Image Restoration

SCC0251/5830 – Image Processing

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Agenda

1. Introduction
2. Noise
   - Sources and models of noise
   - Noise generation
   - Noise reduction
   - Bilateral filtering
3. Blur
   - Degradation functions
   - Inverse and pseudo-inverse filtering
   - Least squares filtering

Agenda

Obtaining better images

Problem — to improve the visual quality of the images

- **Enhancement** $\times$ **Restoration**

Obtaining better images

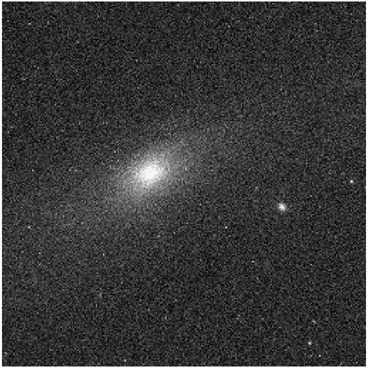
Problem — to improve the visual quality of the images

- **Enhancement** $\times$ **Restoration**
  - Enhancement: subjective method based on operations that supposedly improve image quality
  - Restoration: objective method based on prior knowledge about the image degradation model

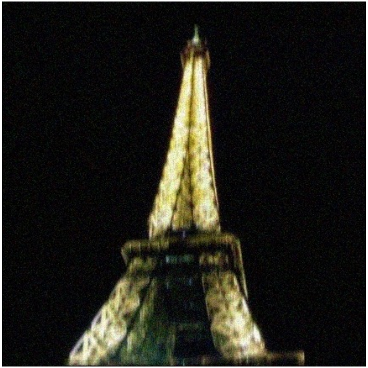
Degradation: blur

Degradation: motion blur

Degradation: noise



Degradation: blur and noise



Problem

$$g(\mathbf{x}) = \mathcal{N}\left\{f(\mathbf{x}) * h(\mathbf{x})\right\}$$

- g — observed (degraded) image
- f — ideal or original image
- $*$ — convolution
- h — degrading function
- $\mathcal{N}()$ — noise generation process

Problem

When the nature of the noise is "additive"

$$g(\mathsf{x}) = f(\mathsf{x}) * h(\mathsf{x}) + n(\mathsf{x})$$

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This equation tries to capture the idea of an imaging system

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Restoration algorithms aim to achieve a restored image $\hat{f}(\mathbf{x})$ that is as similar as possible to the original/ideal image $f(\mathbf{x})$.

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Restoration algorithms aim to achieve a restored image $\hat{f}(\mathbf{x})$ that is as similar as possible to the original/ideal image $f(\mathbf{x})$.

- In order to to that, we use knowledge about the *point spread function* and *noise*.

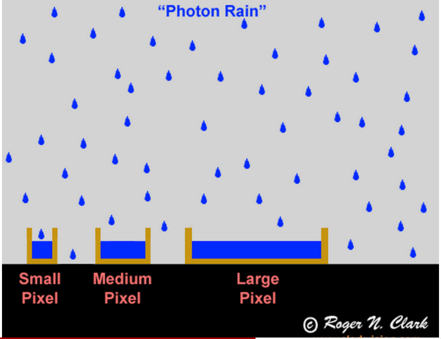
Sources of noise

Generally, the source defines the noise characteristic. Most images has noise that is accumulated through several acquisition steps

- **Photo counting**
- **Thermal**
- **Quantisation**
- **Transmission/display**

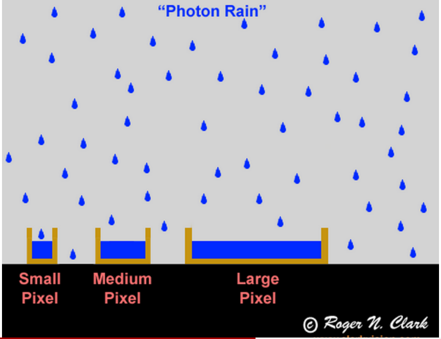
Sources of noise — photon counting

- **Photon counting**: light detection via a sensor is a statistical process, well modeled by a Poisson distribution.
- The precision of the measured signal is proportional to the mean of the signal (the amount of photons).



