Graduate Statistical Mechanics -1st Exam- 08/10/2022

Instruções para a realização da avaliação: A prova deverá ser enviada até 10:00h de 10/10/2022 no meu escaninho, localizado no piso superior do DFGE e/ou para os email `fiorecarlos.cf@gmail.com` e `william2.castilho@usp.br`. Neste último caso, a entrega via papel será obrigatória (até o dia de aula 11/10/2022). Ela pode ser realizada com a consulta de livros e/ou materiais, porém as respostas e interpretações são **individuais**.

Todas as respostas deverão ser devidamente justificadas com base em argumentos físicos e matemáticos adequados. A não justificativa das respostas implicará na não integralidade da nota. Utilizem o idioma que for melhor para você.

Para ajudar na correção e garantir que tudo corra bem, procurem seguir as seguintes recomendações:

- procurem mandar a prova em um único arquivo PDF, pode ser foto ou scan. Sugestão de app: Adobe Scan;
- verifiquem a ordem das páginas, a orientação delas e a legibilidade do texto;
- mandem para o email do professor e do monitor e confiram o envio.

Boa prova!

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1 Extra(Canonical ensemble)

Consider a system of N particles, in which each one can occupy only two levels with energies 0 e ϵ, respectively. The system is placed in contact with a thermal reservoir of temperature T.
a) (0.25) Obtain the partition function $Z(T,N)$ as well as the internal energy and specific heat per site $u(T), c_v(T)$ as a function of T.
b) (0.5) Obtain an equation of state for the pressure $p = p(T,v)$ by introducing the effect of volume $v = V/N$ according to the following relation $\epsilon = a/v^\gamma$, with $a > 0$. Obtain the equation of state relating p, v and u.
c) (0.75) Obtain the expression for the isothermal compressibility $\kappa_T = -\frac{1}{v}(\frac{\partial v}{\partial P})_T$ and its relation with $c_v(T)$.
d) (0.5) Obtain the expression for the thermal expansion coefficient $\alpha = \frac{1}{v}(\frac{\partial v}{\partial T})_P$ as well as their relationships with $c_v(T)$ and κ_T.

2 Fermi ideal gas

Consider a system of ideal fermions with degeneracy γ constrained in a volume $V = L^3$, whose energy spectrum is given by

$$\epsilon_{\vec{k}} = c|\vec{k}|, \tag{1}$$

where $c > 0$.
a) (0.5) Find the prefactor A of the equation of state $pV = AU$ for an arbitrary temperature T;
b) (0.5) Find the Fermi energy as a function of the volume V and the particle number N;
c) (0.5) Obtain expressions for the mean energy, pressure and compressibility of a gas of free fermions at null temperature;
d) (0.5) Find the asymptotic expression for the specific heat at constant volume for $T << T_F$;
e) (1.0) Find the asymptotic expression for the isothermal compressibility $\kappa_T = -\frac{1}{v}(\frac{\partial v}{\partial P})_T$ for $T << T_F$.

3 Ultrarelativistic ideal gas of bosonic particles

Let us consider an ideal gas of bosonic particles (spinless) constrained in a volume $V = L^3$, whose spectrum of energy given by $\epsilon_{\vec{k}} = \hbar c|\vec{k}|$.
a) (0.5) Obtain an expression for the Bose-Einstein condensation temperature T_0;
b) (1.5) Obtain expressions for the pressure, mean energy U, entropy S and molar specific heat c_v in terms of Bose-Einstein functions $g_\nu(z)'s$ and appropriate system parameters at the normal phase;
c) (1.0) Obtain expressions for above quantities for $T \leq T_0$.

4 (1.5) Bosonic particle with internal degree of freedom

Consider an ideal Bose gas consisting of molecules with internal degrees of freedom. Assuming that, besides the ground state $\epsilon_0 = 0$, only the first excited state ϵ_1 of the internal spectrum needs to be taken into account. In other words, assume that the spectrum of energy is given by

$$\epsilon_{\vec{k},\sigma} = \frac{\hbar^2 k^2}{2m} + \epsilon_1 \sigma, \tag{2}$$

where $\sigma = 0, 1$. Determine the (transcendental equation) for the condensation temperature T_0 of the gas as a function of ϵ_1. Obtain an approximate expression for $\beta_0 \epsilon_1 >> 1$.

5 van-der-Waals gas

The van-der-Waals is given by the following equation of state

$$p = \frac{RT}{v - b} - \frac{a}{v^2}. \tag{3}$$

a) (0,5) Find the critical parameters p_c, v_c e T_c.
b) (0,25) Show that the molar Helmholtz free energy is given by $f(T, v) = -RT \ln(v - b) - a/v + f_0(T)$, where $f_0(T)$ solely depends on the temperature.
c) (0,25) Obtain the reduced expression for the equation of state (as shown in my lectures) $\tilde{P} = 8\tilde{T}/(3\tilde{v}-1) - 3/\tilde{v}^2$.
d) (0,5) Show that the Maxwell construction leads to the expression

$$\ln(3\tilde{v_g} - 1) + \frac{9}{4\tilde{T}\tilde{v_g}} - \frac{3\tilde{v_g}}{3\tilde{v_g} - 1} = \ln(3\tilde{v_l} - 1) + \frac{9}{4\tilde{T}\tilde{v_l}} - \frac{3\tilde{v_l}}{3\tilde{v_l} - 1}$$

e) (0,5) Obtain, by performing the Maxwell construction, the values for the pressure, volume of liquid and gas phases ($\tilde{P_l} = \tilde{P_g}, \tilde{v_l}$ and $\tilde{v_g}$) for $\tilde{T} = 0.92$. *Suggestion: Use the mathematica notebook I sent you.*
f) (0,5) For very small temperatures, the van-der-Waals gas isotherms present negatives values for the pressure. Obtain the temperature in which this occurs.