Escola Politécnica da Universidade de São Paulo
Departamento de Engenharia Mecânica

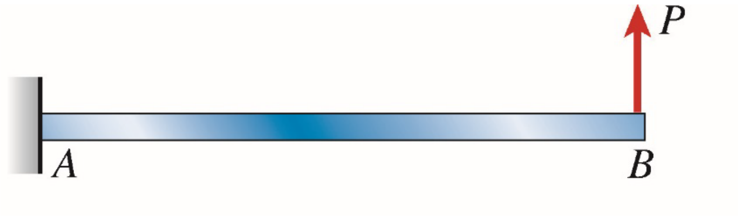
PME-3210 - Mecânica dos Sólidos I

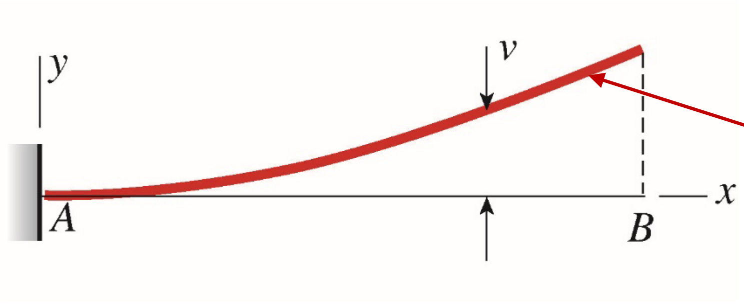
Aula #23

Prof. Dr. Clóvis de Arruda Martins

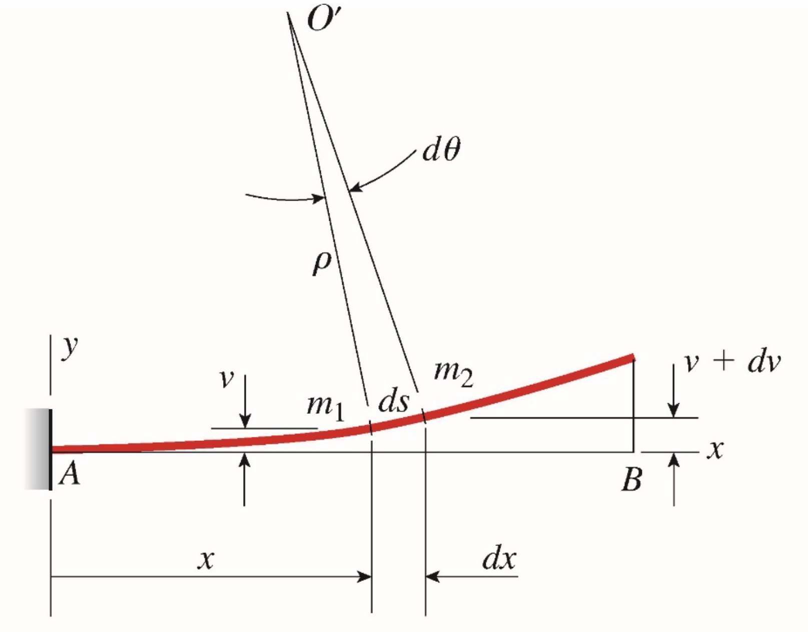
27/06/23

9.2 Equações diferenciais da curva de deflexão





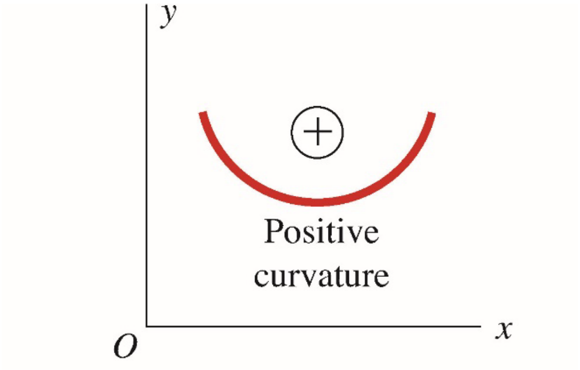
Curva de deflexão ou linha elástica

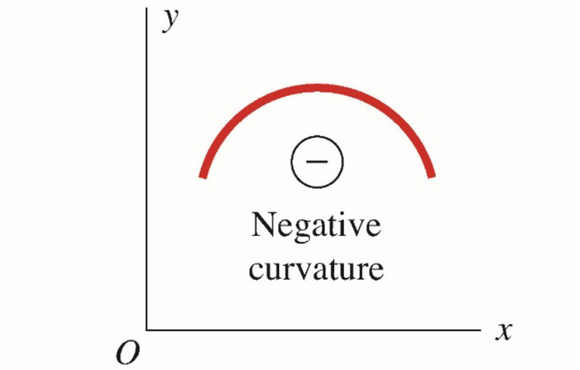


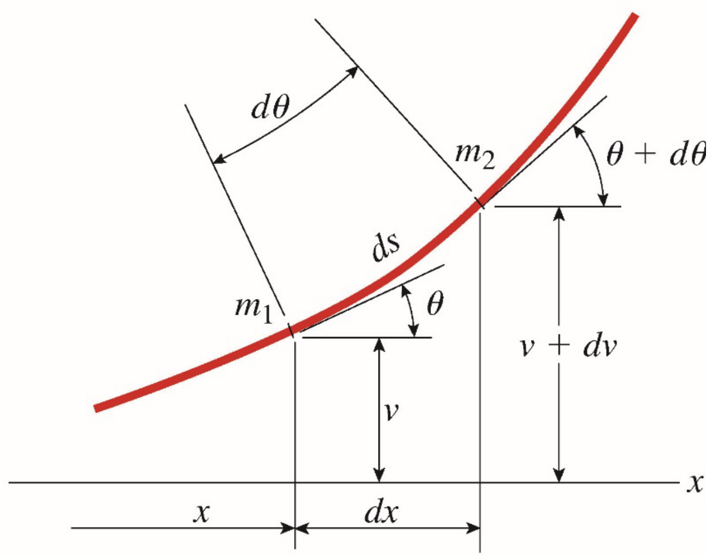
$$\rho d\theta = ds$$

