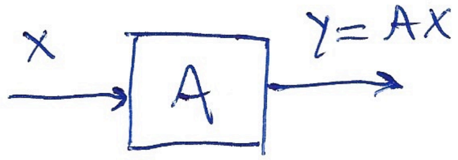
PS15794 – MATRIX ANALYSIS

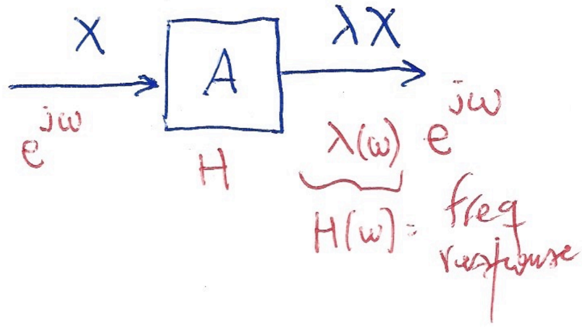
Nov/18/2021
LEC 08

6. Eigenvals, Eigenvecs, Jordan Form, Matrix functions

Eigenanalysis: a method to study the structure of lin transformations.



$A = [\mathcal{L}]_{B_V B_V}$



6.1. FUNDAMENTALS

Def 1: right eigenvectors and left eigenvectors

right evec

$$\boxed{Ax = \lambda x}$$

x is the right col evec associated with eval λ

x acts on the cols of A

(λ, x) is an r pair for A

left evec

$$\boxed{y^* A = \mu y^*}$$

y is the left col evec associated with eval μ

y acts on rows of A

(μ, y) is an l pair for A

For a given A, several pairs (λ_i, x_i) and (μ_k, y_k) may exist.

remarks

a) an evec cannot be the zero vector.

$A0 = \gamma 0$ otherwise any number γ is an eval for any matrix A.

b) On the other hand, ~~an~~ a nonzero evec may be associated with a zero eval

$Ax = 0 \cdot x \quad , \quad A, x \neq 0 \qquad (0, x)$ epair for A

c) $\left.\begin{array}{l} Ax = 0 \cdot x = 0 \\ Ax = 0 \end{array}\right\} \quad x \in \mathcal{N}(A)$

$x \in \mathcal{N}(A) \Rightarrow Ax = 0 \quad , \quad A, x \neq 0$

$Ax = 0 \Rightarrow Ax = 0 \cdot x \Rightarrow (0, x)$ epair for A

$(0, x)$ eigenpair for A $\Leftrightarrow x \in \mathcal{N}(A)$

Lemma 1: (λ, x) rpair for A $\Longleftrightarrow$ $(\bar{\lambda}, x)$ lpair for A^*

$(\Rightarrow)$

$Ax = \lambda x$

$(Ax)^* = (\lambda x)^*$

$x^* A^* = \bar{\lambda} x^*$

(λ, x) r for A $\Rightarrow$ $(\bar{\lambda}, x)$ l for A^*

$(\Leftarrow)$

$x^* A^* = \bar{\lambda} x^*$

$(x^* A^*)^* = (\bar{\lambda} x^*)^*$

$Ax = \lambda x$

$(\bar{\lambda}, x)$ lpair for A^* $\Rightarrow$ (λ, x) rpair for A

$\blacksquare$

Lemma 2: (λ, x) epair for A $\Longleftrightarrow$ $(A - \lambda I)$ singular matrix

$(\Rightarrow)$

$Ax = \lambda x$

$Ax - \lambda x = 0$

$(A - \lambda I)x = 0$

but x is evec, then $x \neq 0$

$\Rightarrow (A - \lambda I)$ has a nontrivial Null space

$\therefore (A - \lambda I)$ is singular

or $x \in \mathcal{N}(A - \lambda I)$

$(\Leftarrow)$

$(A - \lambda I)$ singular

$(A - \lambda I)x = 0$ $(\cdot), x \neq 0$

$Ax - \lambda x = 0$

$Ax = \lambda x$

$\therefore (\lambda, x)$ epair for A

$\blacksquare$

<u>Remark:</u> Lemma 2 provides a tool for calculating the eigenvectors for matrix A, once we know the evals of A.

