1 Introduction

This paper documents the calculations used in Unwedge to determine the safety factor of wedges formed around underground excavations. This involves the following series of steps:

1. Determine the wedge geometry using block theory (Goodman and Shi, 1985).
2. Determine all of the individual forces acting on a wedge, and then calculate the resultant active and passive force vectors for the wedge (section 3).
3. Determine the sliding direction of the wedge (section 4).
4. Determine the normal forces on each wedge plane (section 5).
5. Compute the resisting forces due to joint shear strength, and tensile strength (if applicable) (section 6).
6. Calculate the safety factor (section 7).

If the Field Stress option is used, then the normal and shear forces on each wedge plane are determined from a boundary element stress analysis. See sections 9 and 10 for complete details.

2 Wedge Geometry

The orientations of 3 distinct joint planes must always be defined for an Unwedge analysis. Using block theory, Unwedge determines all of the possible wedges which can be formed by the intersection of the 3 joint planes and the excavation.

The method used for determining the wedges is described in the text by Goodman and Shi, "Block Theory and Its Application to Rock Engineering", (1985).

In general, the wedges which are formed are tetrahedral in nature (ie. the 3 joint planes make up 3 sides of a tetrahedron, and the fourth "side" is formed by the excavation boundary). However, prismatic wedges can also be formed. This will occur if two of the joint planes strike in the same direction, so that the resulting wedge is a prismatic, rather than a tetrahedral shape.

When the wedge coordinates have been determined, the geometrical properties of each wedge can be calculated, including:

- Wedge volume
- Wedge face areas
- Normal vectors for each wedge plane

3 Wedge Forces

All forces on the wedge can be classified as either Active or Passive. In general, Active forces represent driving forces in the safety factor calculation, whereas Passive forces represent resisting forces.

The individual force vectors are computed for each quantity (eg. Wedge weight, bolt force, water force etc), and then the resultant Active and Passive force vectors are determined by a vector summation of the individual forces.

3.1 Active Force Vector

The resultant Active force vector is comprised of the following components:

$$A = W + C + X + U + E$$

$A =$ resultant active force vector
$W =$ wedge weight vector
$C =$ shotcrete weight vector
$X =$ active pressure force vector
$U =$ water force vector
$E =$ seismic force vector

3.1.1 Wedge Weight Vector

The wedge weight is usually the primary driving force in the analysis.

$$W = \left(\gamma_r V\right) \bullet \hat{g}$$

$W =$ wedge weight vector
$\gamma_r =$ unit weight of rock
V = wedge volume
$\hat{g} =$ gravity direction

3.1.2 Shotcrete Weight Vector

This accounts for the weight of shotcrete applied to a wedge. This quantity is sometimes neglected in wedge stability calculations, however it can represent a significant load if the shotcrete thickness is substantial.

$$C = (\gamma_s t a_e) \bullet \hat{g}$$

C = shotcrete weight vector
γ_s = unit weight of shotcrete
t = shotcrete thickness
a_e = surface area of wedge on excavation face
$\hat{g}$ = gravity direction

3.1.3 Pressure Force (Active) Vector

Pressure force is applied with the **Pressure** option in the Support menu, and can be defined as either active or passive.

$$X = \sum_{i=1}^{n} p_i a_i \hat{n}_i$$

X = resultant active pressure force vector
n = number of polygons making up excavation wedge face (see figure below)
p_i = pressure on the i^{th} polygon making up excavation wedge face
a_i = area of the i^{th} polygon
$\hat{n}_i$ = outward (out of excavation) normal of i^{th} polygon

If a wedge intersects a curved or non-linear portion of the excavation perimeter, then the excavation wedge face will be formed of a number of individual polygons. Each polygon is formed by the intersection of the wedge planes with a planar "strip" of the excavation boundary. These are the "polygons" referred to above, in the Pressure Force Vector calculation. See Figure 3.1.3.

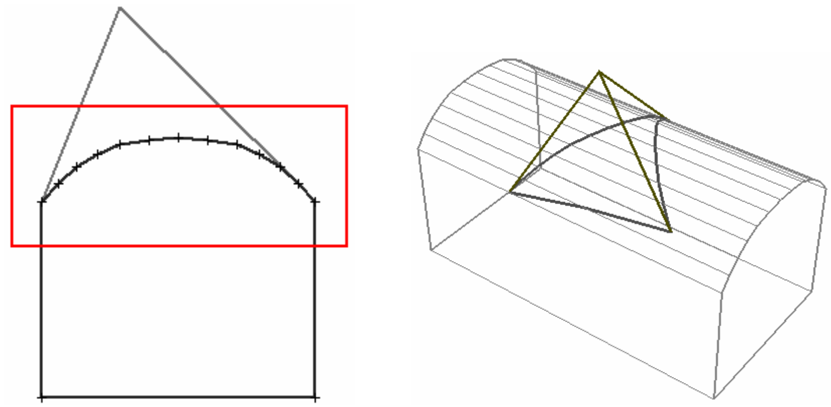


Figure 3.1.3 – example of excavation wedge face formed of multiple polygons.

3.1.4 Water Force Vector

In Unwedge there are two different methods for defining the existence of water pressure on the joint planes – Constant or Gravitational.

3.1.4.1 Constant pressure on each joint

$$U = \sum_{i=1}^{3} u_i a_i \hat{n}_i$$

$U =$ resultant water force vector

$u_i =$ water pressure on the ith joint face

$a_i =$ area of the ith joint face

$\hat{n}_i =$ inward (into wedge) normal of ith joint face

3.1.4.2 Gravitational pressure on each joint

For the Gravitational water pressure option, the water pressure is assumed to vary linearly with depth from a user-specified elevation.

