Dynamic Oppositional Symmetries for Color, Jungian and Kantian Categories*

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Abstract. This paper investigates some classical oppositional categories, like synthetic vs. analytic, posterior vs. prior, imagination vs. grammar, metaphor vs. hermeneutic, metaphysics vs. observation, innovation vs. routine, and image vs. sound, and the role they play in epistemology and philosophy of science. The role these categories play in the *objective cognitive constructivism* epistemological framework is the final aim of these investigations. The path we follow visits some related topics of interest including algebraic lattice structures like the cube and hexagon of opposition, and the role played by oppositional categories and their appropriate representations in the contexts of modern color theory, Kantian philosophy, Jungian psychology, and linguistics.

> *Socrates: Two principles we should be glad to understand: First, that of, in a clear and consistent way, bringing together into one idea scattered particulars... as we just did in our definition of Love. Second, that of dividing things again by classes, where the natural joints are, and not trying to break any part, after the manner of a bad carver.*
>
> Plato; Phaedrus, 265d,e (adapted).

> *There are two methods of treatment: First, the synthetic; Second, the reductive* [or analytic]. *The first develops the material into a process for differentiating the personality. The second traces everything back to primitive instincts... The two methods are complementary.*
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> Jung; Psychological Types, pr.427 (adapted).

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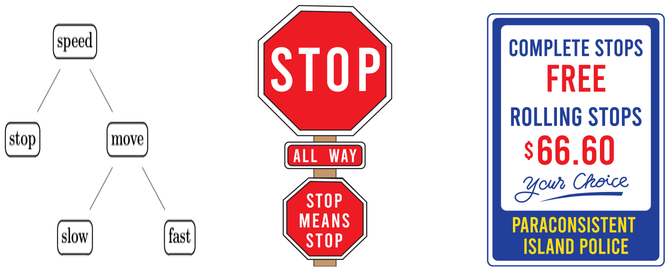


FIGURE 1. Hierarchical conceptual splitting and lexical disambiguation

1. Introduction: A Short Story of Shifting (At)Tentions

Cornell university campus security has a bad reputation for giving tickets for any, even the most frivolous, traffic violations, see Feller (1966, v.II, ch.10, prob.13). However, back in 1987, at a road intersection, a good cop spared me from one after a *fast stop* – that I characterized by null velocity or zero derivative, that is, $dx/dt=0$. Furthermore, after asking why such an instantaneous stop was not good enough, I got (essentially) the following explanation: The Stop sign requires a complete or *slow stop*, characterized by the ceasing of all movement, including the (damped) oscillation of the chassis after the wheels stop. Hence, the car only comes to a slow stop after the energy in the suspension system is dissipated, implying zero integral displacements over a finite time interval, δ, that is, $\mathrm{Int}[t, t+\delta]\, x\, dx = 0$. At the moment, I politely thanked the cop for giving me only a warning but was annoyed by what I felt was malicious nitpicking.

A few months later, while driving on snow for the first time in my life, I tried to do a fast stop only to see it transmute into a slow skidding movement, and barely avoided a serious accident. At the moment, I blamed the (soon after replaced) tires on my car, but saw nothing wrong with my driving skills. Nevertheless, that night I had a dream about the incident and, waking up, realized the wisdom of the formerly received advice and how to avoid similar occurrences when driving on snow. Moreover, I realized the importance of having adequate concepts to think about my experiences and the benefits of having corresponding words to express them in language.

Figure 1 illustrates how important it is to make the distinctions under scrutiny living in Ithaca, NY, where winter is cold and snowy, in contrast to my native country, Brazil, where it never snows. The tree diagram in Figure[1] 1l, from Seuren and Jaspers (2014), depicts the distinctions I initially had in mind at the beginning of this short story. Figure 1b illustrates my perplexity when facing new situations requiring more refined concepts and a richer lexical palette. However, his traffic sign seems like a riddle conceived by the sphinx, stating veiled treats and giving

[1]Figure position locators: t=top, b=bottom, l=left, c=center, r=right.

obscure advice. In contrast, the sign in Figure 1c has a clear warning and educational purposes, for it succinctly explains relevant distinctions, considers implied choices and their consequences, and even introduces pertinent lexical expressions.

This short story also illustrates important attention shifts, structured around classical oppositional structures. As already mentioned, at the beginning of this story I was fully aware of the move vs. stop opposition (due to parking tickets), and the slow vs. fast movement opposition (due to speeding tickets). Later on, I was made aware of the (for me, new) opposition category of fast vs. slow stops (and corresponding tickets), and had to integrate all these oppositions into a coherent structure that was more complex that the one I initially had in mind. For further analyses related to this case, see Béziau (2017,p.183), Jaspers and Seuren (2016, p.621), Jaspers (2017), and Seuren and Jaspers (2014). In the next sections, I will analyze in detail similar oppositional structures related to color, Kantian and Jungian categories.

In the aforementioned dream, I "felt" how difficult it is to dissipate the car's kinetic energy on a slippery road, and understood why *snow-stops* require *slow-stops* (pun intended)[2]. On the one hand, having finer distinctions concerning the concepts of fast, slow, move, and stop, and a richer structure for organizing them, allowed me to better analyze my predicaments. On the other hand, the same concepts afforded me a better synthetic vision, allowing me to see pertinent connections between somewhat similar situations that lead to effective behavioral adjustments. Initially, I had to be always mindful of these adjustments but, after a while, the new behavioral patterns became fully automatic, and driving on snow became a natural and pleasant thing to do. This short story is intended to highlight some analytic, synthetic, logical, and metaphorical powers and properties of language and ontologies – topics to be investigated in the sequel using, as a paradigmatic model, modern color theory.

2. Modern Color Theory

This section gives a brief review of basic notions of modern color theory needed for our considerations in the following sections. In 1704, Isaac Newton publishes the book *Opticks*, where he develops a series of experiments and explanations for the behavior of light. First, Newton showed that a glass prism can decompose a white light ray into a *spectrum* of colors we see in the rainbow, where colors are arranged in a linear segment, progressing, from left to right in Figure 2l, in the following order: Red, Yellow, Green, Cyan, Blue and Violet. An important characteristic of the color spectrum is that colors change gradually along the line segment. Hence the aforementioned color names only mark a few standard location points in the spectrum. Someone could be willing to use additional location points, like orange between Red and Yellow or lime between Yellow and Green, or don't bother to

[2]Analogous sound relations or paronomasia, famously used in Freudian psychoanalysis to study subconscious slips (parapraxis), will play an important role in following sections.

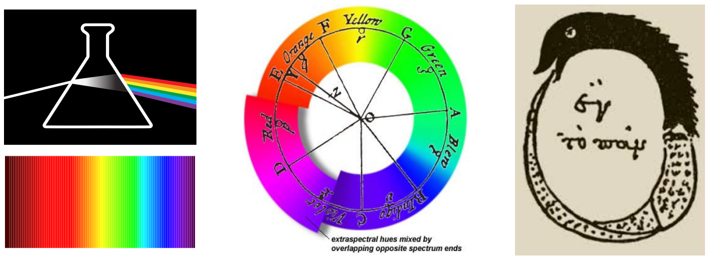


FIGURE 2. Newton (1704) spectra and Ouroboros

have a proper name for Cyan, describing it instead by a compound expression like light blue. However, the aforementioned colors written with capital initials have a special role to play in the sequel.

Using two prisms, Newton experimented with mixing light of two single colors, that is, mixing light taken from narrow bands in the spectrum. Most of the time, these mixtures only generate colors already present in between them in the linear spectrum. However, by mixing different proportions of Red and Blue, Newton obtained additional colors, like Purple (more red than blue), Magenta, and new shades of Violet (more blue than red). Newton used a *color wheel*, similar to the one depicted in Figure 2c, to give a graphical representation for all these compositions. Today, the region between red and blue is called the *paradoxical region* of the color wheel, as highlighted in Figure 2c, including *non-spectral colors*, like purple, magenta, and some shades of violet, and also some other shades of violet at the extreme right of the linear spectrum. Carl Gustav Jung compared the color wheel to the alchemical symbol of the Ouroboros, see Figure 2r, a snake that bites its tail, where the body of the snake spans the linear spectrum, and the paradoxical region is represented by the region joining head to tail.

2.1. Maxwell's Theory of Color Composition

More than a century after Newton's work, in 1860, James Clerk Maxwell was able to show that all colors perceived by the human eye can be obtained by mixing, in different proportions, three specific *primary colors*, namely, Red, Green, and Blue (RGB). The device used by Maxwell in his experiments is remarkably simple, using three discs, one for each primary color, made of paper and having a radial slit, see Figure 3l. The paper discs are then mounted on a supporting metal disc, where angular sections of different sizes expose each of the three colors. Spinning the supporting disc, a viewer has the visual perception of a color mixture in the same proportion of the exposed section of each color disc.

Maxwell's triangle, depicted in Figure 3r, gives a mathematical representation of his experiment. The *compositional* or *barycentric* coordinates, $\langle r, g, b \rangle$, specify the position of a point P inside an equilateral triangle of unitary height by its distances to each of the triangle's sides. By Viviani's theorem, these coordinates must be non-negative and add up to one, that is, $r, g, b \in [0, 1]$ and $r + g + b = 1$.

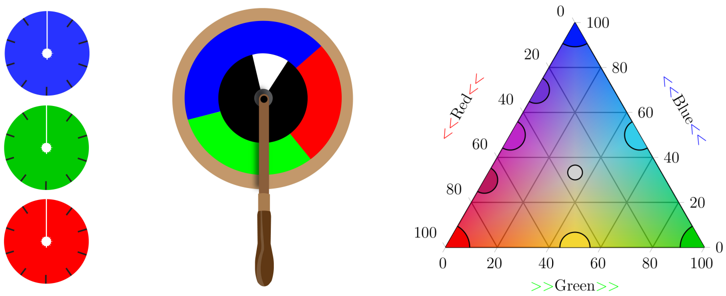


FIGURE 3. Maxwell (1860) composition structure

Physically, P will be at the center of mass or equilibrium point of the triangle if we place at its vertices, $\{R, G, B\}$, corresponding non-negative masses adding up to unity. The following coordinates for some already familiar colors may help the reader get used to this coordinate system: $Y = \langle 1/2, 1/2, 0\rangle$, $C = \langle 0, 1/2, 1/2\rangle$, $M = \langle 1/2, 0, 1/2\rangle$, $P = \langle 3/4, 0, 1/4\rangle$, $V = \langle 1/4, 0, 3/4\rangle$; finally, the coordinates $\langle 1/3, 1/3, 1/3\rangle$ give the gray point at the center of the triangle.

Notice that the barycentric coordinate system only specifies colors by relative weights. If in Maxwell's experiment, we use distinct sets of paper discs with brighter or more obscure colors, we will get lighter or darker shades of gray at the center point of the triangle. Maxwell's device includes a smaller set of blacK and White paper discs close to the spinning center (since the letter B is already used for blue, we use K for black). Regulating the size of exposed segments of RGB and KW discs allows us to match shades of gray obtained at the larger and smaller spinning rings and, in so doing, evaluate relative the brightness of color papers.

Figure 4tl depicts the RGB rectangular coordinate system. In contrast to the barycentric system, the rectangular coordinates are independent. For convenience, they are normalized in the $[0, 1]$ interval, but they are no longer subject to the constraint of adding up to unity. This is a natural coordinate system to use when we can independently regulate the intensity of RGB light sources, like three lamps of these colors illuminating a white wall, or three color LEDs (light-emitting diodes) in a single pixel of a TV screen.