Sources of noise — photon counting

- **Photon counting**: light detection via a sensor is a statistical process, well modeled by a Poisson distribution.
- The precision of the measured signal is proportional to the mean of the signal (the amount of photons).
- The amount of noise can be approximated by the squared root of the number of photons.

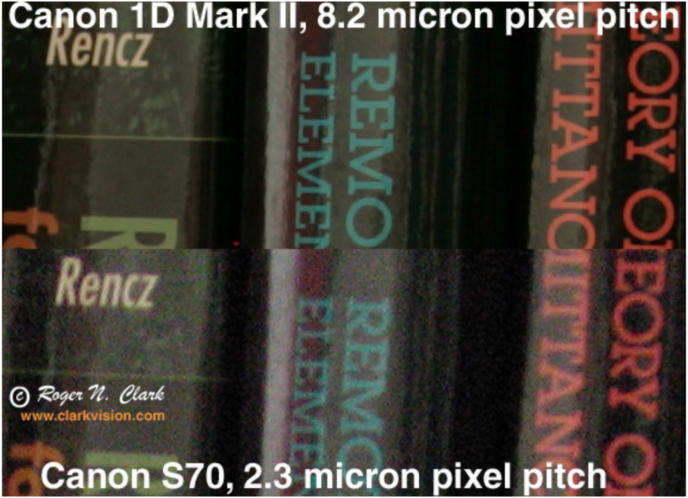


Photons	Noise	NR
9	3	1/3
100	10	1/10
900	30	1/30
10000	100	1/100
90000	300	1/300

Sources of noise — photon counting

- That is why two cameras with the same pixel quantities but different sensor sizes can result in different images.
- Below two images from the same maker, number of pixels, ISO parameter, aperture and shutter speed, but different sensors.

Sources of noise — photon counting



thanks to Roger Clark

http://www.clarkvision.com/articles/telephoto_reach/

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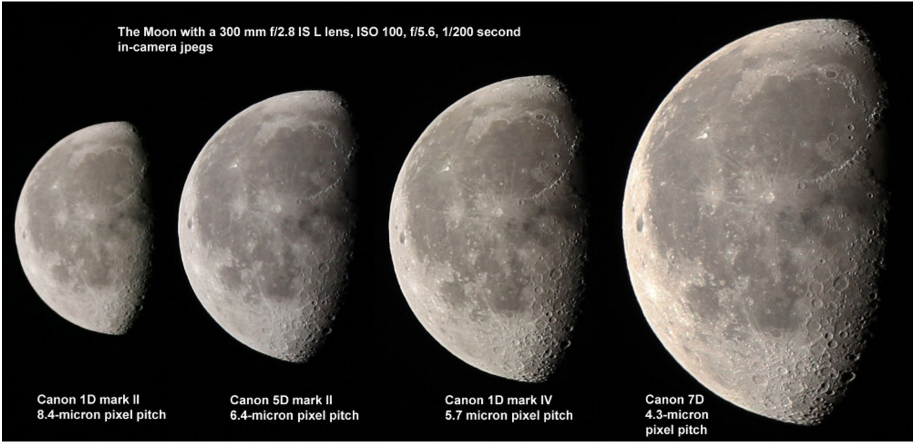
  - Each pixel will have a lower amount of photons.
  - Therefore, a sharper image, but still **noisier**.
- Smaller pixels allow to observe more details, paying the cost of a lower signal-to-noise ratio per pixel.

Sources of noise — photon counting



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Sources of noise — photon counting

- Sparse images, with low exposure time, has noise characterised by Poisson distribution. Examples are:
  - Astronomic images
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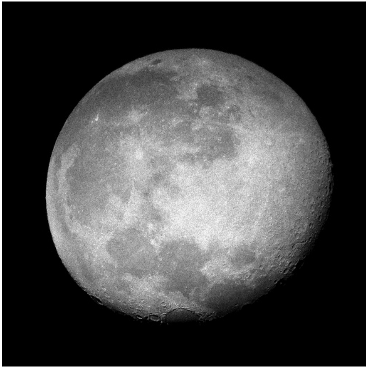
- Noise is **signal dependent** (correlated).
- Its image formation is given by $g(\mathbf{x}) = \mathcal{P}\{f(\mathbf{x}) * h(\mathbf{x})\}$
- When imaging with good illumination conditions and adequate exposure, counting noise is often low and can be neglected.
  - This is because the Poisson distribution approaches the Normal distribution, i.e. $\mathcal{P}(\lambda) \sim \mathcal{N}(\lambda, \lambda)$, as $\lambda \to \infty$.