To obtain an accurate estimate of the total water force on each joint face, each joint face is first triangulated into *n* sub-triangles (3 vertices each). The pressure on each sub-triangle is calculated, and the total water force on each joint face is determined by a summation over all sub-triangles.

$$U = \sum_{i=1}^{3} \sum_{j=1}^{n} \gamma_w h_{ij} a_{ij} \hat{n}_i$$

$U =$ resultant water force vector
$i =$ joint face#, 3 for a tetrahedron
$j =$ triangle# for joint face i
$n =$ number of triangles for joint face i
$\gamma_w =$ unit weight of water
$a_{ij} =$ area of the $\mathrm{j^{th}}$ triangle making up the $\mathrm{i^{th}}$ joint face
$\hat{n}_i =$ inward (into wedge) normal of $\mathrm{i^{th}}$ joint face
$h_{ij} =$ average depth of the 3 triangle vertices below ground surface
$= \frac{1}{3} \sum_{i=1}^{3} \left(gse - y_i \right)$
$gse =$ ground surface elevation
$y_i =$ elevation of the $\mathrm{i^{th}}$ vertex in the triangle

3.1.5 Seismic Force Vector

This determines the seismic force vector if the **Seismic** option is applied. If the seismic coefficients have been specified in terms of orthogonal components (eg. North / East / Up), then the resultant seismic force is the vector sum of the individual force components.

$$E = \left(k \gamma_r V \right) \bullet \hat{e}$$

$E =$ seismic force vector
k = seismic coefficient
$\gamma_r =$ unit weight of rock
V = wedge volume
$\hat{e} =$ direction of seismic force

3.2 Passive Force Vector

The resultant Passive Force Vector is the sum of the bolt, shotcrete and pressure (passive) support force vectors.

$$P = H + Y + B$$

P = resultant passive force vector
H = shotcrete shear resistance force vector
Y = passive pressure force vector
B = resultant bolt force vector

3.2.1 Pressure Force (Passive) Vector

Pressure force is applied with the **Pressure** option in the Support menu, and can be defined as either active or passive.

$$Y = \sum_{i=1}^{n} p_i a_i \hat{n}_i$$

Y = resultant passive pressure force vector
n = number of polygons making up excavation wedge face
p_i = pressure on the ith polygon making up excavation wedge face
a_i = area of the ith polygon
$\hat{n}_i$ = outward (out of excavation) normal of ith polygon

3.2.2 Bolt Force Vector

Bolt forces are always assumed to be Passive in Unwedge.

The resultant Bolt Force Vector is the sum of all individual bolt force vectors. For a description of how the bolt support forces are determined, see the Unwedge Help system (Theory > Support > Bolt Support Force).

3.2.3 Shotcrete Shear Resistance Force Vector

Shotcrete forces are always assumed to be Passive in Unwedge.

For a description of how the shotcrete support force is determined, see the Unwedge Help system (Theory > Support > Shotcrete Support Force).

4 Sliding Direction

Next, the sliding direction of the wedge must be determined. The sliding (deformation) direction is computed by considering active forces only (**A** vector). Passive forces (**P** vector) DO NOT influence sliding direction.

The calculation algorithm is based on the method presented in chapter 9 of "Block Theory and its application to rock engineering", by Goodman and Shi (1985).

For a tetrahedron there are 7 possible directions ($\hat{s}_0, \hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_{12}, \hat{s}_{13}, \hat{s}_{23}$). These represent the modes of: falling / lifting ($\hat{s}_0$), sliding on a single joint plane ($\hat{s}_1, \hat{s}_2, \hat{s}_3$), or sliding along the line of intersection of two joint planes ($\hat{s}_{12}, \hat{s}_{13}, \hat{s}_{23}$).

Calculation of the sliding direction is a two step process: 1) compute all possible sliding directions and 2) determine which one of the possible sliding directions is the actual valid direction.

4.1 Step 1 – Compute list of 7 possible sliding directions

4.1.1 Falling (or Lifting)

$$\hat{s}_0 = \hat{a} = A / \|A\|$$

$\hat{s}_0 =$ falling or lifting direction
$\hat{a} =$ unit direction of the resultant active force
$A =$ active force vector

4.1.2 Sliding on a single face i

$$\hat{s}_i = \frac{(\hat{n}_i \times A) \times \hat{n}_i}{\|(\hat{n}_i \times A) \times \hat{n}_i\|}$$

$\hat{s}_i =$ sliding direction on joint i
$\hat{n}_i =$ normal to joint face i directed into wedge
$A =$ active force vector

4.1.3 Sliding on two faces i and j

$$\hat{s}_{ij} = \frac{\hat{n}_i \times \hat{n}_j}{\left\|\hat{n}_i \times \hat{n}_j\right\|} sign\left(\left(\hat{n}_i \times \hat{n}_j\right) \bullet A\right)$$

$\hat{s}_{ij}$ = sliding direction on joint i and j (along line of intersection)

$\hat{n}_i$ = normal to joint face i directed into wedge

$\hat{n}_j$ = normal to joint face j directed into wedge

A = active force vector

4.2 Step 2 – Compute which of the possible sliding directions is valid

For the following 8 tests, whichever satisfies the given inequalities is the sliding direction of the wedge. If none of these tests satisfies the given inequalities, the wedge is unconditionally stable.

4.2.1 Falling wedge

$$A \bullet \hat{n}_1 > 0$$
$$A \bullet \hat{n}_2 > 0$$
$$A \bullet \hat{n}_3 > 0$$
$$A \bullet W \geq 0$$

A = resultant active force vector

$\hat{n}_i$ = inward normal to joint i

W = weight vector

4.2.2 Lifting wedge

$$A \bullet \hat{n}_1 > 0$$
$$A \bullet \hat{n}_2 > 0$$
$$A \bullet \hat{n}_3 > 0$$
$$A \bullet W < 0$$

4.2.3 Sliding on joint 1

$$A \bullet \hat{n}_1 \le 0$$
$$\hat{s}_1 \bullet \hat{n}_2 > 0$$
$$\hat{s}_1 \bullet \hat{n}_3 > 0$$

4.2.4 Sliding on joint 2

$$A \bullet \hat{n}_2 \le 0$$
$$\hat{s}_2 \bullet \hat{n}_1 > 0$$
$$\hat{s}_2 \bullet \hat{n}_3 > 0$$

4.2.5 Sliding on joint 3

$$A \bullet \hat{n}_3 \le 0$$
$$\hat{s}_3 \bullet \hat{n}_1 > 0$$
$$\hat{s}_3 \bullet \hat{n}_2 > 0$$

4.2.6 Sliding on the intersection of joint 1 and joint 2

$$\hat{s}_{12} \bullet \hat{n}_3 > 0$$
$$\hat{s}_1 \bullet \hat{n}_2 \le 0$$
$$\hat{s}_2 \bullet \hat{n}_1 \le 0$$

4.2.7 Sliding on the intersection of joint 1 and joint 3

$$\hat{s}_{13} \bullet \hat{n}_2 > 0$$
$$\hat{s}_1 \bullet \hat{n}_3 \le 0$$
$$\hat{s}_3 \bullet \hat{n}_1 \le 0$$

4.2.8 Sliding on the intersection of joint 2 and joint 3

$$\hat{s}_{23} \bullet \hat{n}_1 > 0$$
$$\hat{s}_2 \bullet \hat{n}_3 \le 0$$
$$\hat{s}_3 \bullet \hat{n}_2 \le 0$$

5 Normal Force

The calculation of the normal forces on each of the three joint planes for a tetrahedron first requires the calculation of the sliding direction. Once the sliding direction is known, the following equations are used to determine the normal forces given a resultant force vector, F. The force vector, F, is generally either the active or the passive resultant force vector.