However, in many applications, it is convenient to use a coordinate system that uses separate coordinates for absolute brightness and other distinctive aspects of color sensation. Painters and other artists have for centuries tried to accurately describe such essential characteristics. Figure 4b depicts the HSL and HSB variants of the *hexcone* coordinate system, developed to handle independently three separate characteristics of color sensation, namely, *brightness* or *luminosity* value, *hue* and *saturation*. This system has the form of a hexagonal prism or pyramid. The vertical coordinate in this system, ranging from zero to 1 or 100%, conveys brightness or luminosity. Hue relates to a position in the color wheel, and is given by a continuous angular coordinate that starts at 0°, for Red, and reaches, at 60°

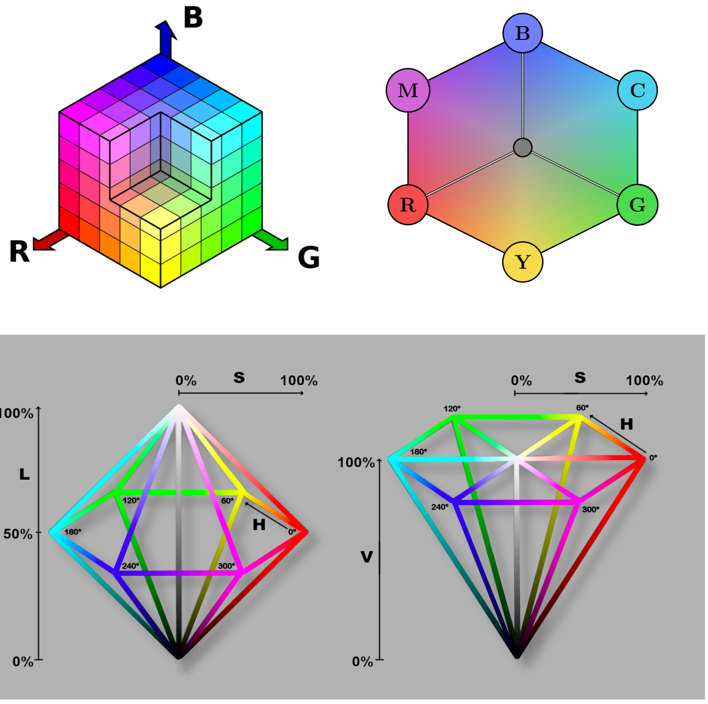


FIGURE 4. RGB rectangular coordinates, cube projection, and Wundt (1892) - Smith (1978) Hue, [0,360°], Saturation, [0,100%], and Luminosity or brightness value, [0,100%], hexcone encoding.

increments, Y, G, C, B, and M. Saturation is the radial coordinate ranging from zero, for a pale or undifferentiated gray at the center, to 1 or 100%, for a pure color at the border. The hexcone system was developed in the mid-XX century for color-TV broadcasting and computer graphics and is now the underlying form of color encoding in most technological applications of the modern world.

Figure 4tr, depicts the *color hexagon*, a helpful key in understanding the geometric relation between these distinct coordinate systems. The color hexagon can be seen as a projection of the RGB cube, depicted in Figure 4tl, along its KW axis, which is perpendicular to the hexagon. A small gray hexagon can be seen at the center of the cube; hidden below it, lays the black corner at the bottom of the cube. The white corner, which should be at the top of the cube, has been removed, together with its entire quadrant, in order to expose some inner layers and make the construction of the cube easier to understand.

2.2. Hering's Theory of Color Opposition

Besides the several alternatives examined in the last section, other coordinate systems are useful in varied applications. For example, instead of the angular

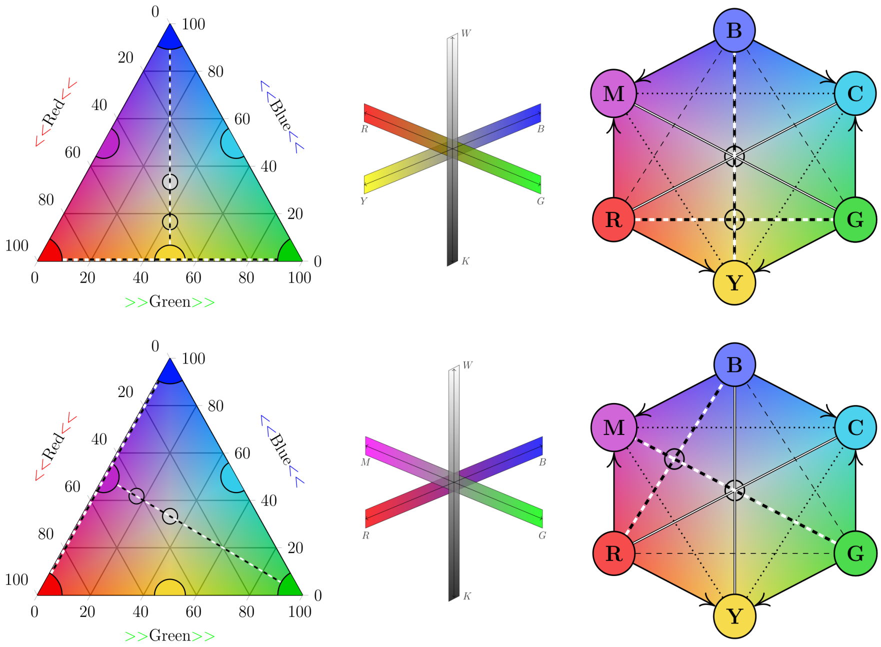


FIGURE 5. Orthogonal axes for Yellow and Magenta -headed kites

and radial polar coordinates used in the hexcone system, one can use orthogonal [0,1] coordinates along perpendicular axes in the hexagon in order to specify the balance between opposing reference colors. For example, Figure 5t uses a vertical coordinate along the YB axis and a horizontal coordinate along the RG axis. Analogously, Figure 5b uses the MG and RB axes of "opposing" colors. Due to their geometrical shape, as depicted in the hexagon, the preceding systems are known, respectively, as the Yellow-headed kite and the Magenta-headed kite, see Jaspers (2017). Some of the reasons for using such oppositional systems are explained in the sequel. The meaning of the arrows and additional lines drawn inside the hexagon will be explained in section 2.3.

Shortly after Maxwell (1860) postulated his tripolar (RGB) compositional theory of color, Ewald Hering (1878) postulated a rival quadripolar oppositional theory, according to which, color vision is based on two antagonistic contrast signals for color proper, namely, R/G and Y/B, plus a third contrast signal for brightness, K/W. Figure 6r depicts orthogonal axes corresponding to these three signals, that coincide with those in Figure 5tc. The circle of oppositional colors shown in Figure 6c is taken directly from Hering (1878) work.

Hering claimed to have empirical evidence supporting his theory based on experiments concerning color constancy of objects displayed against varying backgrounds, visual perception of color contrasts, and color perception by individuals

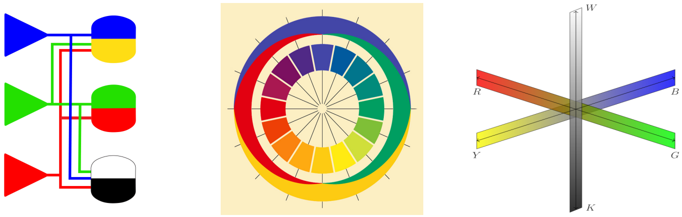


Figure 6. Hering (1878) opposition structure.

with color vision deficiencies. Only in the XX century could neural physiology reveal the mechanisms behind these phenomena. At the eye's retina, there are three distinct types of color receptors, just as predicted by Maxwell's tripolar compositional theory. However, at the optic nerve that connects the human eyes to the brain, there is a neural network that recombines the three types of color signals generated at the retina into antagonistic contrast signals with the same structure predicted by Hering's theory, as shown in Figure 6l. From this neural network architecture, we can understand why the Yellow-headed kite coordinate system shown in Figure 5t is useful when studying the human perception of color contrast and related phenomena.

2.3. Logical Representation of Color (De)Composition

This section provides a logical view of the mereological relations, that is, of the compositional and decompositional properties of colors studied in this section. It does so by presenting abstract diagrammatic representations of these relations that can capture them in compact visual forms that are easy to grasp and understand. These diagrammatic representations are also going to be useful, later on, in the analyses of other categories exhibiting similar logical structures.

Maxwell's experiments demonstrate how primary colors can be added. Mixing light of two primary colors generates one of the *secondary* colors, as shown in the *Hasse diagram* shown at figure Figure 7l, namely, R+B=M, R+G=Y, and B+G=C. This diagram represents an *algebraic lattice*, that also includes a blacK element at its *bottom* and a White element at its *top*, corresponding to the absence or the presence of all three primary colors. As shown by Maxwell, primary colors constitute a *additive basis*, meaning that other color perception can be generated by mixing the three primary colors in the right proportion. In the same way, secondary colors constitute a *subtractive basis*, meaning that other color perception can be generated by filtering, from a white light source, the three secondary colors in the right proportion. A set of stained glass filters can be used for direct empirical demonstration of this property, which is implied by the algebraic lattice structure. The bit-string color representations that appear in Figure 7 are Boolean or $\{0,1\}$ versions of the continuous coordinate systems used to specify color mixtures examined in previous subsections. These bit-strings can capture and concisely display the essential logical properties of this compositional system.

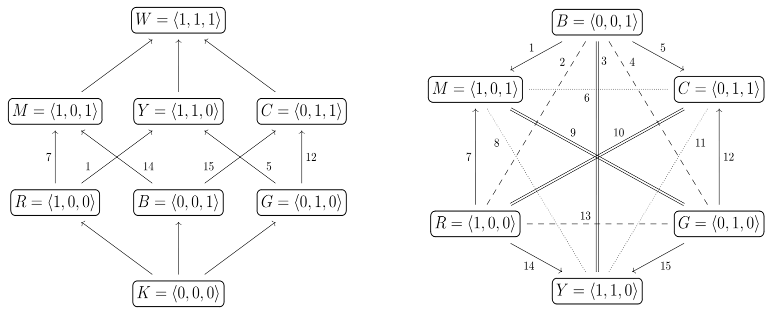


FIGURE 7. Left: Hasse diagram for (transitive) mereological relations of entailment or inferiority ($\longrightarrow$). Right: Aristotelian diagram of opposition for additive (RGB) and subtractive (CMY) colors with corresponding mereological or bit-string relations of complementarity ($=\!=$), contrariety ($- \, -$), sub-contrariety ($\cdots$).

The *Aristotelian diagram* shown in Figure 7r, gives an alternative representation of these logical or mereological relations (i.e., compositional and decompositional properties). It clearly and succinctly displays the following logical relations: Implication or entailment relations (as in the Hasse diagram) are represented by arrows, ($\longrightarrow$). Contrariety relations are represented by dashed lines ($- \, -$); contrary conditions cannot both be valid or true, although they might both be invalid or false. Sub-contrariety relations are represented by dotted lines ($\cdots$); sub-contrary conditions cannot both be false, although they might both be true. Contradiction relations are represented by parallel lines ($=\!=$); contradictory conditions must have opposite validity status or truth-false values.

3. Praxis, Lexicalization, and Archerypes

In Judo and other martial arts, an athlete trains by executing, over and over again, some basic techniques. Repetition commits these techniques to "muscle memory", allowing the required movements to be executed without hesitation in a fully automatic way. In so doing, the athlete builds "new instincts" for her or himself, being able to retrieve and execute such procedures or behavioral patterns in a (quasi) reflexive manner. Afterward, in combat, laborious conscious thinking about what to do and how to do it may be bypassed, resulting in decisiveness and agility that often lead straight to victory. Judo practitioners also want to talk about the things they do, so they create specific names for the basic techniques they use, known in Japanese as Waza, 技 or 技. In time, some of these names become standard words used in language that, later on, may even be enthroned as headwords in the pertinent dictionary or, using the jargon of computer science, become elements of the pertinent *ontology*, see Stern (2014, 2017, 2020) and references therein. Ontologies serve many purposes, for example, the rules of a Judo tournament use the waza ontology to determine how and which techniques can be used. The historical process in which words are created and standardized in the language used by a human community is called *lexicalization*, a main topic discussed in this section.

Judo waza require hard work to be properly introjected. In contrast to these "artificial instincts", human babies display a set of involuntary reflexes that are inherited and inborn. They allow a baby to perform some simple tasks essential to survival, like blinking, sucking, grasping, stepping, etc. These inherited reflexes offer starting points for development processes aiming to build up more complex skills and behaviors. For example, the stepping reflex is a starting point for the development of walking – something that any human being strives for and will eventually do in his own style. Finally, some human behaviors or functions, like breathing or heart beating, can develop and operate in a fully autonomous way, requiring no conscious attention whatsoever. After all, breathing must start, and never stop, from the moment of birth, while heart beating, although it can stop and go for a little while, must begin much earlier.