Sources of noise — photon counting



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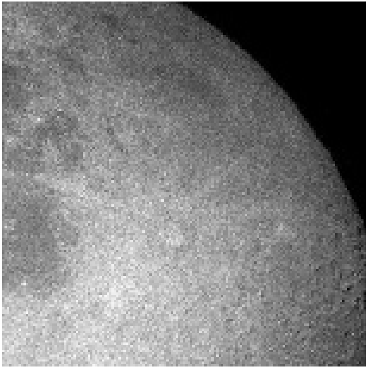
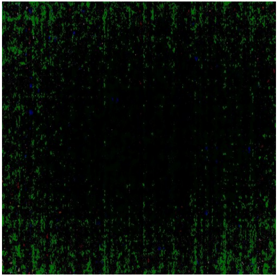


Sources of noise — thermal

- **Thermal**: electrons are generated when the photons are detected. Those will vary given the temperature of the sensor.
- Usually we assume this noise to be Gaussian (Normal) and additive, also called White noise.
  - This noise is independent of the signal.
  - Image formation is given by: $g(\mathbf{x}) = f(\mathbf{x}) * h(\mathbf{x}) + n(\mathbf{x})$

Sources of noise — thermal

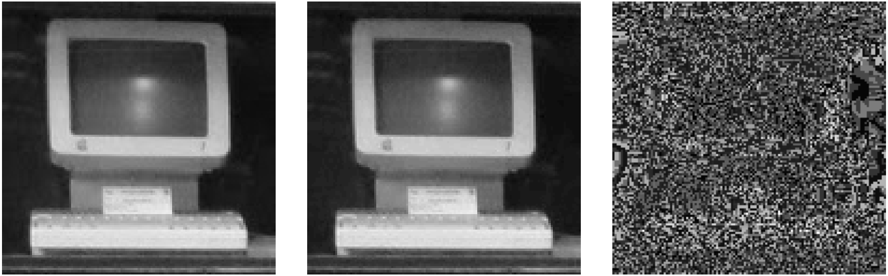
- A possible way to diminish thermal noise is via a Dark Frame capture, an image obtained without light acquisition.
- This image contains a map of the thermal noise. Although it varies with the temperature, it is usually stable after a period.
  - Dark Frame can then be subtracted from acquired images
  - Below: Dark Frames of CCDs from a telescope (left), and a cellphone camera (right), with normalised levels.



Sources of noise — quantisation

- **Quantisation**: noise caused by quantisation of pixels from continuous to unsigned int/char.
  - It often follows uniform distribution.
  - When quantisation level is low, the noise can become signal dependent and correlated to each region of the image (non-uniform).

Sources of noise — quantisation



(a) (b) (c)

(a) 256 level quantisation, (b) 64 level quantisation, (c) quantisation noise with 64 levels

Sources of noise — transmission/display

- Noise often caused by errors in some bits when storing or failure when transmitted.
- Resulting noise is referred to as “impulsive”, but also “salt and pepper”.
    - Can be caused by other processes then transmission/display
    - Affects a smaller number of pixels, but the ones affected are completely destroyed.

