Lecture 10
Hydrodynamics

Part IV

Summary on viscous hydro/Navier-Stokes

$T^{\mu\nu} = T^{\mu\nu}_{ideal} + \Pi^{\mu\nu}$
$\partial_\mu T^{\mu\nu} = 0$
$u_\mu \Pi^{\mu\nu} = 0$ (Landau frame)
$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Delta^{\mu\nu}\Pi$

$\partial_\mu s^\mu \geq 0 \Rightarrow$ Possible choice $\pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>}$ and $\Pi = \zeta\nabla_\alpha u^\alpha$

[Often one defines $\Pi^{\mu\nu} = \pi^{\mu\nu} - \Delta^{\mu\nu}\Pi$ then $\Pi = -\zeta\nabla_\alpha u^\alpha$, we do this thereafter]

where $\nabla_{(\mu}u_{\nu)} \equiv (1/2)\nabla_{<\mu}u_{\nu>} + (1/3)\Delta_{\mu\nu}\nabla_\alpha u^\alpha$

Bjorken model in the Navier-Stokes approximation

It is usual to work directly with (τ, η_s) coordinates.

We must substitute:

$\partial_\mu \longrightarrow d_\mu$ (covarient derivative)

$\partial_\mu u^\nu \longrightarrow d_\mu u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\alpha\mu} u^\alpha$

$\nabla^\alpha = \Delta^{\mu\alpha}\partial_\mu \longrightarrow \Delta^{\mu\alpha} d_\mu$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1/\tau^2 \end{pmatrix}$$

$\partial_\mu T^{\mu\nu} = 0 \longrightarrow d_\mu T^{\mu\nu} = 0 = (1/\tau)\partial_\mu(\tau T^{\mu\nu}) + \Gamma^\nu_{\lambda\mu} T^{\lambda\mu}$

all Γ's are zero except $\Gamma^\tau_{\eta\tau} = \Gamma^\tau_{\tau\eta} = 1/\tau$ and $\Gamma^\tau_{\eta\eta} = \tau$

We get simple forms for

$u^\mu = (1, 0, 0, 0)$

$$T^{\mu\nu}_{ideal} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p/\tau^2 \end{pmatrix}$$

$\Pi = -\zeta \nabla_\alpha u^\alpha = -\zeta d_\alpha u^\alpha = -\zeta/\tau$

$\Delta^{\eta\eta} = -1/\tau^2$

$u_\mu \pi^{\mu\nu} = 0 \Rightarrow \pi^{\tau\nu} = 0$, $\pi^{xx} = \pi^{yy}$, $\pi^\mu_\mu = 0 \Rightarrow \pi^{xx} = -\tau^2 \pi^{\eta\eta}/2$, using definition other $\pi^{\mu\nu} = 0$ and $\pi^{\eta\eta} = -(4/3)\eta/\tau^3$

$d_\mu T^{\mu\nu} = 0 \Rightarrow$ Thermodynamical quantities are independent of x, y, η_s and $\partial_\tau(\tau T^{\tau\tau}) + \tau^2 T^{\eta\eta} = 0$

So we only need to solve:

$$\boxed{\partial_\tau(\tau\epsilon) + p - \frac{4}{3}\frac{\eta}{\tau} - \frac{\zeta}{\tau} = 0}$$

Solution:

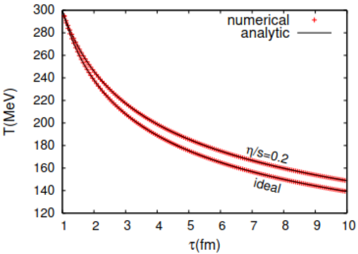
It is common to re-write

$$\frac{\partial \epsilon}{\partial \tau} = -\frac{\epsilon + p}{\tau}\left(1 - \frac{4}{3\tau T}\frac{\eta}{s} - \frac{1}{\tau T}\frac{\zeta}{s}\right) = 0$$

For massless quarks and gluons:

$$T(\tau) = T_0\left(\frac{\tau_0}{\tau}\right)^{1/3}\left[1 + \frac{1}{2\tau_0 T_0}\left(\frac{4}{3}\frac{\eta}{s} + \frac{\zeta}{s}\right)\left(1 - \left(\frac{\tau_0}{\tau}\right)^{2/3}\right)\right]$$

Slower decrease of temperature with time.



