RELATIVISTIC QUANTUM MECHANICS

1. Lorentz group (classical/quantum) & Poincarè group
2. Unitary representations on particles
3. Relativistic wave eqs.
4. Quantum relativistic fields
5. Quantum electrodynamics

Part 1 - The Lorentz group (classical)

1. Notation and properties

Lorentz tr. $\quad x'^\mu = \Lambda^\mu{}_\nu x^\nu$

defined to leave the interval $s^2 \equiv x^\mu x_\mu$ invariant.

Remembering

$$x'^\mu x'_\mu = g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu{}_\alpha x^\alpha \Lambda^\nu{}_\beta x^\beta \overset{!}{=} g_{\alpha\beta} x^\alpha x^\beta$$

we conclude that

$$\boxed{g_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu}} \leftarrow \text{defining relation of Lorentz gr.}$$

Attention: I will use the "west coast" metric

$$g_{\alpha\beta} = \begin{bmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

and not the "East Coast" metric $\quad g_{\alpha\beta} = \begin{bmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix}$.

The defining relation can be written in matrix form calling

$$\{\Lambda^\mu{}_\nu\} \equiv \Lambda \quad ; \quad \{g_{\mu\nu}\} = g$$

$$\Rightarrow \quad g_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu}$$

$$g = \Lambda^T g \Lambda$$

We define the LORENTZ GROUP $SO(1,3) \equiv \{\Lambda \,/\, g = \Lambda^T g \Lambda\}$

This definition is the analog of $O^T O = \mathbb{1}$ for $O(N)$.

Properties of Λ:

(1) $\det g = \det \Lambda^T \det g \det \Lambda = \det g \, (\det \Lambda)^2$

$\Rightarrow \det \Lambda = \pm 1$

(2) From $g_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \, g_{\mu\nu}$, taking $\alpha = 0 = \beta$:

$$+1 = \Lambda^{\mu}{}_{0} \Lambda^{\nu}{}_{0} \, g_{\mu\nu} = (\Lambda^{0}{}_{0})^2 - \sum_{i=1}^{3} (\Lambda^{i}{}_{0})^2$$

$$\Rightarrow (\Lambda^{0}{}_{0})^2 = 1 + \sum_{i=1}^{3} (\Lambda^{i}{}_{0})^2 \geq 1$$

We can use these properties to classify different "sections" of $SO(1,3)$:

	$\det\Lambda = +1$	$\det\Lambda = -1$
$\Lambda^{0}{}_{0} \geq 1$	PROPER ORTHOCRONOUS LORENTZ GROUP $SO(1,3)_+$	Space inversions
$\Lambda^{0}{}_{0} \leq -1$	Time invers.	Time invers.

Transformations belonging to $SO(1,3)_+$ can be close to $\mathbb{1}$

$\Rightarrow$ it is a Lie group

2. Algebra of $SO(1,3)_+$

As all Lie groups, $SO(1,3)_+$'s properties are determined (locally) by its Lie algebra.
We take Λ close to the identity :

$$\Lambda \simeq \mathbb{1} + i\hat{\omega}$$

The Lorentz condition implies

$$g_{\alpha\beta} = (\delta^{\mu}{}_{\alpha} + i\,\hat{\omega}^{\mu}{}_{\alpha})(\delta^{\nu}{}_{\beta} + i\,\hat{\omega}^{\nu}{}_{\beta})\, g_{\mu\nu}$$

$$= g_{\alpha\beta} + i\,\underbrace{(\hat{\omega}_{\beta\alpha} + \hat{\omega}_{\alpha\beta})}_{= 0}$$

We must thus have : (i) antisymmetry $\hat{\omega}_{\beta\alpha} = -\hat{\omega}_{\alpha\beta}$

(ii) from Λ real $\Rightarrow$ $\Lambda^* = \mathbb{1} - i\hat{\omega}^* = \mathbb{1} + i\hat{\omega}$

$\Rightarrow \hat{\omega}^* = -\hat{\omega} \Rightarrow \hat{\omega}$ purely imaginary

The most general matrix satisfying these conditions is

$$\{\hat{\omega}_{\alpha\beta}\} = \begin{bmatrix} 0 & i\omega_{01} & i\omega_{02} & i\omega_{03} \\ -i\omega_{01} & 0 & i\omega_{12} & i\omega_{13} \\ -i\omega_{02} & -i\omega_{12} & 0 & i\omega_{23} \\ -i\omega_{03} & -i\omega_{13} & -i\omega_{23} & 0 \end{bmatrix}$$

with 6 independent parameters $\Rightarrow$ $SO(1,3)_+$ has 6 generators

From special relativity we know that the interval is left invariant by $\begin{cases} \text{3-dim rotations} & (3 \text{ parameters}) \\ \text{boosts} & (3 \text{ parameters}) \end{cases}$

We can identify the generators as follows:

(a) we write ω using the usual matrix notation $\omega^\alpha{}_\beta$:

$$\hat{\omega}^\alpha{}_\beta = g^{\alpha\mu}\,\hat{\omega}_{\mu\beta} = \begin{bmatrix} 0 & i\omega_{01} & i\omega_{02} & i\omega_{03} \\ i\omega_{01} & 0 & -i\omega_{12} & -i\omega_{13} \\ i\omega_{02} & i\omega_{12} & 0 & -i\omega_{23} \\ i\omega_{03} & i\omega_{13} & i\omega_{23} & 0 \end{bmatrix}$$

(b) the antisymmetric ones generate the SO(3) subgroup:

$$(J_1)^\alpha{}_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} ; \quad (J_2)^\alpha{}_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

$$(J_3)^\alpha{}_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfying $\quad [J_i, J_k] = i\,\epsilon_{ikm} J_m$

(c) Boosts act on a 4-vector X^μ as $(c=1)$

$$X^{0\prime} = X^0 + v^i X^i \quad ; \quad X^{i\prime} = X^i + v^i X^0$$

which can be summarized into

$$X^{\mu\prime} = \left(\delta^\mu{}_\alpha - i\,(\vec{v}\cdot\vec{K})^\mu{}_\alpha\right) X^\alpha$$

with

$$(K_i)^{\alpha}{}_{\beta} = \begin{array}{cc} & \begin{array}{cccc} \quad & \quad & i & \quad \end{array} \\ i & \begin{bmatrix} 0 & & i & \\ & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(entry equal to i in the ith position)

The structure constants of $SO(1,3)_+$ can be found from direct computation:

$$[J_i, J_k] = i\,\epsilon_{ikm} J_m$$

$$[J_i, K_m] = i\,\epsilon_{imn} K_n$$

$$[K_i, K_j] = -i\,\epsilon_{ijm} J_m$$

The Lie algebra becomes simpler complexifying and defining

$$J_i^{\pm} = \frac{J_i \pm i K_i}{2} \quad \Rightarrow \quad \underbrace{\begin{array}{l} [J_i^{\pm}, J_j^{\pm}] = i\,\epsilon_{ijk} J_k^{\pm} \\ [J_i^{+}, J_j^{-}] = 0 \end{array}}_{}$$

the Lie algebra of $SO(1,3)_+$ is equivalent to two independent $SU(2)$ algebras

any representation of $SO(1,3)_+$ can be univocally determined assigning two semi-integer numbers (that completely determine the $SU(2)$ representation)

Since $\vec{J} = \vec{J}^{+} + \vec{J}^{-}$, the two semi-integers immediately determine the spin content of the representation using the usual angular momentum sum rules:

given $(m_+, m_-) \leftrightarrow$ spin content is

$$|m_+ - m_-|, \ldots, m_+ + m_-$$

$SO(1,3)_+$	$(0,0)$	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	...
$SO(3)$	0	$\frac{1}{2}$	$\frac{1}{2}$	$0+1$	...

2 inequivalent spin-$\frac{1}{2}$
(LH + RH spinors)

must be spin-1

in 2-dim, the $SU(2)$ Lie algebra is given by

$$\vec{J}^{\pm} = \frac{\vec{\sigma}}{2}$$

LH spinor : $(\frac{1}{2},0) \leftrightarrow \vec{J}^{+} = \frac{\vec{\sigma}}{2}$; $\vec{J}^{-} = 0$

$$\Rightarrow \vec{J} = \frac{\vec{\sigma}}{2}, \quad \vec{K} = \frac{-i\vec{\sigma}}{2}$$

RH spinor : $(0,\frac{1}{2}) \leftrightarrow \vec{J}^{+} = 0$; $\vec{J}^{-} = \frac{\vec{\sigma}}{2}$

$$\Rightarrow \vec{J} = \frac{\vec{\sigma}}{2}, \quad \vec{K} = \frac{i\vec{\sigma}}{2}$$

3. The Poincarè group (and a better notation for the Lorentz generators)

The Poincarè group consists of

LORENTZ TR $\oplus$ SPACE-TIME TRANSLATIONS

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\alpha} + b^{\alpha}$$

This is a group because:

(•) Combination:

$$x \xmapsto{(\Lambda,a)} \Lambda x + a \xmapsto{(\Lambda',a')} \Lambda'(\Lambda x + a) + a' = \Lambda'\Lambda x + \Lambda' a + a'$$

is still a Poincarè transformation

(•) Inverse:

$$x \xmapsto{(\Lambda,a)} \Lambda x + a \xrightarrow{(\Lambda,a)^{-1} = (\Lambda',a')} x$$

if $\Lambda' = \Lambda^{-1}$ and $a' = -\Lambda^{-1} a$ we obtain the inverse.
But this is still a Poincarè transformation

$\Rightarrow \mathcal{P} = \{ (\Lambda, a) / \Lambda \in SO(1,3) \text{ \& } a \in M^4 \}$ is a group

Restricted to $SO(1,3)_+$ we obtain a Lie group.

There is a more convenient notation for the 6 generators of $SO(1,3)_+$
We write $\Lambda \simeq \mathbb{1} + i\hat{\omega}$

with $$\hat{\omega}_{\alpha\beta} = \frac{1}{2}\left(\omega_{\mu\nu} M^{\mu\nu}\right)_{\alpha\beta}$$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$, $M^{\mu\nu} = -M^{\nu\mu}$

(the antisymmetry guarantees the correct # of generators)

There is a compact way to write the generators:

$$(M^{\mu\nu})_{\alpha\beta} = i(\delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\,\delta^{\nu}_{\alpha})$$

We have the following mapping with the usual notation:

$$\underbrace{\vec{K} = \{M^{01}, M^{02}, M^{03}\}}_{K^i = M^{0i}} \quad ; \quad \underbrace{\vec{J} = \{M^{23}, M^{31}, M^{12}\}}_{J^i = \epsilon^{ijk} M^{jk}}$$

4. The Poincaré group in Quantum Mechanics

We now must seek for a unitary operator

$$\mathcal{U}(\mathbb{1} + i\hat{\omega}, \epsilon) \simeq \mathbb{1} + i\,\epsilon_{\mu} P^{\mu} + \frac{i}{2}\,\omega_{\mu\nu} J^{\mu\nu}$$

acting on the Hilbert space. The (quantum) generators of the Lorentz group are denoted by $J^{\mu\nu}$; the (quantum) generator of 4-dim translations are the components of the 4-momentum P^{μ}. The computation of the commutation relations is very tedious (see Weinberg - QFT1) and here we just show the results:

$$\begin{cases} i\,[J^{\alpha\beta}, J^{\mu\nu}] = g^{\beta\mu} J^{\alpha\nu} + g^{\alpha\nu} J^{\mu\beta} - g^{\alpha\mu} J^{\beta\nu} - g^{\beta\nu} J^{\mu\alpha} \\ i\,[P^{\alpha}, J^{\mu\nu}] = g^{\alpha\mu} P^{\nu} - g^{\alpha\nu} P^{\mu} \\ [P^{\mu}, P^{\nu}] = 0 \end{cases}$$

$\Rightarrow$ FUNDAMENTAL RESULT: the components of the 4-momentum operator form a compatible set of operators

Observe that the Hamiltonian $P^0 = H$ does not commute with the Lorentz generators, $\mathcal{U}(\Lambda,0)\, H\, \mathcal{U}^{-1}(\Lambda,0) \neq H$.
How can the Poincaré group be a quantum symmetry? When we say that $\mathcal{P}$ is a symmetry we mean that $\mathcal{U}(\Lambda)$ must commute with the S-matrix:

$$\mathcal{U}(\Lambda)\, S\, \mathcal{U}^{-1}(\Lambda) = S \quad \Rightarrow \quad [\mathcal{U}(\Lambda), S] = 0$$

2 - UNITARY REPRESENTATIONS ON PARTICLE STATES

To describe a particle state we choose to diagonalize the momentum operator $\hat{P}^\mu$:

$$\hat{P}^\mu |p,\sigma\rangle = p^\mu |p,\sigma\rangle \qquad \sigma = \text{additional quantum numbers}$$

Furthermore, we assume the eigenvalue is either timelike ($p^2 > 0$) or null ($p^2 = 0$). The only other Lorentz invariant that can be constructed using p^μ is $\mathrm{sign}(p^0)$. We will assume

$$\mathrm{sign}(p^0) = 1 \text{ on physical states.}$$

It is important to notice that any two momenta with the same values of p^2 and $\mathrm{sign}(p^0)$ are related by a Lorentz tr. Λ.