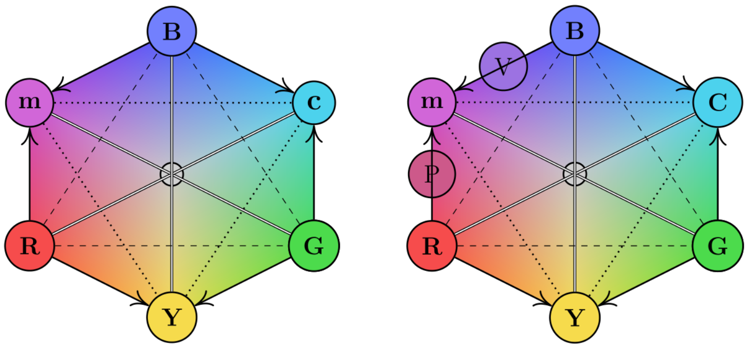
In the Jungian framework, one speaks of an *individuation* process that reenacts in the ontogeny (i.e. personal development) of each individual certain prototypical behaviors or *archetypes*[3]. Also, it is assumed that (basic patterns for) such archetypes are phylogenetically transmitted and genetically inherited by any member of its species. Moreover, it is assumed that archetypes are adaptive, that is, that they have been selected in the evolution of their species for enabling (re)actions that contribute to the individual's survival. Therefore, archetypes are considered *teleological*, that is, we regard them as capable of manifesting themselves in a purposeful manner. Furthermore, in the Jungian framework, archetypes may be unconscious, that is, they may operate partially or totally outside the individual's conscious attention. Finally, even if unconscious, it is assumed that archetypes are capable of manifesting themselves psychologically by *archetypal images*. In so doing, they may bring to an individual's conscious attention something akin to mental representations of Platonic ideal forms, known as $\epsilon\iota\delta\eta$ or $\epsilon\iota\delta\omega\lambda\alpha$.

3.1. Color Archetypes and Neotypes

Let us now return to the topic of human perception and understanding of colors. Section 2 presented a summary of modern compositional and oppositional color theory, as how they were developed and integrated, from the beginning of the XVIII until the late XX century. Let us then consider the following questions:

Could we, humans, somehow have had prior knowledge of essential facts of modern color theory? Could mankind have developed, long ago, a similar Weltanschauung? Could some basic knowledge about the structural organization of color vision have been perceived, presented, and recorded in language *a priori* (i.e., before) the development of modern means and methods of scientific investigation? Furthermore, if the answer to the former questions is in the affirmative, could this pre-scientific knowledge have been kept unconscious, that is, could it always have

[3]Similarly, Ernst Haeckel's law states that, in the embryonic development on an individual's body, ontogeny recapitulates phylogeny, see Gould (1977), Danesi (1993), and Lenoir (1989).



: אָדֹם *adom* = Red; < דָּם *dam* = blood ≈ אָדָם *adam* = man; < אֲדָמָה *adama* = ground, earth, fertile soil.

: צָהֹב *tzahov* = Yellow; to be bright, to shine; ≈ זָהָב *zahav* = gold.

: יָרֹק *yaroq* = Green; < יָרַק *yaraq* = herb, vegetation.

: תְּכֵלֶת *tekhelet* = Cyan (a sky-blue color obtained from Hexaplex dye).

: כָּחֹל *kachol* = Blue; < כָּחָל *kachal* = sublimated antimony pigment, the rectified spirit of any substance obtained as sublimated powder or distilled liquid > כֹּהָל *kohal* = al-cohol.

: סָגֹל *sagol* = Violet; < סְגוּלָה *segula* = grape cluster; > סִגֵּל Sigel = well-acclimated, well-adapted; ≈ סְגֻלָּה *segula* = "a charm that supersedes logic".

: אַרְגָּמָן *argaman* = Purple (bluish-red color obtained from Hexaplex dye used in textiles) < אָרַג *arag* = to weave, wove (a textile).

: תּוֹלַעַת שָׁנִי *tola(at shani* = kermes vermilio crimson, as in Exodus 39:1-3.

FIGURE 8. Top: Traditional color terms in English and Hebrew; Capital (vs. lower case) letters correspond to simple lexical forms. Bottom: Hebrew terms, etymology and correlates in the language.

been kept alive but hidden in plain sight? In the sequel, I present some arguments supporting the preceding conjectures.

The World Color Survey (WCS) and several related projects across the fields of linguistics, anthropology, and neurophysiology of color perception were conducted during the XX century, see Kay et al. (1997, 1999), yielding important conclusions pertinent to the topics under discussion, namely:

(1) All traditional human languages have a small set of words that jointly partition the psychological color space;

(2) Languages evolve by expanding this set along a path that proceeds near a "characteristic encoding order" or "standard lexicalization sequence". Moreover, this sequence follows the hierarchical branching process given next:

(a) Separation between light (White) vs. darkness (blacK);

(b) Separation between warm (Red, Yellow) vs. cold (Green, Blue) colors;

(c) Splitting of warm colors between Red and Yellow (where Red usually retains its primitive lexical form and so gets to be the oldest fundamental color); and
(d) Splitting of cold colors between Green and Blue.

The aforementioned hierarchical branching process can be obtained from the output signals of the neural network displayed in Figure 6l, by aggregating and then disaggregating them again into: One combined W/K signal; Two combined W/K and GB/RY signals; and Three separate W/K + G/R + B/Y signals. Figure 5t gives intuitive representations of the fully disaggregated color perception space. For further details on color perception and lexicalization, see Berlin and Kay (1999), Elliot et al. (2015), Kuehni and Schwarz (2008), and MacLaury et al. (2007).

In this way, the historical evolution of color words in traditional human languages seem to have been able to perceive, present, and record essential information concerning modern color theory. However, such basic lexicons had to be developed before any color theory was available, or even conceivable, for how could we possibly theorize about “things” we could not yet talk about?! (or about concepts we could not yet distinguish?!) A reasonable explanation for all these interconnected phenomena is that the ontogeny of human languages proceeds along tracks laid by the (invariant) architecture of color perception physiology that is genetically encoded and phylogenetically transmitted to each new human being.

Figure 8tl shows, in capital letters, the vertices corresponding to the standard color lexicon of most traditional human languages, including English and German; the close correspondence with Figures 5t and 6l should be obvious. Variations from the standard lexicalization path do occur. For example, on the one hand, Japanese never made the final split between Green and Blue, using for both the same word, Aoi, 靑 or 青. On the other hand, Russian splits blue into light blue (goluboy) and dark blue (siniy). Meanwhile, Hebrew, as shown in Figure 8tr,b, developed a richer palette of traditional color words, for reasons analyzed in the sequel.

3.2. Hebrew Colors: Revealed, Veiled, and Unveiled

Figure 8tl displays a set of four words corresponding to the four basic colors in Hering’s oppositional color theory. As previously discussed, these words constitute the traditional basic color lexicon of English, German, and many other languages (plus blacK and White words for an additional orthogonal axis, as depicted in Figures 4 and 5). Figure 5t depicts an orthogonal coordinate system for the color space that is oriented according to the Yellow-headed kite. This system has four cardinal points that match the four capitalized color words at the vertices of Figure 8tl. The Yellow-headed kite also agrees with the way in which the (non)capitalized letters in Figure 8tl break the logical and geometrical symmetry of the color hexagon.

In contrast, Figure 8tr shows the traditional basic color lexicon of the Hebrew language. Figure 8b explains these words, their meanings, etymological derivations, and some correlates of possible interest. Comparing Figures 8tl and 8tr we notice the inclusion of a word for Cyan, like in Russian, and the inclusion of two more words, for Purple and Violet, near the Magenta vertex of the hexagon. Figure

5b depicts an orthogonal coordinate system oriented according to the Magenta-headed kite. This reorientation corresponds to an attention shift in the color space to the regions covered by the lexical expansions beyond the standard Hering's base, as suggested by Figure 8tr. The Magenta-headed kite also agrees with the way in which the (non)capitalized letters in Figure 8tr break the full logical and geometrical symmetry of the color hexagon.

The expansion of the basic color lexicon to seven words goes several steps beyond the previously discussed standard lexicalization sequence leading to RGBY. There are, however, strong reasons for this expansion motivated by ancient traditional practices. Since early biblical times, Jewish religion made use of colors along the RMBC arch, with specific ritual roles attributed to Purple, Cyan, and also to crimson, see for example Exodus 39:1-3, although this last color, designated by a compound expression (tola'at shani), never became fully lexicalized. For further details and explanations, see Amar (2005), Sterman (2012), and Ziderman (1986).

The rituals of Jewish religion are supposed to have been instructed by the word of God. However, these instructions would make no sense if the words used in revelation could not be understood, or if the designated colors could not be obtained. Therefore it is important for us, in present times, to have precise and reliable information concerning ancient meanings of biblical words (as they were understood at the legendary time of revelation, or at least at pertinent historical times of religious practice) and also concerning equally old color technologies; those are highly non-trivial tasks.

Fortunately, there is ample material to support detailed studies in this area. First, the biblical text itself is one of the largest and best-preserved corpora of the ancient literature of western civilization. Second, every detail and minutia of the biblical text has been the subject of extensive secondary literature developed over millennia. The Talmud is the best-known example thereof, consisting of the Mishnah, compilations of older oral traditions from c.200CE, and the Gemara, compilations of further comments and discussions from c.350CE (Jerusalem version) and c.500CE (Babylonian version). Third, extensive archaeological studies employing, among others, methods of analytical chemistry, give us today a clear picture of the dyeing technologies developed and available in the Mediterranean and near-east regions at the pertinent periods.

Ample archaeological evidence shows that the aforementioned colors were obtained from sea snails, certainly Hexaplex trunculus and possibly also Murex brandaris, see Figure 9bcb. These colors were used by several civilizations in the Mediterranean region, where dying textiles mas a major industry, motivation for commerce, and source of wealth. Figure 9bct shows a silver coin from c.450BCE engraved with an hexaplex shell, attesting the importance of *Tyrian purple* for this Phoenician city. Figure 9brb displays some samples of textiles dyed in this way, exhibiting a wide range of colors in the RPMVBC arch, see Karapanagiotis (2019) and Ramig et al. (2017). Modern chemistry identifies 6,6'-Dibromoindigo, see Figure 9brt, as the key molecule of this ancient business. During the extraction and dyeing process, one or both of the Bromide atoms can be removed from the

Cultural	|	Color	|	Technology
Organization & structure	⇒	Visual information	⇒	Dye industry
⇑				⇓
Systemic functions		Color words & definitions		Textile fabrication
⇑				⇓
Personal roles	⇐	Languages & contexts	⇐	Material media
symbolism	|	meanings	|	know-how

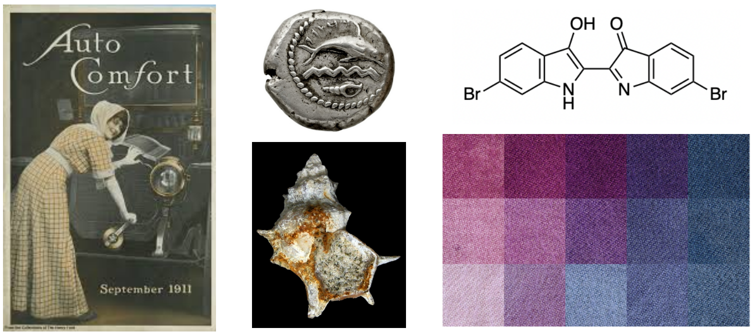


FIGURE 9. Hexaplex dyes and color production diagram

molecule by ultra-violet radiation in sun light rendering, in the last case, the Indigo dye nowadays obtained from a much cheaper vegetable source, *Indigofera tinctoria.* Moreover, hydrogen bonds at the Oxygen and Nitrogen atoms can be activated in different ways of binding the molecule to substrata. All these variations in using this wonder molecule account for endless color variations. The same hydrogen bonds facilitate good fixation of the dye, rendering high-quality and enduring (non-fading) products.

Historical records show that Tyrian purple and similar dyes had a market value of several times their weight in gold, being among the most expensive commodities traded in the ancient world. Such exorbitant prices were explained by the social role these dyes had in many Mediterranean societies, where it was used in clothing and banners to show the high prestige and status of the bearer or, more formally, to visually identify the powerful. For example, in the Roman empire, kings and generals could wear a *toga picta*, a garment fully dyed in *Royal blue* or *Tyrian purple.* Meanwhile, a *toga praetexta*, a white garment edged with appropriate color stripes, identified the role and rank of important members of society, like the military, senators, magistrates, priests, etc., see Pons (2016).

Figure 9t depicts a production diagram for these colors, showing interdependences and reinforcement loops in functional interactions that maintain, strengthen, and perpetuate the industry and its associated culture (or the other way around). Practical know-how of dye chemistry (alchemy) allowed the fabrication of textiles with bright, stable, and well-defined colors that could, in turn, be used as social markers to visually identify users by their role and rank, hence helping the smooth operation of complex organizations. Such visual markers constituted, in effect, tokens of visual communication used in social and religious contexts. Moreover, these markers were immersed in a web of inter-related meanings, and hence provided rich and multi-layered ways of expression.

