}{\alpha\beta}\iint A\left(y', z'\right) e(ik[(Y/\alpha)y' + (Z/\beta)z']/R)\, dy'\, dz'$$

or $E'(Y,Z) = (1/\alpha\beta)\, E(Y/\alpha,\, Z/\beta)$.

11.39 $\displaystyle C_{ff} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A\sin(\omega t + \varepsilon)\, A\sin(\omega t - \omega\tau + \varepsilon)\, dt$

$\displaystyle \quad = \lim_{T\to\infty} \frac{A^2}{2T}\int_{-T}^{T}\left[\frac{1}{2}\cos(\omega\tau) - \frac{1}{2}\cos(2\omega t - \omega\tau + 2\varepsilon)\right] dt,$

since $\cos\alpha - \cos\beta = -2\sin\left[(1/2)(\alpha+\beta)\right]\sin\left[(1/2)(\alpha-\beta)\right]$.

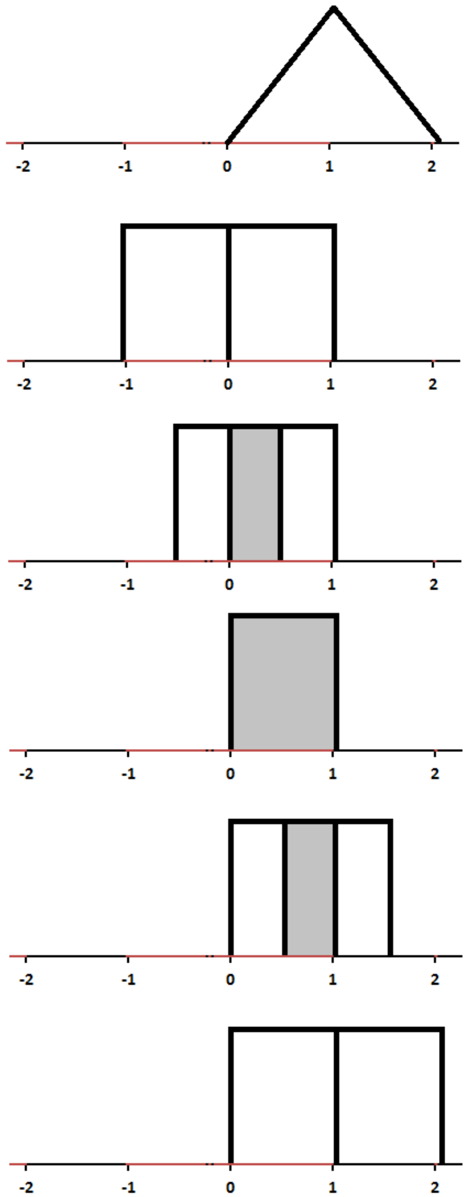
Thus $C_{ff} = (A^2/2)\cos(\omega\tau)$.

11.40 $E(kz) = \int_{-b/2}^{b/2} A_0 \cos(\pi z/b)\exp(ik_Z z)\, dz$

$$= A_0 \int \cos(\pi z/b)\cos(k_Z z)\, dz + iA_0 \int \cos(\pi z/b)\sin(k_Z z)\, dz$$

$$E(k_Z) = A_0 \cos\frac{bk_Z}{2}\left[\frac{1}{\pi/b - k_Z} + \frac{1}{\pi/b + k_Z}\right].$$

11.41 The width of $g(X)$ is the sum of the widths of the functions $f(x)$ and $h(x)$ whose widths are 1: Thus the width is 2. The peak will be at $x = 1$.



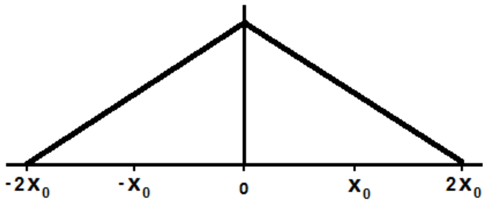
11.42 $c_{ff}(x) = \int_{-\infty}^{\infty} \cos(ku+\varepsilon)\cos(ku+\varepsilon+kx)\, du$

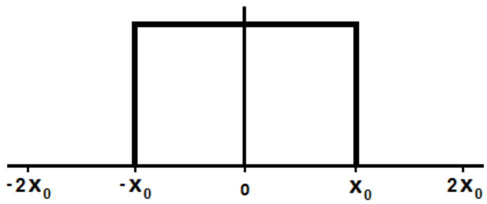
Here x was replaced with $u + x$. Using the identity:

$$\cos A \cos B = \frac{1}{2}\left(\cos(A+B)+\cos(A-B)\right)$$

$$c_{ff}(x) = \frac{1}{2}\int_{-\infty}^{\infty} [\cos(2ku+2\varepsilon+kx)+\cos(kx)]\, du = \frac{1}{2}\cos kx$$

11.43 The width of the autocorrelation function is twice the width of the function, thus the width of $c_{ff}(x)$ is $4x_0$. The function is an even function which starts at $-2x_0$ and ends at $2x_0$.





11.44 Symmetry is a fundamental property of autocorrelation. The maximum overlap occurs at the when $X = x = 0$, thus the correlation function $c_{ff}(x)$ will be symmetric about the origin. The width of the correlation function is twice that of the original function. Since it is symmetrical about the origin, it will stretch from -1 to 1.

11.45 (From 11.52). $h(X) = f(x) \circledast g(x) = \int_{-\infty}^{\infty} f(x)\, g(X-x)\, dx$, so,

$f(x) * g(-x) = h'(x) = \int_{-\infty}^{\infty} f(x) g\left(X+x\right) dx$, which is the form of (11.86),

so $f(x) * g(-x) = f(x) \odot g(x)$.

11.46 The width of the correlation function is twice that of the original function. The width of each individual peak is twice the width of each individual square pulse.

11.47 The autocorrelation is periodic.

11.48 There will be three spots in the autocorrelation function. The first results from the overlap of only one spot from each pair (the original pair and its duplicate). Because $f(x, y)$ is symmetrical, mirroring it has no effect. Thus one just sweeps one circle over the other and records the product at each displacement. The resulting irradiance can be seen in Fig. 11.37b. The second spot will be separated from the first by the same distance as that between the two original spots. This second spot will be brighter than the first because it results from the overlap of both spots from each pair. The resulting irradiance pattern is the same as that for the first except it will be brighter (*i.e.*, it can be seen in Fig. 11.37b). The third spot will be identical to the first spot, and will be located the same distance from the bright spot as the first spot is from the bright spot. The width of the autocorrelation function is twice the width of the original pattern.

11.50 $f(t) = g(t) \odot h(t) = A\cos(\omega_o t) \odot e(-i\omega_o t)$.

$F(\omega) = G(\omega)\, H(\omega)$; $\quad G(\omega) = (A/2)(2\pi/(\omega+\omega_o)+2\pi/(\omega-\omega_o))$;

$H(\omega) = 2\pi/\omega-\omega_o$,

so, $F(\omega) = 2\pi^2 A/(\omega-\omega_o)(2\omega/\omega^2-\omega_o^2)$.