<u>Def 2:</u> Characteristic polyn. & Equation

$\boxed{\phi_A(t) \triangleq \det(tI - A)}$ Char. polynomial

$\boxed{\phi_A(t) = 0}$ Char. equation

$\phi_A(t)$: N^{th} degree poly for $A_{N\times N}$, exactly N roots, possibly repeated, possibly complex.

$$\left\{ \begin{matrix} \text{roots} \\ \phi_A(t) \end{matrix} \right\} \overset{\text{COINCIDE}}{=} \left\{ \begin{matrix} \text{evals} \\ A \end{matrix} \right\}$$

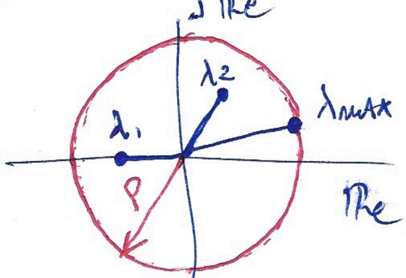
Why coincide? evals may be found via other methods (e.g., algorithm QR) and $\phi_A(t)$ has other uses in matrix Analysis

Def 3: the spectrum of A: set of all evals of A

$$\lambda(A) = \{\lambda \mid \lambda \text{ is eval of } A\}$$

Def 4: spectral radius: maximum nonnegative real number ρ such that

$$\rho(A) = \max_{\lambda(A)} |\lambda|$$



smallest disk centered at 0 that contains $\lambda(A)$

Some Properties

1) $\lambda(A) = \lambda(A^T)$
2) $\lambda(A^*) = \overline{\lambda(A)}$
3) $\lambda^\ell \in \lambda(A^\ell)$, evecs (A^ℓ) = evecs (A)
4) $\lambda^{-1} \in \lambda(A^{-1})$, evecs (A^{-1}) = evecs (A)

Def 5: For a scalar polynomial $f(x) = \sum_{\ell=0}^{L} c_\ell x^\ell$, a <u>matrix polynomial</u> for a square matrix X is

$$f(X) = \sum_{\ell=0}^{L} c_\ell X^\ell = c_0 I + c_1 X + c_2 X^2 \ldots$$

<u>Lemma 3</u>: the right and left spectrum of a matrix A are <u>equal</u>.

$$Ax = \lambda x \qquad\qquad y^*A = \mu y^*$$

$$(A-\lambda I)\,x = 0 \qquad\qquad y^*A - \mu y^* = 0$$

$$|A-\lambda I| = 0 \qquad\qquad y^*(A-\mu I) = 0$$

$$|A-\mu I| = 0$$

they have the same characteristic equation!

<u>Lemma 4</u>: $A \in \mathbb{C}^{N\times N}$ with an rpair (λ_i, x_i) and an lpair (λ_k, y_k), $\lambda_i \neq \lambda_k$ for $i \neq k$.
then $y_k^* x_i = 0$ or $y_k \perp x_i$ for $\lambda_i \neq \lambda_k$ (usual inner prod.)

$$A x_i = \lambda_i x_i$$

$$y_k^* A x_i = \lambda_i y_k^* x_i$$

$$\lambda_k y_k^* x_i = \lambda_i y_k^* x_i$$

$$(\lambda_k - \lambda_i)\, y_k^* x_i = 0$$

but $\lambda_k \neq \lambda_i$, then

$$\boxed{y_k^* x_i = 0}$$

right evecs and left evecs are orthogonal for diff. λ's. However, $\{x_i\}$ evecs are LI and $\{y_k\}$ evecs are LI, they are not necessarily orthog!

