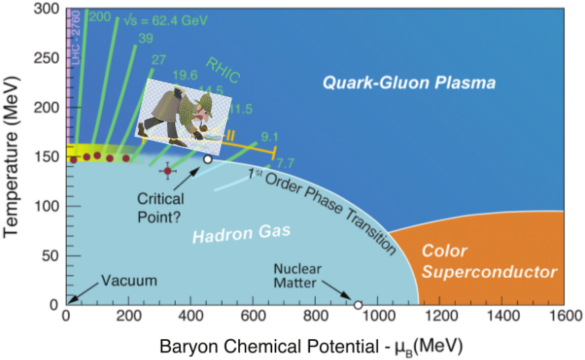
Lecture 24
Exploration of the QCD phase diagram and search for its critical point (part II)



In this lecture, we discuss cumulants expected from some models and compare with data, searching for evidence of the critical point

Susceptibilities

Consider a system in thermal and chemical equilibrium.

The thermodynamic variables such as pressure, density, etc, can be computed from a partition function $\mathcal{Z}$ that encodes the statistical properties of the system.

We then expect that there must be a relationship between the cumulants of a conserved quantity and the partition function. The relationship is (for baryon number):

$$\boxed{C_n = VT^3 \chi_n \text{ where } \chi_n \equiv \frac{\partial^n p / T^4}{\partial(\mu_b / T)^n}}$$

where $p = (T/V) \ln \mathcal{Z}$ and χ_n is the nth order susceptibility.

Derivation (for a classical discrete system):
The grand canonical partition function $\mathcal{Z}$ (exchange heat and particles with environment at fixed T,V, μ) can be written as function of the canonic one (exchange heat with environment at fixed T,V, N):
$\mathcal{Z}(T, V, \hat{\mu}_b) = \sum_N e^{\hat{\mu}_b N} Z_N(T, V)$ where $\hat{\mu}_b \equiv \mu_b / T$

The probability for the system to have N particles is:
$w_N = e^{\hat{\mu}_b N} Z_N(T, V) / \mathcal{Z}(T, V, \hat{\mu}_b)$

So the cumulant generating function is:

$$
\begin{aligned}
K(z) &= \ln < e^{zN} >= \ln \sum w_N e^{zN} \\
&= \ln \sum w_N e^{(\hat{\mu}_b + z)N} \frac{Z_N(T, V)}{\mathcal{Z}(T, V, \hat{\mu}_b)} \\
&= \ln \mathcal{Z}(T, V, \hat{\mu}_b + z) - \ln \mathcal{Z}(T, V, \hat{\mu}_b)
\end{aligned}
$$

and the cumulants are:

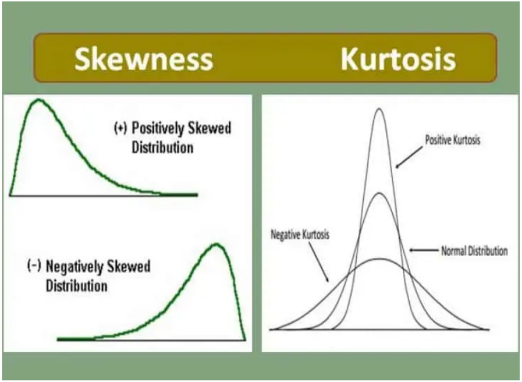
$$
\begin{aligned}
C_n &= \frac{\partial^n}{\partial z^n} K(z)|_{z=0} \\
&= \frac{\partial^n}{\partial z^n} \ln \mathcal{Z}(T, V, \hat{\mu}_b + z)|_{z=0} \\
&= \frac{\partial^n}{\partial \hat{\mu}_b^n} \ln \mathcal{Z}(T, V, \hat{\mu}_b) = VT^3 \chi_n
\end{aligned}
$$

Skewness and kurtosis

In statistics, these two quantities can be used to describe the shape of distributions and they are defined as :

$$S = \frac{<(\delta N)^3>}{<(\delta N)^2>^{3/2}} = \frac{C_3}{C_2^{3/2}} \text{ and } \kappa = \frac{<(\delta N)^4>}{<(\delta N)^2>^2} = \frac{C_4}{C_2^2}$$

Their meaning is shown in the figure below.



In order to remove the volume dependence, it is common to consider the following quantities:

$$\frac{\sigma^2}{\mu} = \frac{C_2}{C_1} = \frac{\chi_2}{\chi_1},\ S\sigma = \frac{C_3}{C_2} = \frac{\chi_3}{\chi_2} \text{ and } \kappa\sigma^2 = \frac{C_4}{C_2} = \frac{\chi_4}{\chi_2}$$

Results for some theoretical models

Hadronic Resonance Gas (HRG)

The HRG can successfully describe the observed particle abundances in heavy ion collisions (cf. lecture 14). To simplify and for illustration, we use the Boltzmann approximation and the pressure can be expressed as (lecture 5):

$\frac{p}{T^4} = \frac{1}{2\pi^2} \sum_i g_i \left(\frac{m_i}{T}\right)^2 K_2\left(\frac{m_i}{T}\right)\ e^{b_i \mu_b / T}$ (b_i=0 for mesons, 1 for baryon and -1 for anti-baryon)