Let us analyze more in detail the representations of the translation operator

$$\underbrace{e^{-i a\cdot P}}_{\mathcal{U}(a)} |p,\sigma\rangle = e^{-i a\cdot p} |p,\sigma\rangle .$$

We now show that if a representation contains e^{-ipa} then it will also contain $e^{-ip'\cdot a}$ with $p' = \Lambda p$.

We expect

$$\mathcal{U}(a)\,\mathcal{U}(\Lambda)\,|p,\sigma\rangle = e^{-i\Lambda p\cdot a}\,\mathcal{U}(\Lambda)|p,\sigma\rangle$$

i.e. that $\mathcal{U}(\Lambda)|p,\sigma\rangle$ is a state with momentum Λp. This claim is true, as can be shown observing that

$$\begin{tikzcd}
p+\Lambda^{-1}a \arrow[r, "\Lambda"] & \Lambda p + a \\
p \arrow[u, "T_{\Lambda^{-1}a}"] \arrow[r, "\Lambda"'] & \Lambda p \arrow[u, "T_a"']
\end{tikzcd}$$

In terms of operators, this is $\mathcal{U}(\Lambda)\,\mathcal{U}(\Lambda^{-1}a) = \mathcal{U}(a)\,\mathcal{U}(\Lambda)$

Applying on $|p,\sigma\rangle$, we obtain

$$\mathcal{U}(a)\,\mathcal{U}(\Lambda)\,|p,\sigma\rangle = \mathcal{U}(\Lambda)\,\mathcal{U}(\Lambda^{-1}a)|p,\sigma\rangle$$

$$= e^{-ip\cdot\Lambda^{-1}a}\,\mathcal{U}(\Lambda)\,|p,\sigma\rangle = e^{-i\Lambda p\cdot a}\,\mathcal{U}(\Lambda)\,|p,\sigma\rangle$$

$$p\cdot(\Lambda^{-1}a) = (\Lambda p)\cdot a$$

Lorentz invariant

which is the result we wanted to establish.

Thus, if a representation contains vectors $|p,\sigma\rangle$, it also contains vectors with any momentum Λp. The vector $\mathcal{U}(\Lambda)|p,\sigma\rangle$ is one such vector.

Now, since we are assuming $\{|p,\sigma\rangle\}_\sigma$ = complete set of states, we

Conclude that

$$\boxed{\mathcal{U}(\Lambda)\,|p,\sigma\rangle = \sum_{\eta} C(p,\sigma,\eta)\,|\Lambda p,\eta\rangle} \qquad \circledast$$

As we can see, states that share the Lorentz invariants p^2 & $\mathrm{sign}(p^0)$ are all connected. We will thus select one such state ("reference momentum") for which computations become simple: $k^\mu \leftrightarrow |k,\sigma\rangle$.
We then <u>define</u>

$$\begin{array}{c} p^\mu = L_p{}^\mu{}_\nu\, k^\nu \\ \updownarrow \\ |p,\sigma\rangle = N_p\, \mathcal{U}(L_p)\,|k,\sigma\rangle \end{array} \qquad \begin{array}{l} L_p = \text{Lorentz tr. that connects} \\ \quad\ \ k \text{ to 4-momentum } p \\ \qquad (**) \end{array}$$

where N_p is a normalization factor.
Once this choice is made, we still do not have any information about the σ's. Looking at $\circledast$, we see that information about how the σ's are connected can be obtained if we can eliminate the information about the momenta, i.e. specializing to those transformations W such that $Wp = p$. These transformations form the <u>LITTLE GROUP</u>.
Let us see how we can write $(**)$ in terms of little group tr.
The first step is to observe that, given k and L, we have

$$\begin{tikzcd} k \arrow[r, bend left, "L_p"] \arrow[rr, bend right, "L_{\Lambda p}"'] & p \arrow[r, bend left, "\Lambda"] & p' \end{tikzcd}$$

$$\Lambda L_p = L_{\Lambda p} \overset{!}{=} L_{\Lambda p}\,\underbrace{(L_{\Lambda p}^{-1}\Lambda L_p)}_{W\,\in\,\text{little group of } k}$$

and thus

$$\begin{aligned}
\mathcal{U}(\Lambda)\,|p,\sigma\rangle &= N_p\ \mathcal{U}(\Lambda)\,\mathcal{U}(L_p)\,|k,\sigma\rangle \\
&\overset{!}{=} N_p\ \mathcal{U}(\Lambda L_p)\,|k,\sigma\rangle \\
&\overset{!}{=} N_p\ \mathcal{U}(L_{\Lambda p})\,\underbrace{\mathcal{U}(L^{-1}_{\Lambda p}\Lambda L_p)\,|k,\sigma\rangle}_{\text{apply } \circledast} \\
&\overset{!}{=} N_p\ \mathcal{U}(L_{\Lambda p})\sum_\eta C(k)_{\sigma\eta}\,|k,\eta\rangle \\
&\overset{!}{=} N_p \sum_\eta C(k)_{\sigma\eta}\,|\Lambda p,\eta\rangle
\end{aligned}$$

once these coefficients are known, the unitary representation of the Lorentz group is completely determined.

But, as we mentioned earlier, the coefficients C are determined specializing $\circledast$ to the little group:

$$\mathcal{U}(W)\,|k,\sigma\rangle = \sum_\eta C(k)_{\sigma\eta}\,|k,\eta\rangle$$

→ matrix representation of the little group transformation

The little group depends on whether the particle is massive or massless, as we are now going to see.

2.1 - CASIMIR OPERATORS OF THE POINCARÈ GROUP

Algebra of Poincaré → see pg. 8

Explicit computation shows that

$P^2 = P_\mu P^\mu$ is A CASIMIR OPERATOR

$$\Rightarrow \quad P^2 |p,\sigma\rangle = p^2 |p,\sigma\rangle = m^2 |p,\sigma\rangle$$

⇒ any irreducible repr. of Poincaré has fixed mass (which makes sense, since a Poincaré tr. does not change the mass of a particle)

There is a second Casimir → obtained from the little group

$$WK = K \quad \Rightarrow \quad (\delta^\alpha_\beta + i\hat{\omega}^\alpha{}_\beta) k^\beta = k^\alpha$$

$$k^\alpha + i\hat{\omega}^\alpha{}_\beta k^\beta = k^\alpha \quad \Rightarrow \quad \boxed{\hat{\omega}^\alpha{}_\beta k^\beta = 0}$$

but $\quad \hat{\omega}_{\alpha\beta} = \frac{1}{2}(\omega_{\mu\nu} M^{\mu\nu})_{\alpha\beta}$

$$(M^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)$$

$$\omega_{\mu\nu}(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha) k^\beta = 0$$

$$\Downarrow$$

$$\omega_{\alpha\beta} k^\beta = 0$$

→ solved by $\quad \omega_{\alpha\beta} = \epsilon_{\alpha\beta\mu\nu} k^\mu h^\nu$

↳ arbitrary

At the quantum level :

$$\mathcal{U}(w) = e^{\frac{i}{2} w_{\mu\nu} J^{\mu\nu}} = e^{\frac{i}{2} \epsilon_{\mu\nu\alpha\beta} k^{\alpha} h^{\beta} J^{\mu\nu}}$$

W_{β} = PAULI – LUBANSKI vector

Properties of W_{α} :

(1) $[W^{\alpha}, P^{\beta}] = 0 \qquad \forall \alpha, \beta$

(2) $W^2 \equiv W_{\alpha} W^{\alpha}$ commutes with all generators of Poincaré

$\Downarrow$

<u>W^2 is the second Casimir</u>

we'll see its physical meaning in the next two sections

2.1 - MASSIVE PARTICLES

For massive particles we can chose $k^\mu = \begin{pmatrix} m \\ \vec{0} \end{pmatrix}$ [rest frame]

in such a way that

$$\text{little group} = SO(3)$$

Representations of $SO(3)$ $\to$ determined by semi-integer s,

dimension $2s+1$

Index σ in $|k,\sigma\rangle$ $\Rightarrow$ takes values $-s \leq \sigma \leq s$

and $\quad C_{\sigma\eta} = D^{(2s+1)}_{\sigma\eta} =$ $(2s+1)$-dim representation of $SO(3)$,

computed as usual (see SYMMETRIES)

Under a Lorentz tr. we thus have

$$\mathcal{U}(\Lambda)|p,\sigma\rangle = N_p \sum_\eta D^{(2s+1)}_{\sigma\eta} |\Lambda p, \eta\rangle$$

$\hookrightarrow$ $(2s+1)$-dim matrix

Pauli-Lubanski:

$$W_\alpha = \frac{1}{2}\epsilon_{\mu\nu\beta\alpha} \underset{\downarrow\ m\,\delta^{\beta 0}}{k^\beta} J^{\mu\nu} = \frac{m}{2} \underset{\downarrow}{\epsilon_{\mu\nu 0\alpha}} J^{\mu\nu}$$

indices μ,ν,α must be spatial

$$= \frac{m}{2}\epsilon_{ij0k} J^{ij}$$

$$= -\frac{m}{2} \underbrace{\epsilon_{ijk0}}_{\text{behaves like } \epsilon_{ijk}} J^{ij} = -m S_k \qquad S_k = \text{spin}$$

$\Rightarrow$ Casimir $W^2 = m^2 \vec{S}^2$

$\Rightarrow$ we need mass + spin to classify massive particles

2.2 - MASSLESS PARTICLES

Things are more complicated.