Sources of noise — transmission/display

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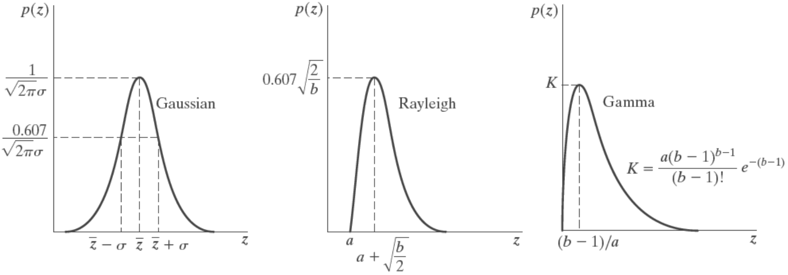
- The mathematical representation of the impulsive noise can seen as two "impulses" (or Dirac functions) in 0 (black) e 255 (white)
  - A random pixel has probability p of been affected by noise, usually $p/2$ for "salt" and $p/2$ for "pepper".

Agenda

Noise generation

- It is possible to simulate noise in images using known distributions.
- Real noise is difficult to simulate, but by knowing the basic image formation system it is possible to obtain a good approximation.
- Implementation consists in generating random numbers and using probability density functions.

Noise generation



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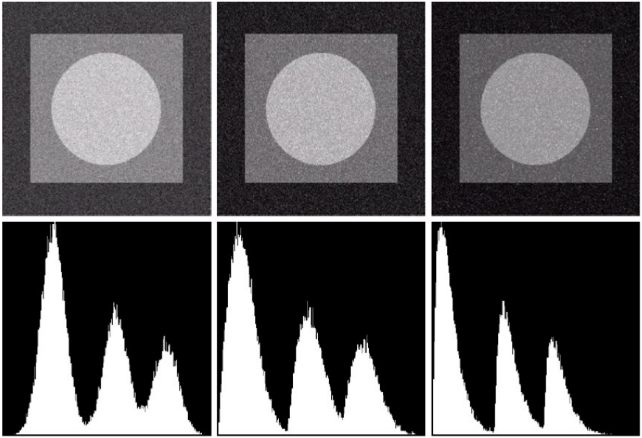

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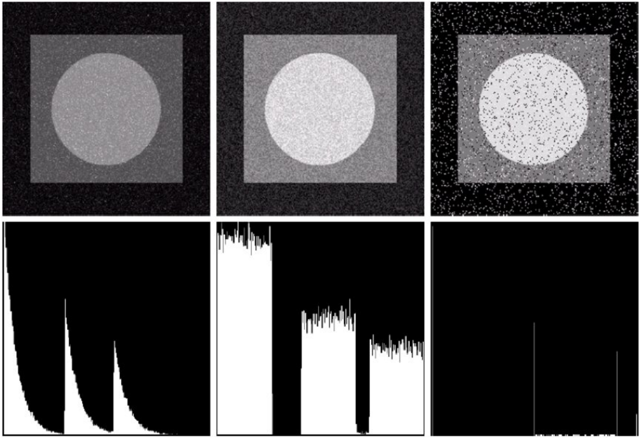

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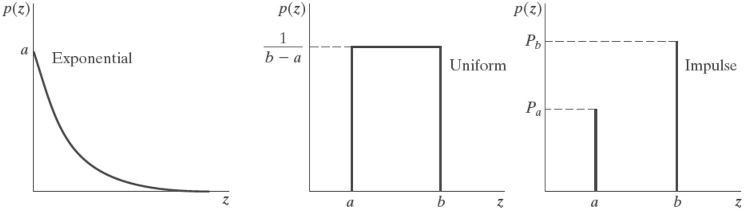

Agenda

Mean filtering

- Smooth out pixels using the contextual information (neighbours),
- Mean operators allow to **reduce the signal variance** and, therefore, noise.
- Variations of mean filtering: arithmetic, geometric, harmonic, weighted.