Knowledge concerning dyeing technology implied competitive advantages and was often guarded as an industrial and trade secret, which probably explains the scarcity and obscurity of contemporary written accounts thereof. Moreover, overexploitation made raw materials scarce, and dye production unreliable. Furthermore, from the first centuries of the common era on, the Roman and Byzantine empires imposed strict controls over the production and commercialization of these dyes, and also prohibited their use by general populations throughout the regions under their domain. Under these conditions, the instability and decline of the Roman and Byzantine empires in late antiquity and the middle ages further hampered this industry. Finally, a variety of new products brought from the east, like Indigo, made dyes obtained sea snails uncompetitive.

At the beginning of the modern era, production of these dyes was exceedingly rare, and the associated know-how was mostly forgotten. Jewish scholars could only lament the loss of this knowledge, and the consequent impossibility of observing and practicing some rituals requiring tekhelet – a cyan, blue, or maybe violet dye obtained from some strange marine creature. In summary, a community of practice that had lost some of its practices had also lost some associated knowledge, see Wenger (1998) and the following quotation. Only in the XXI century were these practices reinstated in some Jewish communities, after sufficient archaeological and chemical research, followed by theological study, debate, and (never consensual) agreements on how to reconstitute these practices.

> *A process cannot be understood by stopping it. Understanding must move with the flow of the process, must join it and flow with it.*
> *The first law of mentat*, from Dune (1965) by Frank Herbert.

The materials studied in this section can give us some hints on how epistemological processes can be entangled in complex dichotomous relations of a priori conditions vs. a posteriori possibilities, or of antecedent pre-requisites vs. subsequent developments. Figure 9bl shows a crank leaver used to manually turn an old car's engine in order to start its combustion cycle and, in this way, get the engine running by itself. It alludes to actions that must be taken, procedures that must be developed, or resources that must be available before (i.e. *a priori*) another process can, later on (i.e. *a posteriori*), be developed, start, or proceed. In the cases at hand, human physiology of color perception seems to be a determinant precondition for the unfolding of the standard lexicalization sequence that, in turn, seems to be a prerequisite for additional lexicalization steps extending the standard color palette. Moreover, the production diagram in Figure 9t hints at even more complex circular relations of positive reinforcement or negative feedback between such processes. Stern (2007; 2008, ch.6) further analyze such complex interdependence relations and their epistemological consequences. Furthermore, the following sections introduce a few more dichotomous relations characterizing epistemological processes that are correlated to those previously mentioned.

4. Language, Synthesis, and Analysis

Let us now pay close attention to the Hebrew words used to denote the seven lexicalized colors appearing in Figure 8. Most of these words point to a direct analogy or abstraction where a color name generalizes the color of something in particular. The ways in which these words are generated deserves close attention.

Hebrew, like all Semitic languages, is based on a system of tri-literal roots, where three consonants, from an alphabet of twenty-two, form basic units of meaning. There is also a small inventory of ancient bi-literal roots (that are very old in the history of the language, but still in full use). Each consonantal root can then be inflected by vocalization and other grammatical alterations, generating specific words in the final form in which they appear in a sentence. Some of such inflections are used to characterize the grammatical function the word has in a sentence, much like it happens in indo-european languages. Other inflections are used to generate clusters of words with closely related meanings. Finally, "small phonetic mutations" can be used to generate new tri-literal roots, where clusters related by *paronomasia*, i.e. that sound alike, are likewise related by meaning.

The color words in Figure 8 give a few good examples of such generative relations: (1a) Red, *adom*, is generated by extending by a single letter the bi-literal root meaning blood, *dam*. (1b) Yellow, *tzahov*, is generated from the word for gold, *zahav*, by a mutation that replaces the first consonant in the root by a similar sounding one. The following pairs of words share the same consonantal root, only using a different vocalization pattern: (1c) Green, *yaroq*, and vegetation, *yaraq*; (1d) Blue, *kachol*, and antimony, *kachal*; (2a) Violet, *sagol*, and grape cluster, *segula*. Finally, (2b) Purple is generated from the same root of the word for textile; and (3) Crimson is just a descriptive compound expression. The enumeration used in this paragraph distinguishes a 1st set in the standard lexicalization sequence, a 2nd set of additional lexicalizations, and a 3rd set of non-lexicalized expressions.

The words in Figure 8 constitute a specialized vocabulary for colors. In computer science, such a vocabulary, organized in a dictionary explaining the *semantic* functions of these words, i.e. their meanings, how to use them, and their interrelations, is known as an *ontology*, see Stern (2014, 2017) and references therein. On the one hand, these words "divide" the continuous color spectrum into a discrete and small set of color regions. The use of this ontology affords simple reference and efficient handling of color in practical life. On the other hand, each one of these words allows us to, in a "unified" way, speak of, refer to, or represent a color property common to many particular things, namely, that of (nearly, in spectral order) sharing the same color. In this article, I will refer to these "dividing" vs. "unifying" powers of such an ontology as its *analytic* vs. *synthetic* aspects.[4]

In the philosophical literature the two poles of the synthetic vs. analytic dichotomy have been interpreted or characterized in many different ways; see Jong (2010) for pertinent comparisons in the works of Kant, Bolzano, and Frege. I make this dichotomy in a way that serves the specific purposes of this article in hope this simple instance can serve as a useful proxy for the discussion of far more sophisticated cases. For example, in Section 7 we discuss how this and other dichotomies of interest find themselves correlated to or entangled with each other.

The synthetic aspect and the etymological or grammatical derivation of each of the Hebrew color words previously examined makes them implicit metaphors. For the purposes of this article, it will be useful to look at a metaphor ($\mu\epsilon\tau\alpha$, meta = across, after; $\varphi\epsilon\rho\omega$, phero = to bear, to carry) from two orthogonal perspectives or axes, namely, an axis of similarities and compatibilities vs. an axis of differences and disparities between two distinct sites, situations, or (sets of) objects. The axis of similarities and compatibilities between the two sites allows or motivates the transport of pertinent ideas, useful knowledge, or relevant meanings across the barrier imposed by the axis of differences and disparities, just like a ferryboat carrying valuable goods between two sides of a river, see Figure 10tl.

The term *metaphorical model* refers to a metaphor explained in some detail along both its orthogonal axes, like the preceding comparison between a metaphor and a ferryboat. The term *analogy* refers to a succinct statement of a model, for example: "A metaphor is, in some aspects, like a ferryboat crossing a river"; or "These yellow flowers look, in color, like gold"; or "The sequence of arithmetic operations implied by an algebraic formula can be represented by the structure of a tree diagram", see Figure 10tc. The term *simile* refers to an analogy described only along its similarity axis, for example: "A metaphor is like a ferryboat". In its most succinct form, a metaphor is reduced to an identity, for example: "A metaphor is a ferryboat"; see Leary (1990, p.136) for further details.

As seen in the previous section, traditional color ontologies are *not* arbitrary: The standard lexicalization sequence corresponds to invariant and preset structures of human physiology for color perception, while additional lexicalizations reflect salient features of the civilization that language serves. This is an essential characteristic concerning the analytic aspect of good ontologies, captured by Socrates' classical quotation opening this article, see also Figure 10bl.

[4] This interpretation is in strict accordance with etymology: Analysis, $\alpha\nu\alpha\lambda\upsilon\sigma\iota\varsigma$, separation of a whole into its constituent parts $< \alpha\nu\alpha$, *ana*, thoroughly + $\lambda\upsilon\omega$, *luo*, to loosen, to separate. Synthesis, $\sigma\upsilon\nu\theta\epsilon\sigma\iota\varsigma$, combining of separate elements into a coherent whole $< \sigma\upsilon\nu$, *sun*, together + $\tau\iota\theta\epsilon\mu\iota$, *tithemi*, to set, to place.

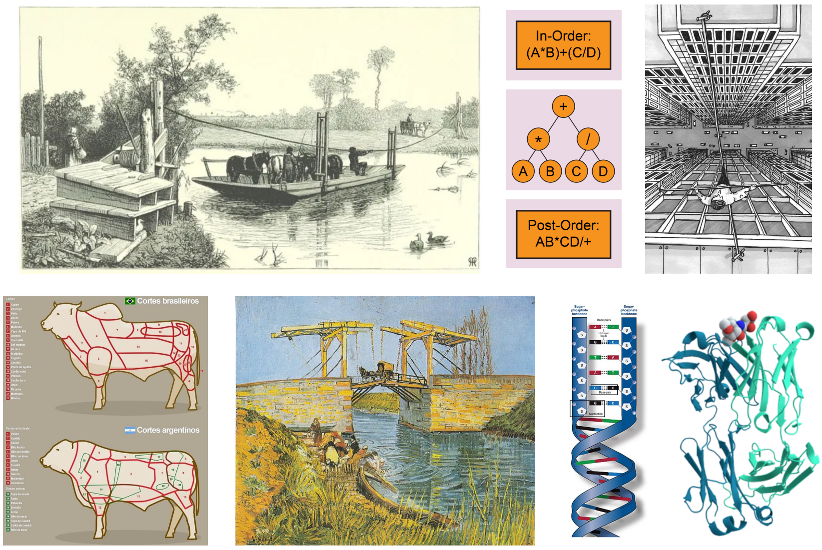


Figure 10. Top: Cable ferry, wire-walk, and an exact law or rule, i.e. a parsing "tree" with its root at the top and leaves at the bottom. Bottom: Analyzing an ox, bridge, or protein at Its precise joints.

Figure 10bl shows two distinct ways, traditional in Argentina and Brazil, to "carve an ox at its joints". Nevertheless, although distinct, both ways follow the same basic principles: First, a good carver knows how to run the knife precisely through cartilage connections at narrow gaps between hard bones of the animal and, in so doing, easily separate major anatomical parts. Second, a good carver knows how to run the knife precisely through thin layers or sheets of soft tissue separating major muscle groups. Figure 10bl displays the resulting major cuts and their names in each of the aforementioned ontologies, see also Campbell el all (2011, Sec.1.1) and Méndez (2020). Finally, the carver knows how to prepare small meat portions with specific culinary qualities. It is important to remark that the intrinsically precise nature of the aforementioned cutting operations, especially in the first and second steps, afford great consistency and stability for the entire process, resulting in final products exhibiting some regular (approximately invariant) characteristics. These regular characteristics are what a consumer implicitly demands when he or she asks for a meat cut by its proper (ontological) name.

The analytic, synthetic, and metaphoric aspects are intrinsic and unavoidable aspects of language, and scientific languages are no exception. In Newtonian physics, the gravitational force applied to a body is supposed to act "like a rope" pulling it, only without the rope! The last statement is an oxymoron – an obvious self-contradiction, an embarrassing situation known in the specialized literature as "action at a distance". We ought to recognize that the concept of Newtonian force is essentially metaphorical and, as previously stated, metaphors must always be regarded from two orthogonal perspectives, namely, that of similarities and that of disparities. The next quotation addresses this point:

> *Scientific metaphors define more abstract, general, or complex concepts in terms of distinct and more concrete, specific, or simpler concepts. Hence, it is unavoidable to have some degree of inconsistency in the underlying properties and functions of the objects involved in these definitions.*
>
> Lakoff and Johnson (1980).

Nevertheless, the concept of Newtonian force is a good metaphor, for it is able to successfully carry valuable goods (meanings, means, and methods) across the differences separating distinct situations. More specifically, from a similarity perspective, these forces can be regarded as "working in the same way", for example: (a) The mathematical rule used to compute the resulting force when multiple ropes are pulling an object, namely, the parallelogram law, is exactly the same rule used in the case of Newtonian forces. This law allows us to analyze complex force systems and, via decomposition and recomposition operations, calculate the system's single resultant, see Stern (2017) for further comments. (b) Important measurable effects, like resulting static deformations or dynamic accelerations, are exactly the same for either direct contact or action at a distant forces. These similarities make the metaphor worthwhile, this is why it makes sense to use, in both situations, the same word – force.

As a last metaphor for this section, "a protein is like the DNA strand from which it was translated". However, Figure 10br shows how different those two things are. Nevertheless, this is a good metaphor – under appropriate conditions, it works. In fact, this metaphor and its associated language make life as we know it possible. At the molecular level, it reduces the production of a complex protein (an object with 3-d geometry) to a well-defined step-by-step assembly (a linearly ordered discrete process) of amino acids according to DNA encoded sequences. At the cellular level, the genetic language (mediated by the protein apparatus it encodes) is used to represent stresses posed by the environment and to teleologically react (by controlling the expression of a discrete and limited set of genes) to these stresses in order to preserve the cell and its resources or, if necessary, to fix what is broken and regenerate the cell's integrity.