Since there is no rest frame $\rightarrow$ choose $k^\mu = \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix}$

Generators little group $\rightarrow$ from explicit computation

$$A = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{array}\right]$$

$$B = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$J_3 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

with algebra

$$\left.\begin{aligned} &[A, B] = 0 \\ &[J_3, A] = iB \\ &[J_3, B] = iA \end{aligned}\right\} \text{ Euclidian group in } d=2$$

A, B compatible $\rightarrow$ common eigenvectors $|a, b\rangle$

$$\begin{cases} A|a,b\rangle = a\,|a,b\rangle \\ B|a,b\rangle = b\,|a,b\rangle \end{cases}$$

Important relations:

$$\mathcal{U}(\theta)A\,\mathcal{U}^{-1}(\theta) = \left(1 + i\,\theta J_3 - \frac{\theta^2}{2}J_3^2 + \dots\right) A \left(1 - i\,\theta J_3 - \frac{\theta^2}{2}J_3^2 + \dots\right)$$

$=$ explicit computation using the algebra

$$= \cos\theta\, A - \sin\theta\, B$$

$$\mathcal{U}(\theta)B\,\mathcal{U}^{-1}(\theta) = \sin\theta\, A + \cos\theta\, B$$

$\Rightarrow$ for arbitrary θ, we can define

$$|a,b\rangle_\theta \equiv \mathcal{U}^{-1}(\theta)\,|a,b\rangle$$

so that

$$\begin{cases} A|a,b\rangle_\theta = (a\cos\theta - b\sin\theta)\,|a,b\rangle_\theta \\ B|a,b\rangle_\theta = (a\sin\theta + b\cos\theta)\,|a,b\rangle_\theta \end{cases}$$

Physical interpretation: if $a,b \neq 0$ we have a <u>continuum of states</u> $|a,b\rangle_\theta$

$\Rightarrow$ we should observe a continuum quantum number for the photon (only known massless particle)

Since no continuum quantum number observed

$\Rightarrow$ we need to admit $\boxed{a = 0 = b}$

$\Downarrow$

J_3 = only relevant generator

$\Downarrow$

the "physical" little group is $SO(2)$
[generated by J_3]

But $SO(2) \subset SO(3) \subset SO(1,3)_+ \Rightarrow$ eigenvalues quantized
$\frac{n}{2}$, $n \in \mathbb{Z}$

What is the Pauli-Lubanski?

$$W_\alpha = \frac{1}{2} \epsilon_{\mu\nu\beta\alpha} k^\beta J^{\mu\nu}$$
$$= -\epsilon_{12\beta\alpha} k^\beta J^3 = \begin{bmatrix} -E J^3 \\ 0 \\ 0 \\ E J^3 \end{bmatrix}$$

Since $k = \begin{bmatrix} E \\ 0 \\ 0 \\ E \end{bmatrix}$

$\Rightarrow W_\alpha = \begin{bmatrix} -\vec{k} \cdot \vec{J} \\ 0 \\ 0 \\ \vec{k} \cdot \vec{J} \end{bmatrix}$ $\Rightarrow$ little group selects as useful quantum number the projection of angular momentum along $\vec{k}$

Consequences :

$$\underbrace{k^2 = 0 \quad ; \quad kW = 0 \quad ; \quad W^2 = 0}$$

$$\Downarrow$$

necessarily $\quad W^\alpha = h\, k^\alpha$

for some h. Which one?

$$W^\alpha = \begin{bmatrix} -\vec{k}\cdot\vec{J} \\ 0 \\ 0 \\ -\vec{k}\cdot\vec{J} \end{bmatrix} = -\frac{\vec{k}\cdot\vec{J}}{|\vec{k}|} \begin{bmatrix} |\vec{k}| \\ 0 \\ 0 \\ |\vec{k}| \end{bmatrix} = \underbrace{-\frac{\vec{k}\cdot\vec{J}}{|\vec{k}|}}_{h \,=\, \text{HELICITY}} k^\alpha$$

Since $\quad [W^\alpha, P^\beta] = 0 \qquad$ COMPATIBLE OBSERVABLES

$\Longrightarrow$ we label the states as $|k, h\rangle$

with

$$\begin{cases} P^\mu |k, h\rangle = k^\mu |k, h\rangle \\ W^\mu |k, h\rangle = h\, k^\mu |k, h\rangle \end{cases}$$

PART 3 : RELATIVISTIC WAVE EQUATIONS

We now use Wigner's classification:

to describe particles of mass m & spin s we will combine representations of the Lorentz group that contain the chosen spin.

How? Since we seek for wave equations, we will allow

① for the operator $P_\mu = i\,\partial_\mu = \begin{pmatrix} i\partial_t \\ i\vec{\nabla} \end{pmatrix}$ to appear

② we will demand for Lorentz covariance (i.e. all observers agree on the form of the eq.)

③ we will allow for a maximum of 2 time derivatives (= at most 2 operators P_μ can appear in the wave eq.) to avoid Ostrogradsky instability (energy is unbounded from below if more than 2 time derivatives appear)

3.1 - SPIN 0

Contained in

scalar	$\phi(x)$	$\leftrightarrow$	$(0,0)$
vector	$\phi^\mu(x)$	$\leftrightarrow$	$(\frac{1}{2},\frac{1}{2})$ $[\phi^0(x) = \text{scalar}]$
tensor	$\phi^{\mu\nu}(x)$	$\leftrightarrow$	$(1,0)$ or $(0,1)$ $[\phi^{00}(x) = \text{scalar}]$

→ to satisfy point ③ above, we consider only

$$\phi(x) \quad \& \quad \phi^\mu(x)$$

Only covariant terms that can be formed:

$$\begin{cases} P^\mu \phi_\mu = m\phi \\ P^\mu \phi = m\phi^\mu \end{cases}$$

the two constants can always be taken equal with wave functions redefinition

$$\Rightarrow \quad P^\mu P_\mu \phi = m P_\mu \phi^\mu = m^2 \phi$$

$$\Downarrow$$

$$(P^2 - m^2)\phi(x) = 0$$

$$\Updownarrow$$

$$(\Box + m^2)\phi(x) = 0 \qquad \text{KLEIN-GORDON EQ.}$$

Once $\phi(x)$ is known, we use

$$\phi^\mu = \frac{1}{m} P^\mu \phi = \frac{i\partial^\mu \phi}{m} \quad \text{to compute } \phi^\mu(x)$$

$\Rightarrow$ ϕ^μ not independent from ϕ

SOLUTIONS TO THE KLEIN-GORDON EQ.

Since KG = wave eq. $\Rightarrow$ plane waves = complete set of solutions

$\forall$ 3-momentum $\vec{k}$ we can write

$$\phi_k(t,\vec{x}) \equiv e^{+i\vec{k}\vec{x}} u_k(t)$$

Using this in KG $\longrightarrow$ $\left(\partial_t^2 - \vec{\nabla}^2 + m^2\right) \phi_k(t,\vec{x}) = 0$

$$e^{i\vec{k}\cdot\vec{x}} u_k''(t) + e^{i\vec{k}\cdot\vec{x}} \vec{k}^2 u_k(t) + m^2 e^{i\vec{k}\cdot\vec{x}} u_k(t) = 0$$

$$\Downarrow$$

$$u_k''(t) + \underbrace{\left(\vec{k}^2 + m^2\right)}_{\text{relativistic energy } E^2} u_k(t) = 0$$

$$\downarrow$$

solutions : $u_k(t) = e^{\pm iEt}$

$$\Downarrow$$

$$\phi_k^{\pm}(t,\vec{x}) = e^{\pm iEt + i\vec{k}\cdot\vec{x}}$$

Physical interpretation : Hamiltonian is $H = P^0 = i\partial_t$

$$\Rightarrow H\phi_k^{\pm} = i\partial_t \phi_k^{\pm} = \mp E \phi_k^{\pm}$$

$\Rightarrow$ ϕ^- has positive energy, ϕ^+ has negative energy

We will discuss later the meaning of this.

Most general solution of the KG eq.

$$\phi(t,\vec{x}) = \int d^3k \left[\varphi_k^+ \, e^{+i(Et+\vec{k}\cdot\vec{x})} + \varphi_k^- \, e^{-i(Et-\vec{k}\cdot\vec{x})} \right]$$

change $\vec{k} \to -\vec{k}$

$$= \int d^3k \left[\varphi_k^+ \, e^{i(Et-\vec{k}\cdot\vec{x})} + \varphi_k^- \, e^{-i(Et-\vec{k}\cdot\vec{x})} \right]$$

$Et - \vec{k}\cdot\vec{x} = k\cdot x$ with $k^\mu = \begin{pmatrix} E \\ \vec{k} \end{pmatrix}$; $x^\mu = \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$

$$= \int d^3k \left[\varphi_k^+ \, e^{ikx} + \varphi_k^- \, e^{-ikx} \right]$$

We'll see later that all this does not make sense interpreting $\phi(x)$ as (relativistic) wave function.

3.2 - SPIN 1/2

According to Wigner, $S=\frac{1}{2}$ contained in

$$\left(\tfrac{1}{2},0\right) \quad \& \quad \left(0,\tfrac{1}{2}\right)$$

To distinguish between the two spinorial representations, we use indices

$$\left(\tfrac{1}{2},0\right) \leftrightarrow \xi^{a} \qquad a=1,2$$

$$\left(0,\tfrac{1}{2}\right) \leftrightarrow \bar{\chi}^{\dot{a}} \qquad \dot{a}=1,2$$

How can P^{μ} contract spinorial indices?

Detour: from spinors to 4-vectors [Borut]

Studying rotations we saw that $\vec{k} \leftrightarrow \hat{k} = \vec{k}\cdot\vec{\sigma}$

Rotation implemented as $\hat{k} \rightarrow \mathcal{U}(R)\,\hat{k}\,\mathcal{U}^{\dagger}(R)$

with $\mathcal{U}(R) \in SU(2)$

Here we do something similar:

$$V^{\mu} \rightarrow \hat{V} = V^{\mu}\sigma_{\mu} \qquad \sigma_{\mu} = (\mathbb{1}, \vec{\sigma})$$

$$= \begin{pmatrix} V_0+V_z & V_x-iV_y \\ V_x+iV_y & V_0-V_z \end{pmatrix}$$

Properties: (1) $\hat{V}$ hermitian

(2) $\det \hat{V} = V^{0^2} - \vec{V}^2 = V^2$

If we apply $\hat{V} \to M V M^\dagger$

(1) automatically satisfied

(2) $\det(M \hat{V} M^\dagger) = \det M \, \det M^\dagger \, \det \hat{V}$

$\to$ requiring $\det M = 1 = \det M^\dagger$

allows to identify $\hat{V} \to M \hat{V} M^\dagger$ as a Lorentz transf.

A matrix M such that $\det M = 1$ is a unimodular matrix $M \in SL(2, \mathbb{C})$

Explicitly

$$\hat{V}' = V'^{\mu} \sigma_\mu = \begin{cases} \Lambda^\mu{}_\nu V^\nu \sigma_\mu \\ V^\nu M \sigma_\nu M^\dagger \end{cases}$$

$$\Rightarrow \boxed{M \sigma_\nu M^\dagger = \Lambda^\mu{}_\nu \sigma_\mu}$$

As for $SO(3) \leftrightarrow SU(2)$, the correspondence is 2:1

$$\Lambda \in SO(1,3) \to \pm M \in SL(2, \mathbb{C})$$

Since M is 2×2 $\rightarrow$ must be one of the spinorial representations.

But which one between $\left(\frac{1}{2},0\right)$ & $\left(0,\frac{1}{2}\right)$?

Remembering that

LH spinor : $\left(\frac{1}{2},0\right) \leftrightarrow \vec{J}^{+} = \frac{\vec{\sigma}}{2}\,;\ \vec{J}^{-}=0$

$$\Rightarrow \vec{J} = \frac{\vec{\sigma}}{2}\,,\quad \vec{K} = -i\frac{\vec{\sigma}}{2}$$

RH spinor : $\left(0,\frac{1}{2}\right) \leftrightarrow \vec{J}^{+} = 0\,;\ \vec{J}^{-}=\frac{\vec{\sigma}}{2}$

$$\Rightarrow \vec{J} = \frac{\vec{\sigma}}{2}\,,\quad \vec{K} = i\frac{\vec{\sigma}}{2}$$

we have

det = 1   identify with

$$\psi_L \longrightarrow \exp\left[i\left(\vec{\alpha}-i\vec{\beta}\right)\frac{\vec{\sigma}}{2}\right]\psi_L \equiv M\psi_L$$

$$\psi_R \longrightarrow \exp\left[i\left(\vec{\alpha}+i\vec{\beta}\right)\frac{\vec{\sigma}}{2}\right]\psi_R = \sigma_2 M^{*}\sigma_2\,\psi_R$$

$$\Downarrow$$

$$\sigma_2\psi_R \longrightarrow M^{*}\sigma_2\psi_R$$

We can always identify

LH spinor $\psi_L \equiv \xi_a$, $\quad \xi_a \rightarrow M_a{}^{b}\,\xi_b$

RH spinor $\sigma_2\psi_R \equiv \bar{\chi}_{\dot a}$, $\quad \bar{\chi}_{\dot a} \rightarrow M^{*}{}_{\dot a}{}^{\dot b}\,\bar{\chi}_{\dot b}$

But then

$$M \sigma_\nu M^\dagger = \Lambda^\mu{}_\nu \sigma_\mu$$

$$\Downarrow$$

$$\Lambda^\mu{}_\alpha (\sigma_\mu)_{a\dot{a}} = M_a{}^b \, M^*_{\dot{a}}{}^{\dot{b}} (\sigma_\nu)_{b\dot{b}}$$

this is indeed $\left(\frac{1}{2}, \frac{1}{2}\right)$ because it carries one undotted & one dotted index

Observe now that, defining
$$\begin{cases} \epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{ab} \\ \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\dot{a}\dot{b}} \end{cases}$$

we have $\epsilon^{ab} \xi_a \xi'_b \to \underbrace{\epsilon^{ab} M_a{}^c M_b{}^d}_{\det M \, \epsilon^{cd}} \xi_c \xi'_d = \epsilon^{cd} \xi_c \xi'_d$

+ same for dotted indices

$\Rightarrow$ ϵ works like a metric tensor

[analogy with $g_{\mu\nu} x^\mu x^\nu$ = invariant]

$\Rightarrow$ can be used to raise/lower indices!