The susceptibilties are easy to compute:

$\chi_{2n} = \frac{1}{2\pi^2} \sum_i (b_i)^{2n} g_i \left(\frac{m_i}{T}\right)^2 K_2\left(\frac{m_i}{T}\right)\ e^{b_i \mu_b / T}$

$\chi_{2n-1} = \frac{1}{2\pi^2} \sum_i (b_i)^{2n-1} g_i \left(\frac{m_i}{T}\right)^2 K_2\left(\frac{m_i}{T}\right)\ e^{b_i \mu_b / T}$

So for a HRG the prediction is simply:

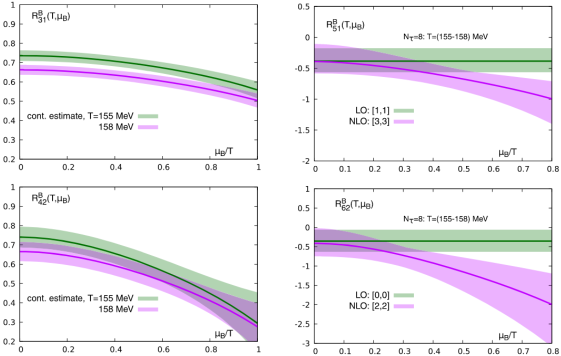
$\frac{\mu}{\sigma^2} = S\sigma = \tanh(\mu_b/T)$ and $\kappa\sigma^2 = 1$

Other versions of the HRG take into account strangeness and electric charge conservations, attractive and repulsive interactions, etc.

Lattice QCD

Lattice QCD is not easy to perform at $\mu_b \neq 0$ (cf. lecture 6), however results computed at $\mu_b = 0$ can be used to obtain those at $\mu_b \neq 0$ with a Taylor expansion, for exemple: $\frac{p(T,\mu_b)-p(T,0)}{T^4} = \sum_n C_{2n}(T)\hat{\mu}_b^{2n}$ where the $C_{2n}(T)$ are computed on the lattice.

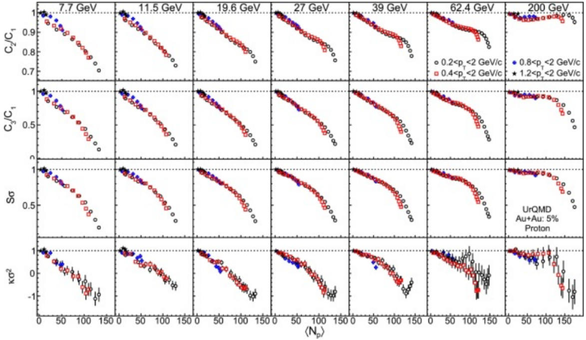
With this method, there is indication that the critical point is not in the region $\mu_b/T \leq 2$.
The susceptibilities are shown below ($R_{nm} \equiv \chi_n/\chi_m$):



HotQCD Collaboration PRD 101 (2020) 074502: $R_{31} > R_{42} > 0 > R_{51} > R_{62}$

UrQMD (Ultra-relativistic Quantum Molecular Dynamics) model

- It is a microscopic transport model to describe hadron-hadron interactions and system evolution.
- Production of particles occurs via fragmentation of strings made of valence quarks of the original colliding hadrons, resonance excitation, and decays.
- There is no quark-hadron phase transition or equation of state implemented. Neverthless cumulants can be computed from the simulated event-by-event net-proton distributions.



S.He and X.Luo PLB774 (2017) 623. In addition C_5 and C_6 >0.

Effects of the critical point

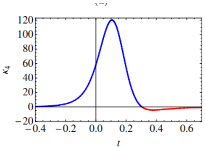
In the theory of critical phenomena

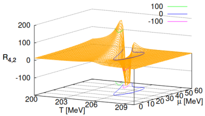
- all singular contributions to the thermodynamic quantities are powers of the correlation length ξ, which diverges at the critical point (ex. χ_n).
- These powers, or critical exponents, are universal, depending only on the degrees of freedom in the theory and the symmetry.
- Very different physical systems may belong to the same universality class, as far as their critical behavior is concerned.
- In the static case, the QCD critical point is in the class of the Ising model (because the order parameter has only one-component): $p(T, \mu_b) = p_{usual} + p_{crit}$ with $p_{crit} = p_{Ising}$
- In the dynamic case, the universality class is determined by the degrees of freedom which define the effective hydrodynamic theory near the critical point and would be the one describing the liquid-gas phase transition (model H)

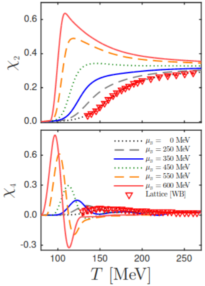
M.Stephanov arXiv:0402115 §2.4

Special features are predicted with models that incorporate the critical point, for exemple the sigma model relevant for the Ising case (Stephanov et al. arXiv:0302002, 1104.1627), the quark-meson (QM) and Polyakov-quark-meson (PQM) (Schaefer and Wagner arXiv:1111.6871), even lattice calculations (R.V. Gavai and S.Gupta PLB696 (2011) 459) and holographic models (R.Critelli et al. 1706.00455))