Mean filtering

- **Arithmetic**: increase the blur by creating a new value based on the average of neighbour pixels $S_{(\mathbf{x})}$, where $(\mathbf{x}) = (x, y)$.
- Neighbourhood is rectangular of size $m \times n$
- when $\lambda_{(s,t)} = 1$ for all s, t, then all pixels have the same weigh

$$\hat{f}(\mathbf{x}) = \frac{1}{nm} \sum_{(s,t) \in S_{\mathbf{x}}} \lambda_{s,t} \cdot g(s, t)$$

Mean filtering

- **Geometric**: can help preserving details when pixel differences are in the order of multiples of a given base (2, 10, etc.), i.e. it is logarithmic.

$$\hat{f}(\mathbf{x}) = \left[\prod_{(s,t) \in S_{\mathbf{x}}} \lambda_{s,t} \cdot g(s,t) \right]^{\frac{1}{nm}}$$

Mean filtering

- **Harmonic**: reduce the influence of outliers.
- This filter is adequate when there is additive noise mixed with salt noise (outlier)

$$\hat{f}(\mathbf{x}) = \frac{mn}{\sum_{(s,t)\in S_{\mathbf{x}}} \frac{1}{g(s,t)}}$$

Order statistic filters

- Given a series of observations of some random variable, the order statistics are obtained by sorting those observations in ascending order.
- In context of images, the observations are pixels in a neighbourhood.
- Result in non-linear filters such as
  - Median
  - Maximum, mininum
  - Mean point

Order statistic filters

- **Median**: widely used in image pre-processing
- Remove texture, preserve edges.
- Very effective to remove impulsive noise.
- The resulting pixel is the percentile 50 of a ordered sequence of numbers

$$\hat{f}(\mathbf{x}) = \text{median}_{(s,t)\in S_{\mathbf{x}}} \{g(s,t)\}$$

Order statistic filters

- **Max**: 100º percentile (maximum value)
- Can be used to locate bright points in the image

$$\hat{f}(\mathbf{x}) = \max_{(s,t)\in S_{\mathbf{x}}} \{g(s,t)\}$$

- **Min**: 0º percentile (minimum value)
- Can be used to locate dark points in the image

$$\hat{f}(\mathbf{x}) = \min_{(s,t)\in S_{\mathbf{x}}} \{g(s,t)\}$$

Order statistic filters

- **Mean point**: combines order statistics with mean
- Usually produces an effect similar to median, but often thickens the borders/edges.

$$\hat{f}(\mathrm{x}) = \frac{1}{2}\left[\max_{(s,t)\in S_{\mathrm{x}}}\{g(s,t)\} + \min_{(s,t)\in S_{\mathrm{x}}}\{g(s,t)\}\right]$$

Adaptive filtering

- Take into account local statistics.
- The objective is to allow smoother results mostly in flat regions (with less detail);
- Any filter can be developed in an adaptive fashion. For example:
  - Adaptive noise reduction using mean and local variance,
  - Adaptive noise reduction using median and local inter-quartile range (IQR).

Adaptive noise reduction using mean and variance

Considering a local region $S_{\mathbf{x}}$, the response of the adaptive filter needs:

1. $g(\mathbf{x})$: the value of noisy image at $\mathbf{x}$
2. σ_η^2 : the variance of noise in the image (global)
3. m_L : local mean of pixels in $S_{\mathbf{x}}$
4. σ_L^2 : local variance of pixels in $S_{\mathbf{x}}$

$$\hat{f}(\mathbf{x}) = g(\mathbf{x}) - \frac{\sigma_\eta^2}{\sigma_L^2}\left[g(\mathbf{x}) - m_L\right]$$

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$$\hat{f}(\mathbf{x}) = g(\mathbf{x}) - \frac{\sigma_\eta^2}{\sigma_L^2}\left[g(\mathbf{x}) - m_L\right]$$

- We need to estimate (or know — strong assumption) the noise variance
    - It is possible to estimate σ_η^2 measuring variance in a flat region of the image.

Adaptive noise reduction using mean and variance

$$\hat{f}(\mathbf{x}) = g(\mathbf{x}) - \frac{\sigma_\eta^2}{\sigma_L^2}\left[g(\mathbf{x}) - m_L\right]$$

The filter behaves in each point as follows:

- if $\sigma_L^2 = 0$, then the response is $g(\mathbf{x})$,
- if $\sigma_L^2 \gg \sigma_\eta^2$, then it approaches $g(\mathbf{x})$,
- if $\sigma_L^2 \approx \sigma_\eta^2$, then the response is the local mean at region $S_\mathbf{x}$.