5.1 Falling or lifting wedge

$$N_1 = 0$$
$$N_2 = 0$$
$$N_3 = 0$$

$N_i =$ normal force on the i$^{\text{th}}$ joint

5.2 Sliding on joint 1

$$N_1 = -F \bullet \hat{n}_1$$
$$N_2 = 0$$
$$N_3 = 0$$

$N_i =$ normal force on the i$^{\text{th}}$ joint
$F =$ force vector
$\hat{n}_1 =$ inward (into wedge) normal of joint plane 1

5.3 Sliding on joint 2

$$N_1 = 0$$
$$N_2 = -F \bullet \hat{n}_2$$
$$N_3 = 0$$

$N_i =$ normal force on the i$^{\text{th}}$ joint
$F =$ force vector
$\hat{n}_2 =$ inward (into wedge) normal of joint plane 2

5.4 Sliding on joint 3

$$N_1 = 0$$
$$N_2 = 0$$
$$N_3 = -F \bullet \hat{n}_3$$

$N_i =$ normal force on the i$^{\text{th}}$ joint
$F =$ force vector
$\hat{n}_3 =$ inward (into wedge) normal of joint plane 3

5.5 Sliding on joints 1 and 2

$$N_1 = -\frac{\left(F \times \hat{n}_2\right) \bullet \left(\hat{n}_1 \times \hat{n}_2\right)}{\left(\hat{n}_1 \times \hat{n}_2\right) \bullet \left(\hat{n}_1 \times \hat{n}_2\right)}$$
$$N_2 = -\frac{\left(F \times \hat{n}_1\right) \bullet \left(\hat{n}_2 \times \hat{n}_1\right)}{\left(\hat{n}_2 \times \hat{n}_1\right) \bullet \left(\hat{n}_2 \times \hat{n}_1\right)}$$
$$N_3 = 0$$

$N_i =$ normal force on the i$^{\text{th}}$ joint
$F =$ force vector
$\hat{n}_1 =$ inward (into wedge) normal of joint plane 1
$\hat{n}_2 =$ inward (into wedge) normal of joint plane 2

5.6 Sliding on joints 1 and 3

$$N_1 = -\frac{(F \times \hat{n}_3) \bullet (\hat{n}_1 \times \hat{n}_3)}{(\hat{n}_1 \times \hat{n}_3) \bullet (\hat{n}_1 \times \hat{n}_3)}$$

$$N_2 = 0$$

$$N_3 = -\frac{(F \times \hat{n}_1) \bullet (\hat{n}_3 \times \hat{n}_1)}{(\hat{n}_3 \times \hat{n}_1) \bullet (\hat{n}_3 \times \hat{n}_1)}$$

$N_i =$ normal force on the i^{th} joint

$F =$ force vector

$\hat{n}_1 =$ inward (into wedge) normal of joint plane 1

$\hat{n}_3 =$ inward (into wedge) normal of joint plane 3

5.7 Sliding on joints 2 and 3

$$N_1 = 0$$

$$N_2 = -\frac{(F \times \hat{n}_3) \bullet (\hat{n}_2 \times \hat{n}_3)}{(\hat{n}_2 \times \hat{n}_3) \bullet (\hat{n}_2 \times \hat{n}_3)}$$

$$N_3 = -\frac{(F \times \hat{n}_2) \bullet (\hat{n}_3 \times \hat{n}_2)}{(\hat{n}_3 \times \hat{n}_2) \bullet (\hat{n}_3 \times \hat{n}_2)}$$

$N_i =$ normal force on the i^{th} joint

$F =$ force vector

$\hat{n}_2 =$ inward (into wedge) normal of joint plane 2

$\hat{n}_3 =$ inward (into wedge) normal of joint plane 3

6 Shear and Tensile Strength

There are 3 joint strength models available in Unwedge: 1) Mohr-Coulomb, 2) Barton-Bandis, and 3) Power Curve.

Shear strength is computed based on the normal stress acting on each joint plane. The normal stress is computed based on the active and passive normal forces computed on the joint planes using the equations in the previous section.

6.1 Compute normal stress on each joint

First compute the stress on each joint plane based on the normal forces computed in section 5.

$$\sigma_{n_i} = N_i / a_i$$

σ_{n_i} = normal stress on the ith joint

N_i = normal force on the ith joint

a_i = area of the ith joint

6.2 Compute shear strength of each joint

Use the strength criteria defined for the joint, and the normal stress, to compute the shear strength.

6.2.1 Mohr-Coulomb Strength Criterion

$$\tau_i = c_i + \sigma_{n_i} \tan\phi_i$$

τ_i = shear strength of the ith joint

c_i = cohesion of the ith joint

σ_{n_i} = normal stress on the ith joint

ϕ_i = friction angle of the ith joint

6.2.2 Barton-Bandis Strength Criterion

$$\tau_i = \sigma_{n_i} \tan\left[JRC_i \log_{10}\left(\frac{JCS_i}{\sigma_{n_i}}\right) + \phi_{b_i} \right]$$

$\tau_i =$ shear strength of the ith joint

$JRC_i =$ joint roughness coefficient of the ith joint

$JCS_i =$ joint compressive strength of the ith joint

$\sigma_{n_i} =$ normal stress on the ith joint

$\phi_{b_i} =$ base friction angle of the ith joint

6.2.3 Power Curve Strength Criterion

$$\tau_i = c_i + a_i \left(\sigma_{n_i} + d_i\right)^{b_i}$$

$\tau_i =$ shear strength of the ith joint

$a_i, b_i, c_i, d_i =$ strength parameters of the ith joint

$\sigma_{n_i} =$ normal stress on the ith joint

6.3 Compute resisting force due to shear strength

Force acts in a direction opposite to the direction of sliding (deformation).

$$J_i = \tau_i a_i \cos\theta_i$$

$J_i =$ magnitude of the resisting force due to the shear strength of joint i

$\tau_i =$ shear strength of the ith joint

$a_i =$ area of the ith joint

$\theta_i =$ angle between the sliding direction and the ith joint

6.4 Compute resisting force due to tensile strength

Tensile strength is only applicable if it has been defined by the user. Tensile strength can only be defined for Mohr-Coulomb or Power Curve strength criteria; it cannot be defined for the Barton-Bandis strength criterion.