- Cumulants diverge as power of ξ, and the highest order cumulants diverge more.
- The kurtosis would change sign from negative for smaller μ_b to positive larger for μ_b, this is equivalent to $\sqrt{s_{NN}} \searrow$



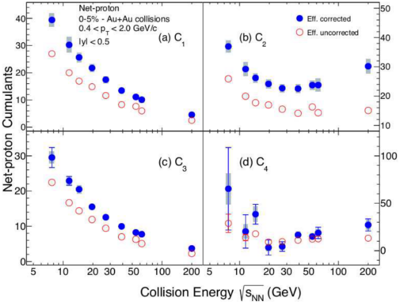




Net-proton data for RHIC and comparison with models

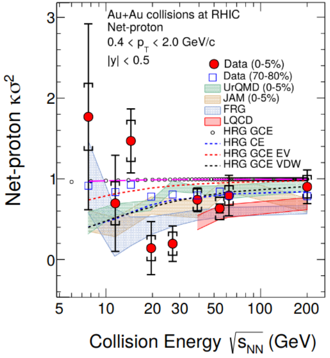
In the following, the comparisons are only indicative: 1) collective expansion is not taken into account (cf. next lecture), 2) experimental cutoffs are not included in general (for exemple in lattice QCD)

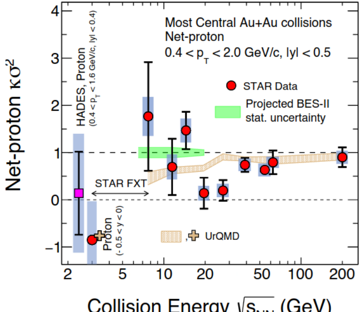
Energy dependence of Cumulants



- C_1 and C_2 increase with decreasing energy
- C_2 and C_4 have non-monotonic dependence with energy
- At large energy, $C_1 \sim C_3$ and $C_2 \sim C_4$ as expected for Skellam distributions (cf. lecture 23/homework)

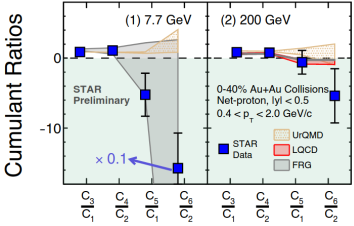
$\rightarrow$ There is some indication of a different behavior at low energy.

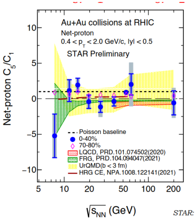




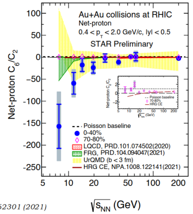
- We know kurtosis is an interesting observable
- In fact, no model reproduces the kurtosis shape at low energy
- There is evidence for something new at low energy $\sim$ 8-30 GeV
- The new fixed-target point at 3 GeV is in agreement with UrQMD (no QGP)
- Soon, additional results from BES-II will decrease the error bars.

Energy dependence of higher order cumulants





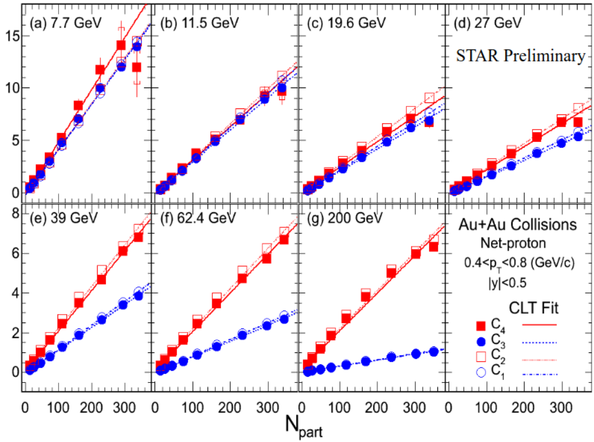
STAR: PRL 127, 262301 (2021)



- Cumulants obey hierarchy predicted by lattice QCD. UrQMD does not follow it.
- C_6 is negative and decreasing with energy as predicted by lattice QCD, the situation is not clear for C_5.

Homework

In this lecture we concentrated on the energy dependence of net-proton cumulants, however results have been obtained for their centrality dependence as well



Could one assume that the fluctations are those of a Skellam distribution?

Other references on this topic

- X.Luo and N.Xu "Search for the QCD Critical Point with Fluctuations of Conserved Quantities in Relativistic Heavy-Ion Collisions at RHIC : An Overview" Nucl. Sci. Tech. 28 (2017) 112, arXiv:1701.02105
- A. Pandava, D. Mallick, B. Mohanty, "Search for the QCD Critical Point in High Energy Nuclear Collisions" Progress in Particle and Nuclear Physics 125 (2022) 103960, arXiv:2203.07817
- B. Mohanty and N. Xu "QCD Critical Point and High Baryon Density Matter " arXiv:2101.09210