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We need that $\sigma_\eta^2 \leq \sigma_L^2$

- if we observe $\sigma_\eta^2 > \sigma_L^2$, then the ratio between the variances must be defined as 1 to avoid spurious values.

Adaptive noise reduction using mean and variance

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- if we observe $\sigma_\eta^2 > \sigma_L^2$, then the ratio between the variances must be defined as 1 to avoid spurious values.
- this condition makes the filter non-linear.

Bilateral filtering

Noise reduction filter with edge preservation that uses the image content in order to avoid averaging across edges. Centered at a pixel **p**, it is given by:

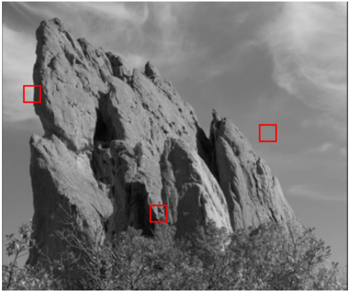
normalisation — A: not new — B: new!

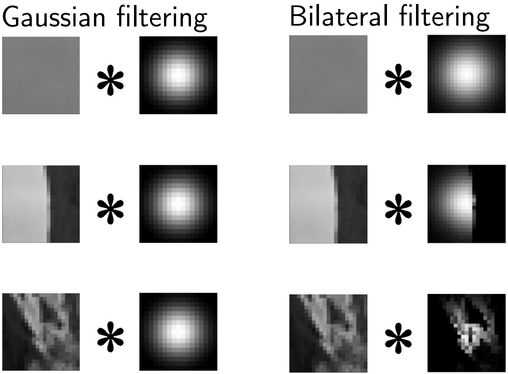
$$BF(g(\mathbf{p})) = \underbrace{\frac{1}{F_{\mathbf{p}}}}_{\text{normalisation}} \sum_{\mathbf{q}} \underbrace{G_{\sigma_s}(||\mathbf{p}-\mathbf{q}||)}_{\text{A: not new}} \underbrace{G_{\sigma_r}(||g_{\mathbf{p}} - g_{\mathbf{q}}||)}_{\text{B: new!}} \, g_{\mathbf{q}}$$

- term A defines the weight in space (difference in coordinates),
- term B controls the range weight (differences in intensities), avoiding filtering over edges.

OBS: removing the normalisation and the term B, we have a Gaussian filter.

Bilateral filtering





Bilateral filtering

$$BF(g(\mathbf{p})) = \frac{1}{F_{\mathbf{p}}} \sum_{\mathbf{q}} G_{\sigma_s}(||\mathbf{p} - \mathbf{q}||)\, G_{\sigma_r}(||g_{\mathbf{p}} - g_{\mathbf{q}}||)\, g_{\mathbf{q}}$$

- σ_s parameter for the size of neighbourhood, e.g. 2% of the image diagonal
- σ_r minimum amplitude to consider presence of an edge, e.g. mean of the image gradient

OBS: because each neighbourhood has a different filter, cannot be precomputed to use with FFT. Naive implementation is slow, but there are approximations with good quality/speed ratio.

Agenda

Problem

Assuming a noise-free scenario, the image formation model is given by:

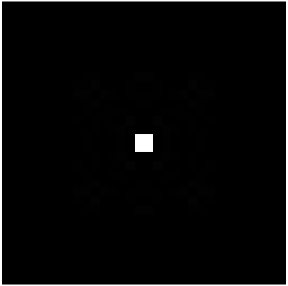
$$g(\mathbf{x}) = f(\mathbf{x}) * h(\mathbf{x})$$

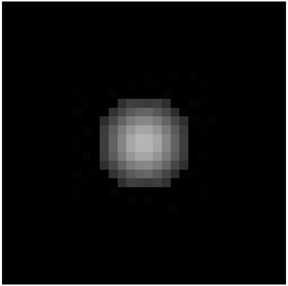
- g — degraded/observed image
- f — ideal or original image
- $*$ — convolution
- h — degradation function

Problem

Function $h(\mathbf{x})$ represents the **impulse response** of the imaging system

- in an image it models how the system responds when the input is a single point (or impulse)