Tensile strength acts in a direction normal to the joint plane. To compute the resisting force, the force is resolved in a direction opposite to the direction of sliding (deformation).

$$T_i = \sigma_{t_i} a_i \sin\theta_i$$

T_i = magnitude of the resisting force due to the tensile strength of joint i

σ_{t_i} = tensile strength of the i^{th} joint

a_i = area of the i^{th} joint

θ_i = angle between the sliding direction and the i^{th} joint

7 Factor of Safety

Unwedge computes 3 separate factors of safety:

i. falling factor of safety
ii. unsupported factor of safety
iii. supported factor of safety

The reported factor of safety is the maximum of the above three factors of safety. The logic of this is simple - support is assumed to never decrease the factor of safety from the unsupported value. The factor of safety can never be less than if the wedge was falling with only support to stabilize it.

The equations are based on 3 joint planes making up a tetrahedral wedge.

The limit equilibrium safety factor calculations only consider force equilibrium in the direction of sliding. Moment equilibrium is not considered.

$$F = \text{Factor of Safety} = \max(F_f, F_u, F_s)$$

F_f = Falling factor of safety
F_u = Unsupported factor of safety
F_s = Supported factor of safety

7.1 Factor of safety definition

$$\text{Factor of safety} = \frac{\text{resisting forces (e.g. shear/tensile strength, support)}}{\text{driving forces (e.g. weight, seismic, water)}}$$

7.2 Falling factor of safety

The falling factor of safety assumes that only passive support and tensile strength act to resist movement. Basically the wedge is assumed to be falling so no influence of the joint planes (shear strength, failure direction) is incorporated. Driving forces are due to the active forces on the wedge as defined in section 3.1. The falling direction is calculated from the direction of the active force vector.

$$F_f = \frac{-P \bullet \hat{s}_0 + \sum_{i=1}^{3} T_i}{A \bullet \hat{s}_0}$$

F_f = Falling factor of safety
P = resultant passive force vector (section 3.2)
A = resultant active force vector (section 3.1)
T_i = magnitude of the resisting force due to the tensile strength of joint i (section 6.4)
$\hat{s}_0$ = falling direction (section 4.1.1)

7.3 Unsupported factor of safety

The unsupported factor of safety assumes that shear and tensile strength act to resist movement. No passive support force is used.

Driving forces are due to the active forces on the wedge as defined in section 3.1. The sliding direction is calculated from the equations in section 4. The shear strength is calculated based on the normal forces from the active force vector only. Normal forces from the passive force vector are not included.

$$F_u = \frac{\sum_{i=1}^{3} \left(J_i^u + T_i \right)}{A \bullet \hat{s}}$$

F_u = Unsupported factor of safety
A = resultant active force vector (section 3.1)
J_i^u = magnitude of the resisting force due to the unsupported shear strength of joint i (section 6.3)
T_i = magnitude of the resisting force due to the tensile strength of joint i (section 6.4)
$\hat{s}$ = sliding direction (section 4)

7.4 Supported factor of safety

The supported factor of safety assumes that passive support forces and shear and tensile strength act to resist movement.

Driving forces are due to the active forces on the wedge as defined in section 3.1. The sliding direction is calculated from the equations in section 4. The shear strength is calculated based on the normal force calculated from the active force vector plus the passive force vector.

$$F_s = \frac{-P \bullet \hat{s} + \sum_{i=1}^{3}\left(J_i^s + T_i\right)}{A \bullet \hat{s}}$$

F_s = Supported factor of safety
P = resultant passive force vector (section 3.2)
A = resultant active force vector (section 3.1)
J_i^s = magnitude of the resisting force due to the supported shear strength of joint i (section 6.3)
T_i = magnitude of the resisting force due to the tensile strength of joint i (section 6.4)
$\hat{s}$ = sliding direction (section 4)

8 Example Calculation

Question: A 3mX3m square tunnel has an axis that plunges at zero degrees and trends exactly north. Three joint planes have a dip and dip direction of 45/0, 45/60, and 45/300. The unit weight of rock is 2.7 tonnes/m3 and all three joint planes have zero cohesion, zero tensile strength, and a 35 degree friction angle. If a 10 tonne rock bolt is placed vertically through the center of the wedge, determine the factor of safety.

Answer: Using block theory as described in Goodman and Shi (1985), the existence of a roof wedge is determined with the block code ULL (011). The actual coordinates of the vertices that form the maximum size block are also determined using the methods described in chapter 8 of the above reference. Using these methods, the volume of the block is calculated to be 3.375m3. The area of each joint face is 5.5114m2. The joint normals are calculated using the following equations (coordinate system is x = East, y = Up, z = South):

$$
\begin{aligned}
n_1 &= \{\sin\alpha_1 \sin\beta_1 \quad \cos\alpha_1 \quad -\sin\alpha_1 \cos\beta_1\} = \{0 \quad 0.7071 \quad -0.7071\} \\
n_2 &= \{-\sin\alpha_2 \sin\beta_2 \quad -\cos\alpha_2 \quad \sin\alpha_2 \cos\beta_2\} = \{-0.6124 \quad -0.7071 \quad 0.3536\} \\
n_3 &= \{-\sin\alpha_3 \sin\beta_3 \quad -\cos\alpha_3 \quad \sin\alpha_3 \cos\beta_3\} = \{0.6124 \quad -0.7071 \quad 0.3536\}
\end{aligned}
$$

