MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.14 Analysis and Design of Feedback Control Systems

Understanding Poles and Zeros

1 System Poles and Zeros

The transfer function provides a basis for determining important system response characteristics without solving the complete differential equation. As defined, the transfer function is a rational function in the complex variable $s = \sigma + j\omega$, that is

$$H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \tag{1}$$

It is often convenient to factor the polynomials in the numerator and denominator, and to write the transfer function in terms of those factors:

$$H(s) = \frac{N(s)}{D(s)} = K\frac{(s-z_1)(s-z_2)\ldots(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_{n-1})(s-p_n)}, \tag{2}$$

where the numerator and denominator polynomials, $N(s)$ and $D(s)$, have real coefficients defined by the system's differential equation and $K = b_m/a_n$. As written in Eq. (2) the z_i's are the roots of the equation

$$N(s) = 0, \tag{3}$$

and are defined to be the system *zeros*, and the p_i's are the roots of the equation

$$D(s) = 0, \tag{4}$$

and are defined to be the system *poles*. In Eq. (2) the factors in the numerator and denominator are written so that when $s = z_i$ the numerator $N(s) = 0$ and the transfer function vanishes, that is

$$\lim_{s \to z_i} H(s) = 0.$$

and similarly when $s = p_i$ the denominator polynomial $D(s) = 0$ and the value of the transfer function becomes unbounded,

$$\lim_{s \to p_i} H(s) = \infty.$$

All of the coefficients of polynomials $N(s)$ and $D(s)$ are real, therefore the poles and zeros must be either purely real, or appear in complex conjugate pairs. In general for the poles, either $p_i = \sigma_i$, or else $p_i, p_{i+1} = \sigma_i \pm j\omega_i$. The existence of a single complex pole without a corresponding conjugate pole would generate complex coefficients in the polynomial $D(s)$. Similarly, the system zeros are either real or appear in complex conjugate pairs.

■ Example

A linear system is described by the differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 2\frac{du}{dt} + 1.$$

Find the system poles and zeros.

Solution: From the differential equation the transfer function is

$$H(s) = \frac{2s+1}{s^2+5s+6}. \tag{5}$$

which may be written in factored form

$$\begin{aligned} H(s) &= \frac{1}{2}\frac{s+1/2}{(s+3)(s+2)} \\ &= \frac{1}{2}\frac{s-(-1/2)}{(s-(-3))(s-(-2))}. \end{aligned} \tag{6}$$

The system therefore has a single real zero at $s = -1/2$, and a pair of real poles at $s = -3$ and $s = -2$.

The poles and zeros are properties of the transfer function, and therefore of the differential equation describing the input-output system dynamics. Together with the gain constant K they completely characterize the differential equation, and provide a complete description of the system.

■ Example

A system has a pair of complex conjugate poles $p_1, p_2 = -1 \pm j2$, a single real zero $z_1 = -4$, and a gain factor $K = 3$. Find the differential equation representing the system.

Solution: The transfer function is

$$\begin{aligned} H(s) &= K\frac{s-z}{(s-p_1)(s-p_2)} \\ &= 3\frac{s-(-4)}{(s-(-1+j2))(s-(-1-j2))} \\ &= 3\frac{(s+4)}{s^2+2s+5} \end{aligned} \tag{7}$$

and the differential equation is

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3\frac{du}{dt} + 12u \tag{8}$$

1.1 The Pole-Zero Plot

A system is characterized by its poles and zeros in the sense that they allow reconstruction of the input/output differential equation. In general, the poles and zeros of a transfer function may be complex, and the system dynamics may be represented graphically by plotting their locations on the complex s-plane, whose axes represent the real and imaginary parts of the complex variable s. Such plots are known as *pole-zero plots*. It is usual to mark a zero location by a circle ($\circ$) and a pole location a cross ($\times$). The location of the poles and zeros provide qualitative insights into the response characteristics of a system. Many computer programs are available to determine the poles and zeros of a system from either the transfer function or the system state equations [8]. Figure 1 is an example of a pole-zero plot for a third-order system with a single real zero, a real pole and a complex conjugate pole pair, that is;

$$H(s) = \frac{(3s+6)}{(s^3+3s^2+7s+5)} = 3\frac{(s-(-2))}{(s-(-1))(s-(-1-2j))(s-(-1+2j))}$$

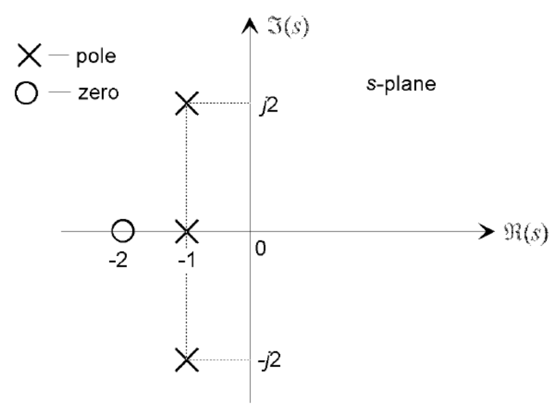

Figure 1: The pole-zero plot for a typical third-order system with one real pole and a complex conjugate pole pair, and a single real zero.