This recurring effort of self-preservation, or *autopoiesis*, creates stable conditions, or *homeostasis*, in which the system's processes can proceed in order to constantly repair and regenerate the machinery that actually performs its operations. In short, homeostasis and autopoiesis constantly and cyclically (re)generate the stable conditions and the invariant structures that characterize the cell as a living organism. For further considerations on autopoiesis, see Eigen and Schuster (1977), Eigen (1992), Foerster (2001, 2003), Krohn et al. (1990), Maturana and Varela (1980), and Zelleny (1980, 1981). Finally, it is interesting to remark on how the considerations in the last paragraphs take the linguistic view developed in this article all the way down to the rock-bottom foundations of life itself.

5. Equality, $=$, the Hallmark of Exact Sciences

Equality is represented in mathematics by the sign $=$. Several variants of this concept may, depending on the context, be distinguished by variations of the basic equality sign, for example: *similarity* , $\sim$; *approximation*, $\approx$; *asymptotic approximation*, $\simeq$; *proportionality*, $\propto$; *value attribution*, $:=$; *value comparison*, $==$; *value-and-type comparison*, $===$; *equivalence*, $\equiv$; *definition*, $\triangleq$; etc. Moreover, other closely related concepts may be distinguished by the scope of a statement, for example, a *functional identity*, like $\cos^2 = 1 - \sin^2$, may indicate a point equality valid over a given functional range, like $\cos^2(x) = 1 - \sin^2(x)$, $x \in [0, 2\pi]$.

Who are the heroes of a book in exact sciences? The answer can be found in the general index of the book, next to the words *equation*, *law*, *rule*, *formula*, *algorithm*, *definition*, etc., usually in reference to keystone statement formulated around the equality sign or some of its variants. This section discusses the nature and importance of such equational statements, also studied in the philosophy of science and foundations of statistics under the titles of *exact laws* and *sharp* or *precise hypotheses*. This discussion is intended to be intuitive and accessible using only high school or basic college science and mathematics, see Section 7.1.

Figure 11tl displays the *normal vibration modes* of a string of length $L = 1$ that is fixed at both extremes, like the strings of a guitar. The n-th normal mode, depicted at the corresponding row of the figure, is also known as the string's n-th *harmonic* or, for n=1, as its *fundamental* mode. The n-th normal mode has $n + 1$ *nodes* or stationary points: The two points at the extremes, 0 and L, plus $n - 1$ points dividing the length of the string in n segments of equal size. The figure also highlights the first node (away from 0) at each normal vibration mode, that are located by the L-scaled harmonic series,

$$\lambda_1 = L, \quad \lambda_2 = L/2, \quad \lambda_3 = L/3, \ \ldots$$

Pythagoras of Samos (c.570-495 BC) and his school already understood that the harmonic series characterized the relative pitch of musical notes produced by string instruments, providing a mathematical basis for musical theory, see for example Benade (1960, 1990) and Josephs (1967).

Let us now return to Socrates' metaphor of dividing things where the natural joints are. Stretching our imagination, we can think of each segment of a normal mode as a "bone", in the sense of being a vibrating unit of the string, and think of each node as a joint or articulation point between them. The pythagorean musical theory uses these basic mathematical elements to analyze the sound produced by musical instruments and the harmony of musical chords and melodies and, in so doing, this theory carves (music's) nature at its joints.

A string can be conceived as a 1-dimensional object, for a position in the string is specified by 1 measurement, namely, a distance along its length. The harmonic series gives *exact* or *precise* locations to the nodes of a normal vibration mode. In practice, these stationary points can only be located within a confidence interval $[\lambda - \delta, \lambda + \delta]$, a.k.a. $\lambda \pm \delta$ or λ plus-or-minus δ, were δ is a tolerance

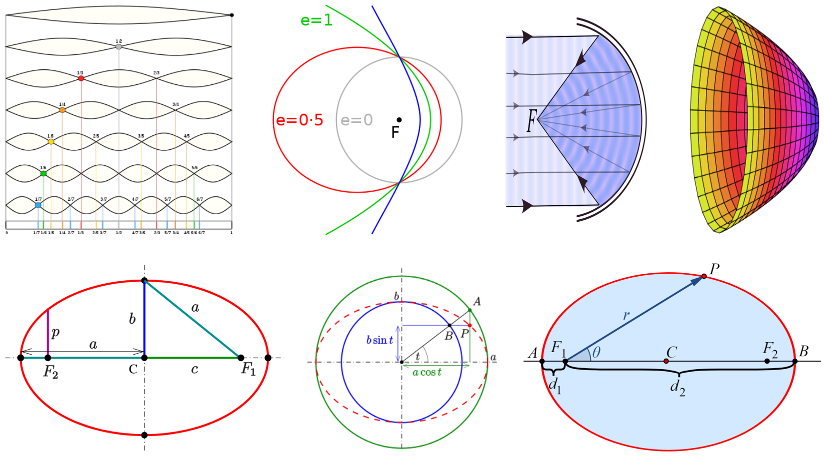


FIGURE 11. Top: Zero measure (length, area, volume) locus of some exact laws: Pythagorean vibrating strings; Kepler's orbits and optics. Bottom: Equations defining an ellipse in the plane.

margin. The better the technology and craftsmanship of a musical instrument, the tighter its tolerance margins, the more precise the musical notes produced by the instrument, and the more beautiful the resulting harmonies. In the idealized case of absolute precision, the tolerance interval has zero length, $\delta = 0$, corresponding to a single point, that is a 0-dimensional object.

5.1. Kepler's Exact Laws and Metaphorical Wire-Walks

Arthur Koestler (1959) places the work of Johannes Kepler (1571-1630) at the watershed marking the beginning of modern science. We will use Kepler's work to illustrate some ideas under discussion and, in so doing, corroborate his position at the watershed in the capacity of a metaphorical wire-walker, see Figure 10tr.

Figure 11tc depicts possible orbits of a planet in Keplerian astronomy that range, according to their eccentricity, from circular ($e = 0$), to elliptic ($e < 1$), parabolic ($e = 1$), and hyperbolic ($e > 1$). Figures 11b and the following equations give two alternative ways of defining the locus of an ellipse in the plane, namely, specifying Cartesian coordinates scaled by the size of the semi-axes, and specifying the radial in terms of the angular polar coordinate.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad r(\theta) = \frac{a(1-e^2)}{1 \pm e\cos(\theta)} \ , \ \text{where} \ \ e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{c}{a} \ , \quad c = a - d_1 \ .$$

There are some aspects of Kepler's law prescribing an elliptical orbit that deserve further attention. First, it is a good example of what is known in the philosophy of science as an *exact law*, corresponding in statistical test theory to the notion of *precise* or *sharp hypothesis*. Each of these equations states a constraint that takes away one degree of freedom from a point in the specified locus or place of movement of the system. Since a free point in the plane has 2 degrees of freedom, that is, its position is specified by two independent coordinates, a point at the ellipse has 1 degree of freedom left, making it a 1-dimensional object.

The standard size measure of a geometrical object in a plane is its area, and the area of any 1-dimensional object is zero. In this sense, we say that a physical or astronomical law stating an elliptical orbit is *precise* or *exact*. In the same way, the standard measure of a geometrical object in a line is its length, and the length of a (0-dimensional) point is zero, as in the previously examined example of the vibrating string. A free point in the 3-dimensional Euclidean space has three degrees of freedom, corresponding, for example, to the three independent Cartesian coordinates, $[x, y, z]$. If we set the ecliptic plane, where all planetary orbits rest, at $z = 0$, a planet moves in space over a locus specified by two equations, each of them taking away 1 degree of freedom. Under these conditions, a Keplerian orbit is conceived as a (3-2=1)-dimensional object in 3-d space. The standard size measure of a geometrical objects in 3-d space is its volume, and the volume of any object of dimension smaller than 3 is zero. So (or even more so), we say that a law stating a 1-d orbit in 3-d space is *precise* or *exact*.

Figures 11tr depict the surface of a parabolic mirror. Kepler's studies in optics showed how lenses and mirrors with quadratic surfaces could be used to concentrate parallel light rays into a single point, namely, the focus, see Stern (2020) and references therein. A quadratic surface (conceived as a deformed plane) is a 2-d object immersed in the 3-d Euclidean space. Moreover, the volume of any 2-dimensional object in 3-d space is zero. Therefore, following arguments similar to those in the last paragraph, we say that Kepler's recipes for building optical devices are another good example of exact or precise laws.

Back to our carving metaphor, the first step consisted of separating major anatomical parts of the ox by cutting through bone joints, while the second step consisted of separating major muscle groups by cutting along thin sheets of soft tissue between them. The surface of a lens or the face of a mirror is an interface between two distinct optical media, for example, the air through which light propagates outside and the glass or metal from which a lens or mirror is made. Hence, stretching our imagination, we can understand the analogy between the mirror's (inter)face and the sheets of tissue guiding the carver.

As in anything done by human hands, absolute precision is unattainable. However, modern astronomical mirrors are manufactured with astonishing precision. For example, NASA's James Webb space telescope has a 6.5 meters primary mirror whose surface is polished to an average roughness of only 20 nanometers, a relative precision of about 3 parts in a billion! Stern (2020) and the references therein describe in detail fundamental techniques developed much earlier in the history of astronomical instrumentation and explain their relevance in the context of the present discussion. Similar to the case of musical instruments, the better the precision of astronomical telescopes, the sharper the resulting images, and the clearer our view of the universe.

Back to the carving metaphor, as technologies improve, the more precise the corresponding cutting operations, the better our abilities to discriminate and analyze objects in our environment, the greater our powers for working with them, and, usually, the clearer our understanding of the science involved. In fact, the other way around, the operational precision of a given technology is arguably one of the most important metrics of its degree of development and also of the development of the scientific theories on which it is based.

The emphasis of this subsection on scientific exactness or precision motivates its title for, stretching our imagination, and also looking at Figures 10tr and 10bc, one can see that a wire-walk offers a very narrow bridge (in the limit, a 1-d line connection), in contrast to a normal bridge that offers a wide (2-d surface) pathway for the user to walk upon.

5.2. Kepler's Data Analyses and Theoretical Syntheses

At the time Kepler started his work, all known astronomical models conceived planetary orbits by a superposition of circular motions. A typical planetary orbit was described by a small *epicycle* carried along a larger *deferent*, with all these circular planetary motions carefully synchronized and placed on a common ecliptic plane. In Greek astronomy, models based on deferents and epicycles were also used to build simulation machines capable of describing past movements of the planets and of predicting their future positions in the sky with great accuracy. The Antikythera mechanism, dated at c.200BC, is a beautiful example of such a machine, found in a shipwreck in 1900. Derek de Solla Price (1974) studied these remains and was able to reverse engineer this complex mechanism composed by more then 30 high precision gears, see Stern (2022) and the references therein. Nevertheless, after the fall of the Hellenic civilization, knowledge concerning this technology was lost and forgotten, consequently limiting the practical motivations for such astronomical models, and also impairing their theoretical understanding.

Kepler struggled for many years trying to find a good explanation for the extremely precise measurements of the astronomical positions of the planet Mars he received from the Danish astronomer Tycho Brahe (1546-1601). At first, he tried to fit Brahe's data using various cycle and epicycle models, but this traditional method (and ontology) proved inadequate for the task at hand. Finally, and quite reluctantly, Kepler developed a new theory, presented in his masterpiece – *Astronomia Nova*, the *New Astronomy, Celestial Physics and the Movements of Planet Mars*, a book published in Prague, in 1609. This book includes Kepler's 1st law, stating that a planet follows an elliptical orbit around the sun – fixed at one of its foci (his 2nd law states that the focus-to-planet radial vector sweeps equal areas in equal times, and his 3rd law states that the square of the orbital period is proportional to the orbit's major axis' length), see Figure 11br.