Important to remember that

$$\epsilon_{ab}\,\epsilon^{bc} = \delta_a{}^c \quad \Rightarrow \quad \epsilon_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{ab}$$

+ analogous for dotted indices

Message: we can apply P^μ on spinors contracting in two ways:

$$P^\mu \to \hat{P}_{a\dot{a}} = P^\mu (\sigma_\mu)_{a\dot{a}}$$

$$P^\mu \to \hat{p}^{\dot{a}a} = P^\mu \underbrace{\epsilon^{\dot{a}\dot{b}}\,\epsilon^{ab}(\sigma_\mu)_{b\dot{b}}}_{\equiv\ (\bar{\sigma}_\mu)^{\dot{a}a} = (\mathbb{1}, -\vec{\sigma})}$$

But then the covariant equations are

$$\begin{cases} \hat{P}_{a\dot{a}}\,\bar{\chi}^{\dot{a}} = m\,\xi_a \\ \hat{p}^{\dot{a}a}\,\xi_a = m\,\bar{\chi}^{\dot{a}} \end{cases}$$

Using one in the other:

$$\hat{P}_{a\dot{a}}\left(\frac{\hat{p}^{\dot{a}b}\,\xi_b}{m}\right) = m\,\xi_a \quad \Rightarrow \quad \hat{P}_{a\dot{a}}\,\hat{p}^{\dot{a}b}\,\xi_b = m^2\,\xi_a$$

Use now

$$\begin{cases} \hat{P}_{a\dot{a}} = P^\mu (\sigma_\mu)_{a\dot{a}} = \begin{bmatrix} P^0 + P^3 & P^1 - iP^2 \\ P^1 + iP^2 & P^0 - P^3 \end{bmatrix} \\ \hat{P}^{\dot{a}a} = P^\mu (\bar{\sigma}_\mu)^{\dot{a}a} = \begin{bmatrix} P^0 - P^3 & -P^1 + iP^2 \\ -P^1 - iP^2 & P^0 + P^3 \end{bmatrix} \end{cases}$$

$$\Longrightarrow \quad \hat{P}_{a\dot{a}} \, \hat{P}^{\dot{a}b} = P^2 \, \delta_a{}^b$$

Analogously $\rightarrow$ $\hat{P}^{\dot{a}a} \, \hat{P}_{a\dot{b}} = P^2 \, \delta^{\dot{a}}{}_{\dot{b}}$

Then

$$\underbrace{(P^2 - m^2)\, \xi_a = 0 \qquad + \qquad (P^2 - m^2)\, \bar{\chi}^{\dot{a}} = 0}_{\text{both spinors satisfy the KG equation}}$$

$\boxed{\text{DIRAC EQUATION}}$

Equivalent and more compact way to present the same physics:

$$\Psi \equiv \begin{bmatrix} \xi_a \\ \bar{\chi}^{\dot{a}} \end{bmatrix}$$

Dirac spinor (or 4-spinor)

$\downarrow$

by construction in the $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$

Then

$$\left[\begin{array}{c|c} -m & i\sigma^\mu\partial_\mu \\ \hline i\bar{\sigma}^\mu\partial_\mu & -m \end{array}\right]\psi = 0$$

$$\Big\{\underbrace{\left[\begin{array}{c|c} 0 & \sigma^\mu \\ \hline \bar{\sigma}^\mu & 0 \end{array}\right]}_{\gamma^\mu}\, i\partial_\mu - m\Big\}\psi = 0$$

γ^μ ("gamma matrices"), Feynman "slash" notation:

$$\not{a} = a_\mu\gamma^\mu = a^\mu\gamma_\mu$$

$$\Downarrow$$

$$(i\not{\partial} - m)\psi = 0 \qquad \text{DIRAC EQUATION}$$

PROPERTIES OF GAMMA MATRICES

(1) $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ [CLIFFORD ALGEBRA]

(2) $\mu = 0 = \nu \Rightarrow (\gamma^0)^2 = \mathbb{1}$

(3) $\mu = i = \nu \Rightarrow (\gamma^i)^2 = -\mathbb{1}$

(4) Hamiltonian form of the Dirac eq.

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \longrightarrow (i\gamma^0\partial_t + i\vec{\gamma}\cdot\vec{\nabla} - m)\psi = 0$$

$\rightarrow \quad i\gamma^0 \partial_t \psi = (-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi$

$\rightarrow$ multiply by γ^0 on the left :

$$i\partial_t \psi = \Big(\underbrace{-i\gamma^0\vec{\gamma}\cdot\vec{\nabla}}_{\equiv\, \vec{\alpha}\cdot\vec{p}} + \underbrace{\gamma^0 m}_{\equiv\, \beta}\Big)\psi \qquad \text{(historical notation)}$$

$\Rightarrow \quad H = \vec{\alpha}\cdot\vec{p} + \beta$

Is this hermitian? (•) From $\gamma^{0\dagger} = \gamma^0 \rightarrow \beta^\dagger = \beta$

(•) From $\gamma^0 \vec{\gamma}^\dagger \gamma^0 = \vec{\gamma} \rightarrow \vec{\alpha}^\dagger = \vec{\alpha}$

$\Downarrow$

$H^\dagger = H$ as wanted

(5) Notice that $\quad (i\not{\partial} + m)(i\not{\partial} - m)\psi = (-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi$

$$= -\left(\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu + m^2\right)\psi$$

$$= -(\Box + m^2)\psi = 0$$

$\Rightarrow$ we recover that solutions of the Dirac eq. satisfy the KG eq.

(6) <u>Helicity & chirality</u> : returning to the $\psi = \begin{pmatrix}\xi \\ \bar{\chi}\end{pmatrix}$ notation, in momentum space we have

$$\sigma^\mu p_\mu \bar{\chi} = (E - \vec{\sigma}\cdot\vec{p})\bar{\chi} = m\xi \,, \quad \bar{\sigma}^\mu p_\mu \xi = (E + \vec{\sigma}\cdot\vec{p})\xi = m\bar{\chi}$$

Obs 1: for $m \neq 0$ we need both ξ & $\bar{\chi}$ to describe a $S=1/2$ particle $\Rightarrow$ we need both chiralities

Obs 2: for $m=0$, $E=|\vec{p}|$ and

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\bar{\chi} = \bar{\chi} \quad , \quad \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\xi = -\xi$$

$\Longrightarrow$ in terms of helicity $h = \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|} = \frac{1}{2}\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$

$$h\bar{\chi} = \frac{1}{2}\bar{\chi} \quad ; \quad h\xi = -\frac{1}{2}\xi$$

[We confirm that in the massless limit the good quantum number is the helicity]

SOLUTIONS OF THE DIRAC EQUATION

Since ψ satisfies the KG eq., we will have both positive & negative energy solutions:

$$\psi_p^+ = u_p\, e^{-ipx} \qquad\qquad \psi_p^- = v_p\, e^{ipx}$$

$$\downarrow \qquad\qquad \downarrow$$

$$(i\cancel{\partial}-m)\psi_p^+ = 0 \qquad\qquad (i\cancel{\partial}-m)\psi_p^- = 0$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(\cancel{p}-m)u_p = 0 \qquad\qquad (\cancel{p}+m)v_p = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\left[\begin{array}{c|c} -m & \sigma\cdot p \\ \hline \bar{\sigma}\cdot p & -m \end{array}\right] u_p = 0 \qquad \left[\begin{array}{c|c} m & \sigma\cdot p \\ \hline \bar{\sigma}\cdot p & m \end{array}\right] v_p = 0$$

To find solutions, go to the rest frame $p^\mu = (m, \vec{0})$:

$$\left[\begin{array}{c|c} -m & m \\ \hline m & -m \end{array}\right] u_p = 0 \qquad \left[\begin{array}{c|c} m & m \\ \hline m & m \end{array}\right] v_p = 0$$

Solutions are constant and of the form

$$u_p = \begin{bmatrix} a \\ a \end{bmatrix} \qquad v_p = \begin{bmatrix} b \\ -b \end{bmatrix}$$

with a, b = any constant 2-component spinor.

Convenient choice to have linearly independent spinors:

$$\underbrace{u_\uparrow = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ u_\downarrow = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad v_\uparrow = \begin{pmatrix} -1 \\ 0 \\ +1 \\ 0 \end{pmatrix}; \ v_\downarrow = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}}_{\substack{\text{will be denoted with} \\ u_s(p),\ v_s(p) \\ s = \uparrow, \downarrow}}$$

with $a_\uparrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -b_\uparrow$

$a_\downarrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b_\downarrow$

To derive a more general expression, solve now for $p^\mu = (E, 0, 0, p_z)$.

Observe that $\quad m^2 = E^2 - p_z^2 = \underbrace{(E+p_z)}_{\beta^2}\underbrace{(E-p_z)}_{\alpha^2}$

$$\Rightarrow \quad p\cdot\sigma = \begin{bmatrix} E-p_z & 0 \\ 0 & E+p_z \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix}$$

$$p\cdot\bar{\sigma} = \begin{bmatrix} E+p_z & 0 \\ 0 & E-p_z \end{bmatrix} = \begin{bmatrix} \beta^2 & 0 \\ 0 & \alpha^2 \end{bmatrix}$$

and the equations we have to solve are

$$\left[\begin{array}{cc|cc} -\alpha\beta & 0 & \alpha^2 & 0 \\ 0 & -\alpha\beta & 0 & \beta^2 \\ \hline \beta^2 & 0 & -\alpha\beta & 0 \\ 0 & \alpha^2 & 0 & -\alpha\beta \end{array}\right] u_p = 0$$

$$\Rightarrow \text{ solution: } u_s(p) = \begin{bmatrix} \alpha\,\xi_1 \\ \beta\,\xi_2 \\ \beta\,\xi_1 \\ \alpha\,\xi_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \sqrt{E-p_z} & 0 \\ 0 & \sqrt{E+p_z} \end{bmatrix} a_s \\ \begin{bmatrix} \sqrt{E+p_z} & 0 \\ 0 & \sqrt{E-p_z} \end{bmatrix} a_s \end{bmatrix} = \begin{bmatrix} \sqrt{p\cdot\sigma}\; a_s \\ \sqrt{p\cdot\bar{\sigma}}\; a_s \end{bmatrix}$$

In the same way we obtain

$$v_s(p) = \begin{bmatrix} \sqrt{p\cdot\sigma}\; b \\ -\sqrt{p\cdot\bar{\sigma}}\; b \end{bmatrix}$$

IMPORTANT OBSERVATION

We have found $\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}$, but <u>ANY</u> matrix satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ is a possible gamma matrix.