Problem

Function $h(\mathbf{x})$ represents the **impulse response** of the imaging system

- in an image it models how the system responds when the input is a single point (or impulse)
- often called point spread function (PSF)

Degradation functions

- h are non-negative due to the physics of image formation,
- if the image is real (yes, there are complex images), PSF is also real,
- imperfections of the imaging system are modelled so that the energy of the signal is preserved:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)dxdy = 1$$

$$\sum_{\mathbf{x}=(0,0)}^{(N-1,M-1)} h(\mathbf{x}) = 1$$

Degradation functions

No blur

$$h(x,y) = \delta(x,y) = \begin{cases} 1, & \text{if } x,y = (0,0) \\ 0, & \text{other positions} \end{cases}$$

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$$h(x,y;R) = \begin{cases} \frac{1}{\pi R^2}, & \text{if } \sqrt{x^2+y^2} \leq R^2, \\ 0, & \text{otherwise} \end{cases}$$

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Motion blur

$$h(x,y;L,\phi) = \begin{cases} \frac{1}{L}, & \text{if} \sqrt{x^2+y^2} \leq \frac{L}{2} \text{ and } \frac{x}{y} = -\tan\phi, \\ 0, & \text{otherwise} \end{cases}$$

Problem

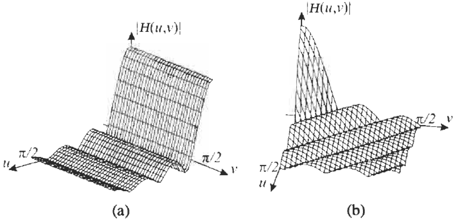

FIGURE 2 PSF of motion blur in the Fourier domain, showing $|H(u, v)|$, for (a) $L = 7.5$ and $\phi = 0$; (b) $L = 7.5$ and $\phi = \pi/4$

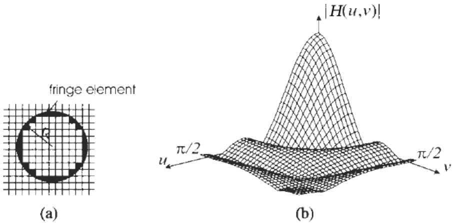

FIGURE 3 (a) Fringe elements of discrete out-of-focus blur that are calculated by integration; (b) PSF in the Fourier domain, showing $|H(u, v)|$, for $R = 2.5$.

Discrete degrading functions

Uniform blur

$$h(\mathbf{x}; R) = \begin{cases} \frac{1}{C} & \text{if } \sqrt{x_1^2 + x_2^2} \leq R^2, \\ 0 & \text{otherwise} \end{cases}$$

where C is a constant so that the sum of the coefficients is 1.

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Motion blur

$$h(\mathbf{x}; L) = \begin{cases} \frac{1}{L} & \text{if } x_1 = 0, |x_2| \leq \lfloor \frac{L-1}{2} \rfloor \\ \frac{1}{2L} \left\{ (L-1) - 2\lfloor \frac{L-1}{2} \rfloor \right\} & \text{if } x_1 = 0, |x_2| = \lfloor \frac{L-1}{2} \rfloor \\ 0, & \text{otherwise} \end{cases}$$

Inverse filtering

We want to invert h, so that:

$$\hat{f}(\mathbf{x}) = g(\mathbf{x}) * h^{-1}(\mathbf{x})$$

Example: Gaussian degradation function 5×5:

0.003	0.014	0.025	0.014	0.003
0.014	0.058	0.095	0.058	0.014
0.025	0.095	0.150	0.095	0.025
0.014	0.058	0.095	0.058	0.014
0.003	0.014	0.025	0.014	0.003

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Example: Gaussian degradation function 5×5:

0.003	0.014	0.025	0.014	0.003
0.014	0.058	0.095	0.058	0.014
0.025	0.095	0.150	0.095	0.025
0.014	0.058	0.095	0.058	0.014
0.003	0.014	0.025	0.014	0.003

Matrix is **singular**, there is no inverse!