At the time Kepler published his astronomical laws, he already had substantial experience with the mathematics of elliptic and other quadratic curves. Five years before, Kepler wrote *Astronomiae pars Optica* – on the Optical part of Astronomy, a book published in Frankfurt in 1604. Section 4 of Chapter IV

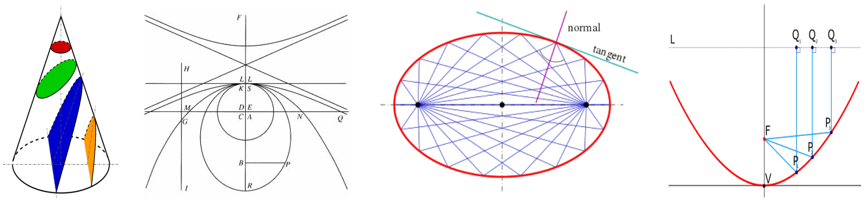


Figure 12. Kepler's continuous transformations and analogies.

of this book, *De Coni sectionibus*, is dedicated to the study of conic sections, the curves defined by the intersection of a cone with a plane, see Figure 12l. Kepler investigates the transformation of conic sections as the conic projection angle gradually increases, showing that, while the form of the intersection curve continuously changes, some important properties of the curve remain identical, due to the existence of some invariant mathematical relationships, see Figure 12cl. Kepler describes his investigation method as a manner of – speaking analogically rather than geometrically, *analogice magis quam geometrice loquendo*.

Kepler explicitly declares how much he likes this analogical method, considering these metaphors as the best way of teaching the secrets of the universe. Furthermore, he considers this metaphorical way of teaching, in itself, as wordy and wonderful as the specific results it renders; Kepler respects his teachers as much as he is grateful for the lesson they teach:

> *Plurimum namque amo analogias, fidelissimos meos magistros, omnium natures arcanorum conscios.*
> Above all, I love analogies, my most reliable teachers, knowledgeable of all secrets of nature.
> Johannes Kepler (1604, Ch.IV, Sec.4), Astronomiae pars Optica.

> *Quippe mihi non multo minus admirandae videntur occasiones, quibus homines in cognitionem rerum coelestium deveniunt; quam ipsa natura rerum coelestium.*
> The roads by which men arrive at their insights in celestial matters seem to me almost as worthy of wonder as these matters in themselves.
> Johannes Kepler (1609, Summary of Ch.45), Astronomia Nova.

As an application of his analogical method, Kepler studies the reflective properties of the ellipse, showing that the light emitted by a lamp placed at one focus is reflected by the ellipse so to converge to the second focus, see Figure 12c. Due to this property, and having in mind mechanical applications, Kepler calls these previously unnamed geometrical points by the name of Focus, *Nos lucis causa, et oculis in Mechanicam intentis ea puncta Focos appellabimus*, using the Latin word for oven, fire-place, or burning-point. Moreover, Kepler uses his analogical method to explain that the circle and the parabola are extreme cases of the ellipse, the circle having the two foci coinciding at the center, and the parabola having one focus taken to an infinite distance. In this way, he characterizes – the parabola and the circle as the most acute and most obtuse of all the possible ellipses, *Etenim omnium Ellipsium, acutissima est parabole, obtusissima circulus.* Apollonius had already used the word Focus to describe the same geometrical point in the parabola, but Kepler boldly expands the scope of this concept and generalizes the meaning of this word. After all, Kepler's 1st law made this most distinguished point the place of the sun, at the center of the universe.

As a second application of his analogical method, Kepler shows how to extend the famous string procedure for drawing a parabola: For drawing an ellipse, one can use a string of length equal to the major axis ($2a$) keeping the endpoints of the string fixed at the foci, see Figure 12c. In the case of the parabola, an analogous construction replaces the focus at the infinite by a *directrix* line perpendicular to the symmetry axis, see Figure 12r, where the respective endpoint of the string can slide. Using these construction methods, it is easy to demonstrate the aforementioned reflection properties.

Kepler also uses and develops his analogical method in later works, like *Nova Stereometria*, published in Linz at 1615. The application of Kepler's analogical method to geometry was further developed in later times by, among others, Gottfried Wilhelm Leibniz (1646-1716) under the name of *lex continui*, and by Jean-Victor Poncelet (1788-1867) under the name of continuity-principle or principle of permanence of mathematical relationships, see Poncelet (1822, p.xiii-xiv), Taylor (1900), Taylor (1911, p.lvii-lix), Gray (2007, p.19), and Struik (1953).

Kepler's method of analogy can be regarded as an early precursor of modern differential and integral calculus, using ideas related to gradual or continuous transformations and, for that purpose, introducing a terminology to denote concepts like: Infinite displacements, *infinito intervallo distant*; Convergence of a curve to its asymptotic lines, *hoc magis rectae seu asymptoto suae fit similis*; and Existence of a necessary limit case, *sic itaque in terminis*; etc; where all these concepts are - Needed to complete an analogy, *tantum ad analogiam complendam.*

Differential and Integral Calculus together with analytic and vector geometry are, arguably, the fundamental (or foundational) languages of modern exact sciences. Arguments of calculus constitute keystones of Isaac Newton's masterpiece, *Principia Mathematica*, published in 1687, although his first systematic treatment of calculus was published only in 1704, as an appendix to his book *Opticks: A Treatise of the Reflexions, Refractions, Inflexions and Colours of Light – Also two treatises of the species and magnitude of curvilinear figures.*

The evolution of astronomy and physics from Kepler to Newton, and the parallel development of the ideas and language of calculus is a fascinating story, for pertinent historical comments see Applebaum (1996), Bell (2005), Boyer (1959), Dugas (1955), Edwards (1979), Gingerich (2002), Kozhamthadam (1994), Lintz (2007, 2012), Martens (1998, 2000), Reyes (2004), Stephenson (1987), and Wilson (1989), and also Boyer and Merzbach (2010), Burton (2010), Crowe (2011), Grattan-Guinness (1980), and Katz (2017). I will only highlight a remarkable coincidence concerning the development of these parallel languages and theories by Kepler and Newton that (at least in logical, if not in chronological order of publication) are first developed in a work on optics, to be later used in a masterpiece on physics or astronomy. It reminds me of an observation made by Einstein in the following quotation:

> *"The task of physics is simply to provide a formal description of the connection between observations." This is the programme of Mach and the positivists that succeeded him. What, however, are 'observations'? In 1926 Einstein said to young Heisenberg: "Only theory can determine what is able to be observed."(Die Teil und das Ganze, p.92). What does that mean? "One can only see that which one knows" was the archaeologist Ludwig Curtius' last statement to one of his pupils.* Weizsäcker (1978, p.418).

The language of Differential and Integral Calculus deals with calculations of tangents (differential rates) to and areas under a curve (integrals). However, an alternative *po(i)etic* interpretation on the name of this language could refer to its powers to serve, on the one hand, as a tool for calculating with small quantities concerning fine distinctions (i.e., for differential analyses) and to serve, on the other hand, as a tool for building metaphors connecting diverse applications and for the abstraction of general theories (i.e., for integrative syntheses). For example, using this analogical language, Newton was able to realize that either a cannonball in parabolic trajectory or a planet in elliptical orbit were both in "free fall", following a curvilinear trajectory of the "same" genus of quadratic curves, see Figure 11tcl. Moreover, a continuous increase of the cannon's power and, consequently, of the cannonball initial velocity, would make it smoothly transition from a parabolic trajectory back to earth to a circular and then to an elliptic orbit around the earth; see Stern (2022, sec.6) for a detailed discussion of this metaphor.

The following quotations, by the great mathematician Gian-Carlo Rota, acknowledge the importance of analogical thinking in mathematical discovery, and also the paradoxical but persistently ungrateful attitude of being ashamed and trying to hide the helpful hand of our graceful teachers:

> *Mathematics is the study of analogies between analogies. All science is. Scientists want to show that things that don't look alike are really the same. That is one of their innermost Freudian motivations. In fact, that is what we mean by understanding.* Gian-Carlo Rota (1997, 214).

> *The enrapturing discoveries of our field systematically conceal, like footprints erased in the sand, the analogical train of thought that is the authentic life of mathematics.* Gian-Carlo Rota (1992, Preface).

What can explain the paradoxical attitude pointed out by Rota of trying (with increasing strength) to hide the footprints of analogy in the way we practice, teach and philosophize about science? In my opinion, this is just a side effect of the double nature of metaphorical thinking, as restated in the following quotation:

> *The greatest thing by far is to be a master of metaphor; it is the one thing that cannot be learnt from others; and it is also a sign of genius, since a good metaphor implies an intuitive perception of the similarity in the dissimilar.*
> Aristotle, Poetics xxii.

The dissimilarity perspective accompanying any metaphor may bring a sensation of vertigo that comes when glimpsing into the abyss of the unknown. The following comments by Jung attest to some psychological needs and consequences of metaphorical paradoxes, see also Jones (2007).

> *This is one of those paradoxes that are the rule: a statement about something metaphysical can only be antinomial.* Jung, CW, IX ii, pr.390.

> *Any content that transcends consciousness, and for which the apperceptive apparatus does not exist, can call forth the same kind of paradoxical or antinomial symbolism.* Jung, CW, XI, pr.277.

Moreover, the equational metaphors used in exact sciences may further exacerbate this vertigo, for wire-walkers are constantly reminded of their precarious equilibrium act over the abyss. This vertigo is hence the presumed origin of the hostility towards metaphorical thinking, producing the side effects of trying to conceal precious insights and intuitions, and risking to trow away our best ladders of ascension. Nevertheless, I see the metaphorical wire-walking of exact sciences as their greatest strength, for it generously opens new ways for symbolic exploration, scientific progress, and evolution of knowledge. For several views of metaphorical and analogical thinking in science and mathematics, see Abelson (1995), Brown (2003), Brown (2021), Ervas (2019), Feest (2010), Forišek and Steinová (2013) Gigerenzer and Murray (1987), Indurkhya (1992), Michalewicz and Fogel (2000), Palma (2016), Polya (1954), Reeves (1993), and Zauderer (2019).

Furthermore, sharp or precise hypotheses are, on the one hand, statements of null (volume) measure and, on the other hand, statements that, using the appropriate statistical methods, can receive strong empirical support. This apparent contradiction is known in the foundations of statistics as the *zero probability paradox*. Logical and statistical inference methods rendering good theoretical and pragmatical solutions to this paradox do exist. Moreover, these inference methods are able to produce strong scientific verification criteria, and engender a viable and constructive notion of objective knowledge; for further details, see the references' section - Objective Cognitive Constructivism and the Bayesian Epistemic Value of Sharp Hypotheses.

6. Tongs Made with Tongs, and other Entangled Pairs of Pairs: Synthetic×Analytic, Posterior×Prior, Imagination×Grammar, Metaphor×Hermeneutic, Metaphysic×Observed, Image×Sound

Pairs of tongs are complicated things to make or even to speak about. A pair is either singular or plural or both simultaneously. In a pair of tongs, it denotes two equals that, opposing each other, can not be the same. In order to make a tong, to manipulate it in the fire of the forge, a blacksmith needs to hold it using a pair of tongs. But who then made the first pair of tongs? In the following quotation, the Hebrew root צְבָת, *tsebat*, can mean, as a noun, pliers, tongs, tweezers, and as a verb, tied, twisted, entangled. Jewish mysticism states that the first pair of tongs, later given to Tubal-Cain, the first blacksmith, were created at twilight, sometime between the end of creation and the beginning of the existing universe.

דְּבָרִים נִבְרְאוּ בְּעֶרֶב ... וְיֵשׁ אוֹמְרִים אַף צְבָת בִּצְבָת עֲשׂוּיָה

Things made at twilight time: ... and also tongs, made with tongs.

Pirkei Avot, compiled c.200 CE, 5:6.

Figure 13 depicts logical diagrams conceived to help us to visualize and understand the theoretical intricacies of interacting tongs related to the synthetic vs. analytic and posterior vs. prior pairs of oppositional concepts. From a logical point of view, the Aristotelian diagram at Figures 7r and 13r have exactly the same structure, known as the *hexagon of opposition*. Meanwhile, the Hasse diagrams in Figures 7l and 13l have very similar structures and interpretations. These diagrams depict the entailment (implication) and oppositional (contrariety, subcontrariety, and contradiction) relations underlying, respectively, the mereological or compositional relations of color theory, and the relations between the basic logical combinations of two (dependent) categories, A and B, and their negations.

Some sub-diagrams of the hexagon of opposition, or the corresponding algebraic sub-latices depicted in the Hasse diagrams, including the well-known square of opposition, were exhaustively studied in medieval logic, see Stern (2022) and references therein. However, the logically complete hexagon of opposition only appeared, independently and almost simultaneously, in the fields of color theory and philosophical logic, in the XX century, a delay that, retrospectively, seems astonishing. This kind of surprising coincidence – like the (almost) simultaneous occurrence of two events not connected by direct causal relations that are, nevertheless connected by important relations of meaning, significance or interpretation – is called, in Jungian psychology, *synchronicity*. In the last statements, it is important to remark that astonishment, surprise, and meaningfulness, as psychological phenomena they are, always lie in the eye of the beholder(s).