Okamoto & Nonaka Eur. Phys. J.C 77 (2017) 383: $T_0 = 300\ MeV$, $\tau_0 = 1\ fm$, $\zeta/s = 0$

Violation of causality in the Navier-Stokes approximation

Perform a small pertubation: $\epsilon = \epsilon_0 + \delta\epsilon(t,x)$, $u^\mu = (1,\vec{0}) + \delta u^\mu(t,x)$

In the direction y, the eq. of motion is (cf. aula 9):

$(\epsilon + p)Du^y - \nabla^y p + \Delta^y_\nu d_\mu \Pi^{\mu\nu} = 0 = (\epsilon_0 + p_0)\partial_t u^y + \partial_x \Pi^{xy} + \mathcal{O}(\delta^2)$

$\Pi^{xy} = \eta(\nabla^x u^y + \nabla^y u^x) + (\zeta - \frac{2}{3}\eta)\Delta^{xy}\nabla_\alpha u^\alpha = -\eta_0 \partial_x \delta u^y + \mathcal{O}(\delta^2)$

$\Rightarrow \partial_t \delta u^y = \frac{\eta_0}{\epsilon_0 + p_0}\partial_x^2 \partial u^y$

Look for sinusoidal perturbation: $\delta u^y(t,x) \propto exp(-\omega t + ikx)$

the following relation must be satisfied:

$\omega = \frac{\eta_0}{\epsilon_0 + p_0}k^2$

The speed of diffusion of mode k is

$\frac{d\omega}{dk} = \frac{2\eta_0}{\epsilon_0 + p + 0}k$

$\Rightarrow$ For large k, $d\omega/dk$ will exceed the speed of light and violate causality

This can be put on a more rigourous footing: see appendix A in Romatschke.

The naive relativistic generalization of the Navier Stokes equations violates causality. This pathology can be cured in various ways

• Israel-Stewart theory: include viscous corrections into the entropy current s^μ, of second order in the gradients, we then must have: $D\Pi \sim -\frac{1}{\tau_\Pi}(\Pi + \zeta D_\alpha u^\alpha)$ and $D\pi^{\mu\nu} \sim -\frac{1}{\tau_\pi}(\pi^{\mu\nu} - \eta \nabla^{<\mu} u^{\nu)})$

W. Israel and J.M. Stewart, Transient relativistic thermodynamics and kinetic theory, Annals of Physics 118 (1979) 341

but there are other possibilities based on different arguments:

• BRSSS

R. Baier, P. Romatschke, D.T. Son, A. O. Starinets, M. A. Stephanov, Relativistic viscous hydrodynamics, conformal invariance, and holography, JHEP 0804 (2008) 100/arXiv:0712.2451

• DNMR

Denicol, Niemi, Molnar, Rischke, Derivation of transient relativistic fluid dynamics from the Boltzmann equation, Phys. Rev. D 85(2012) 114047/arXiv:1202.4551

• etc.

Challenge

Consider a fluid with only bulk viscosity. Using the Bjorken model and the Israel-Stewart theory, show that Π obeys $\frac{\partial \Pi}{\partial \tau} = -\frac{1}{\tau_\Pi}(\Pi + \frac{\zeta}{\tau})$. Solve this equation (for ζ and τ_Π constant) and plot $\Pi(\tau)$ for $\Pi_0 = 0$, $\zeta = 1000\, MeV/fm^2$, $\tau_0 = 1\, fm$ and $\tau_\Pi = 1\, fm$.

Homework

Using the Bjorken model, compute T for $\tau = 5\, fm$ assuming $T_0 = 300\, MeV$, $\tau_0 = 1\, fm$ for a perfect fluid and for a viscous fluid (with $\eta/s = 0.2$, $\zeta/s = 0$) in the Navier-Stokes approximation.

Other references on this topic

- Paul Romatschke “New Developments in Relativistic Viscous Hydrodynamics” Int.J.Mod.Phys.E19 (2010) 1/arXiv:0902.3663
- W. Florkowski, http://ift.uni.wroc.pl/~karp2017/Florkowski_lecture1.pdf http://ift.uni.wroc.pl/~karp2017/Florkowski_lecture2.pdf http://ift.uni.wroc.pl/~karp2017/Florkowski_lecture3.pdf
- W. Florkowski, https://www.youtube.com/watch?v=Cwpx3eaym5Y https://www.youtube.com/watch?v=7EKySqAH3KM https://www.youtube.com/watch?v=kt4cqDevkwo