3.1 A Simple Method for constructing the Magnitude Bode Plot directly from the Pole-Zero Plot

The pole-zero plot of a system contains sufficient information to define the frequency response except for an arbitrary gain constant. It is often sufficient to know the shape of the magnitude Bode plot without knowing the absolute gain. The method described here allows the magnitude plot to be sketched by inspection, without drawing the individual component curves. The method is based on the fact that the overall magnitude curve undergoes a *change* in slope at each break frequency.

The first step is to identify the break frequencies, either by factoring the transfer function or directly from the pole-zero plot. Consider a typical pole-zero plot of a linear system as shown in Fig. 10a. The break frequencies for the four first and second-order blocks are all at a frequency equal to the radial distance of the poles or zeros from the origin of the s-plane, that is $\omega_b = \sqrt{\sigma^2 + \omega^2}$. Therefore all break frequencies may be found by taking a compass and drawing an arc from each pole or zero to the positive imaginary axis. These break frequencies may be transferred directly to the logarithmic frequency axis of the Bode plot.

Because all low frequency asymptotes are horizontal lines with a gain of 0dB, a pole or zero does not contribute to the magnitude Bode plot below its break frequency. Each pole or zero contributes a change in the *slope* of the asymptotic plot of ± 20 dB/decade above its break frequency. A complex conjugate pole or zero pair defines *two* coincident breaks of ± 20 dB/decade (one from each member of the pair), giving a total change in the slope of ± 40 dB/decade. Therefore, at any frequency ω, the slope of the asymptotic magnitude function depends only on the number of break points at frequencies less than ω, or to the left on the Bode plot. If there are Z breakpoints due to zeros to the left, and P breakpoints due to poles, the slope of the curve at that frequency is $20 \times (Z - P)$ dB/decade.

Any poles or zeros at the origin cannot be plotted on the Bode plot, because they are effectively to the left of all finite break frequencies. However, they define the initial slope. If an arbitrary starting frequency and an assumed gain (for example 0dB) at that frequency are chosen, the shape of the magnitude plot may be easily constructed by noting the initial slope, and constructing the curve from straight line segments that change in slope by units of ±20 dB/decade at the breakpoints. The arbitrary choice of the reference gain results in a vertical displacement of the curve.

Figure 10b shows the straight line magnitude plot for the system shown in Fig. 10a constructed using this method. A frequency range of 0.01 to 100 radians/sec was arbitrarily selected, and a gain of 0dB at 0.01 radians/sec was assigned as the reference level. The break frequencies at 0, 0.1, 1.414, and 5 radians/sec were transferred to the frequency axis from the pole-zero plot. The value of N at any frequency is $Z - P$, where Z is the number of zeros to the left, and P is the number of poles to the left. The curve was simply drawn by assigning the value of the slope in each of the frequency intervals and drawing connected lines.

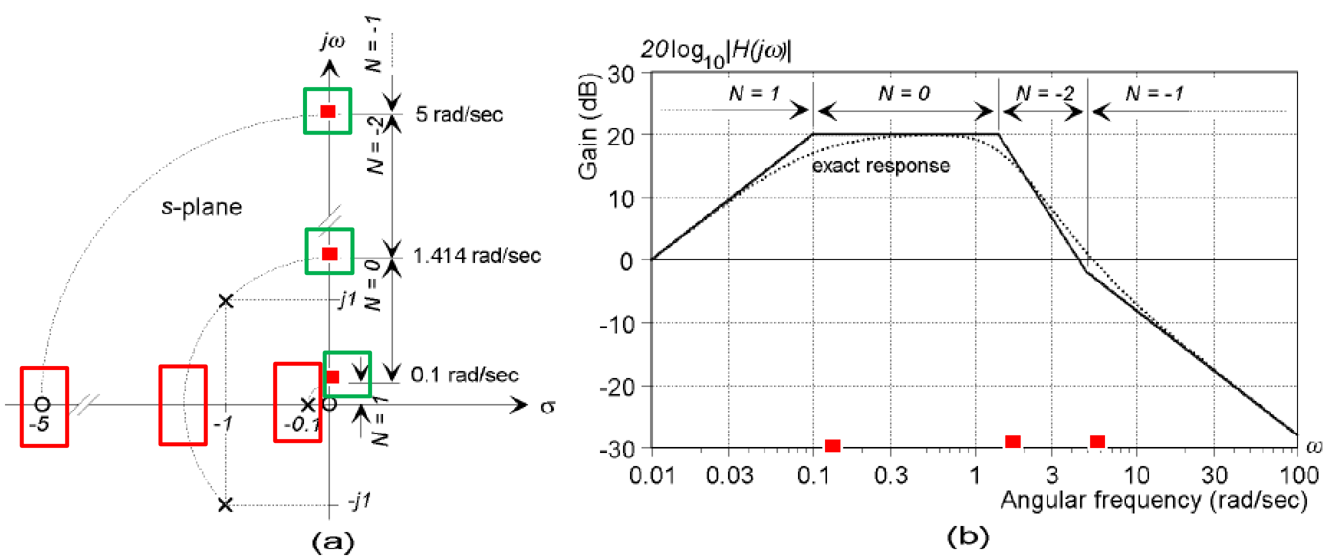

Figure 10: Construction of the magnitude Bode plot from the pole-zero diagram: (a) shows a typical third-order system, and the definition of the break frequencies, (b) shows the Bode plot based on changes in slope at the break frequencies

Break Frequencies (w_b):

$$w_b = \sqrt{\sigma^2 + w^2}$$

The **break frequencies** are all at a frequency equal to the radial distance of the poles or zeros from the origin of the *s*-plane !