In Stern (2022) I make some remarks about the synchronic emergence of the hexagon of opposition in the aforementioned time and fields, but my reference for the first appearance of the logical hexagon was the work of Blanché (1953, 1966). Nevertheless, at the lecture given by Andrew Aberdein at the *7th Square of Opposition* conference, in Leuven, 2022, I had the opportunity to learn about the work of Leonard Nelson (1882-1927), who presented his version of the hexagon of opposition in 1921. The Hasse diagram in Figure 13l is depicted in Nelson (1921; 2011, p.99; 2016, p.80). The corresponding Aristotelian diagram in Figure 13r, although not depicted in graphical form, is described and analyzed in detail in Nelson's text, see also the comments in Aberdein (2016). Nelson used these diagrams to study the Kantian categories of prior vs. posterior and synthetic vs. analytic and their interrelations, see Figure 13 and its caption. After presenting my paper at the *7th Square of Opposition*, I had immediate feedback and subsequent discussions concerning the nature and interrelation of several pairs of oppositional concepts in this section's title, either referring to Stern (2022), or regarding Jungian theory, or in relation to Nelson's work.

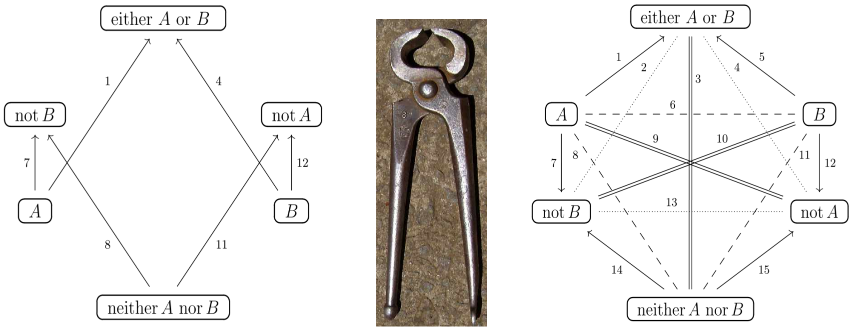


FIGURE 13. Tongs of Tubal-Cain, made at creation's twilight. Nelson (1921) Hasse/Aristotelian diagrams for categories of $A =$ analytic, $B =$ a posteriori, *not* $A =$ synthetic, *not* $B =$ a priori; and logical relations of entailment ($\longrightarrow$), contradiction ($=$), contrariety ($- -$), and sub-contrariety ($\cdots$).

This section discusses the relations between the aforementioned pairs of oppositional concepts in the contexts they were presented in previous sections of this article. These contexts are far more mundane, concrete, simple, and accessible than the technical and abstract context of Kantian philosophy or that of subsequent related works by Bolzano and Frege discussed in Jong (2010). Nevertheless, I hope, these more accessible contexts are still pertinent for carrying on the discussion at hand yielding valuable conclusions. Moreover, the simple instances we use to discuss these categories makes it easier to (a) extend the discussion over all the linguistic scopes considered in previous sections, and (b) show how intricately entangled these tongs can become in the contexts under consideration.

6.1. Cable-Ferries vs. Wire-Walks: Reification, Evolution, Stratification, etc.

Every time an Argentinian and a Brazilian meet at a barbecue, hot ontological debates are likely to arise. It is difficult for one of them to even conceive why not to use the proper ontology (hers or his country's, of course) and corresponding practices. The fellow diner's attention is focused according to hers or his ontological commitments, even if the meat steak on hers or his plate was cut and prepared in a different way. Likewise, the traditional use of different color ontologies may require careful translations and practical adaptations, as in the case of the *aoi* or blueish-green color used in traffic lights in Japan, see Conlan (2005).

Metaphors once alive at their inception, die and become fossils fixed in stone; they become *reified*. That is, they become our routine and automatized theoria and praxis[5], our standard way to look at things and to do things, they point to the beaten path as the (only) way to go, see Lakoff and Johnson (2003). Moreover, it is often the case that an ontology and its associated theories and practices

[5] $\theta\epsilon\omega\rho\iota\alpha < \theta\epsilon\alpha$, *thea*, view + $o\rho\alpha o$, *horao*, I see; $\pi\rho\alpha\xi\iota\varsigma$, activity, practice.

are most effective if jointly adopted by an entire community, a situation that greatly amplifies convergence and stability effects. The DNA language considered in Section 4 offers an extreme example or reification, for alternative coding systems could have been (and probably were) developed and used in the evolution of life. Nevertheless, only one genetic coding system (with minor DNA and RNA code variations) survived and is currently in use on planet earth.

A wire-walker calls the attention of everyone; In contrast, a bucolic cable ferry can go unnoticed, see Figures 10tl and 10tr. In this respect, wire-walking seems to be a metaphor better suited to describe statements made in a young theory, using novelty concepts, with users still struggling to distinguish the orthogonal axes of similarities and disparities, or maybe in a theory that is still in dispute with older ones. Meanwhile, the cable ferry seems to provide a metaphor better suited to describe statements made in a theory after it becomes well accepted and understood, or after it becomes the newly established paradigm.

Reified metaphors are often taken for granted, they become part of the scenery, and are (mis)taken as self-evident, strictly literal, or free of analogical comparisons. Statements formulated using mathematical language are particularly prone to this kind of mistake, as analyzed in Stern (2011a,b).

Finally, let us consider innovation and evolution, as they push life forward, opposing and counteracting the inertia of reification. In biology, evolution makes use of a variety of mechanisms like genetic mutation, horizontal gene transfers, vertical inheritance, sexual recombination, exceptional genetic fusion events, symbiogenesis and symbiosis at multiple ecological levels, etc. All these mechanisms further entangle evolutionary lines in the tree of life into multiple loops and branch fusions. This theme is explored at length in the books of Lynn Margulis and her coauthors listed in the references. The following metaphorical explanation gives a very succinct but understandable summary of the core idea:

> *The branches of animal evolution trees do not just branch but fuse... Animal evolution resembles the evolution of machines, where typewriters and television-like screens integrate into laptops, and internal combustion engines and carriages merge to form automobiles. The principle stays the same: Well-honored parts integrate into startling new wholes.*
>
> Margulis and Sagan (2002a, p.172).

Computational genetic programming allows for controlled experiments in genetic evolution, including important aspects of the genetic coding as a language, like the spontaneous emergents of semantic features in this language and the way in which these features are used in the community of evolving organisms; see Inhasz and Stern (2010) and references therein. These experiments allow us to study how these random evolution mechanisms offer efficient and effective pathways to innovation, always creating and testing new molecular metaphors, trying to further expand already working ontologies, and selecting and preserving those innovations that improve the fitness and further empower the organisms using them.

Finally, a comment on the so called *yoyo* problem. Computer scientists developed clever methods, known as *object-oriented programming* and *class inheritance*, that allow them to easily reuse computer code for alternative purposes, see Budd (2002) and Taenzer (1989). However, the undisciplined use of such methods results in yoyo effects, namely, difficult-to-trace dependencies and side effects that interlink processes operating at distinct hierarchical levels or sharing inherited features in confusing ways. Furthermore, the most undisciplined programmer I know is biology doing its genetic programming in the evolution of life, a behavior that generates massive yoyo entanglements.

6.2. Entangled Pairs of Pairs of Tongs

> *Aristotle introduces a distinction between two types of thinking, namely thinking as considering images* ($\phi\alpha\nu\tau\alpha\sigma\mu\alpha\tau\alpha$) *versus thinking as considering characters* ($\gamma\rho\alpha\mu\mu\alpha\tau\alpha$) *And whereas the former focuses on the visual "form"* ($\epsilon\iota\delta o\varsigma$)*, the latter is rather oriented towards discerning the "formula" or plan* ($\lambda o\gamma o\varsigma$) *that is realized in the actual organism. ... Carl Gustav Jung (1911/1968) likewise distinguished these two types of thinking. While imaginative thinking builds on mental images (Aristotle's* $\phi\alpha\nu\tau\alpha\sigma\upsilon\alpha\tau\alpha$)*, rational thinking is directed by concepts and arguments: by logic. And whereas imaginative thinking is associative and free-floating, rational thinking operates on the basis of linguistic, logical, and mathematical principles (and is therefore more demanding and exhausting, mentally speaking). Finally, whereas imaginative thinking is the oldest form of thinking (more attuned to the spontaneous functioning of the human mind), rational thinking is a more recent acquisition, historically speaking.* Zwart (2018, p.4).

Zwart's quotation aligns several pairs of tongs that appear in this section's title. This alignment may be fine for the perspective taken by Zwart in his specific context of interest. However, the case studies in this paper show several counterexamples for Zwart's suggested couplings; for example, in the case of Hebrew color ontology: (a) The grammatical (letter) and vocalization (sound) mutations used to build color analogies are aligned to their synthetic or imaginative character; and (b) The prior vs. posterior or older vs. more recent qualifications (lexical, biological, technological) in a color ontology are stratified in multiple levels. Furthermore, the ontologies consolidated in older substrata are used, in their full synthetic and analytic powers and capabilities, to support later ontological developments. This complex stratification does not conform to Zwart's simple scheme.

In modern science, it is easy to find similar counterexamples to Zwart's suggested alignments, like *grammatical imagination* or *diagrammatic reasoning.* In fact, I believe arguments of this kind to be extremely relevant in mathematics and exact sciences, see for example the case studies presented in Stern (2022), and also Greaves (2002), Levi (2009), Nelsen (1993, 2000, 2015) and Shive and Weber (1982). Nevertheless, for this article, I have chosen the Hebrew colors example exactly because it can be presented involving no mathematics whatsoever. Like in the color-headed kites studied in Section 2, different jobs or different steps in a job, may benefit or require the use of working tongs in distinct roles, functions, and alignments, with consequent attention shifts and conceptual reorientations.

In statistics, the statement $\Pr(x \in X \,|\, \alpha \in A, \beta \in B, \gamma \in \Gamma, \ldots) = p$ asserts that p is the probability that the stochastic variable x assumes a value in set X, *given the parameters*, *under the conditions* or *assuming the hypotheses* concerning stochastic variables α, β, γ, etc. listed to the right of the *conditional symbol*, $|$. In a statistical model, variables in the *sample space*, conventionally written in Latin letters, are conceived as observed or observable quantities, while variables in the *parameter space*, conventionally written with Greek letters, are conceived as *latent* or *metaphysical* – in the sense of referring to non-observable entities or of having a causal explanatory function. Moreover, statistical hypothesis tests, build over conditional probability statements of this kind, are the standard tools of the trade used in the practice of science for accepting, verifying, and validating (or not) scientific hypotheses; for further details, see the references' section - Objective Cognitive Constructivism and the Bayesian Epistemic Value of Sharp Hypotheses.

The facts in the last paragraph imply that, in a statistical model, even *if* available data can be directly linked to single observed facts, the surrounding conditions, hypotheses, or modeling parameters, must be described in metaphysical terms (in the sense of non-observable references or of explanatory functions). However, any reference to metaphysical concepts is intrinsically metaphorical. This unavoidable entanglement of hard data and abstract concepts is poetically described in the following quotation from Johann Wolfgang Von Goethe (1749-1832):

> *Everything that exists is analogous to all that exists; therefore existence seems to us at the same time isolated and interconnected. If one follows the analogy too far, everything falls together as identical, and if one avoids it completely, everything is scattered into the infinite. In both cases, contemplation stagnates, in one case, as overly alive or, in the other case, as brought to death.*[6] Goethe, Maxims and Reflections (1833).

The probabilist Bruno de Finetti (1906-1985) used to add a *double conditional* symbol, $||$, at the top right corner of the first page of any document he wrote, meaning that any statement within was conditional on all necessary hypotheses, either explicitly stated, or implicitly assumed, possibly including some that he had forgotten, or was not conscious or unaware of. From the considerations in this subsection, I believe "$||\,!$" is a *caveat emptor* that should be posted at the entrance of any mirror-house or at the prolegomena to any future relational theory of tongs (and there are plenty of them going around without this safety warning).