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} \qquad \text{"Weyl representation"}$$

$$\gamma^0 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}, \quad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix} \qquad \text{"Dirac representation"}$$

3.3 - WHAT GOES WRONG WITH THE RELATIVISTIC WAVE EQUATIONS FOR S=0 & S=1/2 ?

$$(\Box + m^2)\,\phi(x) = 0 \qquad\qquad (i\not{\partial} - m)\,\psi(x) = 0$$

Do not make sense as single-particle relativistic equations.

Why ?

(•) KLEIN-GORDON

(a) what to make of negative energies ?

(b) For Schrödinger, we can construct a <u>conserved</u> probability current :

$$\begin{cases} i\hbar \dfrac{\partial \psi}{\partial t} = \left(-\dfrac{\hbar^2 \nabla^2}{2m} + V\right)\psi \\[2ex] -i\hbar \dfrac{\partial \psi^*}{\partial t} = \left(-\dfrac{\hbar^2 \nabla^2}{2m} + V\right)\psi^* \end{cases}$$

$$\Rightarrow\; i\hbar\, \psi^* \frac{\partial \psi}{\partial t} + i\hbar \frac{\partial \psi^*}{\partial t}\psi = \psi^*\left(-\frac{\hbar^2\nabla^2}{2m}\psi\right) + \cancel{V\psi^*\psi}$$

$$+\left(\frac{\hbar^2\nabla^2}{2m}\psi^*\right)\psi - \cancel{V\psi^*\psi}$$

$$i\hbar \frac{\partial}{\partial t}(\psi^*\psi) = -\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \vec{\nabla}^2\psi^*\,\psi\right)$$

$$\frac{\partial}{\partial t}(\psi^*\psi) = \frac{i\hbar}{2m}\,\vec{\nabla}\cdot\left(\psi^*\vec{\nabla}\psi - \vec{\nabla}\psi^*\,\psi\right)$$

$\Rightarrow$ of the form $\dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

For Klein-Gordon things are not so smooth :

$$\frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi + m^2 \phi$$

$$\downarrow$$

$$\phi^* \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi^*}{\partial t^2} \phi = \phi^* \nabla^2 \phi + \cancel{m^2 \phi^* \phi} - \nabla^2 \phi^* \phi - \cancel{m^2 \phi^* \phi}$$

$$= \underbrace{\vec{\nabla} \cdot \left(\phi^* \vec{\nabla} \phi - \vec{\nabla} \phi^* \phi \right)}_{\text{we obtained } \vec{\nabla} \cdot \vec{J}}$$

What to make of the LHS?

$$\phi^* \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi^*}{\partial t^2} \phi = \frac{\partial}{\partial t} \underbrace{\left(\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t} \phi \right)}$$

Cannot be a probability density because it is not positive definite (prob. cannot be negative)

(•) Dirac :

(a) What to make of negative energies?

(b) What about the probability current?

From $\quad i \gamma^\mu \partial_\mu \psi = m \psi$

$$\Rightarrow -i \partial_\mu \psi^\dagger \gamma^0 \gamma^\mu \gamma^0 = m \psi^\dagger$$

$$i \partial_\mu \psi^\dagger \gamma^0 \gamma^\mu = -m \psi^\dagger \gamma^0$$

define $\overline{\Psi} \equiv \Psi^{\dagger}\gamma^0 \Rightarrow i\partial_\mu \overline{\Psi}\gamma^\mu = -m\overline{\Psi}$

But then

$$i\overline{\Psi}\gamma^\mu\partial_\mu\Psi + i\partial_\mu\overline{\Psi}\gamma^\mu\Psi = m\overline{\Psi}\Psi - m\overline{\Psi}\Psi = 0$$

$$i\partial_\mu \overline{\Psi}\gamma^\mu\Psi = 0$$

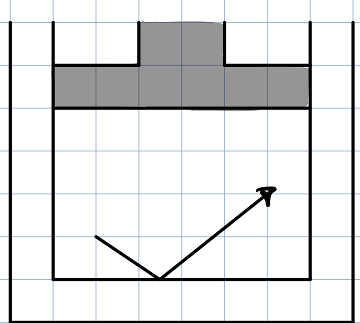
$\Rightarrow$ conserved current is $J^\mu \equiv \overline{\Psi}\gamma^\mu\Psi$

$\Rightarrow \rho = J^0 = \overline{\Psi}\gamma^0\Psi = \Psi^\dagger\gamma^0\gamma^0\Psi = \Psi^\dagger\Psi$

positive definite!

Message: for spin 1/2 we can have a (positive definite) probability density, while for spin 0 apparently we cannot. In both cases, we don't know what to make of the negative energies.

To understand how to move on, let's discuss a "gedanken" experiment by Niels Bohr.



Particle in a box with moveable top.

When I push down the top I localize better and better the particle

$\Rightarrow$ but $\Delta p\, L \gtrsim 1$ (natural units)

When $L \sim \frac{1}{m}$, we have $\Delta p \gtrsim m \Rightarrow \Delta E \gtrsim m$

and there is sufficient energy to create pairs of particles

$\Longrightarrow$ THE ONLY WAY TO MARRY QM WITH RELATIVITY IS TO ADMIT THAT THE THEORY MUST DESCRIBE THE CREATION OF PARTICLES $\Rightarrow$ WE NEED QUANTUM FIELDS

4. RELATIVISTIC QFT

Idea : introduce quantum fields as we did in NRQM

$$A_n(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \, u_n(p,\sigma) \, a_{p\sigma} \, e^{-ipx} \qquad \circledast$$

some appropriate Lorentz index

spin component

with $A_n(x)$ satisfying the wave eqs. we have derived (as the Schrödinger field satisfies Schrödinger eq.)

Expression $\circledast$ is, however, WRONG.

Why ?

(1) We still did not use the negative energy solution;

(2) We saw that relativistic invariance $\Leftrightarrow [\mathcal{U}(\Lambda), S] = 0$

But $$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dt_1 \ldots dt_n \, T\left[V_I(t_1) \cdots V_I(t_n)\right]$$

in a QFT

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n \, T\left[\mathcal{H}(x_1) \cdots \mathcal{H}(x_n)\right]$$

Sufficient conditions for $[S, \mathcal{U}(\Lambda)] = 0$:

(i) $\mathcal{H}(x)$ is a Lorentz scalar : $\mathcal{U}(\Lambda)\,\mathcal{H}(x)\,\mathcal{U}^{-1}(\Lambda) = \mathcal{H}(\Lambda x)$

(ii) $[\mathcal{H}(x), \mathcal{H}(y)] = 0 \qquad \forall\ (x-y)^2 < 0$ MICROCAUSALITY CONDITION

necessary to ensure the invariance of time-ordering : since different observers can disagree on the ordering in time of spacelike separated events, if they are to agree on $T[\mathcal{H}(x)\,\mathcal{H}(y)]$ we must demand that for spacetime separation the ordering of the $\mathcal{H}(x)$'s is irrelevant.

At the level of fields, microcausality implies

$$[A_n(x), A_m(y)]_\pm = 0 = [A_n(x), A_m^\dagger(y)]_\pm \qquad \forall\ (x-y)^2 < 0$$

automatically true if the fields satisfy the same $[\cdot,\cdot]_\pm$ as the particles they describe

will give non-trivial consequences

Take the case of a spinless particle:

$$A(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E(p)}}\, a_p\, e^{ipx}$$

↑ convenient normalization

$$\Downarrow$$

$$[A(x), A^\dagger(y)]_\mp = \frac{1}{(2\pi)^3}\int \frac{d^3p}{\sqrt{2E(p)}}\frac{d^3p'}{\sqrt{2E(p')}} \underbrace{[a_p, a^\dagger_{p'}]_\mp}_{\delta^3(\vec{p}-\vec{p}\,')} e^{ipx-ip'y}$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E(p)} e^{ip(x-y)}$$

$$\equiv \Delta_+(x-y)$$

$=$ (explicit computation)

$$= \begin{cases} \dfrac{m}{4\pi\sqrt{|(x-y)^2|}} K_1\left(m\sqrt{|(x-y)^2|}\right) & (x-y)^2<0 \\ \dfrac{m}{8\pi\sqrt{|(x-y)^2|}} \left[N_1\left(m\sqrt{|(x-y)^2|}\right) + \mathrm{sign}(x^0-y^0)\, i\, J_1\left(m\sqrt{|(x-y)^2|}\right)\right] & (x-y)^2>0 \end{cases}$$

Important: $\Delta_+(x-y) \neq 0$ for $(x-y)^2<0$!

$\Rightarrow$ we loose Lorentz invariance!

But $K_1(z) = K_1(-z)$ for $z^2<0$

Way out $\rightarrow$ use the negative energy solutions!

In addition to $A(x)$, consider also

$$B(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E(p)}}\, b_p\, e^{ipx}$$

and the combination

$$\phi(x) = A(x) + \alpha B^\dagger(x) \qquad \text{for some } \alpha \in \mathbb{C}$$

$$\Rightarrow [\phi(x), \phi^\dagger(y)]_\mp = [A(x) + \alpha B^\dagger(x), A^\dagger(y) + \alpha^* B(y)]_\mp$$

$$\overset{!}{=} [A(x), A^\dagger(y)]_\mp \mp |\alpha|^2 [B(y), B^\dagger(x)]_\mp$$

$$= \Delta_+(x-y) \mp |\alpha|^2 \Delta_+(y-x)$$

$\Rightarrow$ choosing $|\alpha| = 1$ $\oplus$ commutation relations we have

$$[\phi(x), \phi^\dagger(y)] = \Delta_+(x-y) - \Delta_+(y-x) \equiv \Delta(x-y)$$

by construction (because $K_1(z) = K_1(-z)$ for $z^2 < 0$)

$$\Delta(x-y) = 0 \quad \forall\, (x-y)^2 < 0$$

exactly what we wanted!

Messages :

(1) Causality demands that every particle (a) has a partner (b) with the same spin & the same mass (otherwise we cannot construct $\Delta(x-y)$)

(2) Causality & Lorentz invariance force a spin 0 particle to be a <u>BOSON</u>

The result can be generalized for any spin.
Write

$$\varphi_n(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E(p)}} \left[u_n(\vec{p},\sigma)\, a_{p\sigma}\, e^{-ipx} + v_n(\vec{p},\sigma)\, b^{+}_{p\sigma}\, e^{-ipx} \right]$$

and compute $[\varphi_n(x), \varphi^{+}_m(y)]$ for $(x-y)^2 < 0$

The explicit computation (see S. Weinberg "Feynman rules for any spin") gives

$$[\varphi_n(x), \varphi^{+}_m(y)]_{\pm} \propto \int \frac{d^3p}{(2\pi)^3\, 2E(p)} \Pi_{nm}(\vec{p}) \left[e^{ip(x-y)} \pm e^{-ip(x-y)} \right]$$

Some combination of u & v

point important for us : $\Pi_{nm}(p) = (-1)^{2j}\, t_{nm}^{\mu_1 \cdots \mu_{2j}}\, p_{\mu_1} \cdots p_{\mu_{2j}}$.

j = spin

But then

$$\left[\varphi_n(x), \varphi_m^{\dagger}(y)\right]_{\pm} \propto t_{nm}{}^{\mu_1\cdots\mu_{2j}} \int \frac{d^3p}{2E(p)}\, p_{\mu_1}\cdots p_{\mu_{2j}} \left[e^{ip(x-y)} \pm e^{-ip(x-y)} \right]$$

$$\propto t_{nm}{}^{\mu_1\cdots\mu_{2j}}\, \partial_{\mu_1}\cdots\partial_{\mu_{2j}} \int \frac{d^3p}{2E(p)} \left[e^{ip(x-y)} \pm (-1)^{2j} e^{-ip(x-y)} \right]$$

we already know that this integral vanish outside the light cone only if $\pm(-1)^{2j} = -1$

$$\Downarrow$$

$$\underbrace{\begin{array}{l} (+) = (F) \Rightarrow 2j \text{ odd} \Rightarrow j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \\ (-) = (B) \Rightarrow 2j \text{ even} \Rightarrow j = 0, 1, 2, 3, \ldots \end{array}}$$

Spin-statistic connections:

$j = 0, 1, 2, \ldots$ BOSONS

$j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ FERMIONS

Conclusions: microcausality implies

(•) antiparticles must exist (correct interpretation of negative energies)

(•) spin-statistic connection

5. QUANTIZATION ELECTROMAGNETIC FIELD

Now that we know in which sense we need to interpret relativistic wave equations, we construct the QFT of the EM field, for which we already know the form from classical EM.