$n_i =$ unit normal vector of the $\mathrm{i^{th}}$ joint pointing into block

$\alpha_i =$ dip of the $\mathrm{i^{th}}$ joint

$\beta_i =$ dip direction of the $\mathrm{i^{th}}$ joint

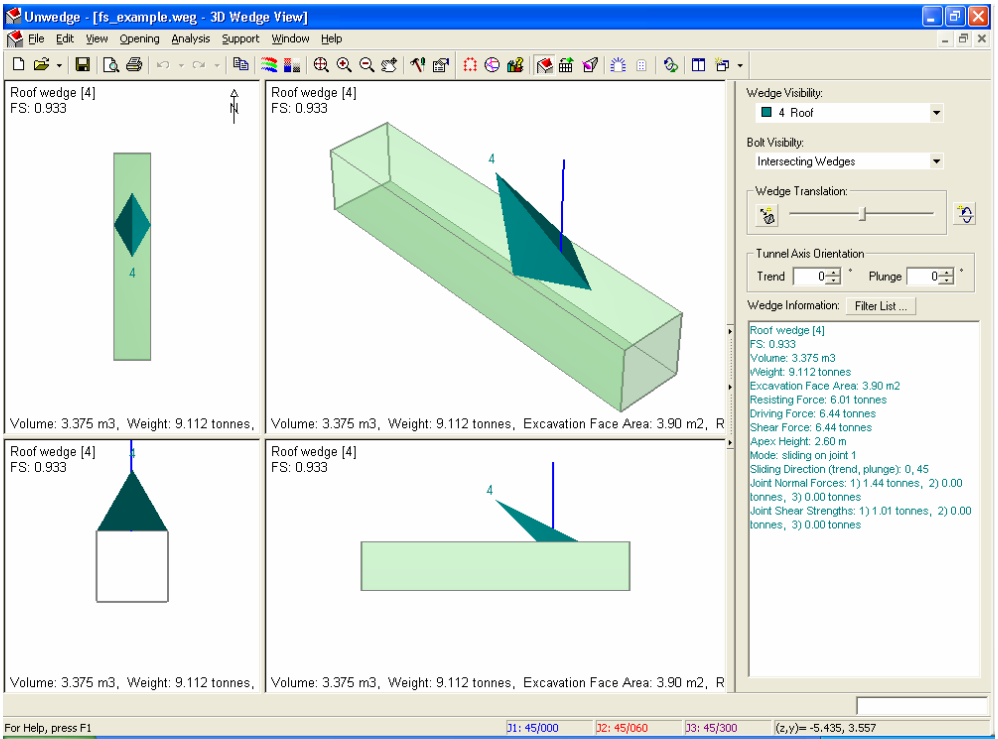


Fig. 8.1 Unwedge Results

Now to determine the factor of safety:

Step 1: Determine active force vector (in this case, only due to the wedge weight)

$$
\begin{aligned}
A = W &= (\gamma_r V) \bullet \hat{g} \\
&= (2.7 * 3.375) \bullet \{0 \quad -1 \quad 0\} \\
&= \{0 \quad -9.1125 \quad 0\}
\end{aligned}
$$

Step 2: Determine passive force vector (in this case, only due to the bolt capacity)

$$P = H + Y + B$$

$$H = Y = \{0 \quad 0 \quad 0\}$$

$$B = \{0 \quad 10 \quad 0\} * e$$

e = bolt orientation efficiency$= -\hat{b} \bullet \hat{s}$ (cosine tension/shear method)

$\hat{b}$ = bolt direction$= \{0 \quad 1 \quad 0\}$

$\hat{s}$ = sliding direction$= \{0 \quad -0.7071 \quad -0.7071\}$

$$e = 0.7071$$

$$P = B = \{0 \quad 10 \quad 0\} * 0.7071 = \{0 \quad 7.071 \quad 0\}$$

Step 3: Determine all possible sliding directions

$$\hat{s}_0 = \frac{A}{\|A\|} = \{0 \quad -1 \quad 0\}$$
$$\hat{s}_1 = \frac{(\hat{n}_1 \times A) \times \hat{n}_1}{\|(\hat{n}_1 \times A) \times \hat{n}_1\|} = \{0 \quad -0.7071 \quad -0.7071\}$$
$$\hat{s}_2 = \frac{(\hat{n}_2 \times A) \times \hat{n}_2}{\|(\hat{n}_2 \times A) \times \hat{n}_2\|} = \{0.6124 \quad -0.7071 \quad -0.3536\}$$
$$\hat{s}_3 = \frac{(\hat{n}_3 \times A) \times \hat{n}_3}{\|(\hat{n}_3 \times A) \times \hat{n}_3\|} = \{-0.6124 \quad -0.7071 \quad -0.3536\}$$
$$\hat{s}_{12} = \frac{\hat{n}_1 \times \hat{n}_2}{\|\hat{n}_1 \times \hat{n}_2\|} sign((\hat{n}_1 \times \hat{n}_2) \bullet A) = \{0.3780 \quad -0.6547 \quad -0.6547\}$$
$$\hat{s}_{13} = \frac{\hat{n}_1 \times \hat{n}_3}{\|\hat{n}_1 \times \hat{n}_3\|} sign((\hat{n}_1 \times \hat{n}_3) \bullet A) = \{-0.3780 \quad -0.6547 \quad -0.6547\}$$
$$\hat{s}_{23} = \frac{\hat{n}_2 \times \hat{n}_3}{\|\hat{n}_2 \times \hat{n}_3\|} sign((\hat{n}_2 \times \hat{n}_3) \bullet A) = \{0 \quad -0.4472 \quad -0.8944\}$$

Step 4: Determine valid sliding direction

It can be shown that the equations for sliding on joint 1 are satisfied:

$$A \bullet \hat{n}_1 = -6.4434 \le 0$$
$$\hat{s}_1 \bullet \hat{n}_2 = 0.25 > 0$$
$$\hat{s}_1 \bullet \hat{n}_3 = 0.25 > 0$$

Therefore the sliding direction is:

$$\hat{s} = \{0 \quad -0.7071 \quad -0.7071\}$$

Step 5: Unsupported shear strength calculation

Unsupported shear strength is a result of active normal force on the sliding plane. Normal force due to passive forces is not included.