[6] *Jedes Existierende ist ein Analogon alles Existierenden; daher erscheint uns das Daseyn immer zu gleicher Zeit gesondert und verknüpft. Folgt man der Analogie zu sehr, so fällt alles identisch zusammen; meidet man sie, so zerstreut sich alles in's Unendliche. In beiden Fällen stagnirt die Betrachtung, einmal als überlebendig, das andere Mal als getötet.* Goethe, Maximen und Reflexionen (1833, v.1,13, p.46).

6.3. Positivist Illusions, and Solipsist or Consensualist Disillusions

Epistemology and philosophy of statistics in the XIX and early XX century were dominated by the schools of *Positivism*, *Logical empiricism*, and their aftermaths. These schools assumed to be possible, desirable, or necessary to make science under very strict conditions, like the following (or variations thereof): (a) Use of *regimented languages* referring exclusively to observable facts, and free of any metaphorical figures; (b) Make a clear-cut and complete separation of analytic and synthetic statements; (c) Make available complete and exhaustive listings of assumed hypotheses or prior conditions to any proposition; (d) Assume the existence of secure and stable alignments (like synthetic and a posteriori to empirical observation) of key categories qualifying scientific statements used for testing, verifying, or validating scientific theories. From the arguments made so far in this article, it should be abundantly clear that I believe such conditions are not only untenable but also essentially misconceived.

Several alternative schools of thought came to denounce the positivist illusions, arriving, however, at very different conclusions – including a pair of alternative "disillusions" named solipsism and consensualism, a.k.a. (inter)subjectivism. They are all-or-nothing, extreme, or radical reactions to the fall of positivism.

Solipsism, *skepticism*, *extreme sujectivism*, and their variations negate any form of objective knowledge. Any truth is only in the eye of the beholder, any certainty, even if quantitatively graduated and reasonably conditioned by its assumed hypotheses, is just a hopeless mirage. Solipsism, in the words of I.J. Good (1983, ch.8, p.93) apud Stern (2017), is an inexpugnable castle. We can only hope that, sooner or later, their reclusive inhabitants will miss the sunlight and the fresh air, and will willingly come outside to have a more significant engagement with science that, in turn, can then play a more meaningful role in their lives.

Consensualism, on the other hand, allows for any form of wishful thinking – as long as it is sanctioned by a qualified endorsing community. Consensualism has the extra advantage of consensual self-validation: If all members of this community agree consensualist epistemology is valid, and that they are a qualified community, then their conclusions must be valid by this (self)validating consensual agreement! I do not deny the role consensualism plays in some forms of social organization, including religious sects and Internet bubbles. However, most people agree science needs, deserves, and can have more reliable demarcation and validation criteria.

7. Conclusions, Final Remarks, and Future Research

In this article we have examined some oppositional categories, like Synthetic vs. Analytic, Posterior vs. Prior, Imagination vs. Grammar, Metaphor vs. Hermeneutics, Metaphysical vs. Observational, Innovation vs. Routine, Image vs. Sound, etc. We examined the utility of these conceptual grasping tools for working at the forges of science and epistemology, and how they can be used to focus our attention and deal with important aspects of entities being manipulated at the forge.

Moreover, narrowly defined conditions in a specific context of application may be able to decouple or fix existing correlations between aspects each of these tongs is designed to hold. Nevertheless, even small variations in context, scope, or conditions of application may require attention shifts, lead to distinct perspectives, and imply different correlations or engender new logical entanglements.

The last conclusions are incompatible with projects in mainstream XIX and XX-century philosophy of science developed by the schools of positivism and logical empiricism. These schools, in turn, served as the epistemological basis for (virtually) all of the philosophy of statistics developed in the XIX and XX century, including either the *frequentist* school developed by Karl Pearson, Ronald Fisher, etc., or the *Bayesian* revival school developed by Bruno de Finetti, Leonard Savage, etc., see Stern (2007). Moreover, statistical test theories provide the universally accepted validation standards used in the practice of science. There have been several negative reactions to this situation, including the resurgence of contemporary forms solipsism or consensualism, or a tendency to take a blasé approach to these issues that deemphasizes foundational studies and critical thinking, focusing instead on technicalities of the daily practice of science. Notwithstanding this state of affairs, we claim that the epistemological framework of *Objective cognitive constructivism* offers a viable alternative to achieve properly qualified objective scientific knowledge, as discussed in the next subsection.

7.1. Objective knowledge, Wire-Walks, and Wonder-Networks

Richard Dawkins, in his foreword for Blackmore (1999), makes a comparison between digital vs. analogical audio or video recording, production, and distribution. The degradation and decay characteristic of analogical media (like vinyl records and magnetic tapes) impose strong limits on the quality and number of processing steps. In contrast, digital technologies do not have intrinsic decay or degradation problems and, consequently, do not impose corresponding restrictions in their production steps or ways of distribution. It is possible to extend this contrasting analysis to other forms of high vs. low fidelity information processing.

Similarly, one could contrast high vs. low-quality roads or pipelines for the transportation of material goods. High-quality networks may impose tolls and taxes but (almost) never damage transported goods (like on bumpy roads), they may require effort and energy to overcome friction and other resistances, but using them should imply (almost) no decay or degradation (like in leaky pipelines). We consider *wonder-networks* as limit cases of very high fidelity information networks or very high-quality transportation networks. Such wonder-networks are conceived as physical analogs to conceptual wire-walk networks whose links are made of equational relations, as those examined in Section 6.

The *objective cognitive constructivism* epistemological framework, *OCogCon* for short, offers a way out of the entrapments posed by, on the one hand, the positivist illusions and, on the other hand, by the solipsist or consensualist disillusions. It fully recognizes, and denounces the naive or untenable nature of the positivist assumptions examined in the last subsection. However, it also avoids the all-or-nothing extremes of solipsism and consensualism and, in so doing, overcomes their obvious inadequacy for the epistemological needs of a working scientist.

Objective cognitive constructivism relies on the possibility of making science using empirical woder-networks corresponding to conceptual networks of equational relations, or vice-versa. OCogCon is an epistemological framework based on the science and technology attested abilities for wire-walking, and for composing multiple wire-walks into complex, far-reaching, and reliable networks.

Accordingly, the OCogCon framework must include statistical tools used to measure how much or how well statistical hypotheses corresponding to these wonder-networks are supported by empirical data. Several measures of statistical support for a hypothesis under scrutiny were developed in the XIX and XX centuries. Among the best-known alternatives developed for this purpose are measures of *statistical significance*, *confidence level*, or *goodness of fit*. Nevertheless, all the aforementioned traditional measures of support have one thing in common – they have serious shortcomings in applications involving sharp or precise statistical hypotheses – the specific kind that is needed for modeling conceptual networks of equational relations, that, in turn, are used to build theoretical representations of scientific or technological wonder-networks.

$\mathrm{ev}\,(H \mid X)$, the *epistemic value* of hypothesis H given the observed data X, a.k.a. the *e-value* of H given X, offers support for the OCogCon epistemological framework providing theoretical and operational tools of mathematical statistics. The e-value is the cornerstone of a comprehensive and coherent theory of statistical testing that was tailor-made for evaluating either a single sharp statistical hypothesis, or a composition of several sharp hypotheses (representing several interconnected links in a wonder-network).

The e-value has intrinsic theoretical characteristics of mathematical simplicity, logical coherence, and statistical exactness (i.e., not relying on asymptotic approximations). Moreover it has excellent operational characteristics, being usually easier to implement, statistically more efficient, and exhibiting better computational performance than traditional alternatives. For technical details, many application examples, mathematical and logical characterizations of the e-value, and discussion of their philosophical and epistemological consequences, see the references' section - Objective Cognitive Constructivism and the Bayesian Epistemic Value of Sharp Hypotheses.

7.2. The Right Science for Philosophy of Science

This article payed great attention to *equational statements*, also known in the philosophy of science as *exact laws* and in foundations of statistics as *sharp* or *precise hypotheses*, considering the role these statements play in epistemology. Such a study needs to be contextualized, including the discussion of concrete examples. There I unexpectedly found myself in the middle of an ongoing war, risking to take some crossfire. Some people want such epistemological discussions to be based on simple prototypical statements like – "The snow is white" or "The sky is blue", while others want to jump straight away to examples in General Relativity or Quantum Mechanics. There are several reasons and arguments supporting these opposing preferences, but I will just clearly state, without further excuses or justifications, that I have avoided some of the perils (and also renounced some of the rewards) that can be found at these extremes, choosing instead to follow the classical advice to take a path of equilibrium going through the middle ground.

The examples used in this article are intended to be sufficiently illustrative of real science but also accessible knowing only high school science at the standards set at that time and place of my life for university candidates in all fields of study, see FUVEST (1977). Our discussion context also includes basic two-years college mathematics, with calculus, geometry, linear algebra, probability, statistics, and programming. For pertinent pedagogical materials used worldwide at that time (and still today), see the reference section – High School Science and Basic Two-Years College Mathematics. This section also includes a few suggestions for up-to-date introductions to computer programming with pertinent applications.

7.3. Future Research: Limited Complexity, Mirroring Laws, and Responsibility

In future research, we intend to investigate additional topics relating logical or algebraic oppositional structures, Jungian psychology and philosophy of science. Recent research in oppositional logic enables the construction of algebraic lattices and related structures of arbitrary dimension. However, most of the applications seen in practice use structures of relatively small dimensions. Jung had already pointed to the human mind's preference for low dimensional representations, relating this fact to the possibility of finding radical solutions only for equations of low degree (up to four), see next quotation. Several lines of research in modern psychology attest to the human mind's preference or necessity of using representation spaces of low dimension. Nelson Cowan (2001) makes the case for psychological representation models using up to four factors, while George Miller (1956) advocates for a practical limit between five and nine. We believe that the relation between the structure and number of explaining factors of a scientific model and its human ineligibility deserves further attention.

> *It is a property of the number four that equations of the fourth degree can be solved, whereas equations of the fifth degree cannot.*
>
> Jung (1989, p.310), Memories, Dreams and Reflections.

Moreover, Jung believed that ideas giving us powerful insights, intuitions or interpretations can always be traced back to archetypes; see next quotation. I believe that the art and science of building intuitive models and their didactic presentation deserves further attention and that, moreover, it can benefit from the desirable quantitative and qualitative characteristics suggested by Jung. Furthermore, this perspective relates to *Mirroring hypothesis* or *Conway's law*, a general principle of system's theory stating that – *The structure of a system reflects the structure of the organization that built it*, see Yourdon and Constantine (1979) and MacCormack et al. (2012). Exploring these connections can further enrich this proposed path for further research.

> *All the most powerful ideas in history go back to archetypes... the central concepts of science, philosophy, and ethics are no exception to this rule*
> Jung(CW, VIII, pr.342).

Finally, we plan to investigate how solipsist or consensualist epistemological attitudes can foster extreme forms of science denial or hostility, as manifested, for example, in climate change negation and anti-vaccine movements. We believe that epistemological frameworks providing better support and stronger forms of validation to objective scientific knowledge can have a positive social impact, including increased awareness and more responsible attitudes concerning science, technology, and their role in daily life. For alternative views of this subject, see Crocker and Sinatra (2021), McIntyre (2019, 2022), Shtulman (2017), Stern (2021), and the references' section - Movies and Film Documentaries.

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Figures:[7] Images I generated using LaTeX and TikZ or pb-diagram: 1l; 2lb; 3r; 4tr; 5(all); 6r; 7(all); 8(all); 9t; 13l,r. Images in Public Domain or distributed under Creative Commons licenses allowing the use in non-commercial *and* commercial applications: 4tl; 9,bct,brt; 10bc,br; 11(all); 12l,cr,r; 13c; Images prepared for this paper by graphic artists Alex freitas and Ian Freitas: 1c; 1r; 2lt; 3l; 4b; 6l; 10tc. Images from books in public domain, more than 70 years past after dates of publication and author's death: 2r, from Cleopatra (1000, 1887); 6c, from Hering (1878); 9bl, from Auto Comfort magazine (1911); 10l, from Robertson (1875); 12cl, from Kepler (1604); Images used with the kind permission given by the author or responsible for copyright: 1c, from Bruce MacEvoy (2005); 9bcb, by Daniel Ledon, image at `http://www.stromboidea.de`; 9brb, by Ioannis Karapanagiotis, from Ramig et al. (2017); 10tr, by Tom McPherson, from *Bird's eye view of tightrope walker* video of Circle Line Art School; 10bl, by `olaargentina.com`.

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[7]Figure position locators: t=top, b=bottom, l=left, c=center, r=right.

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