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds}$$

Convenção de sinais:







Para pequenos ângulos de rotação:

$$ds \approx dx \qquad \theta \approx \tan\theta = \frac{dv}{dx}$$

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx} \qquad \Rightarrow \kappa = \frac{1}{\rho} = \frac{d^2v}{dx^2}$$

$$\frac{dv}{dx} = \tan\theta \qquad \theta = \arctan\frac{dv}{dx}$$

$$\cos\theta = \frac{dx}{ds} \qquad \operatorname{sen}\theta = \frac{dv}{ds}$$

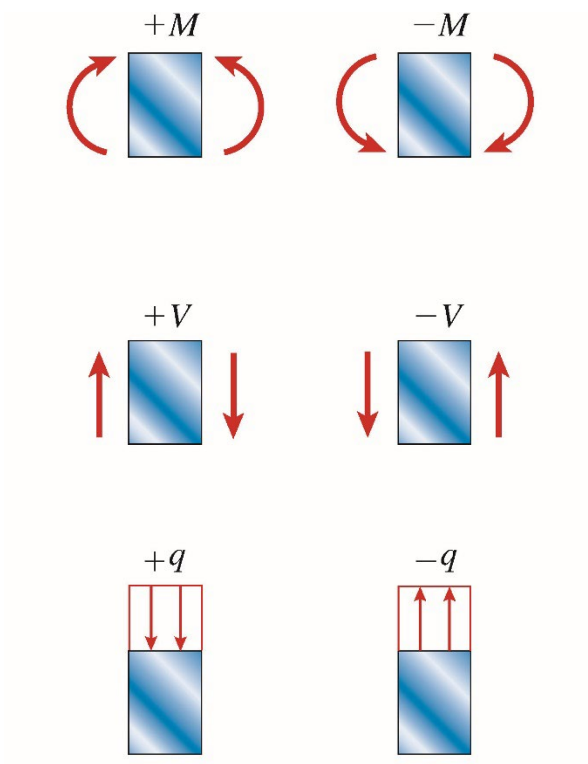
Para material elástico linear:

$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

$$\Rightarrow \frac{d^2v}{dx^2} = \frac{M}{EI} \qquad \Rightarrow EI\frac{d^2v}{dx^2} = M$$

(equação diferencial da curva de deflexão ou equação diferencial da linha elástica)

Convenções de sinais:



Já vimos que:

$$\frac{dV}{dx} = -q \qquad \frac{dM}{dx} = V$$

Como:

$$EI\frac{d^2v}{dx^2} = M$$

Então:

$$\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = \frac{dM}{dx} = V$$

Para viga prismática ($EI = cte$)

$$EI\frac{d^3v}{dx^3} = V$$

Portanto, para uma viga prismática:

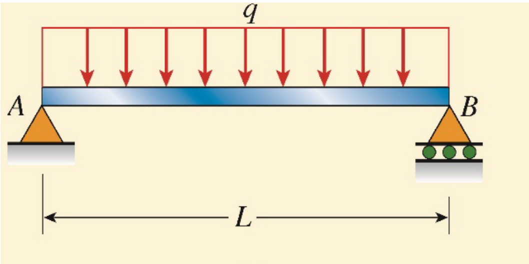
$$EI\frac{d^2v}{dx^2} = M \qquad EI\frac{d^3v}{dx^3} = V \qquad EI\frac{d^4v}{dx^4} = -q$$

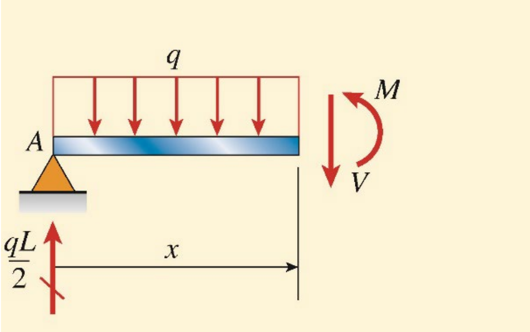
ou, usando a notação de Lagrange,

$$EIv'' = M \qquad EIv''' = V \qquad EIv'''' = -q$$

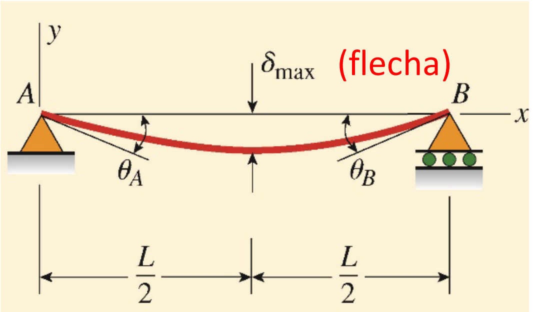
9.3 Deflexões por integração da equação do momento fletor

Exemplo





$$M(x) = \frac{qLx}{2} - \frac{qx^2}{2}$$

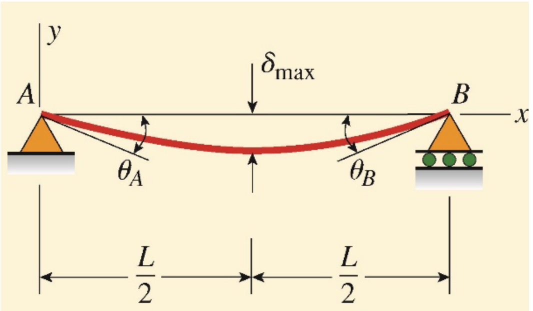


$$EIv'' = M(x)$$

$$EIv'' = \frac{qLx}{2} - \frac{qx^2}{2}$$

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1$$

$$EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_1 x + C_2$$



$$EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_1 x + C_2$$

As constantes C_1 e C_2 são obtidas a partir das condições de contorno (vínculos)

$$v(0) = 0 \Rightarrow 0 = C_2$$

$$v(L) = 0 \Rightarrow 0 = \frac{qL^4}{12} - \frac{qL^4}{24} + C_1 L \Rightarrow C_1 = -\frac{qL^3}{24}$$

Portanto:

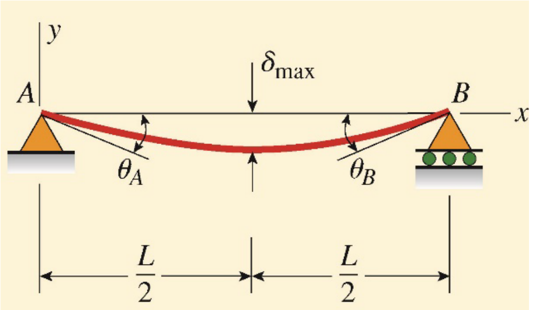
$$v = \frac{1}{EI}\left(\frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^3 x}{24}\right)$$

ou

$$v = -\frac{qx}{24EI}(L^3 - 2Lx^2 + x^3)$$

$$v\left(\frac{L}{2}\right) = -\frac{5\mathrm{qL}^4}{384\mathrm{EI}}$$

$$\delta_{máx} = \left|v\left(\frac{L}{2}\right)\right| = \frac{5\mathrm{qL}^4}{384\mathrm{EI}}$$



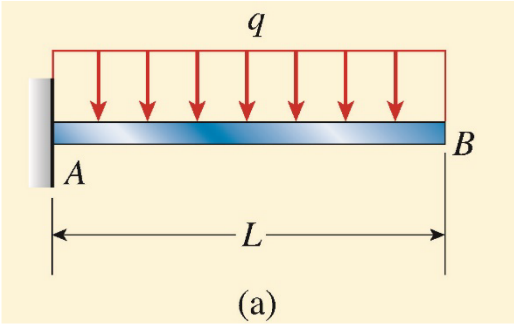
$$v = \frac{1}{EI}\left(\frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^3x}{24}\right)$$

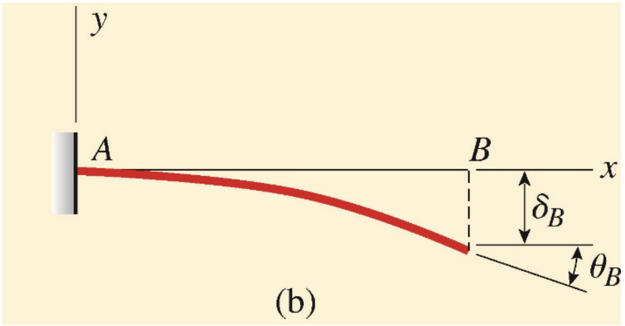
$$v' = \frac{1}{EI}\left(\frac{3qLx^2}{12} - \frac{4qx^3}{24} - \frac{qL^3}{24}\right)$$

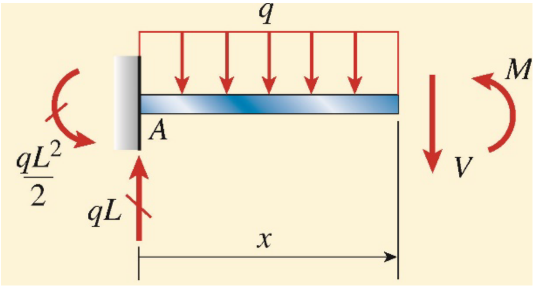
$$\theta_A = -v'(0) \;\; = \frac{qL^3}{24EI}$$