5.1 - Review of EM

Electric & Magnetic fields $\vec{E}, \vec{B}$ $\rightarrow$ 6 d.o.f, not independent.

Is there a more efficient way to represent a EM field with less redundancy?

Maxwell eqs:
$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho & (1) \\ \vec{\nabla}\cdot\vec{B} = 0 & (2) \\ \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 & (3) \\ \vec{\nabla}\times\vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t} & (4) \end{cases}$$

Computing $\frac{\partial}{\partial t}(1) + \vec{\nabla}\cdot(4) \Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ CONTINUITY EQ.

From (2): $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$

From (3): $\vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} = 0 \Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$

$$\Downarrow$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$\Rightarrow$ from $\vec{E}, \vec{B}$ (6 dof) we have reduced to $\phi, \vec{A}$ (4 dof)

$\Rightarrow$ less redundant description <u>but still redundant</u>.

How do we see this?

$\vec{A} \to \vec{A} + \vec{\nabla} w$ keeps $\vec{B} = \vec{\nabla} \times \vec{A}$ unchanged

but changes $\vec{E}$:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \longrightarrow -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\frac{\partial w}{\partial t}$$

$\Longrightarrow$ changing at the same time

$$\phi \to \phi - \frac{\partial w}{\partial t}$$

keeps also $\vec{E}$ unchanged

$$\Rightarrow \begin{cases} \phi \to \phi - \dfrac{\partial w}{\partial t} \\ \vec{A} \to \vec{A} + \vec{\nabla} w \end{cases}$$

is a redundancy of our description that can be used to fix some condition (gauge condition):

$\phi = 0$	(temporal)
$A_z = 0$	(axial)
$\vec{\nabla} \cdot \vec{A} = 0$	(Coulomb or radiation)
$\vec{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t} = 0$	(Lorentz)

Let's take <u>Coulomb gauge</u>, since it will be useful for the rest of the computation.

To make the physics even clearer, we introduce transverse and longitudinal fields as follows:

$$\underbrace{V(x) = V_{\perp}(x)}_{\text{transverse}} + \underbrace{V_{\parallel}(x)}_{\text{longitudinal}} \quad \text{with} \quad \begin{cases} \nabla \times V_{\parallel} = 0 \\ \nabla \cdot V_{\perp} = 0 \end{cases}$$

The decomposition is unique (Helmholtz theorem).

Why are they called transverse / longitudinal?
In Fourier space

$$V(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \, \hat{V}(k) \, e^{ik\cdot x} \quad \Rightarrow \quad \begin{cases} k \times \hat{V}_{\parallel} = 0 \\ k \cdot \hat{V}_{\perp} = 0 \end{cases}$$

$\Rightarrow$ $V_{\perp}$ is transverse to momentum while $V_{\parallel}$ is parallel.

In momentum space is easy to derive explicit expressions:

$$\hat{V}_{\parallel} = \frac{k\cdot\hat{V}}{|k|}\frac{k}{|k|} \qquad \hat{V}_{\perp} = \hat{V} - \hat{V}_{\parallel} = \left(\mathbb{1} - \frac{k\,k}{|k|^2}\right)\hat{V}$$

$$\Downarrow$$

in components

$$(\hat{V}_{\perp})_i = \left(\delta_{ij} - \frac{k_i k_j}{|k|^2}\right)\hat{V}_j$$

From Maxwell eqs

$$\begin{cases} B = B_\perp & \Rightarrow \quad B_\perp = \nabla \times (A_\parallel + A_\perp) = \nabla \times A_\perp \\ E = E_\perp + E_\parallel & \rightarrow \end{cases}$$

to understand how to compute $E_\perp$, $E_\parallel$, we go to momentum space:

$$E = -\nabla\phi - \frac{\partial A}{\partial t} \rightarrow \hat{E} = -ik\hat{\phi} - \frac{\partial \hat{A}}{\partial t}$$

$$\hat{E} \overset{!}{=} -ik\hat{\phi} - \frac{k}{|k|}\frac{\partial \hat{A}_\parallel}{\partial t} - \frac{\partial A_\perp}{\partial t}$$

$\Rightarrow$ all that is proportional to k is longitudinal, the rest is transverse:

$$\begin{cases} E_\perp = -\dfrac{\partial A_\perp}{\partial t} \\ E_\parallel = -\nabla\phi - \dfrac{\partial A_\parallel}{\partial t} \end{cases}$$

All this is completely general.

With transverse/longitudinal fields we also get a new perspective on gauge transformations:

$$A \rightarrow A + \nabla\omega \quad \rightarrow \quad \text{momentum space} \quad \rightarrow \quad \hat{A} \rightarrow \hat{A} + ik\omega$$

$$\Rightarrow \quad A_\parallel + A_\perp \rightarrow (A_\parallel + ik\omega) + A_\perp \qquad \Rightarrow \text{only } A_\parallel \text{ is affected}$$

$\Rightarrow$ different gauge choices correspond to different values for $A_{\parallel}$

In Culomb gauge, $\nabla \cdot A = 0 \Rightarrow k \cdot \hat{A} = 0$

$$\Downarrow$$

$$k \cdot (\hat{A}_{\parallel} + \hat{A}_{\perp}) = 0$$

but $k \cdot A_{\perp} = 0$

$$\Downarrow$$

$$k \cdot \hat{A}_{\parallel} = 0 \Rightarrow \hat{A}_{\parallel} = 0$$

Then $\begin{cases} E_{\parallel} = -\nabla\phi \\ E_{\perp} = -\dfrac{\partial A_{\perp}}{\partial t} \end{cases} \Rightarrow$ complete separation between ϕ & $A_{\perp}$

From Maxwell eqs in Culomb gauge:

$$\nabla \cdot E = -\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot A = -\nabla^2\phi = \rho$$

$\Rightarrow$ ϕ is completely determined,

$$\phi(x,t) = \frac{1}{4\pi}\int d^3y \ \frac{\rho(y,t)}{|x-y|}$$

How many dof? $\quad \underset{1}{\underset{\uparrow}{\phi}}, \ \underset{1}{\underset{\uparrow}{A_{\parallel}}}, \ \underset{2}{\underset{\uparrow}{A_{\perp}}}$

but ϕ not independent, $A_{\parallel} = 0 \Rightarrow$ we are left with $A_{\perp}$ (2 dof)

[see appendix for the counting in Lorentz gauge]

What are the eqs. of motion of $A_\perp$?

→ we start from $$\frac{\partial^2 A}{\partial t^2} - \nabla^2 A + \nabla\left(\nabla\cdot A + \frac{\partial \phi}{\partial t}\right) = J$$

↓

momentum space

↓

$$\frac{\partial^2 \hat{A}}{\partial t^2} + k^2 \hat{A} + ik\left(ik\cdot\hat{A} + \frac{\partial\hat{\phi}}{\partial t}\right) = J$$

↓

apply $\delta_{ij} - \frac{k_i k_j}{|k|^2}$ to project on $\hat{A}_\perp$:

$$\frac{\partial^2 \hat{A}_{\perp i}}{\partial t^2} + k^2 \hat{A}_{\perp i} + \underbrace{i\left(\delta_{ij} - \frac{k_i k_j}{|k|^2}\right) k_j}_{=0}\left(i\, k\cdot\hat{A} + \frac{\partial\hat{\phi}}{\partial t}\right) = 0$$

↓

$$\boxed{\frac{\partial^2 A_\perp}{\partial t^2} - \nabla^2 A_\perp = J_\perp}$$

Solution: $$A_\perp(x,t) = A_\perp^{hom}(x,t) + A_\perp^{P}(x,t)$$

solution homogeneous eq

$\Box A_\perp(x,t) = 0$

⇓

any particular solution

Radiation field $$A_\perp(x,t) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^{3/2}}\left[\alpha_{k\lambda}\, \epsilon_{k\lambda}\, e^{-i(\omega_k t - k\cdot x)} + h.c.\right]$$

the 2 dof

normalization

In Coulomb gauge is simple to compute the energy of the EM field:

$$H = \frac{1}{2}\int d^3x \left(E^2 + B^2\right) = \frac{1}{2}\int d^3x \left[\, |E_{/\!/} + E_\perp|^2 + |B_\perp|^2 \right]$$

but $E_{/\!/} \cdot E_\perp = 0$.

To prove this:

$$\int d^3x \; E^*_{/\!/} \cdot E_\perp = \int d^3x \int \frac{d^3k}{(2\pi)^{3/2}} \frac{d^3q}{(2\pi)^{3/2}} \, \hat{E}^*_{/\!/}(k) \cdot \hat{E}_\perp(q) \, e^{i(-k+q)\cdot x}$$

$$= \int d^3x \int \frac{d^3k}{(2\pi)^{3/2}} \underbrace{\hat{E}^*_{/\!/}(k) \cdot \hat{E}_\perp(k)}_{=0} = 0$$

$$= \frac{1}{2}\int d^3x \left(|E_{/\!/}|^2 + |E_\perp|^2 + |B_\perp|^2\right)$$

$$= H_{radiation} + \underbrace{\frac{1}{2}\int d^3x \, |E_{/\!/}|^2}$$

$$H_{coul} = \frac{1}{2}\int d^3x \, |E_{/\!/}|^2 = \frac{1}{2}\int d^3k \, |\hat{E}_{/\!/}|^2$$

this can be computed as follows:

$$= \frac{1}{2} \int d^3k \ \hat{\rho}^*(k) \frac{|k|^2}{|k|^4} \rho(k)$$

$$= \frac{1}{2} \int d^3k \ \rho^*(k) \rho(k) \frac{1}{|k|^2}$$

$$= \frac{1}{8\pi} \int d^3x \int d^3y \frac{\rho(x)\rho(y)}{|x-y|}$$

5.2 - QUANTUM ELECTRODYNAMICS

What is the quantum version of this story?

We know how to write a relativistic quantum field:

$$\underbrace{A_\perp(x)}_{\text{quantum field}} = \sum_{\lambda=\pm} \sum_{k} \frac{1}{\sqrt{2E_k V}} \epsilon_\lambda(k)\, a_\lambda(k)\, e^{-ikx} + c.c.$$

satisfies $\left[a_\lambda(k), a^\dagger_{\lambda'}(k')\right] = \delta^3(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}$

What is the Hamiltonian that describe the EM and its interactions with matter?