Inverse filtering

If we know the PSF of the imaging system, the image formation can also be considered in frequency domain:

$$G(\mathbf{u}) = F(\mathbf{u})H(\mathbf{u})$$

Inverse and pseudo-inverse filtering

Now we divide the Fourier transform of the observed image by the PSF Fourier transform H, also called OTF (Optical Transfer Function).

$$\hat{F}(\mathbf{u}) = \frac{G(\mathbf{u})}{H(\mathbf{u})}$$

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$$\hat{F}(\mathbf{u}) = \frac{G(\mathbf{u})}{H(\mathbf{u})}$$

When we know the OTF and we have a well-behaved transform (such as the Gaussian function), this operation is possible and approaches a perfect restoration.

Inverse and pseudo-inverse filtering

In a noisy image, we have:

$$\hat{F}(\mathbf{u}) = \frac{H(\mathbf{u})F(\mathbf{u}) + N(\mathbf{u})}{H(\mathbf{u})}$$

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$$\hat{F}(\mathbf{u}) = F(\mathbf{u}) + \frac{N(\mathbf{u})}{H(\mathbf{u})}$$

In this scenario and in those in which H shows values near zero, the ratio $\frac{N(\mathbf{u})}{H(\mathbf{u})}$ dominates the sum, and the resulting image is just noise.

Inverse and pseudo-inverse filtering

In some cases, it is possible to use the pseudo-inverse filtering, changing H below the threshold γ:

$$W(\mathbf{u}) = \begin{cases} H(\mathbf{u}), & H(\mathbf{u}) > \gamma \\ \gamma, & \text{otherwise} \end{cases}$$

The threshold is often between 0.0001 and 0.1. The filter W is then used to achieve the inverse:

$$\hat{F}(\mathbf{u}) = \frac{G(\mathbf{u})}{W(\mathbf{u})}$$

Least squares filtering

The pseudo-inverse filter allows to deal with null or small values, but its formulation does not include explicitly the noise model.

Least squares filters were developed in this context: the constrained least squares filter (CLS) and the Wiener filter are important examples.

Considering image and noise as random variables, this method tries to find an image estimate $\hat{f}$ so that the mean squared error is minimized:

$$e^2 = E\left\{(f - \hat{f})^2\right\}$$

Least squares filtering: Wiener

Assuming:

1. noise is not correlated;
2. noise has zero mean (centered at each pixel);
3. the intensities of the restored image can be written as a linear function of the degraded image.

$$\hat{F}(\mathbf{u}) = \left[\frac{H^*(\mathbf{u}) S_f(\mathbf{u})}{|H(\mathbf{u})|^2 S_f(\mathbf{u}) + S_\eta(\mathbf{u})} \right] \times G(\mathbf{u}),$$

Least squares filtering: Wiener

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- $S_f(\mathbf{u}) = |F(\mathbf{u})|^2$ — power spectrum of the ideal image
- $S_\eta(\mathbf{u}) = |N(\mathbf{u})|^2$ — power spectrum of the noise
- $H^*(\mathbf{u})$ is the complex conjugate of $H(\mathbf{u})$

Least squares filtering: Wiener

How can we know the power spectrum of the ideal/original image and of the additive noise.

- Using the noise variance as parameter, and the direct method of periodogram:
  - $\hat{S}_\eta(\mathbf{u}) = \sigma_\eta^2$ for all $(\mathbf{u})$
  - $\hat{S}_f(\mathbf{u}) = 1/N^2\,[G(\mathbf{u})G^*(\mathbf{u})] - \sigma_\eta^2$

Least squares filtering: Wiener

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There are other methods to obtain S_η and S_f, but it required additional knowledge about image and noise.

Constrained Least squares filtering

From a similar formulation, considering a constraint in the least squares, a method was proposed by regularizing the solution via a Laplacian operator:

$$\hat{F}(\mathbf{u}) = \left[\frac{H^*(\mathbf{u})}{|H(\mathbf{u})|^2 + \gamma |P(\mathbf{u})|^2} \right] \times G(\mathbf{u}),$$

where $P(\mathbf{u})$ is the Fourier transform of a Laplacian operator:

$$p(\mathbf{x}) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

γ controls the influence of the regularization

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