thm 1: Cayley-Hamilton (CLT)

every square matrix A satisfies its characteristic equation $p_A(t)=0$, that is

$$\boxed{p_A(A)=0} \quad (C.L.T.)$$

example: $A=\begin{bmatrix}-7 & -4\\ 8 & 5\end{bmatrix}$, $p_A(t)=t^2+2t-3$

$$p_A(A)=A^2+2A-3=\begin{bmatrix}-7 & -4\\ 8 & 5\end{bmatrix}^2+2\begin{bmatrix}-7 & -4\\ 8 & 5\end{bmatrix}-3I=[0]$$

thm 3: Hermitian matrices have real evals

Hermitian: $A^*=A$

$$Ax_i=\lambda_i x_i$$

$$x_i^* A x_i=\lambda_i x_i^* x_i$$

$$(x_i^* A x_i)^*=(\lambda_i x_i^* x_i)^*$$

$$x_i^* A^* x_i=\bar{\lambda}_i x_i^* x_i$$

but $A^*=A$

$$x_i^* A x_i=\bar{\lambda}_i x_i^* x_i$$

$$x_i^*(\lambda_i x_i)=\bar{\lambda}_i x_i^* x_i$$

$$\lambda_i x_i^* x_i=\bar{\lambda}_i x_i^* x_i$$

x_i is an evec $\therefore x_i\neq 0$

$x_i^* x_i=\|x_i\|^2>0 \quad \forall x_i\neq 0$

$$\lambda_i \cancel{\|x_i\|^2}=\bar{\lambda}_i \cancel{\|x_i\|^2}$$

$$\therefore \boxed{\lambda_i=\bar{\lambda}_i}$$

$i=1,N \quad (A_{N\times N})$

<u>thm 3</u>: Hermitian Matrices have orthogonal evecs for different λ's

(λ_1, x_1) and (λ_2, x_2) and $\lambda_1 \neq \lambda_2$, $x_1 \perp x_2$

study proof LAUB 78

<u>thm 4</u> (General Case): matrix $A \in \mathbb{C}^{N \times N}$ with a set of k distinct evals $1 \leq k \leq N$ has k LI evecs proof: addendum (Beezer book)

<u>Example 1</u>: $\lambda(A) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\} \Rightarrow$ 5 LI evecs

$N = k = 5$

<u>Example 2</u>: $\lambda(A) = \{\underbrace{\lambda_1, \lambda_1, \lambda_1}_{\text{multiplicity}=3}, \lambda_2, \lambda_3\}$

multiplicity = 3 at least 1 evec x_1

evec x_2

evec x_3

x_1, x_2, x_3 evecs are LI

General case: evecs are LI, not necessarily orthog.!
(for different λ's)

If matrix A is Hermitian, or Symmetric, evecs are $\perp$

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Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent
Suppose that A is an $n \times n$ square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

Proof. If $p = 1$, then the set $S = \{\mathbf{x}_1\}$ is linearly independent since eigenvectors are nonzero (Definition EEM), so assume for the remainder that $p \geq 2$.

We will prove this result by contradiction (Proof Technique CD). Suppose to the contrary that S is a linearly dependent set. Define $S_i = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_i\}$ and let k be an integer such that $S_{k-1} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_{k-1}\}$ is linearly independent and $S_k = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k\}$ is linearly dependent. We have to ask if there is even such an integer k? First, since eigenvectors are nonzero, the set $\{\mathbf{x}_1\}$ is linearly independent. Since we are assuming that $S = S_p$ is linearly dependent, there must be an integer k, $2 \leq k \leq p$, where the sets S_i transition from linear independence to linear dependence (and stay that way). In other words, $\mathbf{x}_k$ is the vector with the smallest index that is a linear combination of just vectors with smaller indices.

Since $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k\}$ is a linearly dependent set there must be scalars, $a_1, a_2, a_3, \ldots, a_k$, not all zero (Definition LI), so that

$$\mathbf{0} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_k\mathbf{x}_k$$