We start with a Lagrangian:

$$L = \sum_a \frac{m_a v_a^2}{2} + \frac{1}{2}\int d^3x \left(|E|^2 - |B|^2\right) + \sum_a \left[q_a\, v_a\, A(x_a) - q_a\, \phi(x_a)\right]$$

Defining
$$\begin{cases} \rho(x,t) = \sum_a q_a\, \delta^3(x - x_a(t)) \\ J(x,t) = \sum_a q_a v_a\, \delta^3(x - x_a(t)) \end{cases}$$

we obtain

$$L = \sum_a \frac{m_a v_a^2}{2} + \underbrace{\frac{1}{2}\int d^3x \left(|E|^2 - |B|^2\right) + \int d^3x \left[J(x)\cdot A(x) - \rho(x)\,\Phi(x)\right]}_{\int d^3x\, \mathcal{L}}$$

with $\mathcal{L} = \frac{1}{2}\left(|E|^2 - |B|^2\right) + J\cdot A - \rho\,\Phi$

Going to momentum space [using reality: $\hat{f}(-k) = \hat{f}^*(k)$]

$$\int d^3x \, \mathcal{L} = \int d^3k \left\{ \frac{1}{2}\left(|\hat{E}|^2 - |\hat{B}|^2\right) + J^* \cdot A - \rho^* \Phi \right\}$$

$$= \frac{1}{2} \int d^3k \left\{ |\hat{E}|^2 - |\hat{B}|^2 + J^* \cdot A + J \cdot A^* - \rho^* \phi - \rho \phi^* \right\}$$

We now use $\quad |\hat{E}|^2 = |\hat{E}_{/\!/}|^2 + |\hat{E}_\perp|^2$

To prove this:

$$\int d^3x \, E_{/\!/}^* \cdot E_\perp = \int d^3x \int \frac{d^3k}{(2\pi)^{3/2}} \frac{d^3q}{(2\pi)^{3/2}} \, \hat{E}_{/\!/}^*(k) \cdot \hat{E}_\perp(q) \, e^{i(-k+q)\cdot x}$$

$$= \int d^3x \int \frac{d^3k}{(2\pi)^{3/2}} \underbrace{\hat{E}_{/\!/}^*(k) \cdot \hat{E}_\perp(k)}_{=0} = 0$$

Remembering $\quad \begin{cases} E_{/\!/} = -\nabla \phi & \Rightarrow \quad \hat{E}_{/\!/} = -ik\hat{\phi} \\ E_\perp = -\dot{A}_\perp & \Rightarrow \quad \hat{E}_\perp = -\dot{\hat{A}}_\perp \end{cases}$

From Maxwell eqs: $\nabla \cdot E = \rho \Rightarrow i k \cdot \hat{E}_{/\!/} = \hat{\rho}$

$$\Downarrow$$

$$\hat{E}_{/\!/} = -i \hat{\rho} \frac{k}{|k|^2}$$

$$\Downarrow$$

$$\hat{\Phi} = \frac{\hat{\rho}}{|k|^2}$$

$$\Rightarrow \int d^3x\, \mathcal{L} = \frac{1}{2}\int d^3k \Big\{ \overbrace{\dot{\hat{A}}_\perp^* \cdot \dot{\hat{A}}_\perp}^{|\hat{E}_\perp|^2} - \overbrace{(k\times \hat{A}_\perp^*)\cdot (k\times \hat{A}_\perp)}^{|\hat{B}_\perp|^2}$$

$$+ \frac{\rho^*\rho}{|k|^2} + \hat{J}^*\cdot \hat{A}_\perp + \hat{J}\cdot \hat{A}_\perp^* - 2\frac{\rho^*\rho}{|k|^2} \Big\}$$

$$= \frac{1}{2}\int d^3k \Big\{ \dot{A}_\perp^* \cdot \dot{A}_\perp - (k\times A_\perp^*)\cdot(k\times A_\perp)$$

$$- \frac{\rho^*\rho}{|k|^2} + J^*\cdot A_\perp + J\cdot A_\perp^* \Big\}$$

$$= \underbrace{-\frac{1}{8\pi}\int d^3x\, d^3y\, \frac{\rho(x)\rho(y)}{|x-y|}}_{H_{coul}} + \frac{1}{2}\int d^3x \left[\dot{A}_\perp^2 - (\nabla\times A_\perp)^2\right]$$

$$+ \int d^3x\, J\cdot A_\perp$$

What is, then, the Hamiltonian?

$$H = \underbrace{\sum_a v_a \cdot p_a}_{\text{particles}} + \underbrace{\int d^3x\, \dot{A}_\perp \cdot \Pi_\perp}_{\text{field}} - L$$

Conjugate momenta :

$$\begin{cases} p_a = \dfrac{\partial L}{\partial v_a} = m_a v_a + \underbrace{\dfrac{q}{c} A_\perp(x_a)}_{\text{from } \int d^3x\, J\cdot A_\perp} \Rightarrow v_a = \dfrac{1}{m_a}\left(p_a - \dfrac{q}{c}A_\perp\right) \\[2ex] \Pi_\perp = \dfrac{\partial \mathcal{L}}{\partial \dot{A}_\perp} = \dot{A}_\perp = -E_\perp \end{cases}$$

$$\Rightarrow H = \sum_a \frac{(p_a - q_a A_\perp)^2}{2m_a} + \int d^3x \left[\dot{A}_\perp^2 - \frac{1}{2}\dot{A}_\perp^2 + \frac{1}{2}(\nabla\times A_\perp)^2\right] + H_{coul}$$

$$= \sum_a \frac{(p_a - q_a A_\perp)^2}{2m_a} + \frac{1}{2}\int d^3x \underbrace{\left[\dot{A}_\perp^2 + (\nabla\times A_\perp)^2\right]}_{E_\perp^2 + B_\perp^2} + H_{coul}$$

what happens substituting the quantum field?

There is a problem of ordering in passing from the classical to the quantum expression:

We use

$$AB \longrightarrow \frac{AB+BA}{2} \qquad \text{(symmetric prescription)}$$

$$\Longrightarrow H = \frac{1}{2}\int d^3k \sum_\lambda E_k \left(a^\dagger_{k\lambda} a_{k\lambda} + a_{k\lambda} a^\dagger_{k\lambda}\right)$$

$$a^\dagger_{k\lambda} a_{k\lambda} + \delta^3(0)$$

$$= \underbrace{\frac{1}{2}\,\delta^3(0)\sum_{\lambda}\int d^3k\, E_k}_{\text{vacuum energy } E_0} + \sum_{\lambda}\int d^3k\, E_k\, a^{\dagger}_{k\lambda}\, a_{k\lambda}$$

$\downarrow$

let's study this

5.3 - CASIMIR ENERGY (= VACUUM ENERGY OF THE EM FIELD)

We have found

$$E_0 = \delta^3(0) \int d^3k \, E_k$$

can be written as $\frac{V}{(2\pi\hbar)^3}$

$\Rightarrow$ what we can actually compute is the vacuum energy density

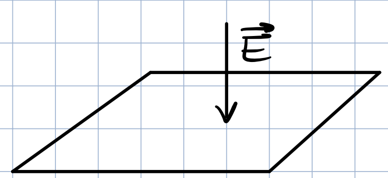
$$\rho_0 \equiv \frac{E_0}{V} = \underbrace{\int \frac{d^3k}{(2\pi)^3} E_k}_{\text{this diverges}}$$

$\downarrow$

to make sense of this integral, we need to regularize it

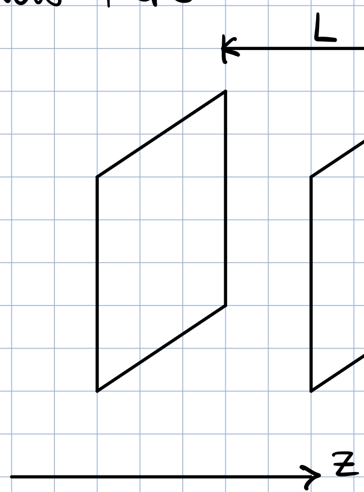
Since ρ_0 is an energy density, for sure it will have a gravitational effect. Can we measure it in the Lab, however?

Imagine we have conducting plates:



$E_{/\!/} = 0$

Now take



we require

$$\vec{A}(x,y,0,t) = 0 = \vec{A}(x,y,L,t)$$

We cannot use the expression for $\vec{A}$ found earlier because now we have boundary conditions.

We can always write the mode functions as

$$e^{-i\omega t} u\ (x,y,z)$$

and take plane waves in the x-y directions :

$$u_\omega(x,y,z) = e^{i\vec{k}\cdot\vec{x}}\ u_{\omega k}(z)$$

$$\vec{k} = k_x\,\hat{e}_x + k_y\,\hat{e}_y$$

The wave eq. becomes

$$\omega^2 u_{\omega k} - \vec{k}^2 u_{\omega k} + \frac{d^2 u_{\omega k}}{dz} = 0$$

$$\Longrightarrow \quad u_{\omega k}(z) = A\ e^{ipz} + B\ e^{-ipz} \qquad\qquad p^2 = \omega^2 - \vec{k}^2$$

To satisfy the boundary conditions we need

$$u_{\omega k}(0) = 0 \quad \Rightarrow \quad u_{\omega k}(z) = A \sin(pz)$$

$$u_{\omega k}(L) = 0 \quad \Rightarrow \quad p_n = \frac{n\pi}{L} \, , \quad n = 1, 2, 3, \ldots$$

The vector potential is thus

$$\vec{A}(x) = \sum_{\lambda=\pm} \sum_{n=1}^{\infty} \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega L}} \left[\epsilon_\lambda(n,k)\, a_{nk\lambda}\, e^{-i\omega_{nk}t + i\vec{k}\cdot\vec{x}} \sin(p_n z) + \text{c.c.} \right]$$

$$\omega_{nk}^2 = k_x^2 + k_y^2 + \left(\frac{\pi n}{L}\right)^2$$

and the Hamiltonian results in

$$H = E_0 + \sum_{\lambda=\pm} \sum_{n=1}^{\infty} \int d^2k \, \omega_{nk} \, a_{nk\lambda}^{\dagger} a_{nk\lambda}$$

$$E_0 = \frac{1}{2} \sum_{\lambda=\pm} \sum_{n=1}^{\infty} \int d^2k \, \omega_{nk} \, \delta^2(0)$$

$$\delta^2(0) \rightarrow \frac{V_2}{(2\pi)^2}$$

V_2 = Volume in the x-y directions

The physical quantity is thus

$$\sigma_0 \equiv \frac{E_0}{V_2} = \sum_{n=1}^{\infty} \int d^2k \; \omega_{nk}$$

To regularize the integral, we use the identity

$$\int_0^{\infty} ds \; s^p e^{-\lambda s} = \lambda^{-1-p} \, \Gamma(1+p)$$

with $\lambda = \omega_{nk}^2$:

$$\omega_{nk} = \sqrt{\lambda} = \lim_{p \to -\frac{3}{2}} \frac{1}{\Gamma(1+p)} \int_0^{\infty} ds \; s^p e^{-\omega_{nk}^2 s}$$

makes the integral over d^2k converge

Then

$$\sigma_0 = \lim_{p \to -\frac{3}{2}} \sum_{n=1}^{\infty} \int \frac{d^2k}{(2\pi)^2} \frac{1}{\Gamma(1+p)} \int_0^{\infty} ds \; s^p e^{-\omega_{nk}^2 s}$$

$$= \lim_{p \to -\frac{3}{2}} \sum_{n=1}^{\infty} \int_0^{\infty} ds \; s^p \frac{1}{\Gamma(1+p)} \underbrace{\int \frac{d^2k}{(2\pi)^2} \exp\left[-\left(\vec{k}^2 + \left(\frac{n\pi}{L}\right)^2\right) s\right]}_{\exp\left[-\frac{n^2\pi^2}{L^2} s\right] \underbrace{\int \frac{d^2k}{(2\pi)^2} e^{-\vec{k}^2 s}}_{\left(\frac{1}{2\pi}\sqrt{\frac{\pi}{s}}\right)^2}}$$

$$= \lim_{p \to -\frac{3}{2}} \frac{1}{4\pi \Gamma(1+p)} \sum_{n=1}^{\infty} \underbrace{\int_0^{\infty} ds \; s^{p-1} e^{-\frac{n^2\pi^2}{L^2} s}}_{\Gamma(p) \left(\frac{n^2\pi^2}{L^2}\right)^{-p}}$$

use $z\Gamma(z) = \Gamma(z+1)$

$$= \lim_{p \to -\frac{3}{2}} \frac{1}{4\pi p} \left(\frac{L}{\pi}\right)^{2p} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{2p}}}_{\zeta(2p)}$$

[Riemann zeta function]

$$= -\frac{\pi^2}{6L^3} \underbrace{\zeta(-3)}_{1/120} = -\frac{\pi^2}{720 L^3}$$

$\Uparrow$

Vacuum energy is negative.

Force on the plates (measurable):

$$p_z = \frac{F_z}{V_2} = -\frac{\partial \sigma_0}{\partial L} = -\frac{\pi^2}{240 L^4}$$

measured and found in accordance with the prediction!