$$\theta_B = v'(L) \;\; = \frac{qL^3}{24EI}$$

Exemplo





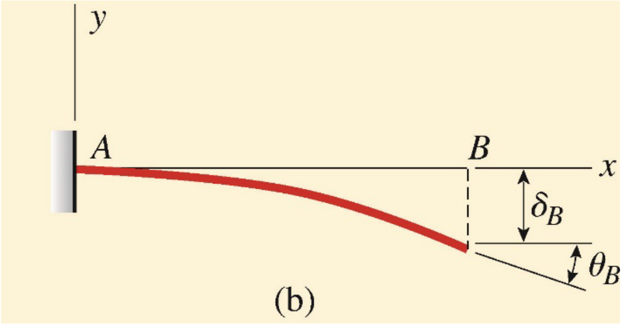


$$M(x) = -\frac{qL^2}{2} + qLx - \frac{qx^2}{2}$$

$$EIv'' = -\frac{qL^2}{2} + qLx - \frac{qx^2}{2}$$

$$EIv' = -\frac{qL^2x}{2} + \frac{qLx^2}{2} - \frac{qx^3}{6} + C_1$$

$$EIv = -\frac{qL^2x^2}{4} + \frac{qLx^3}{6} - \frac{qx^4}{24} + C_1x + C_2$$



$$EIv' = -\frac{qL^2x}{2} + \frac{qLx^2}{2} - \frac{qx^3}{6} + C_1$$

$$EIv = -\frac{qL^2x^2}{4} + \frac{qLx^3}{6} - \frac{qx^4}{24} + C_1x + C_2$$

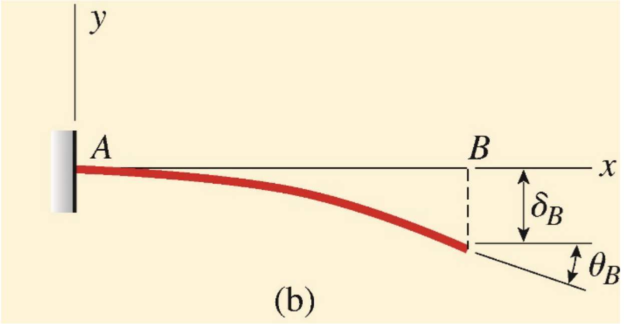
As constantes C_1 e C_2 são obtidas a partir das condições de contorno (vínculos)

$$v(0) = 0 \ \Rightarrow C_2 = 0$$

$$\theta(0) = 0 \ \Rightarrow v'(0) = 0 \Rightarrow C_1 = 0$$

$$v' = -\frac{qx}{6EI}(3L^2 - 3Lx + x^2)$$

$$v = -\frac{qx^2}{24EI}(6L^2 - 4Lx + x^2)$$



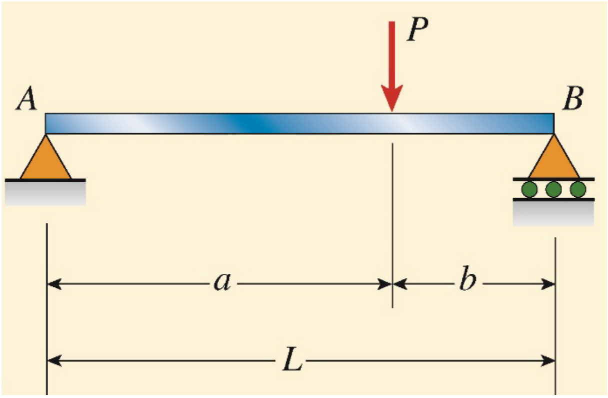
$$v' = -\frac{qx}{6EI}(3L^2 - 3Lx + x^2)$$

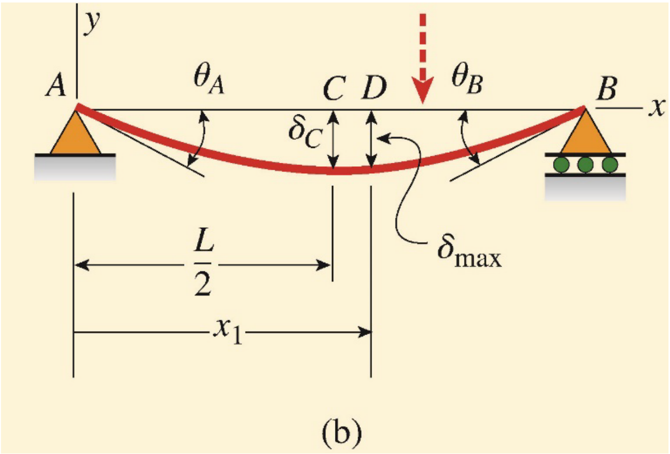
$$v = -\frac{qx^2}{24EI}(6L^2 - 4Lx + x^2)$$