Then,

$$\begin{aligned}
\mathbf{0} &= (A - \lambda_k I_n)\,\mathbf{0} && \text{Theorem ZVSM}\\
&= (A - \lambda_k I_n)\,(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_k\mathbf{x}_k) && \text{Definition RLD}\\
&= (A - \lambda_k I_n)\,a_1\mathbf{x}_1 + \cdots + (A - \lambda_k I_n)\,a_k\mathbf{x}_k && \text{Theorem MMDAA}\\
&= a_1\,(A - \lambda_k I_n)\,\mathbf{x}_1 + \cdots + a_k\,(A - \lambda_k I_n)\,\mathbf{x}_k && \text{Theorem MMSMM}\\
&= a_1\,(A\mathbf{x}_1 - \lambda_k I_n\mathbf{x}_1) + \cdots + a_k\,(A\mathbf{x}_k - \lambda_k I_n\mathbf{x}_k) && \text{Theorem MMDAA}\\
&= a_1\,(A\mathbf{x}_1 - \lambda_k\mathbf{x}_1) + \cdots + a_k\,(A\mathbf{x}_k - \lambda_k\mathbf{x}_k) && \text{Theorem MMIM}\\
&= a_1\,(\lambda_1\mathbf{x}_1 - \lambda_k\mathbf{x}_1) + \cdots + a_k\,(\lambda_k\mathbf{x}_k - \lambda_k\mathbf{x}_k) && \text{Definition EEM}\\
&= a_1\,(\lambda_1 - \lambda_k)\,\mathbf{x}_1 + \cdots + a_k\,(\lambda_k - \lambda_k)\,\mathbf{x}_k && \text{Theorem MMDAA}\\
&= a_1\,(\lambda_1 - \lambda_k)\,\mathbf{x}_1 + \cdots + a_{k-1}\,(\lambda_{k-1} - \lambda_k)\,\mathbf{x}_{k-1} + a_k\,(0)\,\mathbf{x}_k && \text{Property AICN}\\
&= a_1\,(\lambda_1 - \lambda_k)\,\mathbf{x}_1 + \cdots + a_{k-1}\,(\lambda_{k-1} - \lambda_k)\,\mathbf{x}_{k-1} + \mathbf{0} && \text{Theorem ZSSM}\\
&= a_1\,(\lambda_1 - \lambda_k)\,\mathbf{x}_1 + \cdots + a_{k-1}\,(\lambda_{k-1} - \lambda_k)\,\mathbf{x}_{k-1} && \text{Property Z}
\end{aligned}$$

This equation is a relation of linear dependence on the linearly independent set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_{k-1}\}$, so the scalars must all be zero. That is, $a_i\,(\lambda_i - \lambda_k) = 0$ for $1 \leq i \leq k-1$. However, we have the hypothesis that the eigenvalues are distinct, so $\lambda_i \neq \lambda_k$ for $1 \leq i \leq k-1$. Thus $a_i = 0$ for $1 \leq i \leq k-1$.

This reduces the original relation of linear dependence on $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k\}$ to the simpler equation $a_k\mathbf{x}_k = \mathbf{0}$. By Theorem SMEZV we conclude that $a_k = 0$ or $\mathbf{x}_k = \mathbf{0}$. Eigenvectors are never the zero vector (Definition EEM), so $a_k = 0$. So all of the scalars a_i, $1 \leq i \leq k$ are zero, contradicting their introduction as the scalars creating a nontrivial relation of linear dependence on the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k\}$. With a contradiction in hand, we conclude that S must be linearly independent. ■

6.2. Eigenvecs and Similarity transfs

Matrices A and B are similar iff

$$\exists P^{-1} \mid P^{-1}AP = B$$

Def 6: Matrix A is diagonalizable if it is similar to a diagonal matrix Λ.

$$\exists P^{-1} \mid P^{-1}AP = \Lambda$$ A is diagonalizable

thm 5: Matrix $A \in \mathbb{C}^{N\times N}$ is diagonalizable iff it has a full set (N) of LI evecs

N LI evecs $\Longleftrightarrow$ A is diagon.

($\Rightarrow$) $Ax_i = \lambda_i x_i$, then form $P = [x_1\ x_2 \cdots x_N]$

P is full rank (why?) then

$$P^{-1}AP = P^{-1}A[x_1\ x_2 \cdots x_N] = P^{-1}[Ax_1\ Ax_2 \cdots Ax_N]$$
$$= P^{-1}[\lambda_1 x_1\ \lambda_2 x_2 \cdots \lambda_N x_N] = P^{-1}(P\Lambda) = P^{-1}P\Lambda = \Lambda.$$

($\Leftarrow$)

$\exists P^{-1} \mid P^{-1}AP = \Lambda$. Say $P = [p_1\ p_2 \cdots p_N]$.

$$P^{-1}AP = \Lambda$$
$$PP^{-1}AP = P\Lambda$$
$$AP = P\Lambda \rightarrow [Ap_1\ Ap_2 \cdots Ap_N] = [\lambda_1 p_1\ \lambda_2 p_2 \cdots \lambda_N p_N]$$

$\boxed{Ap_i = \lambda_i p_i}$, $i = 1, N$

A has N epairs (λ_i, p_i).

that is, forming P with evecs(A) diagonalizes A via simil. transf.