$$N_1^u = -A \bullet \hat{n}_1 = 6.4434 \text{ tonnes}$$
$$\sigma_{n_1}^u = N_1^u / a_1 = 6.4434 / 5.5114 = 1.1691 \text{ tonnes/m}^2$$
$$\tau_1^u = c_1 + \sigma_{n_1}^u \tan\phi_1 = 0 + 1.26 * \tan(35^\circ) = 0.8186 \text{ tonnes/m}^2$$
$$J_1^u = \tau_1^u a_1 \cos\theta_1 = 0.8823 * 5.5114 * \cos(0^\circ) = 4.5118 \text{ tonnes}$$
$$J_2^u = J_3^u = 0$$

Step 6: Supported shear strength calculation

Supported shear strength is a result of *both* active and passive normal force on the sliding plane.

$$N_1^s = -A \bullet \hat{n}_1 - P \bullet \hat{n}_1 = 6.4434 - 5.0 = 1.4434 \; tonnes$$

$$\sigma_{n_1}^s = N_1^s / a_1 = 1.4434 / 5.5114 = 0.2619 \text{ tonnes/m}^2$$

$$\tau_1^s = c_1 + \sigma_{n_1}^s \tan\phi_1 = 0 + 0.2619 * \tan(35^\circ) = 0.1834 \text{ tonnes/m}^2$$

$$J_1^s = \tau_1^s a_1 \cos\theta_1 = 0.1834 * 5.5114 * \cos(0^\circ) = 1.0107 \text{ tonnes}$$

$$J_2^s = J_3^s = 0$$

Step 7: Factor of safety calculation

$$F_f = \frac{-P \bullet \hat{s}_0 + \sum_{i=1}^{3} T_i}{A \bullet \hat{s}_0} = \frac{-\{0 \quad 7.071 \quad 0\} \bullet \{0 \quad -1 \quad 0\} + 0}{\{0 \quad -9.1125 \quad 0\} \bullet \{0 \quad -1 \quad 0\}} = \frac{7.071}{9.1125} = 0.776$$

$$F_u = \frac{\sum_{i=1}^{3}\left(J_i^u + T_i\right)}{A \bullet \hat{s}} = \frac{4.5118 + 0 + 0}{\{0 \quad -9.1125 \quad 0\} \bullet \{0 \quad -0.7071 \quad -0.7071\}} = \frac{4.5118}{6.4434} = 0.700$$

$$F_s = \frac{-P \bullet \hat{s} + \sum_{i=1}^{3}\left(J_i^s + T_i\right)}{A \bullet \hat{s}} = \frac{-\{0 \quad 7.071 \quad 0\} \bullet \{0 \quad -0.7071 \quad -0.7071\} + 1.0107 + 0 + 0}{\{0 \quad -9.1125 \quad 0\} \bullet \{0 \quad -0.7071 \quad -0.7071\}} = \frac{6.0107}{6.4434} = 0.933$$

$$F = \text{Factor of Safety} = \max(F_f, F_u, F_s) = 0.933$$

Since the supported factor of safety is the maximum value, all forces reported by Unwedge are derived from the supported factor of safety calculation.

9 Field Stress

Unwedge has the ability to incorporate induced stresses around an excavation into the calculation of factor of safety. The induced stresses are a result of an applied constant or gravitational far-field stress. The presence of the excavation causes a re-distribution of stress around the perimeter. In order to compute the induced stress distribution around the excavation, a complete plane strain boundary element stress analysis is performed.

Complete plane strain is well documented in the paper "The boundary element method for determining stresses and displacements around long openings in a triaxial stress field" by Brady and Bray (see references). The method allows for the application of any three-dimensional far-field stress distribution, without restriction, and assumes that the strain along the tunnel axis is zero. A complete three-dimensional stress tensor can then be calculated at any point in the rock mass surrounding the tunnel. The application in Unwedge utilizes the computer code developed for the Examine^{2D} software program developed in the 1980's. As a result, the implementation is well tested and accurate.

The implementation of field stress into the factor of safety calculation influences both the calculation of the active force vector on the wedge and the normal and shear forces on each joint plane.

- The normal forces on each joint plane are calculated from the distribution of stress across each joint plane. Thus, the normal force on each joint plane are specified by the stress analysis and are NOT calculated using the methods in section 5.
- Another difference is that generally there are normal forces on all planes, thus shear strength is incorporated into the resisting forces for all joint planes. The active force vector must also include the normal forces calculated on all joint planes from the stress analysis.

It should be noted that the effect of stress cannot reduce the factor of safety from the value computed without stress in section 7. The reasoning for this is that once any movement of the wedge occurs, contact with the rock mass is lost, and the factor of safety reverts to the unstressed value. As a result, if stress is included in the analysis, both the unstressed and the stressed factors of safety are calculated and the ***maximum*** of the two is reported.

Another point regarding the use of field stress, is that the stress analysis assumes an infinitely long excavation in the direction of the excavation axis. The stress analysis results will be valid, as long as the ratio of the actual excavation length to width is greater than approximately 3. If this ratio is less than 3, then "end effects" will influence the true stress distribution, and the stress analysis results will be less accurate.

Furthermore, because the stress analysis does not calculate the stress distribution around the ends of the excavation, field stress in Unwedge is only applicable for perimeter wedges, and cannot be applied to end wedges.

To calculate the factor of safety using field stress, the following steps are performed:

1. Perform the boundary element stress analysis for the excavation.
2. Determine the wedge geometry using block theory.
3. For each wedge, subdivide each joint plane into a number of triangles. By default, Unwedge uses approximately 100 triangles on each joint plane.
4. Compute the stress tensor at the geometric center of each triangle created in step 3.
5. From the stress tensor, compute the stress vector associated with each triangle on each joint plane.

$$\xi_{ij} = \sigma_{ij} \otimes \hat{n}_i$$

$\xi_{ij} =$ stress vector on the j^{th} triangle on the i^{th} joint

$\sigma_{ij} =$ stress tensor on the j^{th} triangle on the i^{th} joint $= \begin{Bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{Bmatrix}$

$\hat{n}_i =$ normal of the i^{th} joint pointing into the wedge

6. Compute the normal stress magnitude for each triangle, from the stress tensor computed in step 5. Make sure the tensile strength of the joint is utilized in the calculation (tensile failure).

$$\sigma_{n_{ij}} = \xi_{ij} \bullet \hat{n}_i$$

$\sigma_{n_{ij}} =$ normal stress magnitude on the j^{th} triangle on the i^{th} joint