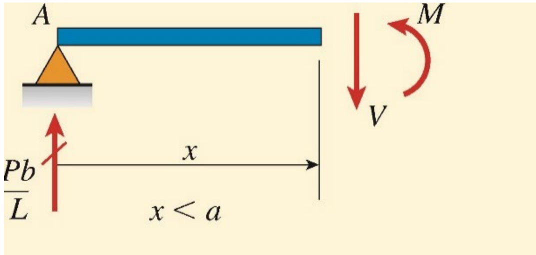
$$\theta_B = -v'(L) = \frac{qL^3}{6EI}$$

$$\delta_B = -v(L) = \frac{qL^4}{8EI}$$

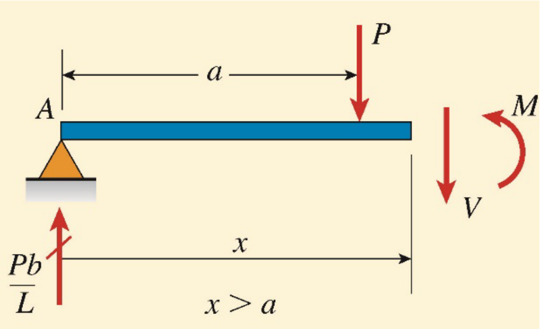
Exemplo







$$M_1 = \frac{Pbx}{L}$$



$$M_2 = \frac{Pbx}{L} - P(x - a)$$

$0 \leq x \leq a$

$$EIv_1'' = M_1 = \frac{Pbx}{L}$$

$$EIv_1' = \frac{Pbx^2}{2L} + C_1$$

$$EIv_1 = \frac{Pbx^3}{6L} + C_1 x + C_3$$

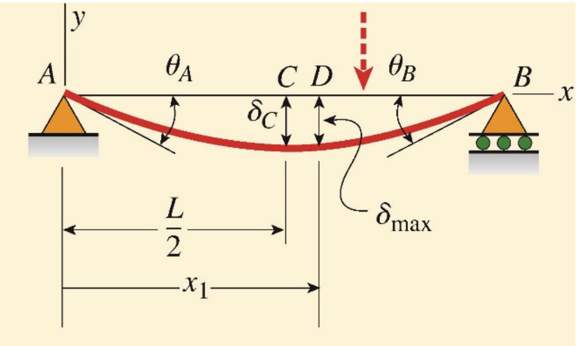
$a \leq x \leq L$

$$EIv_2'' = M_2 = \frac{Pbx}{L} - P(x-a)$$

$$EIv_2' = \frac{Pbx^2}{2L} - \frac{P(x-a)^2}{2} + C_2$$

$$EIv_2 = \frac{Pbx^3}{6L} - \frac{P(x-a)^3}{6} + C_2 x + C_4$$

Condições de contorno:



(b)

i) $v_1(0) = 0$

ii) $v_2(L) = 0$

iii) $v_1(a) = v_2(a)$

iv) $v_1'(a) = v_2'(a)$

$$EIv_1' = \frac{Pbx^2}{2L} + C_1 \qquad EIv_2' = \frac{Pbx^2}{2L} - \frac{P(x-a)^2}{2} + C_2$$

$$EIv_1 = \frac{Pbx^3}{6L} + C_1x + C_3 \qquad EIv_2 = \frac{Pbx^3}{6L} - \frac{P(x-a)^3}{6} + C_2x + C_4$$

i) $v_1(0) = 0 \;\Rightarrow C_3 = 0$

ii) $v_2(L) = 0 \;\Rightarrow 0 = \frac{PbL^2}{6} - \frac{Pb^3}{6} + C_2L + C_4 \;\Rightarrow C_2 = -\frac{Pb}{6}(L^2 - b^2)$

iii) $v_1(a) = v_2(a) \;\Rightarrow \frac{Pba^3}{6L} + C_1a = \frac{Pba^3}{6L} + C_2a + C_4 \;\Rightarrow C_4 = 0$