CONCLUSION:

1) A set of N LI evecs provides a change of basis matrix $P = [x_1\ x_2 \dots x_N]$ that poses the underlying LT $A = [L]_{B_V B_V}$ into a simple form (diagonal) $\Lambda = [L]_{B_Z B_Z}$

2) Similarity transfs change evecs but preserves evals!

$$Ax = \lambda x$$

$$A(PP^{-1})x = \lambda x$$

$$APP^{-1}x = \lambda x$$

$$\underbrace{P^{-1}AP}_{B}\,\underbrace{P^{-1}x}_{y} = \lambda\,\underbrace{P^{-1}x}_{y}$$

$\therefore\ \lambda(B) = \lambda(A)$ or $\lambda(P^{-1}AP) = \lambda(A)$

however $y = P^{-1}x \neq x$

thm : Dyadic Expansion

Let $A \in \mathbb{C}^{N\times N}$ have N distinct evals λ_i, with respective r-pairs and l-pairs

$$A x_i = \lambda_i x_i \quad \text{and} \quad y_k^* A = \lambda_k y_k^* \qquad (1)$$

with normalized evecs such that $x_i^* y_k = \delta_{ik}$ (2) $\quad i,k = 1, N$

Then $A = X \Lambda Y^* = \sum_{i=1}^{N} \lambda_i x_i y_i^*$, $\Lambda = \mathrm{diag}\{\lambda_i\}$ if $x_i^* y_i = a$ then $\frac{1}{a} x_i^* y_i = 1$

Proof: From (1)

$$AX = \Lambda X \iff X^{-1} A X = \Lambda \iff A = X \Lambda X^{-1}$$

from (2)

$$x_i^* y_k = \delta_{ik} \iff y_k^* x_i = \delta_{ik} \iff \begin{bmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_N^* \end{bmatrix} [x_1 \; x_2 \cdots x_N] = I_N$$

$$\iff Y^* X = I \iff Y^* = X^{-1}$$

thus,
$$A = X \Lambda X^{-1} = X \Lambda Y^* = \sum_{i=1}^{N} \lambda_i x_i y_i^*$$

Eigendecomposition

If $A^* = A$ ($A^T = A$ in the real case), it has a complete set of orthogonal right evecs

$$p_k^* p_e = \begin{cases} \alpha_k = \|p_k\|^2, & k = l \\ 0, & k \neq l. \end{cases}$$

If we normalize each evec $p_k \rightarrow \frac{p_k}{\|p_k\|}$ and collect the normalized evecs into matrix $P = [p_1 \cdots p_k \cdots p_N]$, P will be a unitary (orthogonal in the real case) matrix

$$P^{-1} = P^* \qquad (P^T = P \text{ for real entries})$$

How?

If $p_k^* p_e = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$, then $P^* P = I$ (check it!)

Since P is full rank (why?), it is also invertible then:

$$P^* P = I$$
$$P^* P (P^{-1}) = I (P^{-1})$$
$$\boxed{P^* = P^{-1}}$$

and

$$P^{-1} = P^*$$
$$P^{-1}(P) = P^*(P)$$
$$\boxed{I = P^* P}.$$

this allows us to introduce the eigen decomposition for Hermitian (Symmetric in the real case) matrices:

$$A = P\Lambda P^{-1} = P\Lambda P^{*} = \sum_{k=1}^{N} \lambda_k \underbrace{p_k p_k^{*}}_{\text{rank-1 matrix}}$$

or

$$\boxed{A = \sum_{k} \lambda_k p_k p_k^{*}}$$

Which can be viewed as a series decomposition for A. the same holds for the <u>dyadic expansion</u>, but in that case the corresponding matrix P will <u>not</u> be a unitary (orthogonal) matrix and we will have to expand A as

$$A = P\Lambda P^{-1} = P\Lambda Q^{*} = \sum_{k=1}^{N} \lambda_k p_k q_k^{*}$$

with $Q^{*} \triangleq P^{-1}$. Recall that Q will be the set of <u>left</u> evecs of A.

