Lecture 9
Hydrodynamics

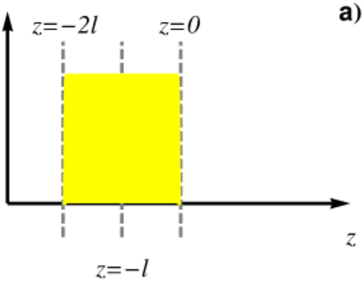
Part III

Landau model

This is another analytical solution for 1+1 expansion with the following initial conditions:

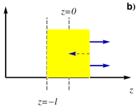
• Initially colliding nuclei form a highly compressed disk of matter with zero fluid velocity.

• This is very different from the Bjorken model where matter forms initially a very long tube (to have boost-invariance) with fluid velocity $v_f = \tanh z/t \Leftrightarrow y = \eta_s$



Initial times

For $t \leq l/c_s$:



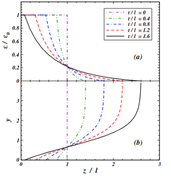
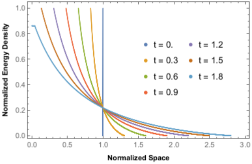
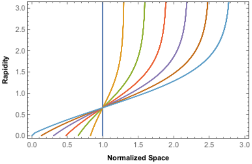


FIG. 1. (Color online) The ratio ϵ/ϵ_0 and flow rapidity y as a function of z/l for different values of $t/l \leq l/c_s$ obtained with the Riemann simple wave solution.





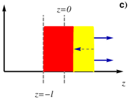
The solution of the hydro eq. is a simple wave, traveling inward with the speed of sound and with expansion outward at light speed

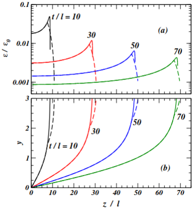
Analytical formulas in Wong et al. Phys. Rev. C 90 (2014) 064907/arXiv:1408.3343 and Florkowski Ch.20

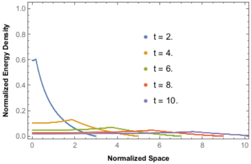
Last 2 figures on the right: V.S.Franção

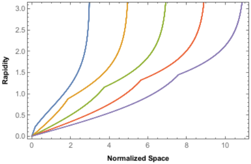
Larger times

For $t > l/c_s$:









The solution of the hydro eq. “inside” (continuous lines) is more complicated (Khalatnikov solution) and “outside” it is still a simple wave (dashed lines)

Exact formulas in Wong et al. Phys. Rev. C 90 (2014) 064907/arXiv:1408.3343 and Florkowski Ch.20

Last 2 figures on the right: V.S.França̋o

Rapidity distribution

In this model, it is argued that the end of the expansion occurs when $t^2 - z^2 = R_f^2$ with $R_f \sim 2R$ and that then $T << T_0$, so:

$$\boxed{s = s_0\, exp(-2L + \sqrt{L^2 - y^2}) \quad \text{where} \quad L \equiv \ln R_f/(2l)}$$

Using $dS = s \times \pi R^2\, R_f\, dy$, we get (for $y << L$)

$\frac{dS}{dy} = \pi R^2\, R_f\, s_0\, e^{-L}\, exp\left(-\frac{y^2}{2L}\right)$

and identifying entropy density with particle density

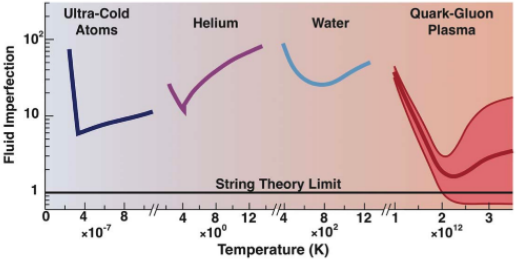
$$\boxed{\frac{dN}{dy} = \frac{N}{\sqrt{2\pi L}}\, exp\left(-\frac{y^2}{2L}\right)}$$

i.e. a Gaussian rapidity distribution comes out naturally.

Classical Navier-Stokes theory

$$\frac{d\vec{v}}{dt} = -\frac{\vec{\nabla}p}{\rho} + \frac{\eta}{\rho}\nabla^2\vec{v} + \frac{(\zeta + \frac{1}{3}\eta)}{\rho}\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) \qquad (1)$$

Acceleration is affected by shear viscosity η and bulk viscosity ζ



The above equation can be rewritten:

$$\begin{aligned}
\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} &= -\frac{1}{\rho}\frac{\partial p}{\partial x^i} - \frac{1}{\rho}\frac{\partial \Pi^{ji}}{\partial x^j}, \\
\Pi^{ji} &= -\eta\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} - \frac{2}{3}\frac{\partial v^k}{\partial x^k}\delta^{ji}\right) - \zeta\frac{\partial v^k}{\partial x^k}\delta^{ji}
\end{aligned}$$

Relativistic Navier-Stokes theory

This derivation follows P.Romatschke Int.J.Mod.Phys.E19 (2010) 1/arXiv:0902.3663

Yet another derivation that $T^{\mu\nu}_{ideal} = (\epsilon + p)u^\mu u^\nu - g^{\mu\nu} p$

$T^{\mu\nu}_{ideal}$ depends on ϵ, p, u^μ, is symmetric and transforms as a tensor under Lorentz transformation:

$T^{\mu\nu}_{ideal} = \epsilon(c_0 g^{\mu\nu} + c_1 u^\mu u^\nu) + p(c_2 g^{\mu\nu} + c_3 u^\mu u^\nu)$