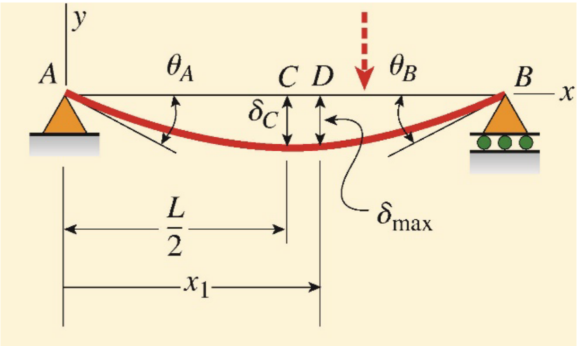
iv) $v_1'(a) = v_2'(a) \;\Rightarrow \frac{Pba^2}{2L} + C_1 = \frac{Pba^2}{2L} + C_2 \;\Rightarrow C_1 = C_2$

$$v_1 = -\frac{Pbx}{6EI}(L^2 - b^2 - x^2)$$

$$v_1' = -\frac{Pb}{6EI}(L^2 - b^2 - 3x^2)$$

$$v_2 = -\frac{Pbx}{6EI}(L^2 - b^2 - x^2) - \frac{P(x-a)^3}{6EI}$$

$$v_2' = -\frac{Pb}{6EI}(L^2 - b^2 - 3x^2) - \frac{P(x-a)^2}{2EI}$$



(b)

$$\theta_A = -v_1'(0) = \frac{Pb}{6LEI}(L^2 - b^2) = \frac{Pab(L+b)}{6LEI}$$

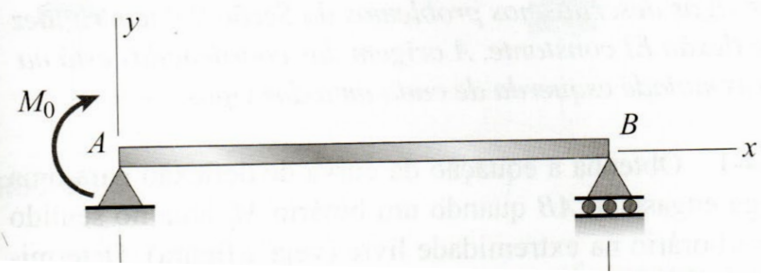
$$\theta_B = v_2'(L) = \frac{Pb}{6LEI}(2L^2 - 3bL + b^2) = \frac{Pab(L+a)}{6EI}$$

Se $a > b$, o ponto *D* corresponde ao ponto em que $v_1' = (0)$, ou seja,

$$x_1 = \frac{\sqrt{L^2 - b^2}}{3}$$

$$\delta_{máx} = -v_1(x_1) = \frac{Pb(`L^2 - b^2)^{\frac{3}{2}}}{9\sqrt{3}LEI}$$

9.3-9 Derive a equação da curva de deflexão para uma viga simples AB carregada por um binário M_0 no suporte esquerdo (veja a figura). Determine também a deflexão máxima δ_{max}. (*Observação:* Use a equação diferencial de segunda ordem da curva de deflexão.)



$$EIv'' = M = M_0\left(1 - \frac{x}{L}\right)$$

$$EIv' = M_0\left(x - \frac{x^2}{2L}\right) + C_1$$

$$EIv = M_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + C_1 x + C_2$$

$$v(0) = 0 \;\Rightarrow C_2 = 0$$

$$v(L) = 0 \;\Rightarrow C_1 = -\frac{M_0 L}{3}$$

$$\Rightarrow v = -\frac{M_0 x}{6LEI}(2L^2 - 3Lx + x^2)$$

$$\Rightarrow v' = -\frac{M_0}{6LEI}(2L^2 - 6Lx + 3x^2)$$

A deflexão é máxima no ponto x_1 para o qual:

$$v'(x1) = 0$$

ou seja,

$$x_1 = L\left(1 - \frac{\sqrt{3}}{3}\right)$$

$$\delta_{máx} = |v(x1)| = \frac{\sqrt{3} M_0 L^2}{27EI}$$

Referência:

Gere, J.M., Goodno, B.J. Mecânica dos Materiais – Tradução da 7ª edição norte-americana. Cengage Learning, 2010, 860p, Capítulo 9.