6.3. Multiplicity of EVALS

Some matrices $A \in \mathbb{C}^{N \times N}$ have $\phi_A(t)$ with repeated roots, implying repeated λ's

$$\boxed{\phi_A(t) = (t-\lambda_1)^{N_1}(t-\lambda_2)^{N_2}\cdots(t-\lambda_k)^{N_k}} \qquad N_1+N_2+\cdots+N_k = N$$

DEF 7 ALGEBRAIC MULTIPLICITY N_i is the number of times λ_i repeats in $\phi_A(t)$

DEF 8 GEOMETRIC MULTIPLICITY M_i is the number of LI evecs provided by the N_i evals $\{\lambda_i\}_N$

$$\boxed{M_i \le N_i} \qquad \text{thus } \sum_{i=1}^{N} M_i \le N$$

if $N_i = 1\ \forall i$ we are ok (N LI evecs)

Example: $A = \begin{bmatrix} 0,5 & 1 \\ 0 & 0,5 \end{bmatrix} \to \phi_A(t) = (t-0,5)^2 \quad N_1(0,5) = 2$

$\lambda(A) = \{\lambda_1, \lambda_1\}$ evecs: $(0,5I - A)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{cases} x_1 = a \\ x_2 = 0 \end{cases} \quad X_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}$

$M_1(0,5) = 1 < N_1(0,5) = 2$ only 1 evec $\therefore$ A not diagon.

Example: $A = \begin{bmatrix} 0,5 & 0 \\ 0 & 0,5 \end{bmatrix} \to \phi_A(t) = (t-0,5)^2 \quad N_1(0,5) = 2$

$\lambda(A) = \{\lambda_1, \lambda_1\}$ evecs: $(0,5I - A)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left\langle X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right.$

$M_1(0,5) = 2 = N_1(0,5)$ $\therefore$ Full set evecs: A is diag.

Note that

$$\boxed{M_i(\lambda_i) = \dim \mathcal{N}(A - \lambda_i I)}$$

proof BACK

Matrices that $M_i < N_i$ are <u>defective</u>: they are <u>not</u> diagonalizable. They are not similar to any Λ.

$$\nexists\ P \mid P^{-1}AP = \Lambda$$

DEFECTIVE $\neq$ SINGULAR (BACK)

$\text{DEFECTIVE} = \overline{\text{DIAGON}}$, $\text{SING} = \overline{\text{NON-SING}}$

DEFECTIVE $\neq$ SINGULAR

$DS = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\qquad D\bar{S} = \begin{bmatrix} 0{,}5 & 1 \\ 0 & 0{,}5 \end{bmatrix}$

$\bar{D}S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\qquad \bar{D}\bar{S} = \begin{bmatrix} 0{,}5 & 0 \\ 0 & 0{,}5 \end{bmatrix}$

$M_i = \dim \mathcal{N}(A - \lambda_i I) = \#$ LI vecs in $\mathcal{N}(\cdot)$

$(A - \lambda_i I) z = 0$: How many LI vecs we find?

$Az - \lambda_i z = 0$

$Az = \lambda_i z$: we can find M_i $\overset{\text{LI}}{\text{evecs}}$ associated with λ_i (by definition)

$\Downarrow$

$(A - \lambda_i I) z = 0$: M_i LI evecs $= \dim \mathcal{N}(A - \lambda_i I)$

6.4. JORDAN CANONICAL FORM (JCF)

For defective matrices we resort to a "relaxed" form of representing a LT $A = L$ ~~via~~ ~~a~~ in a simple form. Instead of trying to bring a defective matrix to a diagonal form Λ, we bring it to the JCF J, which is "almost" diagonal

$$J = \begin{bmatrix} J_{N_1}(\lambda_1) & & & 0 \\ & J_{N_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{N_k}(\lambda_k) \end{bmatrix} \qquad N_1 + N_2 + \cdots + N_k = N$$

J is a block-diagonal matrix where each block $J_{N_i}(\lambda_i)$ contains 1 evec associated with λ_i and it is $N_i \times N_i$

$$J_1(\lambda_i) = [\lambda_i], \quad J_2(\lambda_i) = \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}, \quad J_3(\lambda_i) = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

etc.