$\xi_{ij} =$ stress vector on the j^{th} triangle on the i^{th} joint

$\hat{n}_i =$ normal of the i^{th} joint pointing into the wedge

7. Calculate the resultant normal force vector for all joints by accumulating the normal force vectors for each triangle. The normal force vector for each triangle is simply the normal stress vector calculated in step 6 multiplied by the area of the triangle.

$$Q = \sum_{i=1}^{3} \sum_{j=1}^{n} \left(a_{ij} \sigma_{n_{ij}} \hat{n}_i \right)$$

$Q =$ resultant active force due to stresses on all joint planes

$a_{ij} =$ area of the j^{th} triangle on the i^{th} joint

$\sigma_{n_{ij}} =$ normal stress magnitude on the j^{th} triangle on the i^{th} joint

$\hat{n}_i =$ normal of the i^{th} joint pointing into the wedge

8. Calculate the active force vector that includes the resultant normal force vector for each joint computed in step 7. Use this vector to determine the mode and direction of failure (section 4).

$$A = W + C + X + U + E + Q$$

$A =$ resultant active force vector
$W =$ wedge weight vector
$C =$ shotcrete weight vector
$X =$ active support force vector
$U =$ water force vector
$E =$ seismic force vector
$Q =$ resultant active force due to stresses on all joint planes

9. Using the normal stress on each triangle, compute the shear strength associated with each triangle. Add the shear strength of all triangles to get the total shear strength of each joint (see section 6.2)
10. Compute resisting force due to shear strength according to section 6.3.
11. Incorporate the resisting force due to shear strength and active force in the factor of safety equations in section 7.

10 Example Field Stress Calculation

Question: A 5mX5m square tunnel has an axis that plunges at zero degrees and trends exactly north. Three joint planes have a dip and dip direction of 45/180, 45/60, and 45/300. The unit weight of rock is 2.7 tonnes/m^3.and all three joints have zero cohesion, zero tensile strength, and a 25 degree friction angle. Assume a constant stress tensor equal to:

$$\sigma = \begin{Bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{Bmatrix} = \begin{Bmatrix} 200 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 200 \end{Bmatrix}$$

is computed on the entire face of each joint from the stress analysis (x=East, y=Up and z=South). Determine the factor of safety of the unsupported roof wedge.

Note: a constant stress tensor over the entire area of each joint plane, would (in general) never be computed from an actual stress analysis. This has only been assumed in this example to demonstrate the calculation procedure.

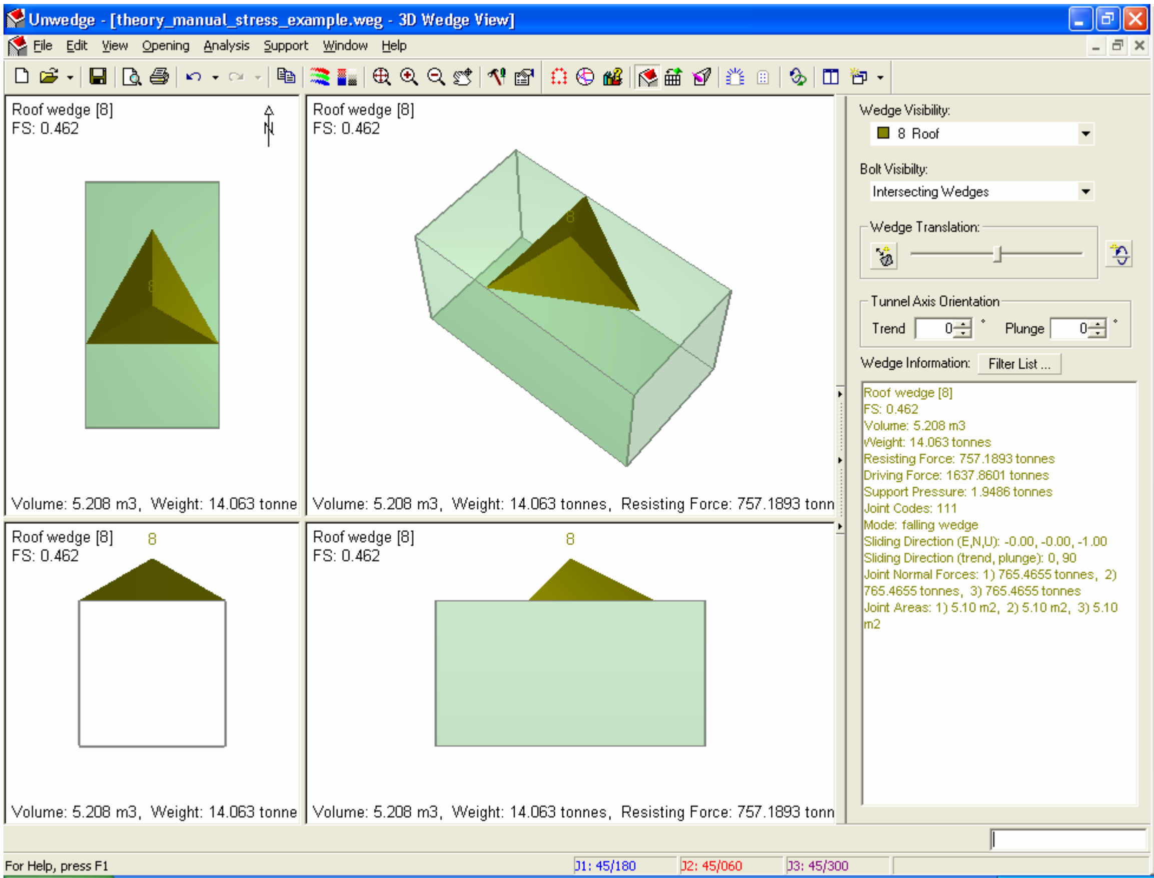
Answer: Using block theory as described in Goodman and Shi, "Block Theory and its application to rock engineering", the existence of a roof wedge is determined with the block code ULL (111). The actual coordinates of the vertices that form the maximum size block are also determined using the methods described in chapter 8 of the above reference. Using these methods, the volume of the block is calculated to be 5.208m^3. The area of each joint face is 5.103m^2. The joint normals are calculated using the following equations (coordinate system is x = East, y = Up, z = South):

$$n_1 = \{-\sin\alpha_1 \sin\beta_1 \quad -\cos\alpha_1 \quad \sin\alpha_1 \cos\beta_1\} = \{0 \quad -0.7071 \quad -0.7071\}$$
$$n_2 = \{-\sin\alpha_2 \sin\beta_2 \quad -\cos\alpha_2 \quad \sin\alpha_2 \cos\beta_2\} = \{-0.6124 \quad -0.7071 \quad 0.3536\}$$
$$n_3 = \{-\sin\alpha_3 \sin\beta_3 \quad -\cos\alpha_3 \quad \sin\alpha_3 \cos\beta_3\} = \{0.6124 \quad -0.7071 \quad 0.3536\}$$

$n_i =$ unit normal vector of the ith joint pointing into block

$\alpha_i =$ dip of the ith joint

$\beta_i =$ dip direction of the ith joint



Now to determine the factor of safety:

Since we are given the stress tensor on each joint plane we can proceed directly to step 5 as defined in section 9. Since the stress tensor is constant over each joint plane, use only one triangle which represents the entire joint face. We do NOT need to subdivide the joint face into 100 triangles!