We know the expression of $T^{\mu\nu}_{ideal}$ in the rest frame and it implies $(c_0 + c_1)\epsilon + (c_2 + c_3)p = \epsilon$ and $-c_0\epsilon - c_2 p = p$

so: $c_0 = 0$, $c_1 = 1$, $c_2 = -1$, $c_3 = 1$ or $T^{\mu\nu}_{ideal} = (\epsilon + p)u^\mu u^\nu - g^{\mu\nu} p$

and another way to take the non-relativistic limit of $\partial_\mu T^{\mu\nu} = 0$

Project the hydro. eq. using u^μ and $\boxed{\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu}$ Note 1: $\Delta^{\mu\nu}$ projects perpendicularly to u^μ: $\Delta^{\mu\nu} u_\mu = 0$. Note 2: Use $u_\nu \partial_\mu u^\nu = 0$

$$\boxed{\begin{aligned} u_\nu \partial_\mu T^{\mu\nu}_{ideal} &= D\epsilon + (\epsilon + p)\partial_\mu u^\mu && = 0 \\ \Delta^\alpha_\nu \partial T^{\mu\nu}_{ideal} &= (\epsilon + p)Du^\alpha - \nabla^\alpha p && = 0 \end{aligned}}$$

with shorthands for proj. of ∂_μ: $\boxed{D = u^\mu \partial_\mu \text{ and } \nabla^\alpha = \Delta^{\mu\alpha}\partial_\mu}$.

For small velocity: $D \sim \partial_t + \vec{v} \cdot \vec{\partial} + \mathcal{O}(|\vec{v}|^2)$ and $\nabla^i \sim \partial^i + \mathcal{O}(|\vec{v}|)$

So for a non-relativistic equation of state $p << \epsilon$ and $\epsilon \sim \rho$, the hydro eq. reduce to entropy conservation and Euler eq.

Relativistic case

Write $T^{\mu\nu} = T^{\mu\nu}_{ideal} + \Pi^{\mu\nu}$

Assume for simplicity no conserved charges.

Multiplying by u_ν or Δ^α_ν, we get:

$$\begin{cases} u_\nu \partial_\mu T^{\mu\nu} & = D\epsilon + (\epsilon + p)\partial_\mu u^\mu + u_\nu \partial_\mu \Pi^{\mu\nu} = 0 \\ \Delta^\alpha_\nu \partial_\mu T^{\mu\nu} & = (\epsilon + p) D u^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} = 0 \end{cases}$$

We use $u_\nu \partial_\mu \Pi^{\mu\nu} = \partial_\mu (u_\nu \Pi^{\mu\nu}) - \Pi^{\mu\nu} \partial_{(\mu} u_{\nu)}$ where $A_{(\mu} B_{\nu)} = (1/2)(A_\mu B_\nu + A_\nu B_\mu)$ and re-write

$$\boxed{\begin{array}{lll} u_\nu \partial_\mu T^{\mu\nu}_{ideal} = D\epsilon + (\epsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu} \nabla_{(\mu} u_{\nu)} & & = 0 \\ \Delta^\alpha_\nu \partial T^{\mu\nu}_{ideal} = (\epsilon + p) D u^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} & & = 0 \end{array}}$$

These are the relativistic viscous hydrodynamics eq. but we still need to specify $\Pi^{\mu\nu}$.

This can be done using the thermodynamic laws/equalities:
$\epsilon + p = Ts$, $Tds = d\epsilon$ and $\partial_\mu s^\mu \geq 0$.
We re-write $\Pi^{\mu\nu}$ in a traceless part $\pi^{\mu\nu}$ and its trace defined as 3Π:
$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Delta^{\mu\nu}\Pi$
Similarly for the tensor $\nabla_{(\mu} u_{\nu)}$:
$\nabla_{(\mu} u_{\nu)} \equiv (1/2)\nabla_{<\mu} u_{\nu>} + (1/3)\Delta_{\mu\nu}\nabla_\alpha u^\alpha$
then
$\partial_\mu s^\mu = Ds + s\partial_\mu u^\mu = \frac{1}{T}D\epsilon + \frac{\epsilon+p}{T}\partial_\mu u^\mu = \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu} u_{\nu)} =$
$\frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu} u_{\nu>} + \frac{1}{T}\Pi\nabla_\alpha u^\alpha \geq 0$
This inequality is fulfilled if
$\boxed{\pi^{\mu\nu} = \eta\nabla^{<\mu} u^{\nu>} \quad \Pi = \zeta\nabla_\alpha u^\alpha}$ with η and $\zeta \geq 0$.
In the non-relativistic limit, we get back the classical Navier-Stokes eq. (slide 6).
This is a beautiful derivation but this theory has some problems.

Challenge

Read Romatschke's paper §I and IIa and work through as many derivations as you can.

Other references on this topic

- Paul Romatschke "New Developments in Relativistic Viscous Hydrodynamics" Int.J.Mod.Phys.E19 (2010) 1/arXiv:0902.3663
- W. Florkowski, http://ift.uni.wroc.pl/~karp2017/Florkowski_lecture1.pdf http://ift.uni.wroc.pl/~karp2017/Florkowski_lecture2.pdf http://ift.uni.wroc.pl/~karp2017/Florkowski_lecture3.pdf
- W. Florkowski, https://www.youtube.com/watch?v=Cwpx3eaym5Y https://www.youtube.com/watch?v=7EKySqAH3KM https://www.youtube.com/watch?v=kt4cqDevkwo