If each $J_{N_i}(\lambda_i)$ is 1-dim $(N_i = 1)$ then $J = \Lambda$

A typical Jordan block $J_{N_i}(\lambda_i)$

$$J_{N_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

there is <u>one</u> evec for <u>each</u> Jordan block.

Order/size of the block

<u>Example</u>: $\lambda(A) = \{\lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_3\}$, corresponding to $\phi_A(t) = (t-\lambda_1)^3 (t-\lambda_2)(t-\lambda_3)$. A is $N \times N$.

$$J = \left[\begin{array}{c|c|c} J_{\lambda_1} & & \\ \hline & J_1(\lambda_2) & \\ \hline & & J_1(\lambda_3) \end{array}\right]$$

the evals $\lambda_2 \neq \lambda_3$ return 2 LI evecs

$J_1(\lambda_2) = [\lambda_2]$

$J_1(\lambda_3) = [\lambda_3]$

The size of J_{λ_1} is $N_1 = 3$, but its INNER structure depends on $M_1(\lambda_1)$, i.e., on the underlying matrix A and has 3 possibilities

$$J_{\lambda_1} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_1 \end{bmatrix}, \quad J_{\lambda_1} = \left[\begin{array}{cc|c} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ \hline 0 & 0 & \lambda_1 \end{array}\right], \quad J_{\lambda_1} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

3 $J_1(\lambda_1)$	1 $J_2(\lambda_1)$, 1 $J_1(\lambda_1)$	1 $J_3(\lambda_1)$
3 LI evecs	2 LI evecs	1 LI evec.

Repeated λ_i's give rise to different JD's, depending on A.

$\boxed{\text{Matrices } \Lambda \text{ and } N \text{ commute: } \Lambda N = N\Lambda}$

~~Note~~ <u>Remark</u>: Matrix J can be written as

$J = \Lambda + N$, where Λ is diagonal and contain the evals and N is nilpotent and contains 1's and 0's on its supradiagonal.

thm 6: Every matrix $A \in \mathbb{C}^{N \times N}$ is similar to a Jordan matrix J. J is unique in the sense that composing blocks are unique, but their order is arbitrary: $P^{-1}AP = J$

proof: HORN 122-128

Also true for Λ matrices: order of evals/evecs is arbitrary

$$\begin{bmatrix} J_2(\lambda_1) & \\ & J_1(\lambda_2) \end{bmatrix} \text{ or } \begin{bmatrix} J_1(\lambda_2) & \\ & J_2(\lambda_1) \end{bmatrix}$$

For a given A
- structure of J (Jordan blocks)
- How to build P, $P^{-1}AP = J$
  - evecs
  - principal vecs (also: generalized evecs) complete the set of N LI vecs for defective matrices

Example: see LAUB 86 sec. 9.3.1 & example 9.33

HOMEWORK (SUGGESTED) Eigen Analysis 2020

1) **Prove**: a) $\lambda(A) = \lambda(A^T)$ b) $\lambda(A^*) = \overline{\lambda(A)}$

c) $\lambda^* \in \lambda(A^*)$, evecs (A^*) = evecs (A)

d) $\lambda^{-1} \in \lambda(A^{-1})$, evecs (A^{-1}) = evecs (A)

2) Create a generic matrix $A_{3\times3}$ and conclude that $A = P \Lambda P^{-1}$

3) Create a 4×4 matrix A and expand it via dyadic expansion

4) Create a Hermitian matrix $A_{3\times3}$ and verify that

a) $\lambda(A) \in \mathbb{R}$ and $x_i \perp x_k$ $i \neq k$

b) Calculate left evecs as well: are they related somehow to the right evecs?

5) Jordan form

a) What is it?

b) Jordan blocks: what are they & how to find

c) Solve problems 4 & 6 LAUB p93

Hint: read section 9.2: thm 9.22 part 2 (Real Jordan)
read section 9.3.1 & Example 9.33 LAUB