Step 5: From the stress tensor, compute the stress vector associated with each triangle on each joint plane.

$$\xi_{11} = \sigma_{11} \otimes \hat{n}_1 = \begin{Bmatrix} 200 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 200 \end{Bmatrix} \begin{Bmatrix} 0 \\ -0.7071 \\ -0.7071 \end{Bmatrix} = \{0 \quad -70.71 \quad -141.4\}$$

$$\xi_{21} = \sigma_{21} \otimes \hat{n}_2 = \begin{Bmatrix} 200 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 200 \end{Bmatrix} \begin{Bmatrix} -0.6124 \\ -0.7071 \\ 0.3536 \end{Bmatrix} = \{-122.5 \quad -70.71 \quad 70.71\}$$

$$\xi_{31} = \sigma_{31} \otimes \hat{n}_3 = \begin{Bmatrix} 200 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 200 \end{Bmatrix} \begin{Bmatrix} 0.6124 \\ -0.7071 \\ 0.3536 \end{Bmatrix} = \{122.5 \quad -70.71 \quad 70.71\}$$

Step 6: Compute the normal stress magnitude for each triangle, from the stress tensor computed in step 5. Make sure the tensile strength of the joint is utilized in the calculation (check tensile failure).

$$\sigma_{n11} = \{0 \quad -70.71 \quad -141.4\}\begin{Bmatrix} 0 \\ -0.7071 \\ -0.7071 \end{Bmatrix} = 150 \text{ tonnes/m}^2$$

$$\sigma_{n21} = \{-122.5 \quad -70.71 \quad 70.71\}\begin{Bmatrix} -0.6124 \\ -0.7071 \\ 0.3536 \end{Bmatrix} = 150 \text{ tonnes/m}^2$$

$$\sigma_{n31} = \{122.5 \quad -70.71 \quad 70.71\}\begin{Bmatrix} 0.6124 \\ -0.7071 \\ 0.3536 \end{Bmatrix} = 150 \text{ tonnes/m}^2$$

Since the friction angle is 25 degrees, the shear strength of all three joint planes is the normal stress (150 tonnes/m^2) multiplied by the tangent of 25 degrees which equals 69.95 tonnes/m2. Since all three normal stresses are positive, there is no tension and the tensile strength check does not have to be done.

Step 7: Calculate the resultant normal force vector for all joints by accumulating the normal force vectors for each triangle. The normal force vectors for each triangle are simply the stress vectors calculated in step 6 multiplied by the area of the triangle.

$$\begin{aligned} Q &= 5.103*150*\{0 \quad -0.7071 \quad -0.7071\} + \\ &\quad 5.103*150*\{-0.6124 \quad -0.7071 \quad 0.3536\} + \\ &\quad 5.103*150*\{0.6124 \quad -0.7071 \quad 0.3536\} + \\ Q &= \{0 \quad -1623.75 \quad 0\} \end{aligned}$$

Step 8: Calculate the active force vector that includes the resultant normal force vector for each joint computed in step 7. Use this vector to determine the mode and direction of failure (section 4).

$$\begin{aligned} A &= W + Q \\ &= (\gamma_r V) \bullet \hat{g} + \{0 \quad -1623.75 \quad 0\} \\ &= (2.7*5.2083) \bullet \{0 \quad -1 \quad 0\} + \{0 \quad -1623.75 \quad 0\} \\ &= \{0 \quad -1637.8 \quad 0\} \end{aligned}$$

The mode of failure is falling in the direction {0 -1 0}

Step 9: Using the normal stress on each triangle, compute the shear strength associated with each triangle. Add the shear strength of all triangles to get the total shear strength of each joint (see section 6.2)

$$\tau_1 = \tau_2 = \tau_3 = 150 \tan 25^\circ = 69.946 \text{ tonnes/m}^2$$

Step 10: Compute resisting force due to shear strength according to section 6.3

$$\theta_1 = \theta_2 = \theta_2 = 45^\circ$$
$$J_1 = \tau_1 a_1 \cos\theta_1 = 69.946 * 5.103 * 0.7071 = 252.39 \text{ tonnes}$$
$$J_2 = \tau_2 a_2 \cos\theta_2 = 69.946 * 5.103 * 0.7071 = 252.39 \text{ tonnes}$$
$$J_3 = \tau_3 a_3 \cos\theta_3 = 69.946 * 5.103 * 0.7071 = 252.39 \text{ tonnes}$$

Step 11: Incorporate the resisting force due to shear strength and active force in the factor of safety equations in section 7.

Since the wedge is unsupported, only the unsupported factor of safety equation needs to be calculated (the falling factor of safety is zero and the supported factor of safety is the same as the unsupported factor of safety).

$$F = F_u = \frac{\sum_{i=1}^{3}\left(J_i^u + T_i\right)}{A \bullet \hat{s}} = \frac{252.39 + 252.39 + 252.39}{\{0 \quad -1637.8 \quad 0\} \bullet \{0 \quad -1 \quad 0\}} = \frac{757.19}{1637.8} = 0.46$$

11 References

Brady, B.H.G. and Bray, J.W. (1978), "The boundary element method for determining stresses and displacements around long openings in a triaxial stress field", Int. J. Rock Mech. Min. Sci. & Geomech., Vol. 15, pp. 21-28.

Goodman, R.E. and Shi, G. (1985), "Block Theory and Its Application to Rock Engineering", Prentice-Hall, London.