$\Rightarrow$ vacuum energy is real

Historically : • effect predicted in 1948 (Hendrik Casimir)

- measured for the 1^{st} time in 1997 (Lamoreaux) within 5% of the prediction with 1 flat plate & 1 plate which is a sphere with large curvature

- measured between 2 parallel plates in 2001 (Bressi, Carugno, Onofrio, Ruoso)

5.4 – Interactions with matter

We saw that, in classical EM,

$$H = \sum_{a} \frac{(p_a - q_a A_\perp)^2}{2m_a} + H_{EM} + H_{Coul}$$

Contains interactions with matter

Formally obtained applying the "minimal substitution"

$$p \to p - q A_\perp$$

In QM, we interpret this as $-i\nabla \to -i\nabla - q A_\perp$

$$\Downarrow$$

$$\nabla \to \nabla - iq A_\perp$$

For the interaction with a Schrödinger field $\psi(x)$ we write

$$H_{matter} = \int d^3x \; \psi^\dagger(x) \left[-\frac{(\nabla - iq A_\perp)^2}{2m} + V(x) \right] \psi(x)$$

Notice that H is gauge invariant provided

$$\begin{cases} \vec{A} \to \vec{A} + \vec{\nabla}\omega \\ \psi \to e^{iq\omega}\psi \end{cases}$$

The whole construction could be justified in terms of symmetries, but we are not going to discuss this here. See any book on QFT

It is customary to define a "covariant derivative"

$$\vec{D}\psi \equiv (\vec{\nabla} - iq\vec{A})\psi$$

Explicitly, we write

$$H_{matter} = H_{\psi} + H_{\psi A} + H_{\psi AA}$$

$$\begin{cases} H_{\psi} = \int d^3x \; \psi^{\dagger}(x)\left[-\frac{\nabla^2}{2m} + V(x)\right]\psi(x) \\ H_{\psi A} = \int d^3x \; A_{\perp} \cdot \left\{ \frac{iq}{2m}\left(\psi^{\dagger}\nabla\psi - \nabla\psi^{\dagger}\psi\right)\right\} \\ H_{\psi AA} = \int d^3x \; \frac{q^2}{2m}\psi^{\dagger}\psi \, A_{\perp}\cdot A_{\perp} \end{cases}$$

where $A_{\perp}$ = relativistic quantum field

Let's have a look at $H_{\psi A}$ in terms of creation/annihilation operators :

$$H_{\psi A} = \frac{iq}{2m}\int_V d^3x \sum_{\lambda=\pm}\sum_{k}\frac{1}{\sqrt{V 2\omega_k}}\Big[\vec{\epsilon}_\lambda(k)\, a_{k\lambda}\, e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} + \vec{\epsilon}_\lambda^{\,*}(k)\, a^\dagger_{k\lambda}\, e^{i\omega_k t + i\vec{k}\cdot\vec{x}}\Big] \times$$

$$\times \sum_{n,m}\Big[u_n^*(\vec{x})\, b_n^\dagger \big(\vec{\nabla} u_m(\vec{x})\big) b_m - \big(\vec{\nabla} u_n^*(\vec{x})\big) b_n^\dagger\, u_m(\vec{x})\, b_m\Big]$$

$$= \frac{iq}{2m}\sum_{\lambda=\pm}\sum_{k}\sum_{n,m}\Big[\frac{\vec{\epsilon}_\lambda(k)}{\sqrt{V2\omega_k}}\, a_{k\lambda}\, e^{-i\omega_k t} + \text{h.c.}\Big] \times b_n^\dagger b_m \overbrace{\int_V d^3x\, e^{i\vec{k}\cdot\vec{x}}\big(u_n^*\vec{\nabla}u_m - \vec{\nabla}u_n^* u_m\big)}^{\vec{I}_{nm}(\vec{k})}$$

$$= \frac{iq}{2m}\sum_{\lambda}\sum_{k}\sum_{n,m}\frac{\vec{\epsilon}_\lambda(k)\cdot\vec{I}_{nm}(\vec{k})\, e^{-i\omega_k t}}{\sqrt{V2\omega_k}}\, a_{k\lambda} b_n^\dagger b_m + \text{h.c.}$$

these interactions are precisely of the type that give rise to coherent states!

Phenomenology :

(1) $H_{\psi A}$ governs emission/absorption of a single photon

(2) $H_{\psi AA}$ governs emission/absorption of 2 photons and, more importantly, photon scattering off the system ($\gamma C_n \to \gamma' C_m$)

5.5 - Worked example : the dipole approximation

Suppose we want to compute the rate for $C_n \to \gamma(k', \lambda') + C_{n'}$.

Matrix element $\to$

$$\langle \gamma(\vec{k}', \lambda')\, C_{n'} |\, H_{\psi A} \,| C_n \rangle = \frac{1}{\sqrt{(2\pi)^3 2|\vec{k}|}}\, \vec{\epsilon}^{\,*}_{\lambda'}(\vec{k}') \cdot \vec{I}_{nn'}(\vec{k})$$

If C_n = atom in state n we can simplify the computation.

Typical mode functions u_n in atoms $\to$ concentrated for

$$r \lesssim a_B = \frac{1}{\alpha m_e}$$

Bohr radius

This means that most of the contribution in $I_{nn'}(\vec{p})$ comes from $|\vec{p}| \sim \frac{1}{a_B} = \alpha m_e$.

Now, the energy of the photon $|\vec{k}|$ that enters $I_{nn'}(-\vec{k})$ is, by energy conservation,

$$|\vec{k}| = E_n - E_{n'} \sim \alpha^2 m_e \sim \alpha\,(\alpha m_e) \ll \frac{1}{a_B}$$

$$\alpha \simeq \frac{1}{137}$$

$\Rightarrow$ wavelenght of photons emitted/absorbed is much larger than the size of the atom

$\Rightarrow$ we can approximate $e^{i\vec{k}\cdot\vec{x}} \simeq 1$

$$\Rightarrow \ I_{nn'}(\vec{k}) \simeq I_{nn'}(0) = \frac{iq}{2m}\int d^3x \left(u_{n'}^* \vec{\nabla} u_n - \vec{\nabla} u_{n'}^* u_n\right)$$

by parts

$$= \frac{iq}{m}\int d^3x \ u_{n'}^* \vec{\nabla} u_n$$

$$= -\frac{q}{m}\int d^3x \ u_{n'}^* \vec{p}\, u_n$$

$$= -\frac{q}{m}\vec{p}_{n'n}$$

matrix element of momentum op.

Now, remember that

$$\left[-\frac{\vec{\nabla}^2}{2m} + V(\vec{x})\right] u_n = E_n u_n$$

and, from an explicit computation,

$$i\left[H, \vec{x}\right] = \frac{i}{2m}\left[p^2, \vec{x}\right] = \frac{i\vec{p}}{m}\left[\vec{p}, \vec{x}\right] = \frac{\vec{p}}{m}.$$

But then

$$\frac{\vec{p}_{n'n}}{m} = \langle n' | \frac{\vec{p}}{m} | n\rangle = i\langle n' | H\vec{x} - \vec{x}H | n\rangle$$

$$= i\left(E_{n'} - E_n\right)\underbrace{\langle n' | \vec{x} | n\rangle}_{\frac{\vec{d}_{n'n}}{q}}$$

with $\vec{d} = q\vec{x}$ = electric dipole moment

But then $\quad \vec{I}_{n'n}(0) = -q\,\dfrac{\vec{P}_{n'n}}{m} = i(E_n - E_{n'})\,\vec{d}_{n'n}$

and

$$\langle \gamma(\vec{k}',\lambda')\, C_{n'} |\, H_{\psi A}\, | C_n \rangle \simeq \frac{1}{\sqrt{(2\pi)^3\, 2|\vec{k}'|}}\, \vec{\epsilon}^{\,*}_{\lambda'}(\vec{k}') \cdot \vec{I}_{n'n}(0)$$

$$= \frac{i\,(E_n - E_{n'})}{\sqrt{(2\pi)^3\, 2\,|\vec{k}'|}}\, \vec{\epsilon}^{\,*}_{\lambda'}(\vec{k}') \cdot \vec{d}_{n'n}$$

$$= \underbrace{\frac{i\,|\vec{k}'|}{\sqrt{(2\pi)^3\, 2\,|\vec{k}'|}}\, \vec{\epsilon}^{\,*}_{\lambda'}(\vec{k}') \cdot}_{\substack{\text{this structure appears} \\ \text{in } -\frac{\partial \vec{A}}{\partial t} = \vec{E}}} \vec{d}_{n'n}$$

$$= \left(\langle \gamma(\vec{k}',\lambda') |\, \vec{E}(\vec{x}=0)\, | 0 \rangle \right) \cdot \vec{d}_{n'n}$$

$\Rightarrow$ the matrix element of $H_{\psi A}$ amounts to the matrix element of the dipole interaction $\vec{d}\cdot\vec{E}$

5.6 - COMMUTATION RELATIONS

What are the commutation relations of the quantum field $A_\perp$?

According to canonical quantization, we have

$$[A_{\perp i}(x,t), \Pi^{\dagger}_{\perp j}(y,t)] = i\,\delta_{ij}\,\delta^3(x-y)$$

This can also be obtained from an explicit computation with $a_\lambda, a^\dagger_\lambda$.

We can rewrite this in terms of $A_i(x,t)$ & $\Pi_j(y,t)$ remembering that

$$\hat{A}_{\perp i}(k) = \left(\delta_{ij} - \frac{k_i k_j}{|k|^2}\right)\hat{A}_j(k) \quad \Rightarrow \quad A_{\perp i}(x,t) = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}\right) A_j(x,t)$$

Then, calling $P_{ij}(x) = \delta_{ij} - \dfrac{\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}}{\nabla_x^2}$,

$$\underbrace{[A_{\perp i}(x,t), \Pi_{\perp j}(y,t)]}_{i\,\delta^3(x-y)\,\delta_{ij}} = P_{im}(x)\,P_{jn}(y)\,[A_m(x,t), \Pi_n(y,t)]$$

Inversion can be done remembering that P is a projector:

$$\begin{aligned}[A_m(x,t), \Pi_n(y,t)] &= i\,P_{mi}(x)\,P_{nj}(y)\,\delta_{ij}\,\delta^3(x-y) \\ &= i\,P_{mi}(x)\,P_{ni}(y)\,\delta^3(x-y) \\ &= i\,P_{mn}(x)\,\delta^3(x-y) = i\left(\delta_{mn} - \frac{\partial_m \partial_n}{\nabla^2}\right)\delta^3(x-y)\end{aligned}$$

APPENDIX 1 : COUNTING DEGREES OF FREEDOM IN LORENTZ GAUGE

In Lorentz gauge $\frac{\partial \phi}{\partial t} + \nabla \cdot A = 0$

$$\begin{cases} \text{From (1)}: & \vec{\nabla}\cdot\vec{E} = -\vec{\nabla}^2\phi - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = \rho \\ \text{From (4)}: & \vec{\nabla}\times\vec{B} = \underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}} = \vec{J} - \frac{\partial}{\partial t}\vec{\nabla}\phi - \frac{\partial^2\vec{A}}{\partial t^2} \end{cases}$$

$$\downarrow$$

$$\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{\partial\phi}{\partial t}\right) = \vec{J}$$

$$\Downarrow \text{ Lorentz gauge}$$

$$\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A} = \vec{J}$$

Counting : $\phi, \vec{A}$ (4 dof) $-$ $\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$ (1 constraint)

$\Rightarrow$ 3 dof left

But we still have some freedom left :

$$\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0 \xrightarrow{\text{gauge}} \frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A} + \underbrace{\vec{\nabla}^2\omega - \frac{\partial^2\omega}{\partial t^2}}_{\text{if } =0\text{, Lorentz gauge is preserved}}$$

$\Rightarrow$ I use this residual gauge freedom to set $\phi = 0$

$\Longrightarrow$ EM field completely described by $\vec{A}$ with $\vec{\nabla} \cdot \vec{A} = 0$

$\Rightarrow$ 2 dof (of the original 6)